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RESEARCH PAPER

HIGH-ORDER ALGORITHMS FOR RIESZ DERIVATIVE AND THEIR APPLICATIONS (III)

Hengfei Ding [1], Changpin Li [2,*]

Abstract

Numerical methods for fractional calculus attract increasing interest due to its wide applications in various fields such as physics, mechanics, etc. In this paper, we focus on constructing high-order algorithms for Riesz derivatives, where the convergence orders cover from the second order to the sixth order. Then we apply the established schemes to the Riesz type turbulent diffusion equation (or, Riesz space fractional turbulent diffusion equation). Numerical experiments are displayed which support the theoretical analysis.

MSC 2010: 65M06, 65M12

Key Words and Phrases: space fractional turbulent diffusion equation, high-order algorithms, stability analysis

1. Introduction

In recent years, fractional calculus has attracted increasing interests due to its applications in physics, mechanics, etc. For more details, see the recent publications [1, 2, 3, 5, 6, 7, 8, 9, 10, 14, 15, 17, 19, 20, 21, 22], and references cited therein. The Riemann-Liouville (R-L) derivative and Caputo derivative are commonly used, respectively defined below:

$$ {}_{RL}D_{a,x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-\xi)^{n-\alpha-1}f(\xi)d\xi,\ x\in(a,b), $$

and

$$ {}_{C}D_{a,x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\xi)^{n-\alpha-1}f^{(n)}(\xi)d\xi,\ x\in(a,b), $$

where $n-1<\alpha<n\in Z^{+}$.

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A linear combination of the left R-L derivative (defined above) and the right R-L derivative given as

$$ {}_{RL}D_{x,b}^{\alpha}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (\xi - x)^{n-\alpha-1}f(\xi)d\xi,\ x \in (a,b), $$

defines a new derivative, i.e., the Riesz derivative,

$$ \frac{d^{\alpha}f(x)}{d|x|^{\alpha}} = -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left({}_{RL}D_{a,x}^{\alpha} + {}_{RL}D_{x,b}^{\alpha}\right)f(x),\ \ n-1 < \alpha < n \in Z^+. $$

The left R-L derivative reflects the dependence on the history, while the right R-L derivative the dependence upon the future. So the Riesz derivative value of $f(x)$ at x relies on the whole space (a,b), but the values are different at different $x \in (a,b)$. From an angle of application, the case with $\alpha \in (0,2)$ is mostly attracted attention.

From the studies available, the high-order numerical algorithms for R-L derivatives were firstly established in [16]. The high-order algorithms for Caputo derivatives were firstly constructed in [12]. Shortly after, some other high-order algorithms for Caputo derivatives were also appeared, for example see [1, 3, 14] and references cited therein. And the high-order algorithms for Riesz derivatives were derived in [5, 6]. In [5], Ding et al. constructed the fourth-order schemes for the Riesz derivative and applied them to the space Riesz fractional diffusion equation. In that paper they established the fourth-order schemes for R-L derivatives [5] from which some similar fourth-order schemes were derived by other people. In [6], Ding et al. continued to establish the sixth-order, eighth-order, tenth-order and twelfth-order schemes for the Riesz derivative by using the Fourier analysis, where the fourth-order compact scheme for R-L derivative was especially highlighted. Then they used the two schemes, among them to the Riesz space fractional reaction-dispersion equation, where the rigorous error analysis was given. The other odd-order (third-order, fifth-order, seventh-order, ninth-order, eleventh-order) schemes can be established from [6] by choosing the different parameters therein. In spite of these, it is absolutely necessary to construct intuitionist and straightforward schemes for the Riesz derivatives. Here we find an interesting and enlightening way to establish high-order schemes (from 2nd-order to 6th-order) by using the corresponding generating functions [16]. Then we use these schemes to solve the Riesz space fractional turbulent diffusion equation.

In the following, we briefly introduce fractional modelling in this respect. From the known first Fick's law

$$ J(x,t) = d_1u(x,t) - d_2\frac{\partial u(x,t)}{\partial x},\ \ d_1 > 0,\ d_2 > 0, $$

one obtain the following advection-diffusion equation,

$$\frac{\partial u(x,t)}{\partial t}=-d_1\frac{\partial u(x,t)}{\partial x}+d_2\frac{\partial^2 u(x,t)}{\partial x^2}.$$

If the advection-diffusion process at any position $x\in(a,b)$ relies on the whole space (a,b) (i.e., long-range interactions), then the classical Fick's law does not work well yet. However, the fractional derivative can well characterize such long-range interactions. Now we generalize the typical Fick's law to a fractional version,

$$J_\alpha(x,t)=d_1u(x,t)-d_2\frac{\partial u(x,t)}{\partial x}-d_\alpha\frac{\partial^{\alpha-1}u(x,t)}{\partial|x|^{\alpha-1}},\quad \alpha\in(0,1),$$

where d_1, d_2, d_α are positive constants. It follows that the Riesz space fractional turbulent diffusion equation (or, Riesz type turbulent diffusion equation) is obtained,

$$\frac{\partial u(x,t)}{\partial t}=-d_1\frac{\partial u(x,t)}{\partial x}+d_2\frac{\partial^2 u(x,t)}{\partial x^2}+d_\alpha\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}.$$

If there is a source term, then one has

$$\frac{\partial u(x,t)}{\partial t}=-d_1\frac{\partial u(x,t)}{\partial x}+d_2\frac{\partial^2 u(x,t)}{\partial x^2}+d_\alpha\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}+s(x,t),\qquad(1.1)$$

where the Riesz partial derivative with order $\alpha\in(0,1)$ is given by

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}=-\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left({}_{RL}D^\alpha_{a,x}+{}_{RL}D^\alpha_{x,b}\right)u(x,t),\ 0<\alpha<1.\qquad(1.2)$$

Equation (1.1) is subject to the following initial value condition,

$$u(x,0)=\varphi(x),\ x\in[a,b],$$

and the boundary value conditions (here choosing the homogeneous condition for brevity),

$$u(a,t)=0,\ \ u(b,t)=0,\qquad t\geq 0.$$

The remainder of this paper is outlined as follows. In Section **2**, five kinds of high-order (2nd-order, 3rd-order, $\cdots$, 6th-order) algorithms for the Riesz derivatives are developed. In the next Section **3**, we apply the derived schemes to the Riesz-type turbulent diffusion equation (1.1). Here we only use the 2nd-order, 4th-order, 6th-order schemes to (1.1). The convergence orders are $\mathcal{O}(\tau^2+h^2)$, $\mathcal{O}(\tau^2+h^4)$, and $\mathcal{O}(\tau^2+h^6)$, where τ and h are temporal and spatial stepsizes, respectively. In Section **4**, numerical examples are presented which support the theoretical analysis. The last Section **5** concludes this article. In Appendices A and B we provide the detailed proofs of the main results from Section **2**.

2. High-order numerical schemes

If $f^{(k)}(a+) = 0$ $(k = 0, 1, \ldots, p-1)$, then it follows from [16] that the left R-L derivative has the following approximations

$$ {}_{RL}D_{a,x}^{\alpha} f(x) = \frac{1}{h^{\alpha}} \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} f(x - \ell h) + \mathcal{O}(h^p), \tag{2.1} $$

in which h is the steplength. Here we only show interests in $p = 2, 3, 4, 5, 6$.

The convolution (or weight) coefficients $\varpi_{p,\ell}^{(\alpha)}$ in the above equations are those of the Taylor series expansions of the corresponding generating functions $W_p^{(\alpha)}(z)$,

$$ W_p^{(\alpha)}(z) = \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} z^{\ell}, \quad \alpha \in (0, 2), $$

where

$$ \begin{aligned} W_2^{(\alpha)}(z) &= \left(\frac{3}{2} - 2z + \frac{1}{2} z^2 \right)^{\alpha}, \\ W_3^{(\alpha)}(z) &= \left(\frac{11}{6} - 3z + \frac{3}{2} z^2 - \frac{1}{3} z^3 \right)^{\alpha}, \\ W_4^{(\alpha)}(z) &= \left(\frac{25}{12} - 4z + 3z^2 - \frac{4}{3} z^3 + \frac{1}{4} z^4 \right)^{\alpha}, \\ W_5^{(\alpha)}(z) &= \left(\frac{137}{60} - 5z + 5z^2 - \frac{10}{3} z^3 + \frac{5}{4} z^4 - \frac{1}{5} z^5 \right)^{\alpha}, \\ W_6^{(\alpha)}(z) &= \left(\frac{147}{60} - 6z + \frac{15}{2} z^2 - \frac{20}{3} z^3 + \frac{15}{4} z^4 - \frac{6}{5} z^5 + \frac{1}{6} z^6 \right)^{\alpha}. \end{aligned} $$

By tedious but direct calculations, one has

$$ \varpi_{2,\ell}^{(\alpha)} = \left(\frac{3}{2} \right)^{\alpha} \sum_{\ell_1=0}^{\ell} \left(\frac{1}{3} \right)^{\ell_1} \varpi_{1,\ell_1}^{(\alpha)} \varpi_{1,\ell-\ell_1}^{(\alpha)}, $$

$$ \varpi_{3,\ell}^{(\alpha)} = \left(\frac{11}{6} \right)^{\alpha} \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\left[\frac{1}{2}\ell_1\right]} \left(\frac{7}{11} \right)^{\ell_1-\ell_2} \left(\frac{2}{7} \right)^{\ell_2} \frac{(-1)^{\ell_2} (\ell_1 - \ell_2)!}{\ell_2! (\ell_1 - 2\ell_2)!} \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)}, $$

$$ \begin{aligned} \varpi_{4,\ell}^{(\alpha)} = \left(\frac{25}{12} \right)^{\alpha} & \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\left[\frac{2}{3}\ell_1\right]} \sum_{\ell_3=\max\{0, 2\ell_2-\ell_1\}}^{\left[\frac{1}{2}\ell_2\right]} \left(\frac{23}{25} \right)^{\ell_1-\ell_2} \left(\frac{13}{23} \right)^{\ell_2-\ell_3} \left(\frac{3}{13} \right)^{\ell_3} \\ & \times \frac{(-1)^{\ell_2} (\ell_1 - \ell_2)!}{\ell_3! (\ell_2 - 2\ell_3)! (\ell_1 + \ell_3 - 2\ell_2)!} \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)}, \end{aligned} $$

$$\varpi_{5,\ell}^{(\alpha)} = \left(\frac{137}{60}\right)^{\alpha} \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\left[\frac{3}{4}\ell_1\right]} \sum_{\ell_3=\max\{0,2\ell_2-\ell_1\}}^{\left[\frac{2}{3}\ell_2\right]} \sum_{\ell_4=\max\{0,2\ell_3-\ell_2\}}^{\left[\frac{1}{2}\ell_3\right]} \left(\frac{163}{137}\right)^{\ell_1-\ell_2}$$
$$\times \left(\frac{137}{163}\right)^{\ell_2-\ell_3} \left(\frac{63}{137}\right)^{\ell_3-\ell_4} \left(\frac{4}{21}\right)^{\ell_4}$$
$$\times \frac{(-1)^{\ell_2} (\ell_1-\ell_2)!\, \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)}}{\ell_4!\,(\ell_3-2\ell_4)!\,(\ell_1+\ell_3-2\ell_2)!\,(\ell_2+\ell_4-2\ell_3)!},$$

and

$$\varpi_{6,\ell}^{(\alpha)} = \left(\frac{147}{60}\right)^{\alpha} \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\left[\frac{4}{5}\ell_1\right]} \sum_{\ell_3=\max\{0,2\ell_2-\ell_1\}}^{\left[\frac{3}{4}\ell_2\right]} \sum_{\ell_4=\max\{0,2\ell_3-\ell_2\}}^{\left[\frac{2}{3}\ell_3\right]} \sum_{\ell_5=\max\{0,2\ell_4-\ell_3\}}^{\left[\frac{1}{2}\ell_4\right]}$$
$$\left(\frac{213}{147}\right)^{\ell_1-\ell_2} \left(\frac{237}{213}\right)^{\ell_2-\ell_3} \left(\frac{163}{237}\right)^{\ell_3-\ell_4} \left(\frac{62}{163}\right)^{\ell_4-\ell_5} \left(\frac{5}{31}\right)^{\ell_5}$$
$$\times \frac{(-1)^{\ell_2} (\ell_1-\ell_2)!\, \varpi_{1,\ell-\ell_1}^{(\alpha)} \varpi_{1,\ell_1-\ell_2}^{(\alpha)}}{\ell_5!\,(\ell_4-2\ell_5)!\,(\ell_1+\ell_3-2\ell_2)!\,(\ell_2+\ell_4-2\ell_3)!\,(\ell_3+\ell_5-2\ell_4)!},$$

$$\ell = 0, 1, \ldots$$

Here $\varpi_{1,j}^{(\alpha)}$ is the first order coefficients defined by $\varpi_{1,j}^{(\alpha)} = \frac{(-1)^j \Gamma(1+\alpha)}{\Gamma(j+1)\Gamma(1+\alpha-j)}$, $j = 0, 1, \ldots$ If $j \geq 2$, then $\varpi_{1,j-1}^{(\alpha)} \leq \varpi_{1,j}^{(\alpha)}$ for $\alpha \in (0,1)$ whilst $\varpi_{1,j-1}^{(\alpha)} \geq \varpi_{1,j}^{(\alpha)}$ for $\alpha \in (1,2)$. See [13] for more information.

On the other hand, if $f^{(k)}(b-) = 0$ $(k = 0, 1, \ldots, p-1)$, then one has the approximations below,

$$_{RL}D_{x,b}^{\alpha} f(x) = \frac{1}{h^{\alpha}} \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} f(x+\ell h) + \mathcal{O}(h^p), \;\; p = 2, \cdots, 6, \tag{2.2}$$

where h is the stepsize.

Based on (2.1) and (2.2), if $f(x)$, together with its derivatives, has homogeneous boundary value conditions, one easily gets

$$\frac{\partial^{\alpha} f(x)}{\partial |x|^{\alpha}} = \frac{-1}{2\cos\left(\frac{\pi\alpha}{2}\right)} \sum_{\ell=0}^{\infty} \varpi_{p,\ell}^{(\alpha)} \frac{f(x-\ell h) + f(x+\ell h)}{h^{\alpha}} + \mathcal{O}(h^p). \tag{2.3}$$

Here, we limit our interests in $\alpha \in (0,1)$. The case $\alpha \in (1,2)$ can be similarly studied. When $\alpha = 1$, $\frac{\partial^{\alpha} f(x)}{\partial |x|^{\alpha}} = f'(x)$ is the trivial case so is omitted here.

The properties of convolution coefficients $\varpi_{p,\ell}^{(\alpha)}$ are very important for constructing effective numerical algorithms for R-L time fractional differential equations and R-L (or Riesz) space fractional differential equations.

For the space fractional differential equations, we use the following properties of coefficients $\varpi_{p,\ell}^{(\alpha)}$ to show the stability and convergence of the derived algorithms.

THEOREM **2.1**. *For* $0<\alpha<1$, *then the following inequalities hold:*

$$\sum_{\ell=0}^{\infty}\varpi_{p,\ell}^{(\alpha)}\cos(\ell\theta)\geq 0,\quad \theta\in[-\pi,\pi],\quad p=2,3,5,6.$$

P r o o f. We only prove $p=2$, the rest cases can be almost similarly shown. Let

$$f_1(\alpha,\theta)=\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\alpha)}\cos(\ell\theta),$$

which can be expanded as

$$\begin{aligned} f_1(\alpha,\theta) &= \sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\alpha)}\cos(\ell\theta)=\frac{1}{2}\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\alpha)}\left(\exp(i\ell\theta)+\exp(-i\ell\theta)\right)\\ &= \frac{1}{2}\left[(1-\exp(i\theta))^{\alpha}\left(\frac{3}{2}-\frac{1}{2}\exp(i\theta)\right)^{\alpha}\right.\\ &\qquad\left.+(1-\exp(-i\theta))^{\alpha}\left(\frac{3}{2}-\frac{1}{2}\exp(-i\theta)\right)^{\alpha}\right]. \end{aligned}$$

Note that $f_1(\alpha,\theta)$ is a real-value and even function, so we need only consider $\theta\in[0,\pi]$.

Using the following equations

$$(1-\exp(\pm i\theta))^{\alpha}=\left(2\sin\frac{\theta}{2}\right)^{\alpha}\exp\left(\pm i\alpha\left(\frac{\theta-\pi}{2}\right)\right)$$

and

$$(x-yi)^{\alpha}=\left(x^2+y^2\right)^{\frac{\alpha}{2}}\exp\left(i\alpha\phi\right),\quad \phi=-\arctan\frac{y}{x},$$

we can rewrite $f_1(\alpha,\theta)$ as

$$f_1(\alpha,\theta)=\left(2\sin\frac{\theta}{2}\right)^{\alpha}\left(\lambda_1^2(\theta)+\mu_1^2(\theta)\right)^{\frac{\alpha}{2}}\cos\alpha\left(\frac{\theta-\pi}{2}+\phi_1\right),$$

where

$$\lambda_1(\theta)=3-\cos\theta,\quad \mu_1(\theta)=\sin\theta,\quad \phi_1=-\arctan\frac{\mu_1(\theta)}{\lambda_1(\theta)}.$$

Let

$$z(\theta)=\frac{\theta-\pi}{2}+\phi_1,\quad 0\leq\theta\leq\pi.$$

Then

$$z'(\theta)=\left(\frac{\theta-\pi}{2}+\phi_1\right)'=\frac{3\sin^2\left(\frac{\theta}{2}\right)}{1+3\sin^2\left(\frac{\theta}{2}\right)}\geq 0.$$

Hence $z(\theta)$ is an increasing function in $[0,\pi]$ and

$$z_{\min}(\theta) = z(0) = -\frac{\pi}{2}, \ \ z_{\max}(\theta) = z(\pi) = 0.$$

Evidently $\alpha \in (0,1)$ and $\theta \in [0,\pi]$ imply $\cos\alpha\left(\frac{\theta-\pi}{2}+\phi_1\right) \geq 0$. Furthermore, one has

$$f_1(\alpha,\theta) = \left(2\sin\frac{\theta}{2}\right)^{\alpha} \left(\lambda_1^2(\theta)+\mu_1^2(\theta)\right)^{\frac{\alpha}{2}} \cos\alpha\left(\frac{\theta-\pi}{2}+\phi_1\right) \geq 0.$$

All this ends the proof. □

For $p = 4$, α can not attain to 1. But we have following theorem.

Theorem **2.2**. *If* $0 < \alpha \leq \dfrac{\pi}{\pi - \arccos\frac{1}{5} + 2\arctan\frac{191\sqrt{6}}{317}} \approx 0.8439$, *then the following inequality holds:*

$$\sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \cos(\ell\theta) \geq 0, \quad \theta \in [-\pi,\pi].$$

P r o o f. Let $f_2(\alpha,\theta) = \sum\limits_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \cos(\ell\theta)$. By almost the same reasoning as those in Theorem **2.1**, we can get

$$f_2(\alpha,\theta) = \left(2\sin\frac{\theta}{2}\right)^{\alpha} \left(\lambda_2^2(\theta)+\mu_2^2(\theta)\right)^{\frac{\alpha}{2}} \cos\alpha\left(\frac{\theta-\pi}{2}+\phi_2\right),$$

where

$$\lambda_2(\theta) = 25 - 23\cos\theta + 13\cos 2\theta - 3\cos 3\theta,$$
$$\mu_2(\theta) = 23\sin\theta - 13\sin 2\theta + 3\sin 3\theta, \quad \phi_2 = -\arctan\frac{\mu_2(\theta)}{\lambda_2(\theta)}.$$

Since

$$\lambda_2(\theta) = 14\left(\cos(\theta) - \frac{1}{2}\right)^2 + 24\cos^2(\theta)\sin^2\left(\frac{\theta}{2}\right) + \frac{17}{2} > 0,$$

and

$$\mu_2(\theta) = \left[12\left(\cos(\theta) - \frac{13}{12}\right)^2 + \frac{71}{24}\right]\sin(\theta) \geq 0,$$

it immediately follows that $\phi_2 \in [-\frac{\pi}{2}, 0]$. We need only consider $0 \leq \theta \leq \pi$, therefore $-\pi \leq \alpha\left(\frac{\theta-\pi}{2}+\phi_2\right) \leq 0$.

Obviously, if $\cos\alpha\left(\frac{\theta-\pi}{2}+\phi_2\right) \geq 0$, then $f_2(\alpha,\theta) \geq 0$. A sufficient condition for $\cos\alpha\left(\frac{\theta-\pi}{2}+\phi_2\right) \geq 0$ is

$$-\frac{\pi}{2} \leq \min_{\theta\in[0,\pi]} \alpha\left(\frac{\theta-\pi}{2}+\phi_2\right) \leq 0,$$

i.e.,

$$0 < \alpha \leq \min_{\theta\in[0,\pi]} \left\{\frac{\pi}{\pi-\theta-2\phi_2}\right\}.$$

Let $y(\theta) = \pi - \theta - 2\phi_2$, then

$$y'(\theta) = \frac{1920\,(5\cos\theta - 1)\sin^4\left(\frac{\theta}{2}\right)}{a_2^2(\theta) + b_2^2(\theta)}.$$

It is clear that $\theta = \arccos\frac{1}{5}$ is a unique maximum point of $y(\theta)$ when $\theta \in [0,\pi]$, i.e.,

$$y_{\max}(\theta) = y_{\max}\left(\arccos\frac{1}{5}\right) = \pi - \arccos\frac{1}{5} + 2\arctan\frac{191\sqrt{6}}{317},$$

it follows that

$$\min_{\theta\in[0,\pi]}\left\{\frac{\pi}{\pi - \theta - 2\phi_2}\right\} = \frac{\pi}{\pi - \arccos\frac{1}{5} + 2\arctan\frac{191\sqrt{6}}{317}} \approx 0.8439,$$

i.e.,

$$0 < \alpha \le 0.8439.$$

This finishes the proof. □

Theorems **2.1** and **2.2** are very suitable for numerically analyzing R-L space fractional partial differential equations. But for numerically analyzing R-L time fractional partial differential equations, the monotonicity of the coefficients $\varpi_{p,\ell}^{(\alpha)}$ $(p = 1,2,3,4,5,6)$ is often necessary. As far as we know, only the monotonicity of the coefficients $\varpi_{1,\ell}^{(\alpha)}$ for $\alpha \in (0,1)$ and $\alpha \in (1,2)$, $\varpi_{2,\ell}^{(\alpha)}$ for $\alpha \in (0,1)$ are available [13]. In the following, we study the rest cases.

THEOREM **2.3**. *The second-order coefficients* $\varpi_{2,\ell}^{(\alpha)}$ $(\ell = 0,1,\ldots)$ *satisfy:*

(1) $\varpi_{2,0}^{(\alpha)} = \left(\frac{3}{2}\right)^{\alpha} > 0,\ \ \varpi_{2,1}^{(\alpha)} = -\frac{4\alpha}{3}\left(\frac{3}{2}\right)^{\alpha} < 0,$

$$\varpi_{2,2}^{(\alpha)} = \frac{\alpha\,(8\alpha-3)}{9}\left(\frac{3}{2}\right)^{\alpha},\quad \varpi_{2,3}^{(\alpha)} = -\frac{4\alpha\,(\alpha-1)\,(8\alpha-7)}{81}\left(\frac{3}{2}\right)^{\alpha},$$

$$\varpi_{2,4}^{(\alpha)} = \frac{\alpha\,(\alpha-1)\left(64\alpha^2 - 176\alpha + 123\right)}{486}\left(\frac{3}{2}\right)^{\alpha},$$

$$\cdots$$

(2) *When* $0 < \alpha < 1$, $\varpi_{2,\ell}^{(\alpha)} < 0$ *and* $\varpi_{2,\ell}^{(\alpha)} < \varpi_{2,\ell+1}^{(\alpha)}$ *hold for* $\ell \ge 4$,

(3) *When* $1 < \alpha < 2$, $\varpi_{2,\ell}^{(\alpha)} > 0$ *and* $\varpi_{2,\ell}^{(\alpha)} > \varpi_{2,\ell+1}^{(\alpha)}$ *hold for* $\ell \ge 5$.

P r o o f. See Appendix A. □

From the results of the above theorem, one can see that: If $\alpha = 1$, then $\varpi_{2,\ell}^{(\alpha)} \neq 0$ for $\ell = 0, 1, 2$, and $\varpi_{2,\ell}^{(\alpha)} = 0$ for $\ell \geq 3$; If $\alpha \in (0, 1)$, then $\varpi_{2,\ell}^{(\alpha)} \neq 0$ for $\ell \geq 0$, $\varpi_{2,\ell}^{(\alpha)} < 0$ and $\varpi_{2,\ell}^{(\alpha)} < \varpi_{2,\ell+1}^{(\alpha)}$ for $\ell \geq 4$. Similarly, if $\alpha = 2$, then $\varpi_{2,\ell}^{(\alpha)} \neq 0$ for $\ell = 0, 1, \ldots, 4$, and $\varpi_{2,\ell}^{(\alpha)} = 0$ for $\ell \geq 5$; If $\alpha \in (1, 2)$, then $\varpi_{2,\ell}^{(\alpha)} \neq 0$ for $\ell \geq 0$, $\varpi_{2,\ell}^{(\alpha)} > 0$ and $\varpi_{2,\ell}^{(\alpha)} > \varpi_{2,\ell+1}^{(\alpha)}$ for $\ell \geq 5$. That is to say, the monotonicity of $\varpi_{2,\ell}^{(\alpha)}$, $\alpha \in (0, 1)$ appears from $\ell = 3$ which is just the number of weigh coefficients for approximating $\frac{\partial u}{\partial x}$ with order 2. The case of $\alpha \in (1, 2)$ has similar explanation.

Besides their monotonicity, studying bounds of these coefficients is also of importance, which can be used to analyze the stability and convergence for time fractional differential equations. In [4], Dimitrov gave the bounds for first-order $\varpi_{1,\ell}^{(\alpha)}, (0 < \alpha < 1)$ below.

Theorem **2.4**. *The first-order coefficients* $\varpi_{1,\ell}^{(\alpha)}$, $(0 < \alpha < 1)$ *satisfy:*

(1) $\widetilde{B}_1^L(\alpha, \ell) < \left|\varpi_{1,\ell}^{(\alpha)}\right| < B_1^R(\alpha, \ell), \quad \ell \geq 3,$

where $\widetilde{B}_1^L(\alpha, \ell) = \exp\left(-(\alpha+1)^2\left(\frac{\pi^2}{6} - \frac{5}{4}\right)\right)\frac{\alpha(1-\alpha)2^\alpha}{\ell^{\alpha+1}},$

$B_1^R(\alpha, \ell) = \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}},$

(2) $\widetilde{S}_1^L(\alpha, \ell) < \sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(\alpha)}\right| < S_1^R(\alpha, \ell), \quad \ell \geq 3,$

where $\widetilde{S}_1^L(\alpha, \ell) = \frac{1-\alpha}{5}\left(\frac{2}{\ell}\right)^\alpha$, $S_1^R(\alpha, \ell) = 2\left(\frac{2}{\ell}\right)^\alpha.$

In this paper, we can give tighter estimates for the lower bounds. See the following theorem.

Theorem **2.5**. *The first-order coefficients* $\varpi_{1,\ell}^{(\alpha)}$, $(0 < \alpha < 1)$ *satisfy:*

(1) $B_1^L(\alpha, \ell) < \left|\varpi_{1,\ell}^{(\alpha)}\right| < B_1^R(\alpha, \ell), \ \ell \geq 3,$

where $B_1^L(\alpha, \ell) = \frac{\alpha(1-\alpha)}{2}\left(\frac{2}{\ell}\right)^{2(\alpha+1)},$

(2) $S_1^L(\alpha, \ell) < \sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(\alpha)}\right| < S_1^R(\alpha, \ell), \ \ell \geq 3,$

where $S_1^L(\alpha, \ell) = \frac{\alpha(1-\alpha)}{2\alpha+1}\left(\frac{2}{\ell}\right)^{2\alpha+1}.$

Next, we list the comparison theorem for $\widetilde{B}_1^L(\alpha,\ell)$ and $B_1^L(\alpha,\ell)$, $\widetilde{S}_1^L(\alpha,\ell)$ and $S_1^L(\alpha,\ell)$, respectively.

Theorem **2.6**. *The following inequalities hold:*

$$\begin{aligned}
&(1)\ \ B_1^L(\alpha,3) < \widetilde{B}_1^L(\alpha,3)\ \ for\ \ 0<\alpha<\frac{12\ln\frac{3}{2}}{2\pi^2-15}-1\approx 0.0267;\\
&\quad B_1^L(\alpha,4) < \widetilde{B}_1^L(\alpha,4)\ \ for\ \ 0<\alpha<\frac{12\ln 2}{2\pi^2-15}-1\approx 0.7551;\\
&\quad B_1^L(\alpha,\ell) < \widetilde{B}_1^L(\alpha,\ell)\ \ for\ \ 0<\alpha<1\ \ and\ \ \ell\geq 5;\\
&(2)\ \ S_1^L(\alpha,\ell) < \widetilde{S}_1^L(\alpha,\ell)\ \ for\ \ 0<\alpha<1\ \ and\ \ \ell\geq 3.
\end{aligned}$$

In the following, we give the bounds for the corresponding coefficients.

Theorem **2.7**. *The first-order coefficients* $\varpi_{1,\ell}^{(1+\alpha)}$, $(0<\alpha<1)$ *satisfy:*

$$\begin{aligned}
&(1)\ \overline{B}_1^L(1+\alpha,\ell) < \left|\varpi_{1,\ell}^{(1+\alpha)}\right| < \overline{B}_1^R(1+\alpha,\ell),\ \ell\geq 4,\ where\\
&\overline{B}_1^L(1+\alpha,\ell) = \frac{(1-\alpha)\alpha(1+\alpha)}{6}\left(\frac{3}{\ell}\right)^{2(2+\alpha)},\\
&\overline{B}_1^R(1+\alpha,\ell) = \frac{\alpha(1+\alpha)}{2}\left(\frac{3}{\ell+1}\right)^{2+\alpha},\\
&(2)\ \overline{S}_1^L(1+\alpha,\ell) < \sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(1+\alpha)}\right| < \overline{S}_1^R(1+\alpha,\ell),\ \ell\geq 4,\ where\\
&\overline{S}_1^L(1+\alpha,\ell) = \frac{(1-\alpha)\alpha(1+\alpha)}{2(3+2\alpha)}\left(\frac{3}{\ell}\right)^{3+2\alpha},\ \overline{S}_1^R(1+\alpha,\ell) = \frac{3\alpha}{2}\left(\frac{3}{\ell}\right)^{1+\alpha}.
\end{aligned}$$

Now we start to show the bounds of the second-order coefficients.

Theorem **2.8**. *The second-order coefficients* $\varpi_{2,\ell}^{(\alpha)}$ *and* $\varpi_{2,\ell}^{(\alpha+1)}$ *for* $0<\alpha<1$ *satisfy:*

$$\begin{aligned}
&(1)\quad B_2^L(\alpha,\ell) < \left|\varpi_{2,\ell}^{(\alpha)}\right| < B_2^R(\alpha,\ell),\ \ell\geq 4,\quad where\\
&B_2^L(\alpha,\ell) = \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha(1-\alpha)}{2}\left(\frac{2}{\ell}\right)^{2(1+\alpha)}\right.\\
&\qquad\left.-\left(1-\left(\frac{1}{3}\right)^{\ell-1}\right)\frac{\alpha^2 2^{2\alpha+1}}{1+(\alpha+1)\ell}\right],
\end{aligned}$$

$$B_2^R(\alpha,\ell) = \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}} - \frac{\alpha^2(1-\alpha)^2 4^{2\alpha+1}}{2}\left(1-\left(\frac{1}{3}\right)^{\ell-1}\right)\left(\frac{2}{\ell}\right)^{4(\alpha+1)}\right].$$

(2) $\overline{B}_2^L(1+\alpha,\ell) < \left|\varpi_{2,\ell}^{(1+\alpha)}\right| < \overline{B}_2^R(1+\alpha,\ell),\ \ell \geq 4,$ *where*

$$\overline{B}_2^L(1+\alpha,\ell) = \left(\frac{3}{2}\right)^{1+\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{(1-\alpha)\alpha(1+\alpha)}{6}\left(\frac{3}{\ell}\right)^{2(2+\alpha)} + \frac{(1-\alpha)^2\alpha^2(1+\alpha)^2}{216}\left(1-\left(\frac{1}{3}\right)^{\ell-3}\right)\left(\frac{6}{\ell}\right)^{4(2+\alpha)} - \frac{\alpha(1+\alpha)^2}{2}\left(\frac{1}{3}+\left(\frac{1}{3}\right)^{\ell-1}\right)\left(\frac{3}{\ell}\right)^{2+\alpha}\right],$$

$$\overline{B}_2^R(1+\alpha,\ell) = \left(\frac{3}{2}\right)^{1+\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha(\alpha+1)3^{\alpha+2}}{2(\ell+1)^{\alpha+2}} + \left(1-\left(\frac{1}{3}\right)^{\ell-3}\right)\frac{\alpha^2(1+\alpha)^2 3^{2(2+\alpha)}}{24(1+(2+\alpha)\ell)} - \frac{(1-\alpha)\alpha(1+\alpha)^2}{6}\left(\frac{1}{3}+\left(\frac{1}{3}\right)^{\ell-1}\right)\left(\frac{3}{\ell-1}\right)^{2(2+\alpha)}\right].$$

All proofs for Theorems **2.4-2.8** are given in Appendix B.

3. Numerical methods for the Riesz-type turbulent diffusion equation

Define $t_k = k\tau$, $k = 0, 1, \ldots, N$, and $\tau = \frac{T}{N}$ for a given $T > 0$, $h = \frac{b-a}{M}$ is the equidistant grid size in space, $x_j = a + jh$, $j = 0, 1, \ldots, M$.

3.1. The 2nd-order scheme in space

Firstly, using the Crank-Nicolson method for the Riesz space fractional turbulent diffusion equation (1.1) in time direction, we obtain

$$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} = \frac{1}{2}\left(d_2\frac{\partial^2 u(x_j,t_{k+1})}{\partial x^2} - d_1\frac{\partial u(x_j,t_{k+1})}{\partial x}\right. \qquad (3.1)$$

$$
\begin{aligned}
&+ d_\alpha \frac{\partial^\alpha u(x_j, t_{k+1})}{\partial |x|^\alpha} + d_2 \frac{\partial^2 u(x_j, t_k)}{\partial x^2} - d_1 \frac{\partial u(x_j, t_k)}{\partial x} \\
&+ d_\alpha \frac{\partial^\alpha u(x_j, t_k)}{\partial |x|^\alpha}\Bigg) + s(x_j, t_{k+\frac{1}{2}}) + \mathcal{O}(\tau^2).
\end{aligned} \tag{3.2}
$$

Secondly, for the first- and second-order derivatives, we use the following approximations, respectively

$$
\frac{\partial u(x_j, t_k)}{\partial x} = \mu_x \delta_x u(x_j, t_k) + \mathcal{O}(h^2), \tag{3.3}
$$

and

$$
\frac{\partial^2 u(x_j, t_k)}{\partial x^2} = \delta_x^2 u(x_j, t_k) + \mathcal{O}(h^2), \tag{3.4}
$$

where $\mu_x \delta_x$ and δ_x^2 are defined by

$$
\mu_x \delta_x u(x_j, t_k) = \frac{u(x_{j+1}, t_k) - u(x_{j-1}, t_k)}{2h},
$$

and

$$
\delta_x^2 u(x_j, t_k) = \frac{u(x_{j+1}, t_k) - 2u(x_j, t_k) + u(x_{j-1}, t_k)}{h^2}.
$$

Next, we choose the second-order formula to approximate Riesz derivative,

$$
\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} &= -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right) h^\alpha} \left(\sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} u(x - \ell h, t) \right. \\
&\qquad \left. + \sum_{\ell=0}^{\infty} \varpi_{2,\ell}^{(\alpha)} u(x + \ell h, t) \right) + \mathcal{O}(h^2).
\end{aligned} \tag{3.5}
$$

Substituting (3.3), (3.4) and (3.5) into (3.2) and removing the high-order term yield

$$
\begin{aligned}
&\left(\frac{d_2}{h^2} + \frac{d_1}{2h}\right) u_{j-1}^{k+1} - \left(\frac{2}{\tau} + \frac{2d_2}{h^2}\right) u_j^{k+1} + \left(\frac{d_2}{h^2} - \frac{d_1}{2h}\right) u_{j+1}^{k+1} \\
&-\nu \left[\sum_{\ell=0}^{j} \varpi_{2,\ell}^{(\alpha)} u_{j-l}^{k+1} + \sum_{\ell=0}^{M-j} \varpi_{2,\ell}^{(\alpha)} u_{j+l}^{k+1} \right] = -\left(\frac{d_2}{h^2} + \frac{d_1}{2h}\right) u_{j-1}^{k} \\
&-\left(\frac{2}{\tau} - \frac{2d_2}{h^2}\right) u_j^k - \left(\frac{d_2}{h^2} - \frac{d_1}{2h}\right) u_{j+1}^k + \nu \left[\sum_{\ell=0}^{j} \varpi_{2,\ell}^{(\alpha)} u_{j-l}^{k} \right. \\
&\left. + \sum_{\ell=0}^{M-j} \varpi_{2,\ell}^{(\alpha)} u_{j+l}^{k} \right] - 2 s_j^{k+\frac{1}{2}}, \;\; j = 1, \ldots, M-1, \; k = 0, 1, \ldots, N-1,
\end{aligned} \tag{3.6}
$$

where $\nu = \dfrac{d_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right) h^\alpha}$.

Next we discuss the stability and convergence of scheme (3.6).

THEOREM **3.1**. *The numerical scheme (3.6) is unconditionally stable and convergent with order $\mathcal{O}(\tau^2 + h^2)$.*

P r o o f. Assume that the solution of equation (1.1) can be zero extended to the whole real line R. Suppose U_j^k is the approximation solution of equation (3.6) and let $\mathcal{E}_j^k = U_j^k - u_j^k$, then the error equation reads as

$$\begin{aligned} &\left(\frac{d_2}{h^2} + \frac{d_1}{2h}\right)\mathcal{E}_{j-1}^{k+1} - \left(\frac{2}{\tau} + \frac{2d_2}{h^2}\right)\mathcal{E}_j^{k+1} + \left(\frac{d_2}{h^2} - \frac{d_1}{2h}\right)\mathcal{E}_{j+1}^{k+1} \\ &-\nu\left[\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\gamma)}\mathcal{E}_{j-l}^{k+1} + \sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\gamma)}\mathcal{E}_{j+l}^{k+1}\right] = -\left(\frac{d_2}{h^2} + \frac{d_1}{2h}\right)\mathcal{E}_{j-1}^{k} \\ &-\left(\frac{2}{\tau} - \frac{2d_2}{h^2}\right)\mathcal{E}_j^{k} - \left(\frac{d_2}{h^2} - \frac{d_1}{2h}\right)\mathcal{E}_{j+1}^{k} + \nu\left[\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\gamma)}\mathcal{E}_{j-l}^{k}\right. \\ &\left.+\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\gamma)}\mathcal{E}_{j+l}^{k}\right], \quad j = 1, \dots, M-1,\ k = 0, 1, \dots, N-1. \end{aligned} \tag{3.7}$$

Let $\mathcal{E}_j^k = \xi^k e^{ij\theta}$ be the solution of equation (3.7), $i = \sqrt{-1}$, where $\theta \in [-\pi, \pi]$ is called the phase angle. The stability condition of scheme (3.6) is $|\xi(\theta)| \le 1$ for all $\theta \in [-\pi, \pi]$.

Substituting $\varepsilon_j^k = \xi^k e^{ij\theta}$ into (3.7) gives

$$\xi(\theta) = \frac{\left(\frac{2h}{\tau} - \frac{4d_2}{h}\sin^2\left(\frac{\theta}{2}\right) - 2\nu h\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\gamma)}\cos(\ell\theta)\right) - id_1\sin(\theta)}{\left(\frac{2h}{\tau} + \frac{4d_2}{h}\sin^2\left(\frac{\theta}{2}\right) + 2\nu h\sum_{\ell=0}^{\infty}\varpi_{2,\ell}^{(\gamma)}\cos(\ell\theta)\right) + id_1\sin(\theta)}.$$

According to Theorem **2.1**, we easily obtain $|\xi(\theta)| \le 1$. So scheme (3.6) is unconditionally stable. It is easy to show the convergence order of scheme (3.6) is $\mathcal{O}(\tau^2 + h^2)$. In effect, the proof is almost the same as that of [5]. □

3.2. The 4th-order scheme in space

Let us first consider the following differential equation

$$d_2\frac{\partial^2 u(x,t)}{\partial x^2} - d_1\frac{\partial u(x,t)}{\partial x} = g(x,t). \tag{3.8}$$

Applying the technique presented in [18], we can obtain a fourth-order difference scheme for solving the above equation

$$\Delta_4 u(x_j,t) = \widetilde{\Delta}_4 g(x_j,t) + \mathcal{O}(h^4), \tag{3.9}$$

where

$$\Delta_4 = \left(d_2 + \frac{d_1^2 h^2}{12 d_2}\right)\delta_x^2 - d_1\mu_x\delta_x, \quad \widetilde{\Delta}_4 = I + \frac{h^2}{12}\left(\delta_x^2 - \frac{d_1}{d_2}\mu_x\delta_x\right),$$

in which I is a unit operator.

Combing (3.2) with (3.8) and (3.9) leads to

$$\begin{aligned} &\frac{2}{\tau}\widetilde{\Delta}_4 \cdot (u(x_j, t_{k+1}) - u(x_j, t_k)) = \Delta_4 \cdot (u(x_j, t_{k+1}) + u(x_j, t_k)) \\ &+ d_\alpha \widetilde{\Delta}_4 \cdot \left(\frac{\partial^\alpha u(x_j, t_{k+1})}{\partial |x|^\alpha} + \frac{\partial u(x_j, t_k)}{\partial |x|^\alpha}\right) + 2\widetilde{\Delta}_4 \cdot s(x_j, t_{k+\frac{1}{2}}) \\ &+ \mathcal{O}(\tau^2 + h^4). \end{aligned} \tag{3.10}$$

For the Riesz derivative in equation (3.10), we use the fourth-order numerical scheme

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right) h^\alpha} \sum_{\ell=0}^{\infty} \varpi_{4,\ell}^{(\alpha)} \left((u(x - \ell h, t) + u(x + \ell h, t)\right) + \mathcal{O}(h^4).$$

Substituting it into (3.10) and ignoring the truncation error term give

$$\begin{aligned} &\left(\frac{2a_1}{\tau} - b_1\right) u_{j-1}^{k+1} + \left(\frac{2a_2}{\tau} - b_2\right) u_j^{k+1} + \left(\frac{2a_3}{\tau} - b_3\right) u_{j+1}^{k+1} \\ &+\nu \left[\sum_{\ell=0}^{j} \varpi_{4,\ell}^{(\alpha)} \left(a_1 u_{j-l-1}^{k+1} + a_2 u_{j-l}^{k+1} + a_3 u_{j-l+1}^{k+1}\right)\right. \\ &\left. + \sum_{\ell=0}^{M-j} \varpi_{4,\ell}^{(\alpha)} \left(a_1 u_{j+l-1}^{k+1} + a_2 u_{j+l}^{k+1} + a_3 u_{j+l+1}^{k+1}\right)\right] \\ &= \left(\frac{2a_1}{\tau} + b_1\right) u_{j-1}^{k} + \left(\frac{2a_2}{\tau} + b_2\right) u_j^{k} + \left(\frac{2a_3}{\tau} + b_3\right) u_{j+1}^{k} \\ &-\nu \left[\sum_{\ell=0}^{j} \varpi_{4,\ell}^{(\alpha)} \left(a_1 u_{j-l-1}^{k} + a_2 u_{j-l}^{k} + a_3 u_{j-l+1}^{k}\right)\right. \\ &\left. + \sum_{\ell=0}^{M-j} \varpi_{4,\ell}^{(\alpha)} \left(a_1 u_{j+l-1}^{k} + a_2 u_{j+l}^{k} + a_3 u_{j+l+1}^{k}\right)\right] \\ &+2a_1 s_{j-1}^{k+\frac{1}{2}} + 2a_2 s_j^{k+\frac{1}{2}} + 2a_3 s_{j+1}^{k+\frac{1}{2}}, \quad j = 1, \ldots, M-1, \\ &k = 0, 1, \ldots, N-1. \end{aligned} \tag{3.11}$$

Here, the parameters in (3.11) are given below,

$$\nu=\frac{d_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right)h^\alpha},\quad a_1=\frac{1}{12}+\frac{d_1h}{24d_2},\quad a_2=\frac{5}{6},\quad a_3=\frac{1}{12}-\frac{d_1h}{24d_2},$$
$$b_1=\frac{d_2}{h^2}+\frac{d_1^2}{12d_2}+\frac{d_1}{2h},\quad b_2=-2\left(\frac{d_2}{h^2}+\frac{d_1^2}{12d_2}\right),\quad b_3=\frac{d_2}{h^2}+\frac{d_1^2}{12d_2}-\frac{d_1}{2h}.$$

THEOREM **3.2**. *When $0<\alpha\le 0.8439$, the numerical scheme (3.11) is unconditionally stable and convergent with order $\mathcal{O}(\tau^2+h^4)$.*

P r o o f. Similar to the above subsection, we can get the error equation of equation (3.11) and let $\mathcal{E}_j^k=\xi^k e^{ij\theta}$, $\theta\in[-\pi,\pi]$. Substituting it into the error equation gives

$$\xi(\theta)=\frac{(s_1-s_2-s_3\nu)-is_4\nu\sin(\theta)}{(s_1+s_2+s_3\nu)+is_4\nu\sin(\theta)},$$

where

$$s_1=\frac{2}{\tau}\left(1-\frac{1}{3}\sin^2\left(\frac{\theta}{2}\right)\right),\quad s_2=2\sin^2\left(\frac{\theta}{2}\right)\left(\frac{2d_2}{h^2}+\frac{d_1^2}{6d_2}\right),$$
$$s_3=2\left(1-\frac{1}{3}\sin^2\left(\frac{\theta}{2}\right)\right)\sum_{\ell=0}^{\infty}\varpi_{4,\ell}^{(\alpha)}\cos(\ell\theta),$$
$$s_4=\left(\frac{d_1h}{6\tau d_2}+\frac{d_1}{h}\right)-\frac{d_1h}{6d_2}\sum_{\ell=0}^{\infty}\varpi_{4,\ell}^{(\alpha)}\cos(\ell\theta).$$

Note that $\nu,s_2,s_3\ge 0$. It follows from Theorem **2.2** that $|\xi(\theta)|\le 1$ if $0<\alpha\le 0.8439$. So scheme (3.11) is unconditionally stable if $0<\alpha\le 0.8439$. For such $\alpha\in(0,0.8439]$, the convergence order of scheme (3.11) is $\mathcal{O}(\tau^2+h^4)$. □

3.3. The 6th-order scheme in space

From Taylor expansion, we have

$$\frac{\partial u(x_j,t)}{\partial x}=\mu_x\delta_x\left(I-\frac{h^2}{6}\delta_x^2\right)u(x_j,t)+\frac{h^4}{30}\frac{\partial^5u(x_j,t)}{\partial x^5}+\mathcal{O}(h^6),\tag{3.12}$$
$$\frac{\partial^2 u(x_j,t)}{\partial x^2}=\delta_x^2\left(I-\frac{h^2}{12}\delta_x^2\right)u(x_j,t)+\frac{h^4}{90}\frac{\partial^6u(x_j,t)}{\partial x^6}+\mathcal{O}(h^6),\tag{3.13}$$
$$\frac{\partial^3 u(x_j,t)}{\partial x^3}=\mu_x\delta_x^3u(x_j,t)+\mathcal{O}(h^2),\tag{3.14}$$
$$\frac{\partial^4 u(x_j,t)}{\partial x^4}=\delta_x^4u(x_j,t)+\mathcal{O}(h^2).\tag{3.15}$$

According to (3.8), one gets

$$\frac{\partial^5 u(x_j,t)}{\partial x^5} = \frac{d_1}{d_2}\frac{\partial^4 u(x_j,t)}{\partial x^4} + \frac{1}{d_2}\frac{\partial^3 g(x_j,t)}{\partial x^3} \tag{3.16}$$

and

$$\frac{\partial^6 u(x_j,t)}{\partial x^6} = \frac{d_1^2}{d_2^2}\frac{\partial^4 u(x_j,t)}{\partial x^4} + \frac{1}{d_2}\frac{\partial^4 g(x_j,t)}{\partial x^4} + \frac{d_1}{d_2^2}\frac{\partial^3 g(x_j,t)}{\partial x^3}. \tag{3.17}$$

From equations (3.12)-(3.17), one has

$$\begin{aligned}\frac{\partial u(x_j,t)}{\partial x} = \;& \mu_x\delta_x\left(I - \frac{h^2}{6}\delta_x^2\right)u(x_j,t) + \frac{d_1 h^4}{30 d_2}\delta_x^4 u(x_j,t)\\ &+\frac{h^4}{30d_2}\mu_x\delta_x^3 g(x_j,t) + \mathcal{O}(h^6),\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\frac{\partial^2 u(x_j,t)}{\partial x^2} = \;& \delta_x^2\left(I - \frac{h^2}{12}\delta_x^2\right)u(x_j,t) + \frac{d_1^2 h^4}{90 d_2^2}\delta_x^4 u(x_j,t)\\ &+\frac{h^4}{90d_2}\delta_x^4 g(x_j,t) + \frac{h^4 d_1}{90 d_2^2}\mu_x\delta_x^3 g(x_j,t) + \mathcal{O}(h^6).\end{aligned} \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.8) gives

$$\Delta_6 u(x_j,t) = \widetilde{\Delta}_6 g(x_j,t) + \mathcal{O}(h^6), \tag{3.20}$$

where

$$\begin{aligned}\Delta_6 &= d_2\delta_x^2\left(I - \frac{h^2}{12}\delta_x^2\right) - d_1\mu_x\delta_x\left(I - \frac{h^2}{6}\delta_x^2\right) - \frac{d_1^2 h^4}{45 d_2}\delta_x^4,\\ \widetilde{\Delta}_6 &= I - \frac{h^4}{90}\delta_x^2\left(\delta_x^2 - \frac{2d_1}{d_2}\mu_x\delta_x\right).\end{aligned}$$

Combining (3.2) with (3.8) and (3.20), we can obtain

$$\begin{aligned}&\frac{2}{\tau}\widetilde{\Delta}_6\cdot\left(u(x_j,t_{k+1}) - u(x_j,t_k)\right) = \Delta_6\cdot\left(u(x_j,t_{k+1}) + u(x_j,t_k)\right)\\ &+ d_\gamma\widetilde{\Delta}_6\cdot\left(\frac{\partial u(x_j,t_{k+1})}{\partial|x|^\gamma} + \frac{\partial u(x_j,t_k)}{\partial |x|^\gamma}\right) + 2\widetilde{\Delta}_6\cdot s(x_j,t_{k+\frac{1}{2}})\\ &+\mathcal{O}(\tau^2 + h^6).\end{aligned} \tag{3.21}$$

For the Riesz derivative, we apply the sixth-order numerical scheme

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)h^\alpha}\sum_{\ell=0}^{\infty}\varpi_{6,\ell}^{(\alpha)}\left(u(x-\ell h,t) + u(x+\ell h,t)\right) + \mathcal{O}(h^6).$$

Substitution in (3.21) gives

$$\begin{aligned}
&\left(\frac{2c_1}{\tau}+e_1\right)u_{j-2}^{k+1}+\left(\frac{2c_2}{\tau}+e_2\right)u_{j-1}^{k+1}+\left(\frac{2c_3}{\tau}+e_3\right)u_{j}^{k+1}\\
&+\left(\frac{2c_4}{\tau}+e_4\right)u_{j+1}^{k+1}+\left(\frac{2c_5}{\tau}+e_5\right)u_{j+2}^{k+1}+\nu\left[\sum_{\ell=0}^{j}\varpi_{6,\ell}^{(\alpha)}\right.\\
&\left(c_1u_{j-l-2}^{k+1}+c_2u_{j-l-1}^{k+1}+c_3u_{j-l}^{k+1}+c_4u_{j-l+1}^{k+1}+c_5u_{j-l+2}^{k+1}\right)\\
&+\sum_{\ell=0}^{M-j}\varpi_{6,\ell}^{(\alpha)}\left(c_1u_{j+l-2}^{k+1}+c_2u_{j+l-1}^{k+1}+c_3u_{j+l}^{k+1}+c_4u_{j+l+1}^{k+1}\right.\\
&\left.\left.+c_5u_{j+l+2}^{k+1}\right)\right]=\left(\frac{2c_1}{\tau}-e_1\right)u_{j-2}^{k}+\left(\frac{2c_2}{\tau}-e_2\right)u_{j-1}^{k}\\
&+\left(\frac{2c_3}{\tau}-e_3\right)u_{j}^{k}+\left(\frac{2c_4}{\tau}-e_4\right)u_{j+1}^{k}+\left(\frac{2c_5}{\tau}-e_5\right)u_{j+2}^{k}\\
&-\nu\left[\sum_{\ell=0}^{j}\varpi_{6,\ell}^{(\alpha)}\left(c_1u_{j-l-2}^{k}+c_2u_{j-l-1}^{k}+c_3u_{j-l}^{k+1}+c_4u_{j-l+1}^{k}\right.\right.\\
&\left.+c_5u_{j-l+2}^{k}\right)+\sum_{\ell=0}^{M-j}\varpi_{6,\ell}^{(\alpha)}\left(c_1u_{j+l-2}^{k}+c_2u_{j+l-1}^{k}+c_3u_{j+l}^{k}\right.\\
&\left.\left.+c_4u_{j+l+1}^{k}+c_5u_{j+l+2}^{k}\right)\right]+2c_1f_{j-2}^{k+\frac{1}{2}}+2c_2f_{j-1}^{k+\frac{1}{2}}\\
&+2c_3f_{j}^{k+\frac{1}{2}}+2c_4f_{j+1}^{k+\frac{1}{2}}+2c_5f_{j+2}^{k+\frac{1}{2}},\quad j=2,\ldots,M-2,\\
&k=0,1,\ldots,N-1.
\end{aligned}\tag{3.22}$$

Here

$$\begin{aligned}
&\nu=\frac{d_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right)h^\alpha},\ \ c_1=-\frac{1}{90}\left(1+\frac{d_1h}{d_2}\right),\ \ c_2=\frac{1}{90}\left(4+\frac{2d_1h}{d_2}\right),\\
&c_3=\frac{14}{15},\ \ c_4=\frac{1}{90}\left(4-\frac{2d_1h}{d_2}\right),\ \ c_5=-\frac{1}{90}\left(1-\frac{d_1h}{d_2}\right),\\
&e_1=\frac{d_2}{12h^2}+\frac{d_1}{12h}+\frac{d_1^2}{45d_2},\ \ e_2=-\left(\frac{4d_2}{3h^2}+\frac{2d_1}{3h}+\frac{4d_1^2}{45d_2}\right),\\
&e_3=\frac{5d_2}{2h^2}+\frac{2d_1^2}{15d_2},\ e_4=-\left(\frac{4d_2}{3h^2}-\frac{2d_1}{3h}+\frac{4d_1^2}{45d_2}\right),\ e_5=\frac{d_2}{12h^2}-\frac{d_1}{12h}+\frac{d_1^2}{45d_2}.
\end{aligned}$$

THEOREM **3.3**. *The numerical scheme (3.22) is unconditionally stable and convergent with order $\mathcal{O}(\tau^2 + h^6)$.*

P r o o f. Let $\mathcal{E}_j^k = \xi^k e^{ij\theta}$, $\theta \in [-\pi, \pi]$. Substituting it into the error equation of the equation (3.22)

$$\begin{aligned}
&\left(\frac{2c_1}{\tau} + e_1\right)\mathcal{E}_{j-2}^{k+1} + \left(\frac{2c_2}{\tau} + e_2\right)\mathcal{E}_{j-1}^{k+1} + \left(\frac{2c_3}{\tau} + e_3\right)\mathcal{E}_{j}^{k+1} + \left(\frac{2c_4}{\tau} + e_4\right)\mathcal{E}_{j+1}^{k+1} \\
&+ \left(\frac{2c_5}{\tau} + e_5\right)\mathcal{E}_{j+2}^{k+1} + \nu\left[\sum_{\ell=0}^{j}\varpi_{6,\ell}^{(\alpha)}\left(c_1\mathcal{E}_{j-l-2}^{k+1} + c_2\mathcal{E}_{j-l-1}^{k+1} + c_3\mathcal{E}_{j-l}^{k+1} + c_4\mathcal{E}_{j-l+1}^{k+1}\right.\right. \\
&\left.\left. + c_5\mathcal{E}_{j-l+2}^{k+1}\right) + \sum_{\ell=0}^{M-j}\varpi_{6,\ell}^{(\alpha)}\left(c_1\mathcal{E}_{j+l-2}^{k+1} + c_2\mathcal{E}_{j+l-1}^{k+1} + c_3\mathcal{E}_{j+l}^{k+1} + c_4\mathcal{E}_{j+l+1}^{k+1} + c_5\mathcal{E}_{j+l+2}^{k+1}\right)\right] \\
&= \left(\frac{2c_1}{\tau} - e_1\right)\mathcal{E}_{j-2}^{k} + \left(\frac{2c_2}{\tau} - e_2\right)\mathcal{E}_{j-1}^{k} + \left(\frac{2c_3}{\tau} - e_3\right)\mathcal{E}_{j}^{k} + \left(\frac{2c_4}{\tau} - e_4\right)\mathcal{E}_{j+1}^{k} \\
&+ \left(\frac{2c_5}{\tau} - e_5\right)\mathcal{E}_{j+2}^{k} - \nu\left[\sum_{\ell=0}^{j}\varpi_{6,\ell}^{(\alpha)}\left(c_1\mathcal{E}_{j-l-2}^{k} + c_2\mathcal{E}_{j-l-1}^{k} + c_3\mathcal{E}_{j-l}^{k+1} + c_4\mathcal{E}_{j-l+1}^{k}\right.\right. \\
&\left.\left. + c_5\mathcal{E}_{j-l+2}^{k}\right) + \sum_{\ell=0}^{M-j}\varpi_{6,\ell}^{(\alpha)}\left(c_1\mathcal{E}_{j+l-2}^{k} + c_2\mathcal{E}_{j+l-1}^{k} + c_3\mathcal{E}_{j+l}^{k} + c_4\mathcal{E}_{j+l+1}^{k} + c_5\mathcal{E}_{j+l+2}^{k}\right)\right], \\
&\quad j = 2, \ldots, M-2, \quad k = 0, 1, \ldots, N-1,
\end{aligned}$$

leads to

$$\xi(\theta) = \frac{(w_1 - w_2 - w_3\nu) - iw_4\sin(\theta)}{(w_1 + w_2 + w_3\nu) + iw_4\sin(\theta)}.$$

Here

$$\begin{aligned}
w_1 &= \frac{2}{45\tau}\left[45 - 8\sin^4\left(\frac{\theta}{2}\right)\right], \; w_2 = \frac{2d_2}{3h^2}(7-\cos\theta)\sin^2\left(\frac{\theta}{2}\right) + \frac{16d_1^2}{45d_2}\sin^4\left(\frac{\theta}{2}\right), \\
w_3 &= \frac{2}{45}\left[45 - 8\sin^4\left(\frac{\theta}{2}\right)\right]\sum_{\ell=0}^{\infty}\varpi_{6,\ell}^{(\alpha)}\cos(\ell\theta), \\
w_4 &= -\frac{8d_1h}{45\tau d_2}\sin^2\left(\frac{\theta}{2}\right) + \frac{d_1}{3h}(4 - \cos\theta) + \frac{8d_1h\nu}{45d_2}\sin^2\left(\frac{\theta}{2}\right)\sum_{\ell=0}^{\infty}\varpi_{6,\ell}^{(\alpha)}\cos(\ell\theta).
\end{aligned}$$

It is clear that $\nu, w_2, w_3 \geq 0$. By using Theorem **2.1**, one also has

$$|\xi(\theta)| \leq 1.$$

So scheme (3.22) is unconditionally stable. It can be shown that the convergence order of scheme (3.22) for equation (1.1) is $\mathcal{O}(\tau^2 + h^6)$. □

In the next section, we present several numerical examples.

4. Numerical examples

We now test the higher-order schemes for Riesz derivatives.

Example 1. Consider the function $f_p(x) = x^p(1-x)^p$, $x \in [0,1]$, $p = 2, 3, 4, 5, 6$.

The Riesz derivative of the above function is analytically expressed as

$$\frac{\partial^\alpha f_p(x)}{\partial |x|^\alpha} = \frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)} \sum_{\ell=0}^{p} \frac{(-1)^\ell p!(p+l)!}{\ell!(p-l)!\Gamma(p+\ell+1-\alpha)} \left[x^{p+\ell-\alpha} + (1-x)^{p+\ell-\alpha}\right].$$

We numerically solve $f_p(x)$ by using numerical scheme (2.3). The numerical results are presented in Tables 1-5. From these tables, the experimental orders are in line with the theoretical orders p ($p = 2, 3, 4, 5, 6$).

α	h	the absolute error	the convergence order
0.2	$\frac{1}{20}$	2.381267e-004	—
	$\frac{1}{40}$	5.900964e-005	2.0127
	$\frac{1}{80}$	1.460639e-005	2.0144
	$\frac{1}{160}$	3.628491e-006	2.0092
	$\frac{1}{320}$	9.039358e-007	2.0051
0.4	$\frac{1}{20}$	7.097814e-004	—
	$\frac{1}{40}$	1.696639e-004	2.0647
	$\frac{1}{80}$	4.123703e-005	2.0407
	$\frac{1}{160}$	1.014980e-005	2.0225
	$\frac{1}{320}$	2.516782e-006	2.0118
0.6	$\frac{1}{20}$	1.638369e-003	—
	$\frac{1}{40}$	3.728453e-004	2.1356
	$\frac{1}{80}$	8.824023e-005	2.0791
	$\frac{1}{160}$	2.141683e-005	2.0427
	$\frac{1}{320}$	5.272483e-006	2.0222
0.8	$\frac{1}{20}$	3.782418e-003	—
	$\frac{1}{40}$	7.826735e-004	2.2728
	$\frac{1}{80}$	1.747182e-004	2.1634
	$\frac{1}{160}$	4.102708e-005	2.0904
	$\frac{1}{320}$	9.923174e-006	2.0477

TABLE 1. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with $p = 2$.

α	h	the absolute error	the convergence order
0.2	$\frac{1}{40}$	3.146678e-007	—
	$\frac{1}{60}$	1.576085e-007	1.7052
	$\frac{1}{80}$	7.991483e-008	2.3608
	$\frac{1}{100}$	4.501080e-008	2.5726
	$\frac{1}{120}$	2.761993e-008	2.6786
0.4	$\frac{1}{40}$	4.349687e-006	—
	$\frac{1}{60}$	1.491902e-006	2.6391
	$\frac{1}{80}$	6.709309e-007	2.7779
	$\frac{1}{100}$	3.560502e-007	2.8394
	$\frac{1}{120}$	2.108270e-007	2.8742
0.6	$\frac{1}{40}$	2.194879e-005	—
	$\frac{1}{60}$	6.996810e-006	2.8196
	$\frac{1}{80}$	3.050074e-006	2.8861
	$\frac{1}{100}$	1.590859e-006	2.9169
	$\frac{1}{120}$	9.316704e-007	2.9347
0.8	$\frac{1}{40}$	1.067572e-004	—
	$\frac{1}{60}$	3.282279e-005	2.9088
	$\frac{1}{80}$	1.407258e-005	2.9439
	$\frac{1}{100}$	7.270149e-006	2.9598
	$\frac{1}{120}$	4.231299e-006	2.9688

TABLE 2. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with $p = 3$.

Next we test the numerical schemes for the equations which have the form of equation (1.1).

Example 2. Consider the following equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + s(x,t),\ 0 \le x \le 1,\ 0 \le t \le 1,$$

in which

α	h	the absolute error	the convergence order
0.2	$\frac{1}{20}$	3.254967e-006	—
	$\frac{1}{25}$	1.421712e-006	3.7121
	$\frac{1}{30}$	7.061638e-007	3.8381
	$\frac{1}{35}$	3.863551e-007	3.9123
	$\frac{1}{40}$	2.279637e-007	3.9509
0.4	$\frac{1}{20}$	1.307893e-005	—
	$\frac{1}{25}$	5.433994e-006	3.9362
	$\frac{1}{30}$	2.610192e-006	4.0217
	$\frac{1}{35}$	1.395202e-006	4.0635
	$\frac{1}{40}$	8.089264e-007	4.0821
0.6	$\frac{1}{20}$	4.165885e-005	—
	$\frac{1}{25}$	1.647646e-005	4.1569
	$\frac{1}{30}$	7.642179e-006	4.2137
	$\frac{1}{35}$	3.980841e-006	4.2309
	$\frac{1}{40}$	2.261840e-006	4.2336
0.8	$\frac{1}{20}$	1.466022e-004	—
	$\frac{1}{25}$	5.460563e-005	4.4258
	$\frac{1}{30}$	2.420431e-005	4.4625
	$\frac{1}{35}$	1.216174e-005	4.4647
	$\frac{1}{40}$	6.707129e-006	4.4568

TABLE 3. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with $p = 4$.

$$
\begin{aligned}
f(x,t) = & \exp(t)x^4(1-x)^4(x^4+10x^3-149x^2+138x-30) + \frac{\exp(t)}{2\cos\left(\frac{\pi}{2}\alpha\right)} \\
& \left\{ \frac{\Gamma(7)}{\Gamma(7-\alpha)}\left[x^{6-\alpha}+(1-x)^{6-\alpha}\right] - \frac{6\Gamma(8)}{\Gamma(8-\alpha)}\left[x^{7-\alpha}+(1-x)^{7-\alpha}\right] \right. \\
& + \frac{15\Gamma(9)}{\Gamma(9-\alpha)}\left[x^{8-\alpha}+(1-x)^{8-\alpha}\right] - \frac{20\Gamma(10)}{\Gamma(10-\alpha)}\left[x^{9-\alpha}+(1-x)^{9-\alpha}\right] \\
& + \frac{15\Gamma(11)}{\Gamma(11-\alpha)}\left[x^{10-\alpha}+(1-x)^{10-\alpha}\right] - \frac{6\Gamma(12)}{\Gamma(12-\alpha)}\left[x^{11-\alpha}+(1-x)^{11-\alpha}\right] \\
& \left. + \frac{\Gamma(13)}{\Gamma(13-\alpha)}\left[x^{12-\alpha}+(1-x)^{12-\alpha}\right] \right\}.
\end{aligned}
$$

α	h	the absolute error	the convergence order
0.2	$\frac{1}{80}$	3.254967e-006	—
	$\frac{1}{100}$	1.421712e-006	3.7121
	$\frac{1}{120}$	7.061638e-007	3.8381
	$\frac{1}{140}$	3.863551e-007	3.9123
	$\frac{1}{160}$	2.279637e-007	3.9509
0.4	$\frac{1}{80}$	8.731739e-010	—
	$\frac{1}{100}$	3.348011e-010	4.2959
	$\frac{1}{120}$	1.473288e-010	4.5023
	$\frac{1}{140}$	7.232096e-011	4.6160
	$\frac{1}{160}$	3.867341e-011	4.6877
0.6	$\frac{1}{80}$	5.385482e-009	—
	$\frac{1}{100}$	1.900398e-009	4.6680
	$\frac{1}{120}$	7.985621e-010	4.7554
	$\frac{1}{140}$	3.806168e-010	4.8071
	$\frac{1}{160}$	1.994064e-010	4.8412
0.8	$\frac{1}{80}$	3.005433e-008	—
	$\frac{1}{100}$	1.023908e-008	4.8256
	$\frac{1}{120}$	4.211445e-009	4.8727
	$\frac{1}{140}$	1.978515e-009	4.9008
	$\frac{1}{160}$	1.025811e-009	4.9192

TABLE 4. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with $p = 5$.

Its analytical solution is $u(x,t) = \exp(t)x^6(1-x)^6$ and satisfy the corresponding initial and boundary values conditions.

We solve this problem with the numerical schemes (3.6) and (3.11) for different values of τ, h and α. The absolute error, temporal and spatial convergence orders are listed in Tables 6 and 7, which display that the numerical results are in line with our theoretical analysis.

In the following, we give a slightly different example.

Example 3. Consider the following equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - 2\frac{\partial u(x,t)}{\partial x} + \alpha^2 \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + s(x,t), \quad 0 \le x \le 1,\ 0 \le t \le 1,$$

in which

α	h	the absolute error	the convergence order
0.2	$\frac{1}{20}$	3.783855e-008	—
	$\frac{1}{40}$	2.310553e-009	4.0335
	$\frac{1}{80}$	4.258651e-011	5.7617
	$\frac{1}{160}$	6.690552e-013	5.9921
	$\frac{1}{320}$	1.035841e-014	6.0133
0.4	$\frac{1}{20}$	3.116503e-007	—
	$\frac{1}{40}$	1.031745e-008	4.9168
	$\frac{1}{80}$	1.651582e-010	5.9651
	$\frac{1}{160}$	2.418476e-012	6.0936
	$\frac{1}{320}$	3.582209e-014	6.0771
0.6	$\frac{1}{20}$	1.564617e-006	—
	$\frac{1}{40}$	3.647148e-008	5.4229
	$\frac{1}{80}$	5.062291e-010	6.1708
	$\frac{1}{160}$	6.799919e-012	6.2181
	$\frac{1}{320}$	9.539537e-014	6.1555
0.8	$\frac{1}{20}$	8.311643e-006	—
	$\frac{1}{40}$	1.432632e-007	5.8584
	$\frac{1}{80}$	1.663072e-009	6.4287
	$\frac{1}{160}$	1.942938e-011	6.4195
	$\frac{1}{320}$	2.433601e-013	6.3190

TABLE 5. The absolute error, convergence order of Example 1 by numerical scheme (2.3) with $p = 6$.

$$
\begin{aligned}
s(x,t) = {} & x^6(1-x)^6\left[\cos(t)(x^4-2x^3+x^2)+\sin(t)(32x^3-288x^2\right.\\
& \left.+256x-56)\right] + \frac{\alpha^2}{2}\sin(t)\sec\left(\frac{\pi}{2}\alpha\right)\left\{\frac{\Gamma(9)}{\Gamma(9-\alpha)}\left[x^{8-\alpha}+(1-x)^{8-\alpha}\right]\right.\\
& -\frac{8\Gamma(10)}{\Gamma(10-\alpha)}\left[x^{9-\alpha}+(1-x)^{9-\alpha}\right]+\frac{28\Gamma(11)}{\Gamma(11-\alpha)}\left[x^{10-\alpha}+(1-x)^{10-\alpha}\right]\\
& -\frac{56\Gamma(12)}{\Gamma(12-\alpha)}\left[x^{11-\alpha}+(1-x)^{11-\alpha}\right]+\frac{70\Gamma(13)}{\Gamma(13-\alpha)}\left[x^{12-\alpha}+(1-x)^{12-\alpha}\right]\\
& -\frac{56\Gamma(14)}{\Gamma(14-\alpha)}\left[x^{13-\alpha}+(1-x)^{13-\alpha}\right]+\frac{28\Gamma(15)}{\Gamma(15-\alpha)}\left[x^{14-\alpha}+(1-x)^{14-\alpha}\right]\\
& \left.-\frac{8\Gamma(16)}{\Gamma(16-\alpha)}\left[x^{15-\alpha}+(1-x)^{15-\alpha}\right]+\frac{\Gamma(17)}{\Gamma(17-\alpha)}\left[x^{16-\alpha}+(1-x)^{16-\alpha}\right]\right\}.
\end{aligned}
$$

α		the maximum errors	temporal convergence orders	spatial convergence orders
0.2	$h=\frac{1}{10},\tau=\frac{1}{10}$	2.581219e-005	—	—
	$h=\frac{1}{20},\tau=\frac{1}{20}$	6.217660e-006	2.0536	2.0536
	$h=\frac{1}{40},\tau=\frac{1}{40}$	1.536085e-006	2.0171	2.0171
	$h=\frac{1}{80},\tau=\frac{1}{80}$	3.844480e-007	1.9984	1.9984
0.3	$h=\frac{1}{10},\tau=\frac{1}{10}$	2.514418e-005	—	—
	$h=\frac{1}{20},\tau=\frac{1}{20}$	6.061411e-006	2.0525	2.0525
	$h=\frac{1}{40},\tau=\frac{1}{40}$	1.498825e-006	2.0158	2.0158
	$h=\frac{1}{80},\tau=\frac{1}{80}$	3.755193e-007	1.9969	1.9969
0.4	$h=\frac{1}{10},\tau=\frac{1}{10}$	2.416676e-005	—	—
	$h=\frac{1}{20},\tau=\frac{1}{20}$	5.842791e-006	2.0483	2.0483
	$h=\frac{1}{40},\tau=\frac{1}{40}$	1.448271e-006	2.0123	2.0123
	$h=\frac{1}{80},\tau=\frac{1}{80}$	3.636282e-007	1.9938	1.9938
0.5	$h=\frac{1}{10},\tau=\frac{1}{10}$	2.270557e-005	—	—
	$h=\frac{1}{20},\tau=\frac{1}{20}$	5.532646e-006	2.0370	2.0370
	$h=\frac{1}{40},\tau=\frac{1}{40}$	1.379247e-006	2.0041	2.0041
	$h=\frac{1}{80},\tau=\frac{1}{80}$	3.477777e-007	1.9876	1.9876
0.6	$h=\frac{1}{10},\tau=\frac{1}{10}$	2.043415e-005	—	—
	$h=\frac{1}{20},\tau=\frac{1}{20}$	5.079755e-006	2.0082	2.0082
	$h=\frac{1}{40},\tau=\frac{1}{40}$	1.283408e-006	1.9848	1.9848
	$h=\frac{1}{80},\tau=\frac{1}{80}$	3.264917e-007	1.9749	1.9749
0.7	$h=\frac{1}{10},\tau=\frac{1}{10}$	1.664449e-005	—	—
	$h=\frac{1}{20},\tau=\frac{1}{20}$	4.378341e-006	1.9266	1.9266
	$h=\frac{1}{40},\tau=\frac{1}{40}$	1.145019e-006	1.9350	1.9350
	$h=\frac{1}{80},\tau=\frac{1}{80}$	2.972789e-007	1.9455	1.9455

TABLE 6. The absolute errors, temporal and spatial convergence orders of Example 2 by difference scheme (3.6).

Its exact solution is $u(x,t)=\sin(t)x^8(1-x)^8$ and satisfy the corresponding initial and boundary values conditions.

The absolute error, temporal and spatial convergence orders are listed in Table 8 by numerical scheme (3.22). The numerical results agree with the theoretical results.

α		the maximum errors	temporal convergence orders	spatial convergence orders
0.2	$h=\frac{1}{4},\tau=\frac{1}{4}$	1.151043e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{16}$	4.361384e-006	2.3610	4.7220
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.346706e-007	2.1081	4.2161
	$h=\frac{1}{32},\tau=\frac{1}{256}$	2.036847e-008	1.7631	3.5262
0.3	$h=\frac{1}{4},\tau=\frac{1}{4}$	1.128702e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{16}$	4.362181e-006	2.3468	4.6935
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.461975e-007	2.0736	4.1472
	$h=\frac{1}{32},\tau=\frac{1}{256}$	2.396154e-008	1.6805	3.3610
0.4	$h=\frac{1}{4},\tau=\frac{1}{4}$	1.088961e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{16}$	4.433621e-006	2.3091	4.6183
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.870612e-007	1.9746	3.9491
	$h=\frac{1}{32},\tau=\frac{1}{256}$	2.910816e-008	1.6509	3.3019
0.5	$h=\frac{1}{4},\tau=\frac{1}{4}$	1.018053e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{16}$	4.654112e-006	2.2256	4.4512
	$h=\frac{1}{16},\tau=\frac{1}{64}$	3.607253e-007	1.8447	3.6895
	$h=\frac{1}{32},\tau=\frac{1}{256}$	3.674011e-008	1.6478	3.2955
0.6	$h=\frac{1}{4},\tau=\frac{1}{4}$	8.897936e-005	—	—
	$h=\frac{1}{8},\tau=\frac{1}{16}$	5.201584e-006	2.0482	4.0964
	$h=\frac{1}{16},\tau=\frac{1}{64}$	4.980729e-007	1.6923	3.3845
	$h=\frac{1}{32},\tau=\frac{1}{256}$	4.862623e-008	1.6783	3.3566
0.7	$h=\frac{1}{4},\tau=\frac{1}{4}$	6.521192e-005	—	—
	$h=\frac{1}{8},\tau=\frac{1}{16}$	6.540139e-006	1.6588	3.3177
	$h=\frac{1}{16},\tau=\frac{1}{64}$	7.724068e-007	1.5410	3.0819
	$h=\frac{1}{32},\tau=\frac{1}{256}$	6.867979e-008	1.7457	3.4914

TABLE 7. The maximum errors, temporal and spatial convergence orders of Example 2 by difference scheme (3.11).

5. Conclusions

In this paper, we construct high-order (from 2nd-order to 6th-order) numerical schemes to approximate the Riesz derivatives. Next, we develop three kinds of difference schemes for the Riesz space fractional turbulent diffusion equation. The stability of the derived numerical algorithms are shown by the Fourier method. The convergence orders are $\mathcal{O}(\tau^2+h^2)$, $\mathcal{O}(\tau^2+h^4)$ and $\mathcal{O}(\tau^2+h^6)$, respectively. Finally, numerical results confirm the theoretical analysis.

α		the maximum errors	temporal convergence orders	spatial convergence orders
0.2	$h=\frac{1}{8},\tau=\frac{1}{8}$	1.360207e-007	—	—
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.071201e-009	2.0124	6.0372
	$h=\frac{1}{32},\tau=\frac{1}{512}$	3.348089e-011	1.9837	5.9510
	$h=\frac{1}{64},\tau=\frac{1}{4096}$	5.235085e-013	1.9997	5.9990
0.3	$h=\frac{1}{8},\tau=\frac{1}{8}$	1.356431e-007	—	—
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.092867e-009	2.0061	6.0182
	$h=\frac{1}{32},\tau=\frac{1}{512}$	3.254863e-011	2.0022	6.0067
	$h=\frac{1}{64},\tau=\frac{1}{4096}$	4.855887e-013	2.0222	6.0667
0.4	$h=\frac{1}{8},\tau=\frac{1}{8}$	1.348600e-007	—	—
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.146379e-009	1.9911	5.9734
	$h=\frac{1}{32},\tau=\frac{1}{512}$	2.972961e-011	2.0580	6.1739
	$h=\frac{1}{64},\tau=\frac{1}{4096}$	3.852828e-013	2.0899	6.2698
0.5	$h=\frac{1}{8},\tau=\frac{1}{8}$	1.335205e-007	—	—
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.246228e-009	1.9645	5.8934
	$h=\frac{1}{32},\tau=\frac{1}{512}$	2.200601e-011	2.2245	6.6735
	$h=\frac{1}{64},\tau=\frac{1}{4096}$	2.816274e-013	2.0960	6.2880
0.6	$h=\frac{1}{8},\tau=\frac{1}{8}$	1.322258e-007	—	—
	$h=\frac{1}{16},\tau=\frac{1}{64}$	2.372915e-009	1.9334	5.8002
	$h=\frac{1}{32},\tau=\frac{1}{512}$	1.827817e-011	2.3401	7.0204
	$h=\frac{1}{64},\tau=\frac{1}{4096}$	4.506292e-013	1.7807	5.3420
0.7	$h=\frac{1}{8},\tau=\frac{1}{8}$	1.357968e-007	—	—
	$h=\frac{1}{16},\tau=\frac{1}{64}$	3.805230e-009	1.7191	5.1573
	$h=\frac{1}{32},\tau=\frac{1}{512}$	6.671019e-011	1.9446	5.8339
	$h=\frac{1}{64},\tau=\frac{1}{4096}$	1.819328e-012	1.7321	5.1964

TABLE 8. The maximum errors, temporal and spatial convergence orders of Example 3 by difference scheme (3.22).

Appendix A

Proof of Theorem **2.3**.

(1) Direct calculations can finish it so we omit the details.

(2) See [13] for details.

(3) Now we show the case $\alpha \in (1,2)$. We firstly show that $\varpi_{2,j}^{(\alpha)} > 0$ for $\ell \geq 5$. For convenience, denote $\alpha = 1+\gamma$, where $0 < \gamma < 1$. Lengthy calculations give

$$\varpi_{2,\ell}^{(\alpha)} = \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=0}^{\ell} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)}$$

$$= \left(\frac{3}{2}\right)^{1+\gamma} \left[\varpi_{1,0}^{(1+\gamma)} \varpi_{1,\ell}^{(1+\gamma)} + \frac{1}{3}\varpi_{1,1}^{(1+\gamma)} \varpi_{1,\ell-1}^{(1+\gamma)} + \frac{1}{9}\varpi_{1,2}^{(1+\gamma)} \varpi_{1,\ell-2}^{(1+\gamma)}\right.$$

$$\left. + \left(\frac{1}{3}\right)^{\ell} \varpi_{1,0}^{(1+\gamma)} \varpi_{1,\ell}^{(1+\gamma)} + \left(\frac{1}{3}\right)^{\ell-1} \varpi_{1,1}^{(1+\gamma)} \varpi_{1,\ell-1}^{(1+\gamma)}\right]$$

$$+ \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=3}^{\ell-2} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)}$$

$$= \left(\frac{3}{2}\right)^{1+\gamma} \left[1 - \frac{\gamma+1}{3} \frac{\ell}{(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)}{18} \frac{\ell(\ell-1)}{(\ell-2-\gamma)(\ell-3-\gamma)}\right.$$

$$\left. + \left(\frac{1}{3}\right)^{\ell} \left(1 - \frac{3\ell(\gamma+1)}{(\ell-2-\gamma)}\right)\right] \varpi_{1,\ell}^{(1+\gamma)} + \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=3}^{\ell-2} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)}$$

$$\geq \left(\frac{3}{2}\right)^{1+\gamma} \left[\frac{244}{243} - \frac{28(\gamma+1)}{81} \frac{\ell}{(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)}{18} \frac{\ell(\ell-1)}{(\ell-2-\gamma)^2}\right] \varpi_{1,\ell}^{(1+\gamma)}$$

$$+ \left(\frac{3}{2}\right)^{1+\gamma} \sum_{m=3}^{\ell-2} \left(\frac{1}{3}\right)^m \varpi_{1,m}^{(1+\gamma)} \varpi_{1,\ell-m}^{(1+\gamma)}, \quad \ell \geq 5.$$

Let

$$F(x,\gamma) = \frac{244}{243} - \frac{28(\gamma+1)}{81} \frac{x}{(x-2-\gamma)} + \frac{\gamma(\gamma+1)}{18} \frac{x(x-1)}{(x-2-\gamma)^2}, \ x \geq 5,$$

and

$$G(x,\gamma) = 486(x-2-\gamma)^2 F(x,\gamma)$$

$$= 488(x-2-\gamma)^2 - 168(\gamma+1)x(x-2-\gamma) + 27\gamma(\gamma+1)x(x-1), \ x \geq 5.$$

Then

$$G_x(x,\gamma) = 976(x-2-\gamma) - 168(\gamma+1)(2x-2-\gamma) + 27\gamma(\gamma+1)(2x-1)$$

and

$$G_{xx}(x,\gamma) = 54\gamma^2 - 282\gamma + 640.$$

Obviously,

$$G_{xx}(x,\gamma) \geq 0 \ \text{ for } \ 0 < \gamma < 1,$$

it immediately follows that $G_x(x,\gamma)$ is an increasing function and $G_x(x,\gamma) \geq G_x(5,\gamma)$ for $x \geq 5$.

Note that

$$G_x(5,\gamma) = 411\gamma^2 - 1909\gamma + 1585 > 0, \quad 0 < \gamma < 1.$$

Hence $G(x,\gamma)$ is an increasing function too, and $G(x,\gamma) \geq G(5,\gamma)$ if $x \geq 5$. Simple calculations yields

$$G(5,\gamma) = 1868\gamma^2 - 3068\gamma + 1872 \geq G_{\min}(5,\gamma) = G\left(5, \frac{767}{934}\right) = 612\frac{131}{467},$$

so, $G(x,\gamma) \geq 0$. Therefore, the following inequality holds

$$F(x,\gamma) = \frac{G(x,\gamma)}{486(x-2-\gamma)^2} \geq 0,$$

which means $\varpi_{2,j}^{(1+\gamma)} > 0$ for $\ell \geq 5$. Here we used the positivity of the coefficient $\varpi_{1,\ell}^{(1+\gamma)} (\ell \geq 2)$.

Next, we show that $\varpi_{2,\ell}^{(\alpha)} > \varpi_{2,\ell+1}^{(\alpha)}$ for $\ell \geq 5$. Note that

$$\begin{aligned}
&\varpi_{2,\ell}^{(\alpha)} - \varpi_{2,\ell+1}^{(\alpha)} = (2+\gamma)\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell}\sum_{\ell_1=0}^{\ell}\frac{3^{\ell_1}}{\ell_1+1}\varpi_{1,\ell_1}^{(1+\gamma)}\varpi_{1,\ell-\ell_1}^{(1+\gamma)} \\
&-\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell+1}\left(1-\frac{2+\gamma}{\ell+1}\right)\varpi_{1,\ell}^{(1+\gamma)} \\
&=\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell}\left[(2+\gamma)\sum_{\ell_1=0}^{\ell}\frac{3^{\ell_1}}{\ell_1+1}\varpi_{1,\ell_1}^{(1+\gamma)}\varpi_{1,\ell-\ell_1}^{(1+\gamma)} - \frac{1}{3}\varpi_{1,\ell}^{(1+\gamma)}\right] \\
&+\frac{2+\gamma}{\ell+1}\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell+1}\varpi_{1,\ell}^{(1+\gamma)} \\
&\geq (2+\gamma)\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell}\left[\sum_{\ell_1=0}^{\ell}\frac{3^{\ell_1}}{\ell_1+1}\varpi_{1,\ell_1}^{(1+\gamma)}\varpi_{1,\ell-\ell_1}^{(1+\gamma)} - \frac{1}{6}\varpi_{1,\ell}^{(1+\gamma)}\right] \\
&+\frac{2+\gamma}{\ell+1}\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell+1}\varpi_{1,\ell}^{(1+\gamma)} \\
&= (2+\gamma)\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell}P(\ell,\gamma)\varpi_{1,\ell}^{(1+\gamma)} + \frac{2+\gamma}{\ell+1}\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell+1}\varpi_{1,\ell}^{(1+\gamma)} \\
&+(2+\gamma)\left(\frac{3}{2}\right)^{1+\gamma}\left(\frac{1}{3}\right)^{\ell}\sum_{\ell_1=3}^{\ell-3}\frac{3^{\ell_1}}{\ell_1+1}\varpi_{1,\ell_1}^{(1+\gamma)}\varpi_{1,\ell-\ell_1}^{(1+\gamma)}.
\end{aligned}$$

Here

$$
\begin{aligned}
P(\ell,\gamma) \;\; = \;\; & \left[\frac{5}{6} - \frac{3\ell(\gamma+1)}{2(\ell-2-\gamma)} + \frac{3\gamma(\gamma+1)\ell(\ell-1)}{2(\ell-2-\gamma)(\ell-3-\gamma)}\right] \\
& + \frac{3^{\ell}}{\ell+1}\left[1 - \frac{(\ell+1)(\gamma+1)}{3(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)\ell(\ell+1)}{18(\ell-2-\gamma)(\ell-3-\gamma)}\right].
\end{aligned}
$$

Obviously, the last two terms in the right-hand side of the last equality are both nonnegative, so we only need prove that the factor $P(\ell,\gamma)$ in the first term is nonnegative.

Let

$$
\begin{aligned}
P_1(\ell,\gamma) &= \frac{5}{6} - \frac{3\ell(\gamma+1)}{2(\ell-2-\gamma)} + \frac{3\gamma(\gamma+1)\ell(\ell-1)}{2(\ell-2-\gamma)(\ell-3-\gamma)}, \\
P_2(\ell,\gamma) &= 1 - \frac{(\ell+1)(\gamma+1)}{3(\ell-2-\gamma)} + \frac{\gamma(\gamma+1)\ell(\ell+1)}{18(\ell-2-\gamma)(\ell-3-\gamma)},
\end{aligned}
$$

then

$$
P(\ell,\gamma) = P_1(\ell,\gamma) + \frac{3^{\ell}}{\ell+1} P_2(\ell,\gamma).
$$

If $\ell = 5$, then

$$
P_2(5,\gamma) = 1 - \frac{2(\gamma+1)}{(3-\gamma)} + \frac{5\gamma(\gamma+1)}{3(3-\gamma)(2-\gamma)} > 0.
$$

Now we consider the case $\ell \geq 6$. Let

$$
Q(x,\gamma) = 18(x-2-\gamma)(x-2.5-\gamma)P_3(x,\gamma), \;\; x \in [6,\infty),
$$

where

$$
P_3(x,\gamma) = 1 - \frac{(x+1)(\gamma+1)}{3(x-2-\gamma)} + \frac{\gamma(\gamma+1)x(x+1)}{18(x-2-\gamma)(x-2.5-\gamma)}.
$$

By simple calculations, one has

$$
Q_{xx}(x,\gamma) = 2\gamma^2 - 10\gamma + 24 > 0, \;\; 0 < \gamma < 1.
$$

So $Q_x(x,\gamma)$ is an increasing function and

$$
Q_x(x,\gamma) \geq Q_x(6,\gamma) = 19\gamma^2 - 80\gamma + 72 > 0, \; x \geq 6, \; 0 < \gamma < 1.
$$

It immediately follows that $Q(x,\gamma)$ is an increasing function with respect to x and

$$
Q(x,\gamma) \geq Q(6,\gamma) = 102\gamma^2 - 198\gamma + 105 > 0, \;\; 0 < \gamma < 1,
$$

i.e., $P_3(x,\gamma) > 0$. Noticing $P_2(\ell,\gamma) > P_3(\ell,\gamma)$ yields

$$
P_2(\ell,\gamma) > 0, \;\; \ell \in [5,\infty).
$$

Therefore,

$$\begin{aligned} P(\ell,\gamma) &= P_1(\ell,\gamma) + \frac{3^\ell}{\ell+1}P_2(\ell,\gamma) \ge P_1(\ell,\gamma) + \frac{81}{2}P_2(\ell,\gamma) \\ &= \frac{124}{3} - \frac{3(\gamma+1)(10\ell+9)}{2(\ell-2-\gamma)} + \frac{3\ell\gamma(\gamma+1)(5\ell+1)}{4(\ell-2-\gamma)(\ell-3-\gamma)} \\ &> \frac{124}{3} - \frac{3(\gamma+1)(10\ell+9)}{2(\ell-2-\gamma)} + \frac{3\ell\gamma(\gamma+1)(5\ell+1)}{4(\ell-2-\gamma)^2}. \end{aligned}$$

When $\ell = 5$, we easily know that $P(\ell,\gamma) > 0$ by direct calculations. Next, we discuss the case $\ell \ge 6$. Let

$$P_4(\ell,\gamma) = \frac{124}{3} - \frac{3(\gamma+1)(10\ell+9)}{2(\ell-2-\gamma)} + \frac{3\ell\gamma(\gamma+1)(5\ell+1)}{4(\ell-2-\gamma)^2},$$

and

$$R(x,\gamma) = 12(x-2-\gamma)^2 P_4(x,\gamma),\ x \ge 6.$$

Differentiating twice with respect to x gives

$$R_{xx}(x,\gamma) = 90\gamma^2 - 270\gamma + 632,$$

which is positive when $\gamma \in (0,1)$.

So, $R_x(x,\gamma)$ is an increasing function and

$$R_x(x,\gamma) > R_x(6,\gamma) = 729\gamma^2 - 2225\gamma + 2006 > 0.$$

Furthermore, $R(x,\gamma)$ is an increasing function as well and

$$R(x,\gamma) > R(6,\gamma) = 3412\gamma^2 - 6020\gamma + 2968 > 0, \quad x \in [6,\infty).$$

So, $P_4(\ell,\gamma) > 0$ implies $P(\ell,\gamma) > 0$. It follows that $\varpi_{2,\ell}^{(\alpha)} \ge \varpi_{2,\ell+1}^{(\alpha)}$ for $\ell \ge 5$.

All this completes the proof. □

Appendix B

First, we list a lemma which comes from [4, 11].

Lemma A.

(*i*) $1 - x < \exp(-x)$ *holds for* $0 < x < 1$.

(*ii*) $1 - x > \exp(-2x)$ *holds for* $0 < x \le 0.7968$.

(*iii*) $(1+x)^\alpha \ge 1 + \alpha x$ *holds for* $\alpha \le 0$ *or* $\alpha \ge 1$,

(*iv*) $(1 + x_1 + x_2 + x_3 + \cdots + x_n) \le (1+x_1)(1+x_2)(1+x_3)\cdots(1+x_s)$, *where* $x_m \ge -1$ *and* $sign(x_m) = sign(x_n)$ *for* $\forall\, 1 \le m, n \le s$.

P r o o f o f T h e o r e m **2.5**.

(1) In view of (ii) of Lemma A, one has

$$\left|\varpi_{1,\ell}^{(\alpha)}\right| = \left(1-\frac{\alpha+1}{\ell}\right)\left|\varpi_{1,\ell-1}^{(\alpha)}\right|$$
$$= \left(1-\frac{\alpha+1}{\ell}\right)\left(1-\frac{\alpha+1}{\ell-1}\right)\cdots\left(1-\frac{\alpha+1}{3}\right)\left|\varpi_{1,2}^{(\alpha)}\right|$$
$$\geq \exp\left(-\frac{2(\alpha+1)}{\ell}\right)\exp\left(-\frac{2(\alpha+1)}{\ell-1}\right)\cdots\exp\left(-\frac{2(\alpha+1)}{3}\right)\left|\varpi_{1,2}^{(\alpha)}\right|$$
$$= \exp\left(-2(\alpha+1)\sum_{k=3}^{\ell}\frac{1}{k}\right)\left|\varpi_{1,2}^{(\alpha)}\right|.$$

Note that

$$\sum_{k=3}^{\ell}\frac{1}{k} < \int_2^{\ell}\frac{1}{x}\mathrm{d}x = \ln\frac{\ell}{2},$$

so we get

$$\left|\varpi_{1,\ell}^{(\alpha)}\right| > \exp\left(-2(\alpha+1)\ln\frac{\ell}{2}\right)\left|\varpi_{1,2}^{(\alpha)}\right| = \frac{\alpha(1-\alpha)}{2}\left(\frac{2}{\ell}\right)^{2(\alpha+1)}.$$

From [4], we know that

$$\left|\varpi_{1,\ell}^{(\alpha)}\right| < \frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}},$$

i.e.,

$$B_1^L(\alpha,\ell) < \left|\varpi_{1,\ell}^{(\alpha)}\right| < B_1^R(\alpha,\ell),\ \textit{where}\ \ B_1^L(\alpha,\ell) = \frac{\alpha(1-\alpha)}{2}\left(\frac{2}{\ell}\right)^{2(\alpha+1)},$$
$$B_1^R(\alpha,\ell) = \alpha\left(\tfrac{2}{\ell+1}\right)^{\alpha+1},\quad \ell\geq 3.$$

(2) From (1), we have

$$\sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(\alpha)}\right| > \alpha(1-\alpha)2^{2\alpha+1}\sum_{k=\ell}^{\infty}\frac{1}{k^{2(\alpha+1)}}.$$

Because of

$$\sum_{k=\ell}^{\infty}\frac{1}{k^{2(\alpha+1)}} > \int_{\ell}^{\infty}\frac{1}{x^{2(\alpha+1)}}\mathrm{d}x = \frac{1}{(2\alpha+1)\ell^{2\alpha+1}},$$

we get

$$\sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(\alpha)}\right| > \frac{\alpha(1-\alpha)}{2\alpha+1}\left(\frac{2}{\ell}\right)^{2\alpha+1}.$$

In addition, we also have [4]

$$\sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(\alpha)}\right| < 2\left(\frac{2}{\ell}\right)^{\alpha},$$

it immediately follows that

$$S_1^L(\alpha,\ell) < \sum_{k=\ell}^{\infty}\left|\varpi_{1,k}^{(\alpha)}\right| < B_1^R(\alpha,\ell),\ where\ \ S_1^L(\alpha,\ell) = \frac{\alpha(1-\alpha)}{2\alpha+1}\left(\frac{2}{\ell}\right)^{2\alpha+1},$$

$$B_1^R(\alpha,\ell) = 2\left(\frac{2}{\ell}\right)^{\alpha},\quad \ell \geq 3.$$

□

Proof of Theorem **2.6**.

(1) Let

$$\alpha_\ell = \frac{12\ln\frac{\ell}{2}}{2\pi^2-15} - 1.$$

Then from $\alpha < \alpha_\ell$, we can get

$$\left(\frac{\ell}{2}\right)^{1+\alpha}\exp\left(-(\alpha+1)^2\left(\frac{\pi^2}{6}-\frac{5}{4}\right)\right) > 1,$$

i.e.,

$$\frac{B_1^L(\alpha,\ell)}{\widetilde{B}_1^L(\alpha,\ell)} < 1.$$

Obviously, the conditions for the above inequality are

$$\alpha_3 \approx 0.0267,\ \alpha_4 \approx 0.7551,\ \text{ and } \ \alpha_5 \geq \frac{5}{2}\ln\frac{5}{2} - 1 > \frac{9}{4} - 1 > 1.$$

(2) Let

$$f(\alpha,\ell) = \frac{\widetilde{B}_1^L(\alpha,\ell)}{B_1^L(\alpha,\ell)} = \frac{2\alpha+1}{5\alpha}\left(\frac{\ell}{2}\right)^{1+\alpha},$$

and

$$g(\alpha,\ell) = \ln f(\alpha,\ell) = \ln(2\alpha+1) - \ln(5\alpha) + (1+\alpha)\ln\frac{\ell}{2}.$$

Then

$$g_\alpha(\alpha,3) = \frac{2}{2\alpha+1} - \frac{1}{\alpha} + \ln\frac{3}{2},$$

and

$$g_{\alpha\alpha}(\alpha,3) = \frac{4\alpha+1}{\alpha^2(2\alpha+1)^2} > 0,$$

that is to say that the function $g(\alpha,3)$ has a minimum value and from the equation $g_\alpha(\alpha,3) = 0$ we know that the minimum point is

$$\alpha^\star = \frac{-1+\sqrt{1+\frac{8}{\ln\frac{3}{2}}}}{4}.$$

Therefore,

$$\begin{aligned} f(\alpha,\ell) &> f(\alpha,3) \geq f(\alpha^\star,3) = \exp(g(\alpha^\star,3)) \\ &= \frac{2\alpha^\star+1}{5\alpha^\star}\left(\frac{3}{2}\right)^{1+\alpha^\star} \approx 1.3443 > 1, \end{aligned}$$

i.e., $B_1^L(\alpha,\ell) < \widetilde{B}_1^L(\alpha,\ell)$. □

P r o o f o f T h e o r e m **2.7**: is almost the same as that of Theorem **2.5**.

P r o o f o f T h e o r e m **2.8**.

(1) Note that $\varpi_{2,\ell}^{(\alpha)} \leq 0$ for $\ell \geq 3$ and $\varpi_{1,\ell}^{(\alpha)} \leq 0$ for $\ell \geq 1$ $(0<\alpha<1)$, then

$$\begin{aligned} \left|\varpi_{2,\ell}^{(\alpha)}\right| &= \left(\frac{3}{2}\right)^\alpha \left|\sum_{\ell_1=0}^{\ell}\left(\frac{1}{3}\right)^{\ell_1}\varpi_{1,\ell_1}^{(\alpha)}\varpi_{1,\ell-\ell_1}^{(\alpha)}\right| \\ &= \left(\frac{3}{2}\right)^\alpha \left|\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\varpi_{1,\ell}^{(\alpha)} + \sum_{\ell_1=1}^{\ell-1}\left(\frac{1}{3}\right)^{\ell_1}\varpi_{1,\ell_1}^{(\alpha)}\varpi_{1,\ell-\ell_1}^{(\alpha)}\right| \\ &= \left(\frac{3}{2}\right)^\alpha \left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\left|\varpi_{1,\ell}^{(\alpha)}\right| - \sum_{\ell_1=1}^{\ell-1}\left(\frac{1}{3}\right)^{\ell_1}\left|\varpi_{1,\ell_1}^{(\alpha)}\right|\left|\varpi_{1,\ell-\ell_1}^{(\alpha)}\right|\right]. \end{aligned}$$

From Lemma A, we know that

$$\begin{aligned} &(1+\ell_1)^{1+\alpha}(1+(\ell-\ell_1))^{1+\alpha} \geq (1+(\alpha+1)\ell_1)(1+(\alpha+1)(\ell-\ell_1)) \\ &\geq 1+(\alpha+1)\ell \end{aligned}$$

and

$$\ell_1^{2(1+\alpha)}(\ell-\ell_1)^{2(1+\alpha)} \leq \left(\left(\frac{\ell_1+(\ell-\ell_1)}{2}\right)^2\right)^{2(1+\alpha)} = \left(\frac{\ell}{2}\right)^{4(\alpha+1)}.$$

So,

$$\left|\varpi_{2,\ell}^{(\alpha)}\right| = \left(\frac{3}{2}\right)^\alpha \left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\left|\varpi_{1,\ell}^{(\alpha)}\right| - \sum_{\ell_1=1}^{\ell-1}\left(\frac{1}{3}\right)^{\ell_1}\left|\varpi_{1,\ell_1}^{(\alpha)}\right|\left|\varpi_{1,\ell-\ell_1}^{(\alpha)}\right|\right]$$

$$\leq \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}}-\sum_{\ell_1=1}^{\ell-1}\left(\frac{1}{3}\right)^{\ell_1}\frac{\alpha^2(1-\alpha)^2 4^{2\alpha+1}}{(\ell-\ell_1)^{2(\alpha+1)}\ell_1^{2(\alpha+1)}}\right]$$

$$\leq \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha 2^{\alpha+1}}{(\ell+1)^{\alpha+1}}-\frac{\alpha^2(1-\alpha)^2 4^{2\alpha+1}}{2}\right.$$

$$\left.\times\left(1-\left(\frac{1}{3}\right)^{\ell-1}\right)\left(\frac{2}{\ell}\right)^{4(\alpha+1)}\right],$$

and

$$\left|\varpi_{2,\ell}^{(\alpha)}\right| = \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\left|\varpi_{1,\ell}^{(\alpha)}\right|-\sum_{\ell_1=1}^{\ell-1}\left(\frac{1}{3}\right)^{\ell_1}\left|\varpi_{1,\ell_1}^{(\alpha)}\right|\left|\varpi_{1,\ell-\ell_1}^{(\alpha)}\right|\right]$$

$$\geq \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha(1-\alpha)}{2}\left(\frac{2}{\ell}\right)^{2(1+\alpha)}\right.$$

$$\left.-\alpha^2 2^{2(\alpha+1)}\sum_{\ell_1=1}^{\ell-1}\left(\frac{1}{3}\right)^{\ell_1}\frac{1}{(1+\ell-\ell_1)^{\alpha+1}(1+\ell_1)^{\alpha+1}}\right]$$

$$\geq \left(\frac{3}{2}\right)^{\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha(1-\alpha)}{2}\left(\frac{2}{\ell}\right)^{2(1+\alpha)}\right.$$

$$\left.-\left(1-\left(\frac{1}{3}\right)^{\ell-1}\right)\frac{\alpha^2 2^{2\alpha+1}}{1+(\alpha+1)\ell}\right].$$

(2) Note that $\varpi_{2,\ell}^{(1+\alpha)} \geq 0$ for $\ell \geq 4$ and $\varpi_{1,\ell}^{(1+\alpha)} \geq 0$ for $\ell \geq 2$ $(0 < \alpha < 1)$, then

$$\left|\varpi_{2,\ell}^{(1+\alpha)}\right| = \left(\frac{3}{2}\right)^{1+\alpha}\left|\sum_{\ell_1=0}^{\ell}\left(\frac{1}{3}\right)^{\ell_1}\varpi_{1,\ell_1}^{(1+\alpha)}\varpi_{1,\ell-\ell_1}^{(1+\alpha)}\right|$$

$$= \left(\frac{3}{2}\right)^{1+\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\left|\varpi_{1,\ell}^{(1+\alpha)}\right|+\sum_{\ell_1=2}^{\ell-2}\left(\frac{1}{3}\right)^{\ell_1}\left|\varpi_{1,\ell_1}^{(1+\alpha)}\right|\left|\varpi_{1,\ell-\ell_1}^{(1+\alpha)}\right|\right.$$

$$\left.-\left(\frac{1}{3}+\left(\frac{1}{3}\right)^{\ell-1}\right)(1+\alpha)\left|\varpi_{1,\ell-1}^{(1+\alpha)}\right|\right].$$

So,

$$
\begin{aligned}
&\left|\varpi_{2,\ell}^{(1+\alpha)}\right| \le \left(\frac{3}{2}\right)^{1+\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha(\alpha+1)3^{\alpha+2}}{2(\ell+1)^{\alpha+2}}\right. \\
&+\sum_{\ell_1=2}^{\ell-2}\left(\frac{1}{3}\right)^{\ell_1}\frac{\alpha^2(1+\alpha)^2 3^{2(2+\alpha)}}{4(\ell_1+1)^{\alpha+2}(\ell-\ell_1+1)^{2+\alpha}} \\
&\left.-\frac{(1-\alpha)\alpha(1+\alpha)^2}{6}\left(\frac{3}{\ell-1}\right)^{2(2+\alpha)}\left(\frac{1}{3}+\left(\frac{1}{3}\right)^{\ell-1}\right)\right] \\
&\le \left(\frac{3}{2}\right)^{1+\alpha}\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{\alpha(\alpha+1)3^{\alpha+2}}{2(\ell+1)^{\alpha+2}}+\left(1-\left(\frac{1}{3}\right)^{\ell-3}\right)\cdot\right. \\
&\left.\frac{\alpha^2(1+\alpha)^2 3^{2(2+\alpha)}}{24(1+(2+\alpha)\ell)}-\frac{(1-\alpha)\alpha(1+\alpha)^2}{6}\left(\frac{1}{3}+\left(\frac{1}{3}\right)^{\ell-1}\right)\left(\frac{3}{\ell-1}\right)^{2(2+\alpha)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\varpi_{2,\ell}^{(1+\alpha)}\right| \ge \left(\frac{3}{2}\right)^{1+\alpha}&\left[\left(1+\left(\frac{1}{3}\right)^{\ell}\right)\frac{(1-\alpha)\alpha(1+\alpha)}{6}\left(\frac{3}{\ell}\right)^{2(2+\alpha)}\right. \\
&+\frac{(1-\alpha)^2\alpha^2(1+\alpha)^2}{216}\left(1-\left(\frac{1}{3}\right)^{\ell-3}\right)\left(\frac{6}{\ell}\right)^{4(2+\alpha)} \\
&\left.-\frac{\alpha(1+\alpha)^2}{2}\left(\frac{1}{3}+\left(\frac{1}{3}\right)^{\ell-1}\right)\left(\frac{3}{\ell}\right)^{2+\alpha}\right].
\end{aligned}
$$

The proof is thus complete. □

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[1] *School of Mathematics and Statistics, Tianshui Normal University*
Tianshui 741001, CHINA

e-mail: dinghf05@163.com

[2] *Department of Mathematics, Shanghai University*
Shanghai 200444, CHINA

e-mail: lcp@shu.edu.cn *Received: April 2, 2015*

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