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High Momentum Behavior of Geometric Bremsstrahlung in the Expanding Universe

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We discuss various kinds of geometric bremsstrahlung processes in the spatially flat Robertson-Walker universe. Despite the fact that the temperature of the universe is much higher than particle masses and the Hubble parameter, the transition probability of these processes does not vanish. It is also pointed out that explicit forms of the probability possess a new duality with respect to the scale factor of the background geometry.

§ 1. Introduction

Particles in the early universe undergo severe redshift as a result of the cosmic expansion and become decelerated extraordinarily in comoving coordinates. Consequently radiation or massless particles may be emitted from the decelerated particle. This process induced by the geometry of the universe is regarded as a kind of bremsstrahlung. We call this a geometric bremsstrahlung. This effect may bring about many sorts of decay and emission processes which are prohibited kinematically in flat spacetime.

Despite its ability to realize such a process in the classical mechanical sense, there is a nontrivial aspect of the existence of quantum geometric bremsstrahlung. Temperature in many interesting situations of the early universe is much higher than particle masses and the Hubble parameter. Since the momenta of the particles are comparable with the temperature, one might naively think that we can neglect all the mass parameters even in calculation of the transition probabilities. Hence it might be expected that the probability of the process is equal to that calculated in massless theories in the flat spacetime, and thus exactly zero. However, this naive expectation turns out to be wrong under careful analysis.

In Ref. 1), the first precise analysis was performed in the four dimensional Robertson-Walker universe. The authors of that paper showed that even in the high momentum limit there remains the nonvanishing probability of photon emission via geometric bremsstrahlung.

In this paper we give an extended analysis of several kinds of processes of geometric bremsstrahlung. This includes analyses of the ϕ^3 theory in arbitrary dimensional spacetimes, the theory with a Yukawa interaction, and the massive vector field theory. It will be shown that the high momentum limit, or the high temperature limit, does not result in the termination of the geometric bremsstrahlung process in a rather wide class of interactions and in arbitrary dimension. It is also stressed that a new type of duality can be found in the forms of the transition probabilities for renormalizable interactions.

In § 2 we introduce the spatially flat Robertson-Walker universe with past and future asymptotic flat regions and explain why we consider it in particular. In § 3 we give a review on free fields in our model of the universe. In § 4 the geometric bremsstrahlung induced by the Yukawa interaction is analyzed in detail. In § 5 the ϕ^3 theory in arbitrary dimensional spacetimes is surveyed. In § 6 we give explicit forms of the transition probability of the process including a massive vector field.

§ 2. Robertson-Walker universe with past and future Minkowskian regions

Here let us comment on our model of the Robertson-Walker universe. In this paper only the spatially flat model is discussed, but we believe that our results can be extended to open and closed models. The geometry of the spacetime is expressed by a metric tensor whose form is written as $g_{\mu\nu}=a(t)^2\eta_{\mu\nu}$, where $a(t)$ is a scale factor function and $\eta_{\mu\nu}$ is the Minkowskian metric. The argument of the scale factor, t, is conformal time, and proper time τ can be defined by $d\tau=a(t)dt$. From the metric, the Christoffel symbol is easily manipulated as follows:

$$\Gamma^{\alpha}_{\beta\gamma}=\delta_{\beta}{}^{\alpha}\partial_{\gamma}\ln a+\delta_{\gamma}{}^{\alpha}\partial_{\beta}\ln a-\eta_{\beta\gamma}\partial^{\alpha}\ln a\,, \tag{1}$$

where $\partial^{\alpha}=\eta^{\alpha\beta}\partial_{\beta}$. This yields an explicit form of the scalar curvature

$$R=-\frac{1}{a^2}\left[2(n-1)\frac{d^2}{dt^2}\ln a+(n-1)(n-2)\left(\frac{d}{dt}\ln a\right)^2\right], \tag{2}$$

where n is the dimension of the spacetime. The Hubble parameter is defined as usual by the scale factor as

$$H(t)=\frac{1}{a}\frac{da}{d\tau}=\frac{1}{a^2}\frac{da}{dt}\,.$$

In later sections we must take account of interactions in the expanding universe. Then a well-defined asymptotic free field is required to construct S matrix elements rigorously. However, in the construction of general specetimes, we often encounter some difficulties.[3)]

For example, the universe accompanied with the big-bang possesses a definite birth time and an initial singularity. Near the birth time more pieces of information (maybe quantum gravity) are required to specify how to define the asymptotic in-field.

The simplest prescription to avoid such problems is to restrict ourselves to analysis in the universe equipped with Minkowskian past and future regions. Clearly this enables us to define both asymptotic in- and out- fields just like those in the flat spacetime. It should be emphasized that physically meaningful results derived from this model must be independent of artificial detail of the model, that is, how the universe gets into and out of the expanding era. It will be found that results obtained later satisfy this criterion.

Owing to the appearance of asymptotic flat regions the scale factor must satisfy two constraints. By virtue of the time rescaling invariance, one of these constraints is expressed, without loss of generality, as

$$a(t\sim\infty)=1\,.$$

Now let b denote the ratio of initial scale factor to final scale factor. Then another constraint is written as

$$a(t\sim-\infty)=b\,.$$

It is useful to grasp the qualitative behavior of Fourier components of $a(t)^n$,

$$F_n(\omega)=\int_{-\infty}^{\infty}dta(t)^n e^{i\omega t}\,.$$

Now let $\omega_{\min}^{(n)}<\omega_1{}^{(n)}<\omega_2{}^{(n)}<\cdots<\omega_{\max}^{(n)}$ denote typical frequencies characterizing detailed evolution of $a(t)^n$ with the following condition satisfied:

$$F_n(\omega_i{}^{(n)})\sim O(1)\,.$$

It is natural to think that all the $\omega_i{}^{(n)}$ are of the order of the Hubble parameter in the expansion era.

For $\omega\ll\omega_{\min}^{(n)}$ it can be shown that

$$F_n(\omega\sim 0)\sim\frac{i}{\omega}(1-b^n)\,. \tag{3}$$

Note that this relation holds for rather arbitrary types of cosmic evolution.

On the other hand, for $\omega\gg\omega_{\max}^{(n)}$ F_n behaves as

$$F_n(\omega\sim\infty)\sim 0\,.$$

§ 3. Free fields in the Robertson-Walker universe

In this section free fields in the universe introduced in § 2 are reviewed for later convenience.

Let us first review the free scalar field in the expanding universe. The action with the conformal coupling term is written as

$$S_{\text{scalar}}=\int d^n x\sqrt{|g|}\left[\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}\left(m^2-\frac{n-2}{4(n-1)}R\right)\phi^2\right]. \tag{4}$$

Next let us change the field variable ϕ into $\tilde{\phi}=a(t)^{n/2-1}\phi$. Then the action is reduced to

$$S_{\text{scalar}}=\int d^n x\left[\frac{1}{2}(\partial\tilde{\phi})^2-\frac{1}{2}(ma(t))^2\tilde{\phi}^2\right]. \tag{5}$$

This is just a free field action with time dependent mass $ma(t)$ in the flat spacetime.

Boundary conditions for asymptotic fields can be described in terms of the rescaled field $\tilde{\phi}$. It is worthwhile to point out that spatially flat spacetimes possess isometry in the spatial section. Consequently, the Fourier transformation can be used in the equation of motion derived from Eq. (5). Therefore what we need is a solution whose form is such that

$$\tilde{\phi}(t,\boldsymbol{x})=u_{\boldsymbol{p}}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\,, \tag{6}$$

where $\boldsymbol{p}$ is the conserved conformal momentum. Its related physical momentum is expressed as $\boldsymbol{p}_{\rm phys}=\boldsymbol{p}/a(t)$. It is easily shown that this u_p satisfies a Schrödinger-type equation,

$$\left[-\frac{d^2}{dt^2}-m^2a(t)^2\right]u_p=p^2u_p\,, \tag{7}$$

where $p=|\boldsymbol{p}|$. The in-mode function of this equation is specified with the boundary condition

$$u_p{}^{\rm in}(t\sim-\infty)=\frac{1}{\sqrt{(2\pi)^{n-1}2\sqrt{p^2+m^2b^2}}}e^{-it\sqrt{p^2+m^2b^2}}\,. \tag{8}$$

On the other hand, the out-mode function satisfies another boundary condition,

$$u_p{}^{\rm out}(t\sim\infty)=\frac{1}{\sqrt{(2\pi)^{n-1}2\sqrt{p^2+m^2}}}e^{-it\sqrt{p^2+m^2}}\,. \tag{9}$$

The in-(out-) mode function may have reflection wave terms in $t\to\infty(t\to-\infty)$ induced by the nontrivial potential term $-m^2a(t)^2$. Usually such existence of the reflection wave represents particle creation from the vacuum state in the field theoretical context. Typical energy of the created particle is of the order of the typical Hubble parameter. Hence the energy can be thought to be much smaller than the temperature of the universe. This effect is of some interest, but it is not our target in this report. In fact we shall concentrate our attention on particles with high momentum nearly equal to the temperature. Consequently the reflection wave term is negligible in the following argument.

Now we can also exhibit results for a soluble example with a step scale factor $a(t)=b\Theta(-t)+\Theta(t)$. For example, the analytic form of the in-mode function is expressed as

$$u_p{}^{\rm in}(t<0)=\frac{1}{\sqrt{(2\pi)^{n-1}2\sqrt{p^2+m^2b^2}}}e^{-it\sqrt{p^2+m^2b^2}}\,,$$

$$u_p{}^{\rm in}(t>0)=\frac{1}{\sqrt{(2\pi)^{n-1}2\sqrt{p^2+m^2b^2}}}\left[Ae^{-it\sqrt{p^2+m^2}}+Be^{it\sqrt{p^2+m^2}}\right],$$

where

$$A=\frac{1}{2}\left(1+\sqrt{\frac{p^2+m^2b^2}{p^2+m^2}}\right), \tag{10}$$

$$B=\frac{1}{2}\left(1-\sqrt{\frac{p^2+m^2b^2}{p^2+m^2}}\right). \tag{11}$$

This will be used in § 4. As it should be, the reflection coefficient B vanishes as $p\to\infty$.

Here we also comment on the WKB amplitudes of Eq. (7). When the high momentum condition $p^3\gg m^2a(da/dt)$ or $p^3_{\rm phys}\gg m^2H$ is satisfied, the WKB approximation is sufficiently validated in Eq. (7). Then the reflection coefficient can be neglected, and both in- and out-mode functions are written in the same form as follows:

$$u_p \sim \frac{1}{\sqrt{(2\pi)^3 2E(p,t)}} e^{-i\int dt E(p,t)}, \tag{12}$$

where $E(p,t)=\sqrt{p^2+m^2a(t)^2}$. In the early universe situation, this form gives us a reliable estimation of the wavefunction.

Next let us review the free propagation of the spinor field. Writing down the action requires a vierbein $e_\mu{}^a$ related with the metric tensor like $g_{\mu\nu}=e_\mu{}^a e_{a\nu}$. For the spatially flat Robertson-Walker universe, the vierbein reads

$$e_\mu{}^a=a(t)\delta_\mu{}^a, \quad e_a{}^\mu=a(t)^{-1}\delta_a{}^\mu. \tag{13}$$

Then spin connection is obtained from this vierbein such that

$$\omega_\mu{}^{ab}=e_\lambda{}^a \nabla_\mu e^{b\lambda}=\delta_\mu{}^a\partial^b \ln a-\delta_\mu{}^b\partial^a \ln a. \tag{14}$$

The action of the free spinor field reads

$$S_{\rm spinor}=\int d^n x \det[e_\mu{}^a][\bar{\Psi} i\gamma^\mu \nabla_\mu \Psi - m\bar{\Psi}\Psi], \tag{15}$$

where

$$\gamma^\mu=e_a{}^\mu\gamma^a, \tag{16}$$

$$\{\gamma_a, \gamma_b\}=2\eta_{ab}, \tag{17}$$

$$\nabla_\mu\Psi=\left(\partial_\mu-\frac{i}{4}\omega_\mu{}^{ab}\sigma_{ab}\right)\Psi, \tag{18}$$

$$\sigma_{ab}=\frac{i}{2}[\gamma_a, \gamma_b]. \tag{19}$$

Subsequently, by defining a rescaled field $\tilde{\Psi}=a^{(n-1)/2}\Psi$ we rewrite the action as

$$S_{\rm spinor}=\int d^n x[\bar{\tilde{\Psi}} i\gamma^a\partial_a\tilde{\Psi}-ma(t)\bar{\tilde{\Psi}}\tilde{\Psi}], \tag{20}$$

where we have used the relations $\gamma^a\gamma_a=n$ and $\gamma^a\gamma^b\gamma_a=(2-n)\gamma^b$. Thus, like the scalar field, the theory can also be reduced into just a free theory with time dependent mass in the fiat spacetimes.

Next let us introduce mode functions more explicitly for $n=4$. Here we adopt the standard representation for the gamma matrices,

$$\gamma^0=\begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix}, \quad \boldsymbol{\gamma}=\begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix}. \tag{21}$$

Also we introduce $\alpha_{h,p}$ and $\beta_{h,p}$ defined by the equations

$$\tilde{\Psi}=e^{ip\cdot x}\begin{bmatrix} \alpha_{h,p}(t)\xi(h,\boldsymbol{p}) \\ \beta_{h,p}(t)\xi(h,\boldsymbol{p}) \end{bmatrix}, \quad \boldsymbol{\sigma}\cdot\boldsymbol{p}\xi(h,\boldsymbol{p})=h|\boldsymbol{p}|\xi(h,\boldsymbol{p}), \tag{22}$$

where $h=\pm 1$ and $h/2$ is helicity of the particle. Then $\alpha_{h,p}$ and $\beta_{h,p}$ satisfy equations such that

$$\beta_{h,p}(t)=\frac{1}{h|\boldsymbol{p}|}\left[i\frac{d}{dt}-ma(t)\right]\alpha_{h,p}(t)\,. \tag{23}$$

$$\left[\frac{d^2}{dt^2}+|\boldsymbol{p}|^2+m^2a(t)^2+im\frac{d}{dt}a\right]\alpha_{h,p}(t)=0\,. \tag{24}$$

The in- and out- mode functions for $\alpha_{h,p}$ are specified by imposing the boundary conditions

$$\alpha_{h,p}^{\mathrm{in}(\pm)}(t\sim-\infty)=\frac{\sqrt{\sqrt{p^2+m^2b^2}\pm mb}}{\sqrt{(2\pi)^3 2\sqrt{p^2+m^2b^2}}}e^{\mp it\sqrt{p^2+m^2b^2}}\,, \tag{25}$$

$$\alpha_{h,p}^{\mathrm{out}(\pm)}(t\sim\infty)=\frac{\sqrt{\sqrt{p^2+m^2}\pm m}}{\sqrt{(2\pi)^3 2\sqrt{p^2+m^2}}}e^{\mp it\sqrt{p^2+m^2}}\,, \tag{26}$$

where the sign $+(-)$ in $\alpha^{(\pm)}$ corresponds to the particle (antiparticle) wavefunction.

For a step evolution like $a(t)=b\Theta(-t)+\Theta(t)$, exact analysis is possible, and for example, the analytic in$(+)$-mode function is given by

$$\alpha_{h,p}^{\mathrm{in}(+)}(t<0)=\frac{\sqrt{\sqrt{p^2+m^2b^2}+mb}}{\sqrt{(2\pi)^3 2\sqrt{p^2+m^2b^2}}}e^{-it\sqrt{p^2+m^2b^2}}\,,$$

$$\alpha_{h,p}^{\mathrm{in}(+)}(t>0)=\frac{\sqrt{\sqrt{p^2+m^2b^2}+mb}}{\sqrt{(2\pi)^3 2\sqrt{p^2+m^2b^2}}}[A_f e^{-it\sqrt{p^2+m^2}}+B_f e^{it\sqrt{p^2+m^2}}]\,,$$

where

$$A_f=\frac{1}{2}\left(1+\sqrt{\frac{p^2+m^2b^2}{p^2+m^2}}+\frac{m(1-b)}{\sqrt{p^2+m^2}}\right), \tag{27}$$

$$B_f=\frac{1}{2}\left(1-\sqrt{\frac{p^2+m^2b^2}{p^2+m^2}}-\frac{m(1-b)}{\sqrt{p^2+m^2}}\right). \tag{28}$$

For the spinor field, the WKB approximation can also be justified when $p\gg ma$ or $p_{\mathrm{phys}}\gg m$. Then the following amplitude is obtained:

$$\alpha_{h,p}^{(\pm)}(t)\sim\frac{\sqrt{p\pm ma}}{\sqrt{(2\pi)^3 2p}}e^{\mp i(pt+(m^2/2p)\int dt a(t)^2)}\,. \tag{29}$$

In will also be useful in later sections to introduce U and V spinors corresponding to particle and antiparticle as follows:

$$U(h,\boldsymbol{p},a)=\begin{bmatrix}\sqrt{E(p,a(t))+ma}\,\xi(h,\boldsymbol{p})\\ h\sqrt{E(p,a(t))-ma}\,\xi(h,\boldsymbol{p})\end{bmatrix},$$

$$V(h,\boldsymbol{p},a)=\begin{bmatrix}-h\sqrt{E(p,a(t))-ma}\,\eta(h,\boldsymbol{p})\\ \sqrt{E(p,a(t))+ma}\,\eta(h,\boldsymbol{p})\end{bmatrix},$$

where $E(\boldsymbol{p},a(t))=\sqrt{p^2+m^2a(t)^2}$ and $\eta(h,\boldsymbol{p})=-i\sigma^2\xi^*(h,\boldsymbol{p})$. Using a polar parametrization such that

$$\boldsymbol{p}=p(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \tag{30}$$

the explicit forms of ξ and η are written as

$$\xi(1, \boldsymbol{p})=\begin{bmatrix} e^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ e^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \quad \xi(-1, \boldsymbol{p})=\begin{bmatrix} -e^{-i\phi/2}\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi/2}\cos\left(\frac{\theta}{2}\right) \end{bmatrix}, \tag{31}$$

$$\eta(1, \boldsymbol{p})=\begin{bmatrix} -e^{-i\phi/2}\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi/2}\cos\left(\frac{\theta}{2}\right) \end{bmatrix}, \quad \eta(-1, \boldsymbol{p})=\begin{bmatrix} -e^{-i\phi/2}\cos\left(\frac{\theta}{2}\right) \\ -e^{i\phi/2}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}. \tag{32}$$

Next let us give a review of the massive vector field in the four dimension. The original equation of motion is written as

$$\nabla^\mu F_{\mu\nu}+m^2 A_\nu=0\,, \tag{33}$$

where $F_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu=\partial_\mu A_\nu-\partial_\nu A_\mu$. Now using the conformal flatness $g_{\mu\nu}=a(t)^2\eta_{\mu\nu}$, the equation is rewritten as

$$[\partial^2+m^2a^2]A_\mu-\partial_\mu(\partial A)=0\,. \tag{34}$$

The transverse wave solution can be introduced such that

$$[A_\mu{}^{(h)}]=e^{ip\cdot x}\tilde{u}_p{}^{(h)}(t)\begin{bmatrix} 0 \\ -\boldsymbol{n}^{(h)} \end{bmatrix}, \tag{35}$$

where $h=\pm1$, and $\boldsymbol{n}^{(h)}$ is a unit vector satisfying $\boldsymbol{p}\cdot\boldsymbol{n}^{(h)}=0$. The equation of motion requires

$$\left[\frac{d^2}{dt^2}+p^2+m^2a(t)^2\right]\tilde{u}_p{}^{(h)}(t)=0\,.$$

On the other hand, the form of the longitudinal wave solution is

$$[A_\mu{}^{(L)}]=e^{ip\cdot x}\begin{bmatrix} \alpha^{(L)}(t) \\ -\frac{\boldsymbol{p}}{p}\beta^{(L)}(t) \end{bmatrix}. \tag{36}$$

Time dependent factors $\alpha^{(L)}$ and $\beta^{(L)}$ need to satisfy

$$\alpha^{(L)}=\frac{ip}{p^2+m^2a(t)^2}\frac{d\beta^{(L)}}{dt}\,. \tag{37}$$

Rescaling such that

$$\beta^{(L)}=\sqrt{\frac{p^2+m^2a^2}{m^2a^2}}\,\tilde{\beta}\,, \tag{38}$$

the following Schrödinger type equation also should hold:

$$\left[\frac{d^2}{dt^2}+p^2+m^2a^2+\frac{3m^2p^2}{(p^2+m^2a^2)^2}\left(\frac{da}{dt}\right)^2-\frac{p^2}{p^2+m^2a^2}\frac{1}{a}\frac{d^2a}{dt^2}\right]\tilde{\beta}(t)=0\,.$$

In the high momentum situation, $p\gg ma,\ da/adt$, the WKB solution takes the form

$$\alpha^{(L)}\sim\frac{1}{\sqrt{(2\pi)^3 2p}}\frac{p}{ma}\left(1-\frac{i}{pa}\frac{da}{dt}+O(p^{-2})\right)e^{-i\int dt[p+(m^2a^2/2p)-(1/2pa)(d^2a/dt^2)]}\,,$$

$$\beta^{(L)}\sim\frac{1}{\sqrt{(2\pi)^3 2p}}\frac{p}{ma}(1+O(p^{-2}))e^{-i\int dt[p+(m^2a^2/2p)-(1/2pa)(d^2a/dt^2)]}\,. \tag{39}$$

Thus the longitudinal component is estimated in the high momentum limit as

$$A_\mu{}^{(L)}\sim\frac{1}{\sqrt{(2\pi)^3 2p}}\left[i\partial_\mu\Lambda-\delta_{0\mu}\frac{ma}{p}\exp\left(i\boldsymbol{p}\cdot\boldsymbol{x}-i\int dt\left(p+\frac{m^2a^2}{2p}-\frac{1}{2pa}\frac{d^2a}{dt^2}\right)\right)\right], \tag{40}$$

where

$$\Lambda=\frac{1}{ma}e^{i\boldsymbol{p}\cdot\boldsymbol{x}-i\int dt[p+(m^2a^2/2p)-(1/2pa)(d^2a/dt^2)]}\,.$$

These solutions will be used in §§ 5 and 6.

§ 4. Yukawa interaction in the expanding universe

In this section we discuss the high energy limit of transition probabilities via Yukawa interaction in the four dimensional universe. The interaction is expressed by

$$S_{\text{Yukawa}}=\lambda\int d^4x\sqrt{-g}\,\phi(\bar{\Psi}_1\Psi_2+\bar{\Psi}_2\Psi_1)$$
$$=\lambda\int d^4x\tilde{\phi}(\bar{\tilde{\Psi}}_1\tilde{\Psi}_2+\bar{\tilde{\Psi}}_2\tilde{\Psi}_1)\,, \tag{41}$$

where $\Psi_1(\Psi_2)$ is a spinor field with mass m_1 (m_2) and ϕ is a scalar field with mass μ. The tilded fields are the rescaled field of § 2.

This action generates several types of processes. For a Ψ_1 particle with conformal momentum $\boldsymbol{p}$ and helicity 1/2 to decay into a Ψ_2 particle with $\boldsymbol{q}$ and a ϕ particle with $\boldsymbol{k}$, the amplitude is written as

$$\text{Amp}_{\Psi_1}=-i\lambda\int d^4x e^{i(\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{x}}u_{\boldsymbol{k}}{}^{\text{out}*}(t)\bar{\tilde{\Psi}}_2^{\text{out}}(h_f,\boldsymbol{q},t)\tilde{\Psi}_1^{\text{in}}(h_i=1,\boldsymbol{p},t)\,, \tag{42}$$

where $u_{\boldsymbol{k}}{}^{\text{out}}(t)$ is an out-mode function of $\tilde{\phi}$, $\tilde{\Psi}^{\text{in/out}}$ is an in-/out- mode function of $\tilde{\Psi}$, and $h_f/2$ is the helicity of the created spinor particle. In Eq. (42), we have omitted the contribution from the Bogoliubov coefficients of the vacuum polarization which is suppressed in the high temperature limit. In the following argument we assume that $m_1<m_2+\mu$, so no decay occurs in the flat spacetime limit. Even in the expanding spacetime, the decay is prohibited, particularly in the past and future flat regions. Thus the decay is physically interpreted to occur only in the era of the cosmic

expansion.

We expect that this assumption, $m_1 < m_2 + \mu$, is not essential for our results when the decay time we obtain is much shorter than that calculated ordinarily in the flat spacetimes, $E_1/(m_1 \Gamma_{\rm flat})$.

In the expanding universe, its scale factor is actually time dependent. Hence the meaning of probability *per unit time* seems ambiguous. Therefore we adopt the transition probability itself to obtain a clear interpretation. The transition probability is defined by

$$\Sigma|S|^2 = \int \frac{L^3 d^3 k}{(2\pi)^3} \frac{L^3 d^3 q}{(2\pi)^3} \frac{1}{N^{\rm out} N^{\rm in} N_\phi} |{\rm Amp}_{\Psi_1}|^2 , \tag{43}$$

where

$$N^{\rm in/out} = \int_{L^3} dx^3 \bar{\tilde{\Psi}}^{\rm in/out} \gamma^0 \tilde{\Psi}^{\rm in/out} , \tag{44}$$

$$N_\phi = \int_{L^3} dx^3 i (u_k^* \partial_t u_k - \partial_t u_k^* u_k) . \tag{45}$$

Now our purpose is to analyze the high momentum limit ($p \to \infty$) of $\Sigma|S|^2$. As a warm-up, let us first calculate a simple case with the scale factor $a(t) = b\Theta(-t) + \Theta(t)$. It is possible to calculate the limit straightforwardly. Substituting the explicit form of mode functions into the definition of $\Sigma|S|^2$, we obtain

$$\begin{aligned} & W_{\Psi_1}^{(\rm step)}(1/2 \to h_f/2) \\ & = \lim_{p\to\infty} \Sigma|S|^2 \\ & = \lim_{p\to\infty} \frac{\lambda^2}{64\pi^3} \int d^3 q \\ & \times \left| \int_0^\infty dt \frac{e^{-it(\sqrt{p^2+m_1^2} - \sqrt{q^2+m_2^2} - \sqrt{(p-q)^2+\mu^2})}}{[(p^2+m_1^2)(q^2+m_2^2)((\boldsymbol{p}-\boldsymbol{q})^2+\mu^2)]^{1/4}} \bar{U}_2(h_f, \boldsymbol{q}, a=1) U_1(1, \boldsymbol{p}, a=1) \right. \\ & \left. + \int_{-\infty}^0 dt \frac{e^{-it(\sqrt{p^2+m_1^2 b^2} - \sqrt{q^2+m_2^2 b^2} - \sqrt{(p-q)^2+\mu^2 b^2})}}{[(p^2+m_1^2 b^2)(q^2+m_2^2 b^2)((\boldsymbol{p}-\boldsymbol{q})^2+\mu^2 b^2)]^{1/4}} \bar{U}_2(h_f, \boldsymbol{q}, a=b) U_1(1, \boldsymbol{p}, a=b) \right|^2 , \end{aligned}$$

where we have used the fact that the reflection component of the wavefunction vanishes in the limit. After some tedious manipulation the final analytic forms of W turns out to be as follows:

$$W_{\Psi_1}^{(\rm step)}(1/2 \to h_f = -1/2) = \frac{\lambda^2}{32\pi^2} \left[\frac{1+b^2}{1-b^2} \ln \frac{1}{b^2} - 2 \right] , \tag{46}$$

$$W_{\Psi_1}^{(\rm step)}(1/2 \to h_f = 1/2) = \frac{\lambda^2}{8\pi^2} \left[1 + \frac{b \ln b^2}{1-b^2} \right] F(m_1, m_2, \mu) , \tag{47}$$

where

$$F(m_1, m_2, \mu) = \int_0^1 dy \frac{(1-y)(m_1 y + m_2)^2}{\mu^2 y + m_2^2 (1-y) - m_1^2 y (1-y)} . \tag{48}$$

Next we attempt to obtain W for arbitrary $a(t)$, excluding $a(\infty)=1$ and $a(-\infty)=b$. Our strategy comes from the fact that approximate energy conservation holds for high momentum reactions, as seen below. We now concentrate on manipulating the contribution from the phase space region where $p\gg ma$ and q and $k=|\boldsymbol{p}-\boldsymbol{q}|$ are of order of p. As a result of this restriction the WKB amplitude can be justified for both initial and final mode functions. Consequently, W turns out to possess a factor like

$$\Delta(h_f)=\int_{-\infty}^{\infty}dt\bar{U}_2(h_f,\boldsymbol{q},a(t))U_1(1,\boldsymbol{p},a(t))$$
$$\times\frac{\exp[i\int dt(\sqrt{q^2+m_2{}^2a(t)^2}+\sqrt{(\boldsymbol{p}-\boldsymbol{q})^2+\mu^2a(t)^2}-\sqrt{p^2+m_1{}^2a(t)^2})]}{[(p^2+m_1{}^2a(t)^2)(q^2+m_2{}^2a(t)^2)((\boldsymbol{p}-\boldsymbol{q})^2+\mu^2a(t)^2)]^{1/4}}\,. \tag{49}$$

In case $h_f=1$ it is very suggestive to roughly estimate it by neglecting masses as follows:

$$\Delta(h_f=1)\sim f(\boldsymbol{p},\boldsymbol{q})\times\int dte^{it(q+|\boldsymbol{p}-\boldsymbol{q}|-p)}$$
$$=f(\boldsymbol{p},\boldsymbol{q})\times(2\pi)^3\delta(q+|\boldsymbol{p}-\boldsymbol{q}|-p)\,. \tag{50}$$

Therefore the following relation must be satisfied at least to this leading order:

$$p\sim q+|\boldsymbol{p}-\boldsymbol{q}|=q+\sqrt{(p-q)^2+2pq(1-\cos\theta)}\,, \tag{51}$$

where $\boldsymbol{p}\cdot\boldsymbol{q}=pq\cos\theta$. This is an approximate energy conservation law and an important clue for us to calculate W_{Ψ_1}. From this "conservation" law, only the phase space region where $0<q<p$ and $\theta\sim0$ hold can contribute to W_{Ψ_1}. This fact tempts us to introduce a tiny but unspecified constant $\theta_o\ll1$ and a small constant ϵ satisfying m_1, m_2, $\mu\ll p\epsilon\ll p$. Now consider only a portion of the phase space with $p\epsilon\le q\le p(1-\epsilon)$ and $0\le\theta\le\theta_o$. Consequently the following expansions are valid:

$$\sin\theta\sim\theta\,,$$
$$1-\cos\theta\sim\frac{1}{2}\theta^2\,.$$

Furthermore, as a result of the high momentum limit, we can expand

$$\sqrt{p^2+m_1{}^2a^2}\sim p+\frac{m_1{}^2a^2}{2p}\,, \tag{52}$$

$$\sqrt{q^2+m_2{}^2a^2}\sim q+\frac{m_2{}^2a^2}{2q}\,, \tag{53}$$

$$\sqrt{(\boldsymbol{p}-\boldsymbol{q})^2+\mu^2a^2}\sim p-q+\frac{pq\theta^2+\mu^2a^2}{2(p-q)}\,. \tag{54}$$

From these useful expressions, we obtain for arbitrary $a(t)$

$$\lim_{p\to\infty}\sum|S(h_f=1)|^2=W_{\Psi_1}(h_f=1)\,,$$

$$W_{\Psi_1}=\lim_{p\to\infty}\frac{\lambda^2}{32\pi^3}\int_{p\epsilon}^{p(1-\epsilon)}dq\frac{q^2}{(p-q)}\int_0^{\theta_0}d\theta\theta^3$$
$$\times\left|\int_{-\infty}^{\infty}dte^{-i\int_0^t dt((m_1{}^2a(t)^2/2p)-(m_2{}^2a(t)^2/2q)-(pq\theta^2+\mu^2a(t)^2)/2(p-q))}\right|^2.$$

Next change the integral variables as follows:

$$q=py\,, \tag{55}$$

$$\theta=\frac{m_1}{p}z\,, \tag{56}$$

$$t=\frac{2p}{m_1{}^2}\eta\,. \tag{57}$$

Then the p dependence appears only in the upper bound of z and the argument of the scale factor. In fact, the result is expressed such that

$$W_{\Psi_1}(1/2\to1/2)=\lim_{p\to\infty}\frac{\lambda^2m_1{}^2}{8\pi^2}\int_{\epsilon}^{1-\epsilon}dy\frac{y^2}{1-y}\int_0^{\theta_0(p/m_1)}dzz^3$$
$$\times\left|\int_{-\infty}^{\infty}d\eta\exp\left[\frac{1}{i}\int_0^{\eta}d\eta\left(a^2-\frac{m_2{}^2a^2}{m_1{}^2y}-\frac{m_1{}^2yz^2+\mu^2a^2}{m_1{}^2(1-y)}\right)\right]\right|^2. \tag{58}$$

To evaluate W_{Ψ_1} the following relation is worth proving:

$$\lim_{p\to\infty}a=\lim_{p\to\infty}a\left(\frac{2p\eta}{m_1{}^2}\right)=b\Theta(-\eta)+\Theta(\eta)\,. \tag{59}$$

Actually, in the limit, $2p\eta/m_1{}^2$ approaches ∞ when $\eta>0$, while $-\infty$ when $\eta<0$. From the fact that $a(\infty)=1$ and $a(-\infty)=b$, one can derive easily Eq. (59). By substituting (59) into (58) we can proceed with calculation of W_{Ψ_1}.

Because we take $p\to\infty$, θ_0p/m can be replaced by ∞. Hence no dependence of θ_0 remains in the final form of W_{Ψ_1}. Moreover, we can take $\epsilon\to0$ because there is no appearance of infra-red divergence in the integral. These replacements yield

$$W_{\Psi_1}(1/2\to1/2)=\frac{\lambda^2m_1{}^2}{8\pi^2}\int_0^1dy(1-y)\int_0^{\infty}dzz^3$$
$$\times\left[\frac{1}{z^2+A(m_1;\,m_2,\mu,y)}-\frac{1}{z^2+b^2A(m_1;\,m_2,\mu,y)}\right]^2,$$

where

$$A(m_1;\,m_2,\mu,y)=\frac{1}{y^2}\left[\frac{m_2{}^2}{m_1{}^2}(1-y)+\frac{\mu^2}{m_1{}^2}y-y(1-y)\right]. \tag{60}$$

After integration with respect to y and z, the right-hand side ends up with the same form of $W^{(\mathrm{step})}$:

$$W_{\Psi_1}(h_1=1)=W_{\Psi_1}^{(\mathrm{step})}(h_f=1)\,. \tag{61}$$

It is a prominent feature that this relation (61) holds for arbitrary $a(t)$. This property implies the existence of notable universality of $W_{\Psi_1}(h_f=1)$.

For $h_f=-1$ the story changes somewhat. The delta factor in Eq. (49) can be estimated roughly as

$$\Delta(h_f=-1)\sim f'(\boldsymbol{p},\boldsymbol{q})\times\int dta(t)e^{it(q+|\boldsymbol{p}-\boldsymbol{q}|-p)}, \tag{62}$$

$$\sim f'(\boldsymbol{p},\boldsymbol{q})F_1(\Delta E), \tag{63}$$

where $\Delta E=q+|\boldsymbol{p}-\boldsymbol{q}|-p$ and

$$F_1(\Delta E)=\int dta(t)e^{it\Delta E}.$$

This scale factor contribution comes from

$$\bar{U}_2(-1,\boldsymbol{q},a(t))U_1(1,\boldsymbol{p},a(t))\sim\sqrt{pq}\left(\frac{m_1}{p}+\frac{m_2}{q}\right)a(t). \tag{64}$$

If $0<q<p$ and $\theta\ll1$ do not hold, $\Delta E\sim O(p)\gg\omega_{\max}^{(1)}$ is inevitably satisfied. Therefore, as mentioned in § 2, $F_1\sim0$ holds. Thus approximate energy conservation becomes valid again:

$$0<q<p, \tag{65}$$

$$\theta\sim0. \tag{66}$$

In this case it should be noted that θ is not so large that ΔE becomes larger than $\omega_{\max}^{(1)}$. Only the phase space region with

$$\theta<\theta_o=\sqrt{\frac{\omega_{\max}^{(1)}}{p}} \tag{67}$$

contributes to W_{Ψ_1}. Estimation of W_{Ψ_1} for $h_f=-1$ is also possible taking Eq. (67) into account. Thanks to the high momentum limit, the scale factor can be replaced the step factor as $a(t)=b\Theta(-t)+\Theta(t)$. After manipulation we can show

$$W_{\Psi_1}(h_f=-1)=W_{\Psi_1}^{(\mathrm{step})}(h_f=-1).$$

Consequently Eqs. (46) and (47) turn out to be correct for arbitrary evolution of $a(t)$. Equation (59) is the key point giving birth of the universality. However, in the actual universe p does not take an infinite value, but remains finite and is of the order of the temperature. It should be noted that for a large but finite value of p the replacement of $a(t)$ into the step evolution is valid only when a high momentum condition like

$$\omega_{\min}>\frac{m^2}{p} \tag{68}$$

is satisfied. Here $\omega_{\min}$ denotes the lowest typical frequency of $a(t)$ and is assumed to be of the same order as the minimum value of the Hubble parameter. The condition (68) enables us to regard $a(t)$ as $b\Theta(-t)+\Theta(t)$. The inequality (68) can also be rewritten as

$$\frac{1}{m} > \left[\frac{m}{p}\right] \frac{1}{\omega_{\min}}, \tag{69}$$

where $1/m$ represents the Compton length and $1/\omega_{\min}$ denotes the maximum radius of the Hubble horizon. The factor $m/p \sim m/E$ can be interpreted as the Lorentz contraction factor. Imagine a particle with Compton length $1/m$ running with high momentum p in the expanding universe. Also suppose that the universe begins to expand when the particle reaches a point A and that the universe ceases to expand when the particle arrives at a point B. The length between A and B can be naively considered of the order of the maximum Hubble horizon ($\sim 1/\omega_{\min}$). Then the particle can receive excitation energy from the gravitational field only while running between A and B. Meanwhile as p becomes larger, the length between A and B becomes Lorentz contracted as $[m/p](1/\omega_{\min})$ from the particle's viewpoint. Thus when the relation in Eq. (69) holds, the particle cannot see details of the way in which the universe has expanded. Consequently, this yields the above universality.

The universality is actually a very powerful tool to estimate the high momentum limit of the transition probability. It can be shown that the universality also appears in other kinds of interactions and in any dimension of spacetime. However, it should be noted that we must take some care of its treatment when the probability grows to infinity as $p \to \infty$. Using the universality we can naively manipulate the limit explicitly. However, it might diverge. In the next section such a phenomenon can be observed explicitly, and then we should cut p off at a certain large value of the order of the temperature of the universe.

It is a rather straightforward application to estimate ϕ particle decay probability into a Ψ_1 particle with helicity h_1 and a Ψ_2 particle with helicity h_2. Here we must assume that $\mu < m_1 + m_2$ in order to suppress the transition in the flat spacetime. The final forms of the transition probability are listed as follows:

$$W_\phi(h_1=1, h_2=1) = \frac{\lambda^2}{16\pi^2}\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2} - 2\right], \tag{70}$$

$$W_\phi(h_1=1, h_2=-1) = \frac{\lambda^2}{8\pi^2}\left(1 + \frac{b}{1-b^2}\ln b^2\right) G(\mu, m_1, m_2), \tag{71}$$

where

$$G(\mu, m_1, m_2) = \int_0^1 dy \frac{m_1^2(1-y)^2 - 2m_1 m_2 y(1-y) + m_2^2 y^2}{m_1^2(1-y) + m_2^2 y - \mu^2 y(1-y)}. \tag{72}$$

Interestingly, it is found that all of these forms of the probabilities possess a certain kind of dual symmetry. Changing b to $1/b$ leaves the forms in Eqs. (46), (47), (70), (71) unchanged. This implies that physics of the expanding universe is related with that of the contracting universe. Note that the symmetry is not merely a time reversal summetry, because initial one particle decays into two particles for both cases. This might be related with a kind of hidden duality.

Finally we discuss the decay rate of the particles. For example, let us consider the decay of the Ψ_1 particle. The decay is expected to occur when $W_{\Psi_1} \sim 1$. For small b, W_{Ψ_1} behaves just as

$$W_{\Psi_1} \sim \frac{N^* \lambda^2}{32\pi^2} \ln \frac{1}{b^2}, \tag{73}$$

where N^* is the number of final modes which contribute to the decay. Thus, when the universe expands enough so that

$$\frac{a_f}{a_i} = \frac{1}{b} \sim e^{(16\pi^2/N^*\lambda^2)} \tag{74}$$

is satisfied, the particle decays. Assuming a radiation dominant universe, b can satisfy

$$b = \frac{a_i}{a_f} = \sqrt{\frac{\tau_i}{\tau_f}}. \tag{75}$$

Consequently, we obtain the decay rate,

$$\Gamma_f = \frac{1}{\tau_f} \sim 2e^{-(32\pi^2/N^*\lambda^2)} H_i, \tag{76}$$

where $H_i = 1/2\tau_i$ is the Hubble parameter at production time of the Ψ_1 particle.

§ 5. Decay due to three point vertex in arbitrary dimension

The geometric bremsstrahlung process is naturally expected to occur in arbitrary dimensional spacetimes. Furthermore, its exitence is not considered to depend on whether couplings of the reaction have dimension or are dimensionless. Here we give a rather simple example showing us this feature. Let us consider the ϕ^3 theory with mass m in the n-dimensional spacetimes. The action reads

$$S = \int d^n x \sqrt{|g|} \left[\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} \left(m^2 - \frac{n-2}{4(n-1)} R \right) \phi^2 - \frac{1}{3!} \lambda \Lambda^{3-n/2} \phi^3 \right], \tag{77}$$

where we have introduced a mass parameter Λ. Hence the coupling constant λ is dimensionless. In $n \le 6$ the vertex is merely a renormalizable interaction. Meanwhile, in $n > 6$, the interaction is not renormalizable, and then Λ represents some cutoff scale of the theory. The transition probability of a particle decay with conformal momentum p is straightforwardly given at the tree level as

$$\Sigma |S|^2 = (2\pi)^{2(n-1)} \lambda^2 \Lambda^{6-n} \int d^{n-1} q \left| \int dt a^{3-n/2} u_q^* u_{|p-q|}^* u_p \right|^2.$$

It turns out soon that this expression itself converges for arbitrary fixed p. However, taking $p \to \infty$, this grows to infinity for $n > 6$. This comes from the fact that the vertex is nonrenormalizable for $n > 6$. However, there exists no trouble in the real universe, because p does not take an infinite value but that of the order the universe temperature. Thus in the following argument we treat p finite but large compared with m and the Hubble parameter. Thus the probability is also finite. To obtain a useful lower bound of the probability in $p \gg m$, we again pick up only the contribution from the phase space region with $0 \le \theta \le \theta_o \ll 1$ and $p\epsilon \le q \le p(1-\epsilon)$. This implies that

$$\Sigma|S|^2 \geq W=(2\pi)^{2(n-1)}\lambda^2\Lambda^{6-n}\frac{2\pi^{n/2-1}}{\Gamma(n/2-1)}$$
$$\times\int_{p\epsilon}^{p(1-\epsilon)}dqq^{n-2}\int_0^{\theta_0}d\theta\sin^{n-3}\theta\left|\int dta^{3-n/2}u_q^* u_{|p-q|}^* u_p\right|^2 .$$

For $n\leq 6$ other phase space contributions vanishes in the high momentum limit and W gives us the probability itself, not merely a lower bound.

Here assuming that $p\epsilon\gg m$, we can use the WKB wavefunctions for both in- and out- mode functions. Replacing the integral variables like

$$q=py\,, \tag{78}$$

$$\theta=\frac{m}{p}z\,, \tag{79}$$

$$t=\frac{2p}{m^2}\eta\,, \tag{80}$$

we obtain

$$W=\lambda^2\left(\frac{m}{\Lambda}\right)^{n-6}\frac{\pi^{n/2-1}}{(2\pi)^{n-1}\Gamma(n/2-1)}\int_\epsilon^{1-\epsilon}dy\frac{y^{n-3}}{1-y}\int_0^{\theta_0(p/m)}dzz^{n-3}$$
$$\times\left|\int d\eta a^{3-n/2}e^{i\int d\eta(a^2(1/y)+(yz^2+a^2)/(1-y)-a^2)}\right|^2 .$$

By imposing $m^2/p\ll\omega_{\min}^{(3-n/2)}$ the replacement $a=b\Theta(-\eta)+\Theta(\eta)$ is again validated. Then the integration with respect to η can be easily performed, and we obtain

$$W=\lambda^2\left(\frac{m}{\Lambda}\right)^{n-6}\frac{\pi^{n/2-1}}{(2\pi)^{n-1}\Gamma(n/2-1)}\int_\epsilon^{1-\epsilon}dyy^{n-5}(1-y)$$
$$\times\int_0^{\theta_0(p/m)}dzz^{n-3}\left[\frac{1}{z^2+C}-\frac{b^{3-n/2}}{z^2+b^2C}\right]^2, \tag{81}$$

where $C=A(m;\, m,\, m,\, y)=(1-y+y^2)/y^2$.

Unfortunately, for $n=2$ this expression is not suitable, and we must calculate separately. However, the estimation is also possible analytically, and it results in simply

$$W(n=2)=0\,.$$

From $n=3$ to $n=6$ we can take $p\to\infty$ and $\epsilon\to 0$ in Eq. (81). The results are as follows:

$$W(n=3)=\frac{\lambda^2}{8\pi}\left(\frac{4}{3}-\ln 3\right)\left(\frac{\Lambda}{m}\right)^3\left[1-\frac{2\sqrt{b}}{1+b}\right],$$

$$W(n=4)=\frac{\lambda^2}{8\pi^2}\left(\frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)-1\right)\left(\frac{\Lambda}{m}\right)^2\left[1+\frac{b}{1-b^2}\ln b^2\right],$$

$$W(n=5)=\frac{\lambda^2}{32\pi^2}\left(\frac{5}{4}\ln 3-1\right)\frac{\Lambda}{m}\left[1-\frac{2\sqrt{b}}{1+b}\right],$$

$$W(n=6)=\frac{\lambda^2}{384\pi^3}\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right].$$

Note that all of these expressions are invariant under the transformation $b\to 1/b$.

As mentioned previously, W diverges for $n>6$ in $p\to\infty$. Keeping $\omega_{\min}^{(3-n/2)}>m^2/p$ in mind, a lower bound with $\theta_o=m/p$ is obtained from Eq. (81). The explicit form with $b\sim 0$ is as follows:

$$W(n>6,\, b\sim 0)\geq\frac{\pi^{n/2-1}}{(2\pi)^{n-1}\Gamma(n/2-1)}\frac{\lambda^2}{(n-3)(n-4)(n-6)}\left(\frac{m}{\Lambda b}\right)^{n-6}.$$

Therefore after the universe expands sufficiently to satisfy

$$1/b\sim O\left(\lambda^{-2/(n-6)}\frac{\Lambda}{m}\right),$$

the particle decays via the geometric bremsstrahlung.

After all we arrive at the conclusion that in spite of the high temperature of the universe, the transition probability due to the Yukawa geometric bremsstrahlung does not vanish even in the n-dimensional spacetimes, except for $n=2$.

§ 6. Decay process including massive gauge field

In this section we survey decay processes including a massive vector particle A_μ with mass μ.

Let us first consider this particle to interact with fermions Ψ_1 and Ψ_2 with mass m_1 and m_2. The interaction term can be expressed as

$$S_{\Psi\Psi A}=g\int d^4x\sqrt{-g}\,\overline{\Psi}_1(c_V+c_A\gamma^5)\gamma^\mu\Psi_2 A_\mu+\text{c.c.}$$
$$=g\int d^4x\overline{\tilde{\Psi}}_1(c_V+c_A\gamma^5)\gamma^a\tilde{\Psi}_2\tilde{A}_a+\text{c.c.}\,, \tag{82}$$

where $A_\mu=e_\mu{}^a A_a=a\delta_\mu{}^a A_a$ and $A_a=a^{-1}\tilde{A}_a$. In the high momentum limit we can use again the WKB approximation of § 2. This enables us to calculate W explicitly.

Let us first consider decay of a transverse component of A_μ possessing conformal momentum p into Ψ_1 and $\overline{\Psi}_2$ particles. Just as the Yukawa interaction case, we take into account only contributions from restricted phase space region as follows:

$$W_T(h,\bar{h})=\lim_{p\to\infty}\frac{g^2}{32\pi^2}\int_{p\epsilon}^{p(1-\epsilon)}dk\frac{k}{p(p-k)}\int_0^{\theta_0}d\theta\theta$$
$$\times\left|\int dt\bar{U}_1(h,\boldsymbol{p}-\boldsymbol{k},a)(c_V+c_A\gamma^5)\gamma^\mu V_2(\bar{h},\boldsymbol{k},a)\epsilon_\mu{}^{(T)}(\boldsymbol{p},a)\right.$$
$$\left.\times e^{-i\int dt(\mu^2a^2/2p-(pk\theta^2+m_1{}^2a^2)/2(p-k)-m_2{}^2a^2/2k)}\right|^2,$$

where

$$\epsilon_\mu{}^{(\pm)}(\boldsymbol{p},a)=\pm\frac{1}{\sqrt{2}}[0,1,\pm i,0]\,. \tag{83}$$

For any combination of $(h, \bar{h})$, it can be straightforwardly shown that W_T converges as $p\to\infty$ and $\epsilon\to 0$. After a straightforward manipulation we obtain the following results:

$$W_T(h=1,\ \bar{h}=1)=\frac{g^2}{4\pi^2}\left(1+\frac{b}{1-b^2}\ln b^2\right)H(m_1,\ m_2,\ \mu)\,,$$

where

$$H=\int_0^1 dy\frac{|c_V+c_A|^2 m_1{}^2 y^2+|c_V-c_A|^2 m_2{}^2(1-y)^2+2(|c_V|^2-|c_A|^2)m_1 m_2 y(1-y)}{m_1{}^2 y+m_2{}^2(1-y)-\mu^2 y(1-y)}\,,$$

$$W_T(h=-1,\ \bar{h}=-1)=0\,,$$

$$W_T(h=1,\ \bar{h}=-1)=\frac{g^2}{24\pi^2}|c_V-c_A|^2\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right],$$

$$W_T(h=-1,\ \bar{h}=1)=\frac{g^2}{24\pi^2}|c_V+c_A|^2\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right]. \tag{84}$$

To obtain W for longitudinal component decay, we assume the photon helicity vector satisfies

$$\epsilon_\mu{}^{(L)}\sim\frac{k}{\mu a}[1,\,0,\,0,\,-1]\,. \tag{85}$$

After some manipulation, we find that W_L in fact converges with $h=1$ and $\bar{h}=-1$ as

$$W_L(h=1,\ \bar{h}=-1)=\frac{g^2}{8\pi^2}\left[1+\frac{b\ln b^2}{1-b^2}\right]K(m_1,\ m_2,\ \mu,\ c_V,\ c_A)\,,$$

where

$$K(m_1,\ m_2,\ \mu,\ c_V,\ c_A)$$
$$=\int_0^1 dy\frac{y(1-y)}{m_1{}^2 y+m_2{}^2(1-y)-\mu^2 y(1-y)}$$
$$\times\left|\frac{c_V(m_1-m_2)-c_A(m_1+m_2)}{\mu}m_1\sqrt{\frac{y}{1-y}}\right.$$
$$\left.-\frac{c_V(m_1-m_2)+c_A(m_1+m_2)}{\mu}m_2\sqrt{\frac{1-y}{y}}-2(c_V+c_A)\mu\sqrt{y(1-y)}\right|^2. \tag{86}$$

$$W_L(h=-1,\ \bar{h}=1)=\frac{g^2}{8\pi^2}\left[1+\frac{b\ln b^2}{1-b^2}\right]K(m_1,\ m_2,\ \mu,\ c_V,\ -c_A)\,. \tag{87}$$

$W_L(h=\pm 1,\ \bar{h}=\pm 1)$ is given as

$$W_L(h=\pm 1,\ \bar{h}=\pm 1)=\frac{g^2}{16\pi^2}\frac{|c_V(m_1-m_2)\pm c_A(m_1+m_2)|^2}{\mu^2}\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right].$$

It is also possible to estimate W when a Ψ_1 particle with momentum p and helicity 1/2 decays into a Ψ_2 particle and a A_μ particle. Substituting the wavefunction forms in § 2 into W, this is written down explicitly for emission of transverse component such that

$$W_{\Psi_1}(h_i;\, h_f,\, T)$$
$$=\lim_{p\to\infty}\frac{g^2}{32\pi^2}\int_{p\epsilon}^{p(1-\epsilon)}dk\frac{k}{p(p-k)}\int_0^{\theta_0}d\theta\theta$$
$$\times\left|\int dt\bar{U}_2(h_f,\, \boldsymbol{p}-\boldsymbol{k},\, a)(c_V+c_A\gamma^5)\gamma^\mu U_1(h_i,\, \boldsymbol{p},\, a)\epsilon_\mu^{*(T)}(\boldsymbol{k},\, a)e^{i\int dt\Delta E}\right|^2,$$

where

$$\Delta E=\frac{\mu^2a^2}{2k}+\frac{pk\theta^2+m_2{}^2a^2}{2(p-k)}-\frac{m_1{}^2a^2}{2p}.$$

For cases with $h_i=1$ and $h_f=1$ our WKB results include an infra-red divergence.

$$W_\Psi(1;\, 1,\, 1)=\frac{g^2}{8\pi^2}|c_V-c_A|^2\left(\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right)\int_\epsilon^1\frac{dy}{y}, \tag{88}$$

$$W_\Psi(1;\, 1,\, -1)=\frac{g^2}{8\pi^2}|c_V-c_A|^2\left(\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right)\int_\epsilon^1\frac{dy}{y}(1-y)^2, \tag{89}$$

In spite of the appearance in Eqs. (88) and (89) of the infra-red divergence with respect to $p\epsilon$, we believe that exact form of the probability converges as a result of the existence of natural infra-red cutoff μ. This cutoff works only in the low momentum region where the WKB approximation is invalid. Because we have discussed only the case where the WKB approximation is valid, our treatment should include such a superficial infra-red divergence. From this point of view, it can be naively expected that $\epsilon\sim\mu/p$.

However, if μ is exactly zero, no natural infra-red cutoff appears. This infra-red divergence really exists owing to the existence of the soft particle. The number of soft particles is counted only to the accuracy of observations. Thus for very soft particles, one cannot discriminate between virtual emission which contribute to the particle mass operator and real emission. This enables us to add a part of the mass operator contribution to the emission probability. Then we hope the theory will have no real infra-red divergence in the physical interpretation, similar to the flat spacetime case.[4]

For other cases of transverse emission, W converges, and final results are given by

$$W_{\Psi_1}(1;\, -1,\, 1)=\frac{g^2}{4\pi^2}\left[1+\frac{b}{1-b^2}\ln b^2\right]R(m_1,\, m_2,\, \mu), \tag{90}$$

where

$$R=\int_0^1 dy\frac{y(|c_V-c_A|^2m_2{}^2-2(|c_V|^2-|c_A|^2)m_1m_2(1-y)+|c_V+c_A|^2m_1{}^2(1-y)^2)}{m_2{}^2y+\mu^2(1-y)-m_1{}^2y(1-y)},$$

$$W_{\Psi_1}(1;\, -1,\, -1)=0.$$

For emission of the longitudinal component, we obtain

$$W_{\Psi_1}(h_i;\ h_f, L)$$
$$=\lim_{p\to\infty}\frac{g^2}{32\pi^2}\int_{p\epsilon}^{p(1-\epsilon)}dk\frac{k}{p(p-k)}\int_0^{\theta_0}d\theta\theta$$
$$\times\left|\int dt\bar{U}_2(h_f, \boldsymbol{p}-\boldsymbol{k}, a)(c_V+c_A\gamma^5)\gamma^\mu U_1(h_i, \boldsymbol{p}, a)\epsilon_\mu^{*(L)}(\boldsymbol{k}, a)e^{i\int dt(\Delta E+(1/2k)(da/adt)^2}\right|^2.$$

Note that $W_{\Psi_1}(1{:}\ 1, L)$ possesses an infra-red divergence with respect to $\epsilon\to 0$.

$$W_{\Psi_1}(1;\ 1, L)=\frac{g^2}{8\pi^2}\left(1+\frac{b\ln b^2}{1-b^2}\right)Q(m_1, m_2, \mu)\,,$$

where

$$Q=\int_\epsilon^1 dy\frac{y(1-y)}{\mu^2(1-y)+m_2{}^2y-m_1{}^2y(1-y)}$$
$$\times\left|\frac{c_V(m_1-m_2)+c_A(m_1+m_2)}{\mu}\frac{m_2}{\sqrt{1-y}}\right.$$
$$\left.+\frac{c_V(m_1-m_2)-c_A(m_1+m_2)}{\mu}m_1\sqrt{1-y}-2(c_V-c_A)\mu\frac{\sqrt{1-y}}{y}\right|^2, \tag{91}$$

and it is expected that $\epsilon\sim\mu/p$.

Finally, it can be shown that $W_\Psi(1;\ -1, L)$ converges and takes the following form:

$$W_\Psi(1;\ -1, L)=\frac{g^2}{32\pi^2}\left|\frac{c_V(m_1-m_2)-c_A(m_1+m_2)}{\mu}\right|^2\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right].$$

Let us next comment on a three-point interaction between the massive vector field with mass μ and complex scalar fields φ_1 and φ_2 with mass m_1 and m_2. The interaction is described by

$$S_{\bar{\varphi}\varphi A}=g\int d^4x\sqrt{-g}\,A_\mu i(\bar{\varphi}_1\nabla^\mu\varphi_2-\nabla^\mu\bar{\varphi}_1\varphi_2)+\text{c.c.}$$

We can also give in this case explicit forms of W in each process as follows.

For decay of the transverse component, we have

$$W^{(T)}=\frac{g^2}{48\pi^2}\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right],$$

and for the longitudinal component decay,

$$W^{(L)}=\frac{g^2}{8\pi^2\mu^2}\left(1+\frac{b\ln b^2}{1-b^2}\right)$$
$$\times\int_{-1/2}^{1/2}dx(1-4x^2)\frac{(m_1{}^2-m_2{}^2)^2+4\mu^2(m_1{}^2-m_2{}^2)+4\mu^4x^2}{2m_1{}^2+2m_2{}^2-\mu^2+4(m_1{}^2-m_2{}^2)x+4\mu^2x^2}\,.$$

For the transverse vector particle emission from a scalar particle, we have

$$W_{\varphi_1}^{(T)}=\frac{g^2}{8\pi^2}\ln\frac{1}{\delta}\left[\frac{1+b^2}{1-b^2}\ln\frac{1}{b^2}-2\right],$$

and for the longitudinal vector particle emission out of a scalar particle,

$$W_{\varphi_1}^{(L)}=\frac{g^2}{8\pi^2\mu^2}\left(1+\frac{b\ln b^2}{1-b^2}\right)\int_\delta^1\frac{dy}{y}\frac{(1-y)[(m_1{}^2-m_2{}^2)y+(2-y)\mu^2]^2}{\mu^2(1-y)+m_2{}^2y-m_1{}^2y(1-y)}$$

$$\sim\frac{g^2}{2\pi^2}\ln\frac{1}{\delta}\left(1+\frac{b\ln b^2}{1-b^2}\right).$$

Thus the probability of geometric bremsstrahlung does not vanish, even including the vector field.

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References

1) T. Futamase, M. Hotta, H. Inoue and M. Yamaguchi, Prog. Theor. Phys. **96** (1996), 113.
2) One can see a standard review of the early universe from the viewpoint of the high energy particle physics, for example, in E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, 1990).
3) N. D. Birrell, Ph. D Thesis, King's College, London (1979).
N. D. Birrell and J. G. Taylor, J. Math. Phys. **21** (1980), 1740.
N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, 1982).
4) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54.