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It’s a Long Way to the Top
(If You Wanna Rock ’N’ Roll)

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Chapter 1

Introduction

Stochastic analysis has played a key role in the development of financial markets. It has provided practitioners with scientific tools that helped them agreeing on which is the fair price of any derivative contract traded and therefore, standardize the financial industry's activity. The simplest derivative contracts are financial instruments that involve two counterparties. These two participants agree today on the form of a future payment, often referred to as the payoff, that the first counterparty will pay to the second one on a future given date known as the contract's maturity. The payoff of this contract will be linked to the performance of an underlying asset during the time to maturity, and some fixed price negotiated between both counterparties, also known as the strike price. The difficulty arises when trying to give a fair price to this derivative contract today. The question can be posed in simple terms as follows:

"How much should the counterparty A pay today to the counterparty B, to acquire the right that the counterparty B pays back the agreed payoff in a future date?"

This, a priory, simple question had remained unanswered for many years and is still today hard to answer. The following subsection provides the reader with a quick introduction to option pricing and a chronological historic background. This will lead into some of the questions that this thesis attempts to answer, through a collection of research articles developed over the past three years. Each of the papers are presented along Chapters 2 to 5, some of which are a bit more theoretical and others try to answer more applied questions. All of the research is somehow connected to the previous key question.

The rest of the subsections in this introduction provide the reader with the techniques and references used later in the articles for a better understanding. Special attention will be put on Malliavin calculus and fractional Brownian motion among other technical results. A summary of the papers is also provided to resume what each of the articles attempt to answer, helping the reader move through the different parts of this work with extra agility. Finally, some ideas on how to perform further research, and how to extend the results presented in this work will be sketched.

1.1 Option Pricing Framework and Historical Background

Many different researchers have been key to the development of mathematical models for financial markets. The first of them was Louis Bachelier (1900), who introduced the basis of mathematical finance in his PhD thesis [5] and set an agenda for the future of probability theory and stochastic analysis. He was the first to use a mathematical model to reproduce the behavior of asset prices after conjecturing that asset prices followed a normal distribution. In modern probability theory, his idea would formally be stated through the following concepts. Mainly, a complete probability space $\left(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, a natural (right-continuous) filtration given by $\mathbb{F} \triangleq \{\mathcal{F}_t, t \in [0,T]\}$ and generated by a standard Brownian Motion $\{W_t, t \in [0,T]\}$. Assuming a constant interest rate equal to zero, and letting $\{S_t, t \in [0,T]\}$ be the stock price process, Bachelier suggested the following equation according to the normality hypothesis.

$$S_t = S_0 + \sigma W_t, \qquad 0 \le t \le T, \tag{1.1.1}$$

where S_0 is the current price of the stock. An obvious deficiency in Bachelier's model is that stock prices being normal, can lead to negative prices at any time $t \in [0,T]$. To overcome this problem, Samuelson introduced in [37] (1955) the geometric Brownian motion model (GBM), in which the stock price S_t, in the risk-neutral setup given by the equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ (see Theorem 1.4.6), is given by

$$S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t}, \tag{1.1.2}$$

where r, σ are constants. Since $W_t \sim N(0,t)$ we have that $\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma W_t}\right] = e^{\frac{1}{2}\sigma^2 t}$. Therefore, the expectation of S_t is $\mathbb{E}^{\mathbb{Q}}[S_t] = S_0 e^{rt}$. This implies that the expected growth rate of the stock is r. This is key to risk-neutral pricing as proposed later. Note that the parameter σ, known as volatility, measures the standard deviation of log-returns, i.e. the standard deviation of $\log(S_{t+h}/S_t)$ is $\sigma\sqrt{h}$. This volatility parameter is the cornerstone of the later developments in financial modeling that serve as motivation for this thesis. In parallel Kiyoshi Itô developed the concept of stochastic integral in 1951, see for instance [23], where he gives an interpretation to the following expression:

$$I_t(X) \triangleq \int_0^t X_s dW_s, \tag{1.1.3}$$

where $\{X_s, s \in [0,t]\}$ is an adapted stochastic process, integrated with respect to a Brownian motion. He also provided formalization of a continuous time stochastic evolution given by the following stochastic differential equation (SDE):

$$dS_t = a(t, S_t)\, dt + b(t, S_t)\, dW_t, \qquad t \in [0,T], \tag{1.1.4}$$

where a, b are sufficiently regular functions. The Itô formula was also being developed during that time, allowing to establish a relationship between a SDE for some independent variable S_t and a SDE for a function of that variable, i.e. $f(t, S_t)$ for a certain family of functions. Itô's formula is a stochastic version of the classical chain rule of differentiation and prescribes how a function of a stochastic process $f(t, S_t)$ changes stochastically as time changes.

Theorem 1.1.1. (Itô formula). Let S_t be the process given in (1.1.4) and consider a function $f(t, x) \in C^{1,2}([0, T] \times \mathbb{R}_+)$. Then

$$\begin{aligned} df(t, S_t) &= \left(\partial_t f(t, S_t) + \partial_x f(t, S_t)\, a(t, S_t) + \frac{1}{2}\partial^2_{xx} f(t, S_t)\, b(t, S_t)^2\right) dt \\ &+ \partial_x f(t, S_t)\, b(t, S_t)\, dW_t. \end{aligned} \tag{1.1.5}$$

By 1960 the Itô integral, SDE's and the connection with the heat equation were already understood and this propitiated that in 1973, Fisher Black, Myron Scholes and Robert Merton derived the celebrated Black-Scholes option pricing formula in two separate papers, both in 1973 (Black-Scholes in [12] and Merton in [31]). Their study awarded Scholes and Merton the Nobel Prize for Economics in 1997. Black had died in 1995. Using a model based on the geometric Brownian motion from equation (1.1.2), and given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1.6}$$

where S is the asset price, $\mu \in \mathbb{R}$ is the drift parameter and $\sigma \in \mathbb{R}_+$ is the volatility or diffusion parameter, which was assumed to be constant. The authors managed to derive the Black-Scholes partial differential equation that an option value V_t had to fulfill, this is,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1.7}$$

where r is the interest rate from a riskless asset, also referred to as the bank account. The previous equation provides the condition for an investor to be indifferent to either a risky or riskless investment. So far, all the previous work had been addressed towards pricing equity options, but little had been done on interest rates. In 1977, Oldrich Vasicek developed a framework for pricing interest rate options in [39]. A first model for short-term interest rate using the geometric Brownian motion from equation (1.1.2) led to an SDE of the form

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t. \tag{1.1.8}$$

The bond pricing equation was postulated as a parabolic partial differential equation (PDE), similar to the Black-Scholes equity counterparty formula.

In 1981 following the analogy established by Black, Scholes and Merton, between risky and riskless assets, Harrison and Pliska introduced the risk-neutrality concept for pricing contingent claims. In [20], the authors develop the risk-neutral pricing formula by means of martingale theory, which is an essential tool of stochastic calculus.

Theorem 1.1.2. (Risk-Neutral Pricing Formula). In an arbitrage-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, such that for all $r \in \mathbb{R}_+$ and $t \in [0, T]$, the price at time t, of a contingent claim $h(S_t)$ is given by

$$V_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[h(S_T) \mid \mathcal{F}_t \right].$$

A basic requirement on any option pricing model is to match observed market prices at any given time t. In order to achieve the previous requirement, the model parameters are chosen such that model prices fit observed market prices. One would also require any good model, to capture the main features in the observed prices. Recalling from [13] that a European call option on an asset S_t that pays no dividends, with maturity date $T > 0$ and strike price K, is given by the payoff $(S_T - K)^+$, the Black-Scholes formula that provides the value of this call option is given by

$$C^{BS}(S_t, K, \tau, \sigma) = S_t \Phi(d_+) - K e^{-r\tau} \Phi(d_-),$$

$$d_+ = \frac{-\ln(S_t/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad d_+ = d_- + \sigma\sqrt{\tau},$$

where $\tau = T - t$ is the time to maturity, Φ is a standard normal cumulative distribution function and $\sigma \in \mathbb{R}_+$ is the constant parameter for volatility. Given that the Black-Scholes function is a strictly increasing function with respect to the volatility $\sigma \in (0, +\infty)$, one can find the theoretical value $\Sigma_t(\tau, k)$ of the volatility parameter, such that Black-Scholes model prices match the observed market prices, i.e. $C^{BS}_{mkt} = C^{BS}(S_t, K, \tau, \Sigma_t(\tau, k))$, where the function

$$\Sigma_t : (\tau, k) \longrightarrow \Sigma_t(\tau, k),$$

is called the implied volatility surface at a fixed date t and $k = \log(S_t/K)$ is often referred to as log-moneyness. The following Figure 1.1, found in [6], shows an implied volatility surface for the options on S&P500 as of August 14, 2003. While the Black-Scholes model from (1.1.6) assumed the implied volatility to be constant, i.e.

$$\Sigma_t(\tau, k) = \sigma,$$

it is clearly observed from empirical data that there exists a strong dependence[1] of implied volatility with respect to strike prices K and time to maturity τ. In other words, one cannot properly calibrate the model to observed market prices with a constant value of σ. Therefore the assumption that volatility is constant, made in the Black-Scholes model, seems to be no longer realistic.

At this point stochastic volatility (SV) models come to play, allowing the volatility parameter σ to vary in a random fashion, see [17] for a detailed

[1] This dependence is often referred to as "skew" if the implied volatility is a decreasing function of strike prices, or "smile" if implied volatility is U-shaped with respect to strike prices.

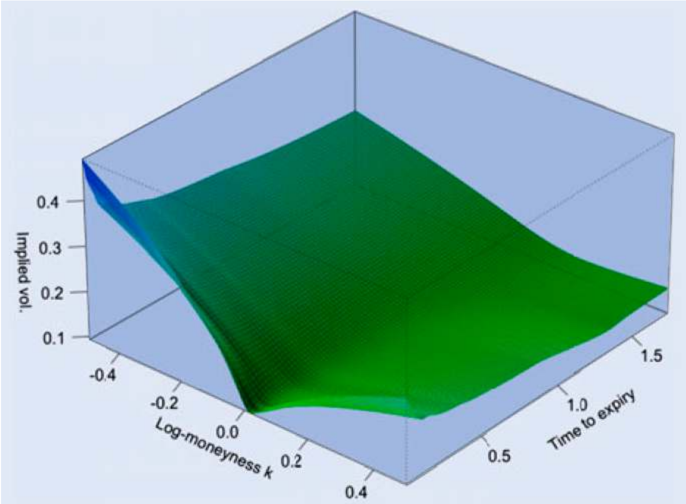

Figure 1.1: SPX volatility surface as of August 14, 2013. Time measured in years. *(Source: C. Bayer, P. Friz and J. Gatheral in [6].)*

introduction. These models are useful since they manage to describe in a consistent way the previous empirical observation, i.e. why options with different strike prices and expirations have different values of implied volatility. These new family of models assume an SDE to describe the dynamics of the volatility. One of the most popular among SV models is due to Steven Heston in 1993. In [21], the author provides joint dynamics for both the underlying asset price S_t and the volatility σ_t through the following SDEs

$$dS_t = \mu_t S_t dt + \sqrt{\sigma_t} S_t dW_t^1, \tag{1.1.9}$$

$$d\sigma_t = -\kappa \left(\sigma_t - \theta\right) dt + \nu \sqrt{\sigma_t} dW_t^2, \tag{1.1.10}$$

where κ is often called the speed of mean reversion, θ is the long term mean value, ν is the volatility of volatility and W^1 and W^2 are two Brownian motions with correlation $\rho \in [-1, 1]$. Equation (1.1.10) is a version of a Cox-Ingersoll-Ross (CIR) process, see for instance [14] and must fulfill the Feller condition [16], given by $2\kappa\theta > \nu^2$, in order to ensure the positivity of the process σ_t. The first main drawback in SV models such as the Heston model is their associated market incompleteness as a consequence of the fact that instantaneous volatility σ is not tradable nor observable at each time t. A second drawback arises from using this class of models to reproduce the term structure of ATM (at-the-money) skew, this is

$$\psi\left(\tau\right) \triangleq \left| \frac{\partial}{\partial k} \Sigma_t \left(\tau, k\right) \right|_{k=0}.$$

Note that $k = 0$ turns out to be the ATM. Estimates of $\psi\left(\tau\right)$ are very sensitive to the choice of the SDE for the volatility dynamics in a SV model. Gatheral *et*

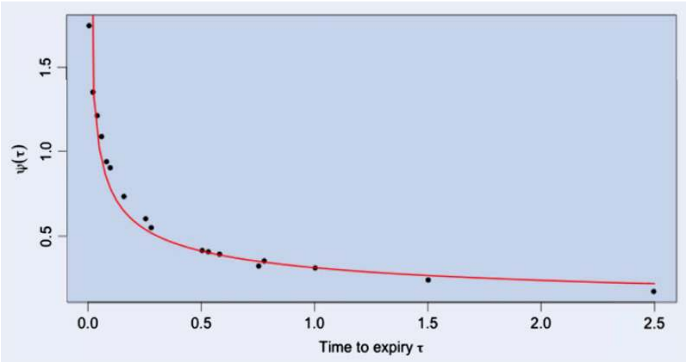

Figure 1.2: The black dots are non-parametric estimates of the S&P500 ATM volatility skew as of June 2013; the red curve is the power law fit $\psi(\tau) = A\tau^{-0.4}$. Time is measured in years. *(Source: J. Gatheral, T. Jaisson and M. Rosenbaum in [18].)*

al. show in 2018 in [18] how none of the classic SV models can fit non-parametric estimates of the S&P500 ATM volatility skews. Indeed, SV models can only generate a term structure for ATM skew such that it is constant for small τ. Instead, fractional stochastic volatility models, this is a SV model where the volatility is driven by a fractional Brownian motion (fBm) with Hurst exponent $H \in (0, 1/2)$, can generate ATM volatility skews of the form $\psi(\tau) \sim \tau^{H-1/2}$, that fit observed data. This is shown in Figure 1.2. A brief introduction to fBm can be found later in Section 1.6.

Alternatively, this sort of explosion in $\psi(\tau)$ as $\tau \to 0$ can also be mimicked to a certain extent, using jump diffusion processes, but can never be achieved using a Brownian motion. Models with jumps, not only produce a huge variety of smile and skew patterns but can also help explain the distinction between skew and smile, in terms of asymmetry of jumps anticipated by the market. This is, the difference in price of index options across different strike prices as a consequence of fear of a large downward jump. The previous fact, often leads to downward skews as shown in [13].

All the previous developments made having closed-form pricing formulas almost impossible. Even when possible, these formulas do not allow in general for fast model calibration of the parameters. Giving a physical interpretation is also sometimes difficult. In view of these difficulties, a race towards the development of approximating formulas for pricing derivatives started. These approximation formulas or decomposition formulas, give better understanding of the model parameters role, since they are the sum of the classical Black-Scholes equation plus a Taylor-type expansion with respect to these parameters. Special decomposition formulas for call option prices in the Heston model using Malliavin calculus and Itô calculus were first developed by Elisa Alòs in [2] and

[1], respectively. Let $T > 0$ be the time horizon, and let W and $\tilde{W}$ be two independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denoting by $\mathcal{F}^W$ and $\mathcal{F}^{\tilde{W}}$, the completed natural filtrations generated by W and $\tilde{W}$, respectively. Set $\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_t^{\tilde{W}}$, $t \in [0,T]$. We will consider the log-price process $X_t = \log S_t$, as it is more convenient for better tractability. Therefore, from the general Itô formula (1.1.5) from Theorem 1.1.1, we have that

$$dX_t = \left(r - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t \left(\rho dW_t + \sqrt{1-\rho^2} d\tilde{W}_t\right). \tag{1.1.11}$$

It will be very useful at this point to define the projected future variance as

$$v_t^2 \triangleq \frac{1}{T-t} \int_t^T \mathbb{E}\left[\sigma_s^2 \mid \mathcal{F}_t\right] ds, \tag{1.1.12}$$

since this will set the basis to define the forward variance as shown later in this section.

Theorem 1.1.3. (Decomposition formula Alòs 2012) Assume the model given by equations (1.1.11) and (1.1.10), where the volatility process $\sigma = \{\sigma_s, s \in [0,T]\}$ satisfies the Feller condition. Then, for all $t \in [0,T]$, the price V_t of an European call option with payoff $\left(e^{X_T} - K\right)^+$ can be written as follows,

$$\begin{aligned} V_t &= BS\left(t, X_t; v_t\right) \\ &+ \frac{\rho}{2}\mathbb{E}\left[\int_t^T e^{-r(s-t)} \Lambda\Gamma BS\left(s, X_s, v_s\right) \sigma_s d\langle M, W\rangle_s \mid \mathcal{F}_t\right] \\ &+ \frac{1}{8}\mathbb{E}\left[\int_t^T e^{-r(s-t)} \Gamma^2 BS\left(s, X_s, v_s\right) d\langle M, M\rangle_s \mid \mathcal{F}_t\right], \end{aligned} \tag{1.1.13}$$

where

$$\begin{aligned} &\Lambda \triangleq \partial_x, \\ &\Gamma \triangleq \left(\partial_x^2 - \partial_x\right); \qquad \Gamma^2 = \Gamma \circ \Gamma = \left(\partial_x^4 - 2\partial_x^3 + \partial_x^2\right), \end{aligned}$$

and $BS\left(t, x, \sigma\right) \triangleq e^x \Phi\left(\tilde{d}_+\right) - K e^{-r(T-t)} \Phi\left(\tilde{d}_-\right)$ is the Black-Scholes function written in terms of log-prices, with $\tilde{d}_\pm \triangleq \frac{X_t - \ln(K) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.

In 1994 Bruno Dupire introduced a new class of SV models, see for instance [15], that were later refined by Lorenzo Bergomi [8, 9] in the 2000s. Once forward variance had been defined as the expectation under the pricing measure of future instantaneous variance, given by,

$$\xi_t^u \triangleq \mathbb{E}\left[\sigma_u \mid \mathcal{F}_t\right]; \qquad u \geq t,$$

forward variance models took over since ξ_t had a deep connection with a whole new class of derivative products that started to become popular among practitioners. Known as variance swaps, these products exchange the realized variance of a given asset with some fixed amount during a period of time until expiry T. Therefore, acquiring a long position in such contract would grant the holder the following payoff:

$$\frac{1}{T-t}\sum_{i=1}^{N}\left(\log\left(S_{t_{i+1}}\right)-\log\left(S_{t_i}\right)\right)^2-V_t^T, \tag{1.1.14}$$

where $\sum_{i=1}^{N}\left(\log\left(S_{t_{i+1}}\right)-\log\left(S_{t_i}\right)\right)^2$ is the realized variance of the asset S and V_t^T is the strike price, such that the initial price of the variance swap is zero.

Definition 1.1.4. Let $T>0$ be the maturity of a variance swap V_t^T. We define the instantaneous forward variance as

$$\xi_t^T \triangleq \frac{d}{dT}\left((T-t)\,V_t^T\right), \qquad t \leq T. \tag{1.1.15}$$

Therefore, we have that

$$V_t^T = \frac{1}{(T-t)}\int_t^T \xi_t^u du. \tag{1.1.16}$$

.

Now that the connection between forward variance and variance swaps is clear, it is known that any Markovian SV model can be rewritten in forward variance form, see for instance Chapter 7 in [10]. Therefore, one would preferably want to work with the later, given that instantaneous volatility σ_t is not tradable but variance swaps are indeed tradable. In such way, one does no longer need to consider an incomplete market.

1.1.1 The Insurance Framework

A lot less has been said on insurance over the years, despite the challenges that this industry has faced were never smaller. Usually, insurance products have longer time horizons than financial products, increasing the importance of certain assumptions that can be simplified or even neglected in finance. The risk derived from interest rates in such long term products acquires higher relevance as well as the modeling of mortality rates of clients, or the distributions of house fires, car accidents and natural catastrophes. We will follow Chapters 2 and 8 in [27], for a quick introduction to mortality transition rates and an analysis of unit-linked policies, respectively.

The studies performed in this thesis that are connected to insurance are centered in a specific insurance product named “unit-linked” policies. The value of such product is tied to the performance of a fund or the value of a stock S_t. The insurer pays an agreed payoff to the insured in case an insured event takes place. These policies have the characteristic feature that the benefits (endowments or death benefits) are not deterministic, but random. A unit-linked policy is usually financed by a single premium due to management of this policies. In the case studied, we model the payoff of such contract at time t, as the maximum between the price of the stock or fund value S_t and a deterministic guarantee $G \in \mathbb{R}_+$,

$$C_t = \max\left\{S_t, G\right\}.$$

We have already seen in Theorem 1.1.2 that, in order for us to give a fair price to this future payment, there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, such that there is no arbitrage opportunity. Therefore, the price at time t of a unit-linked policy with death benefit C_T, is given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\left\{S_T, G\right\} \mid \mathcal{F}_t\right] \cdot {}_T p_x,$$

where ${}_T p_x$ is the probability that an x-year old individual survives for the next $T-t$ years and $\mathbb{Q}$ is an equivalent measure to the historical measure $\mathbb{P}$, such that the discounted value of the underlying fund or stock price is a martingale.

In order to properly introduce the mortality transition rates and probabilities, we will need to introduce some definitions.

Definition 1.1.5. Let $X = \{X_t, t \in [0,T]\}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathcal{S}$ and $T \in \mathbb{R}$, where $\mathcal{S}$ is a countable set consisting of all possible health states of the insured. The process X is called *Markov chain*, if for all

$$n \geq 1,\ t_1 < t_2 < \cdots < t_{n+1} \in [0,T],\ i_1, i_2, \ldots, i_{n+1} \in \mathcal{S}$$

with

$$\mathbb{P}\left[X_{t_1} = i_1, X_{t_2} = i_2, \ldots, X_{t_n} = i_n\right] > 0,$$

the following holds:

$$\mathbb{P}\left[X_{t_{n+1}} = i_{n+1} \mid X_{t_k} = i_k \forall k \leq n\right] = \mathbb{P}\left[X_{t_{n+1}} = i_{n+1} \mid X_{t_n} = i_n\right]. \quad (1.1.17)$$

We will focus mainly on two following health states in this work, represented by $*$ =”alive” and † =”deceased”. Now, it will be convenient to define the following processes:

$$I_i^X(t) = \begin{cases} 1, & \text{if } X_t = i \\ 0, & \text{if } X_t \neq i \end{cases}, \quad i \in \mathcal{S},$$

$$N_{ij}^X(t) = \#\left\{s \in (0,t) : X_{t-} = i, X_t = j\right\}, \qquad i,j \in \mathcal{S}, \quad i \neq j.$$

Here, # denotes the counting measure and $X_{t-} \triangleq \lim_{u \to t} X_u$ the left limit of X at time t. The random variable $I_i^X(t)$ tells us whether the insured is in state i at time t and $N_{ij}^X(t)$ tells us the number of transitions from state i to state j in the whole period $(0,t)$.

Definition 1.1.6. Let X be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$p_{i,j}(s,t) \triangleq \mathbb{P}[X_t = j \mid X_s = i], \qquad \text{where } s \leq t \text{ and } i, j \in \mathcal{S},$$

is called the conditional probability to switch from state i at time s, to state j at time t. These probabilities are often referred to as *transition probabilities.*

The following theorem of Chapman and Kolmogorov states the relation between $P(s,t)$, $P(t,u)$ and $P(s,u)$ for $s \leq t \leq u$, where $P(s,t) = \{p_{ij}(s,t)\}_{(i,j)\in\mathcal{S}\times\mathcal{S}}$ is the matrix of all transition probabilities between all the possible states in $\mathcal{S}$.

Theorem 1.1.7. Let X be a Markov chain. For $s \leq t \leq u \in [0,T]$ and $i, k \in \mathcal{S}$ such that $\mathbb{P}[X_s = i] > 0$, the following equations hold:

$$p_{ik}(s,u) = \sum_{j\in S} p_{i,j}(s,t)\, p_{j,k}(t,u),$$

or in matrix notation,

$$P(s,u) = P(s,t) \times P(t,u).$$

This shows, that one can get $P(s,u)$ by matrix multiplication of $P(s,t)$ and $P(t,u)$, for $s \leq t \leq u \in [0,T]$.

Definition 1.1.8. A family $(p_{ij}(s,t))_{(i,j)\in\mathcal{S}\times\mathcal{S}}$ is called *transition matrix*, if the following four properties hold:

1. $p_{ij}(s,t) \geq 0$.
2. $\sum_{j\in\mathcal{S}} p_{ij}(s,t) = 1$.
3. $p_{ij}(s,t) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$ if $\mathbb{P}[X_s = i] > 0$.
4. $p_{ik}(s,u) = \sum_{j\in\mathcal{S}} p_{ij}(s,t)\, p_{jk}(t,u)$ for $s \leq t \leq u$ and $\mathbb{P}[X_s = i] > 0$.

Theorem 1.1.9. Let X be a Markov chain. Then, $(p_{ij}(s,t))_{(i,j)\in\mathcal{S}\times\mathcal{S}}$ is a transition matrix.

Definition 1.1.10. A Markov Chain X is called *homogeneous*, if it is time homogeneous, i.e. the following equation holds for all $s, t \in \mathbb{R}$, $h > 0$ and $i, j \in \mathcal{S}$ such that $\mathbb{P}[X_s = i] > 0$ and $\mathbb{P}[X_t = i] > 0$:

$$\mathbb{P}[X_{s+h} = j \mid X_s = i] = \mathbb{P}[X_{t+h} = j \mid X_t = i].$$

For a homogeneous Markov chain, we use the notation:

$$p_{ij}(h) \triangleq p_{ij}(s, s+h),$$
$$P(h) \triangleq P(s, s+h).$$

Definition 1.1.11. Let X be a Markov chain in continuous time with finite state space $\mathcal{S}$. Then, X is called *regular*, if

$$\mu_i(t) = \lim_{\Delta t \searrow 0} \frac{1 - p_{ii}(t, t+\Delta t)}{\Delta t} \text{ for all } i \in \mathcal{S},$$
$$\mu_{ij}(t) = \lim_{\Delta t \searrow 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t} \text{for all } i \in \mathcal{S},$$

exist, are finite and continuous with respect to t. The functions μ_i and μ_{ij} are called *transition rates* of the Markov chain.

Theorem 1.1.12. Let X be a regular Markov chain on a finite state space $\mathcal{S}$. Then, the following statements hold:

1. (Backward differential equations)

$$\frac{d}{ds} p_{ij}(s,t) = \mu_i(s)\, p_{ij}(s,t) - \sum_{k \neq i} \mu_{ij}(s)\, p_{kj}(s,t),$$
$$\frac{d}{ds} P(s,t) = -\Lambda(s)\, P(s,t).$$

2. (Forward differential equations)

$$\frac{d}{dt} p_{ij}(s,t) = -p_{ij}(s,t)\, \mu_j(t) + \sum_{k \neq j} p_{ik}(s,t)\, \mu_{kj}(t),$$
$$\frac{d}{dt} P(s,t) = P(s,t)\, \Lambda(t).$$

Definition 1.1.13. Let X be a regular Markov chain on a finite state space $\mathcal{S}$. Then, we denote the conditional probability to stay in state j during the time interval (s,t), by

$$\bar{p}_{jj}(s,t) \triangleq \mathbb{P}\left[\bigcap_{\xi \in [s,t]} \{X_\xi = j\} \mid X_s = j \right],$$

where $s, t \in [0,T]$, $s \leq t$ and $j \in \mathcal{S}$.

This can be used to calculate the probability that the insured survives the next T-years. The following theorem ends this section by illustrating how this probability can be calculated based on the transition rates, showing that everything needed in order to derive these probabilities, is to properly model the transition rates.

Theorem 1.1.14. Let X be a regular Markov chain. Then the probability of being in state $j \in \mathcal{S}$ at time s and staying in the same state at time t, is given by

$${}_T p_x \triangleq \bar{p}_{jj}(t, t+T) = \exp\left\{ -\sum_{k \neq j} \int_t^{t+T} \mu_{jk}(\tau)\, d\tau \right\},$$

where $s \leq t$, provided that $\mathbb{P}[X_s = j] > 0$. For example, this gives us the probability that a t-year old individual who is alive $j = * \in \mathcal{S}$, is still alive after the next T years.

1.2 Basics of Stochastic Analysis

This section provides a brief introduction to the basic concepts of stochastic analysis for both the continuous and discontinuous cases. Both cases are treated respectively in two different subsections.

1.2.1 The Continuous Case: Brownian Motion

We will closely follow [7] for a quick introduction to this topic given that the content in this subsection corresponds to the concepts given in any introduction to stochastic analysis. One can find in [24], a detailed guide to this topic. In order to build the option pricing theory of Black and Scholes, it will be very convenient to study the basics of stochastic analysis. Concepts such as the Itô integral and the Itô formula, constitute the foundation of this mathematical discipline. The so-called martingale processes, or simply martingales, constitute an important class of stochastic processes. In mathematical finance they are one of the main building blocks for deriving option prices and hedging strategies. In order to fully understand the definition of a martingale it is essential to clarify the concept of *conditional expectation*.

Definition 1.2.1. Assume that Z is a random variable. Then the *conditional expectation* $\mathbb{E}[Z \mid \mathcal{F}_s]$ is defined as the unique $\mathcal{F}_s$-adapted random variable X satisfying

$$\mathbb{E}[\mathbf{1}_A X] = \mathbb{E}[\mathbf{1}_A Z], \qquad \forall A \in \mathcal{F}_s. \tag{1.2.1}$$

In order for the conditional expectation to exist, we need to impose a moment condition on Z: one can only define the conditional expectation of random variables Z for which $\mathbb{E}[|Z|] < \infty$.

Definition 1.2.2. A stochastic process M_t is called a *martingale* if it is an adapted process, $\mathbb{E}[M_t] < \infty$ and

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s, \tag{1.2.2}$$

for every $0 \leq s \leq t < \infty$.

On the left-hand side of (1.2.2) we take the expectation of M_t conditioned on all the information the Brownian motion can give us up to time s. This information is encapsulated in the notation $\mathcal{F}_s$.

The Itô integral defines what one should understand by integration of a stochastic process with respect to a Brownian motion (or any stochastic process acting as an integrator). The whole purpose is to give an interpretation to the expression (1.1.3). One would like to define the stochastic integral in (1.1.3), as the following limit

$$\int_0^t X(s,\omega)\,dB(s,\omega) = \lim_{n\to\infty}\sum_{i=1}^{n-1} X(s_i,\omega)\left(B(s_{i+1},\omega) - B(s_i,\omega)\right). \tag{1.2.3}$$

Note that we take the limit for each fixed ω. The problem is that for almost all $\omega \in \Omega$, this limit in general does not exist. The function $s \to B(s,\omega)$ is extremely volatile for almost all ω. It is an example of a continuous, but nowhere differentiable function. Indeed is a function of infinite variation. This fact does not allow to construct a pathwise integral in the sense of Riemann-Stieltjes for all continuous path integrands. However, Itô took advantage of the martingale properties of Brownian motion and the fact that their paths have finite quadratic variation, to construct an integral in the L^2 sense.

We conclude the discussion with the definition of the Itô integral.

Definition 1.2.3. A stochastic process X_s is called Itô integrable on the interval $[0,t]$ if:

1. X_s is adapted for every $s \in [0,t]$, and
2. $\int_0^t \mathbb{E}\left[X_s^2\right] ds < \infty$.

The Itô integral is defined as the random variable

$$\begin{aligned}\int_0^t X_s dB_s &\triangleq \int_0^t X(s)\,dB(s) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n-1} X(s_i)\left(B(s_{i+1}) - B(s_i)\right),\end{aligned} \tag{1.2.4}$$

where the limit is taken in $L^2(\Omega)$.

Theorem 1.2.4. The expectation and variance of the Itô integral are

$$\mathbb{E}\left[\int_0^t X_s dB_s\right] = 0, \qquad \mathrm{Var}\left(\int_0^t X_s dB_s\right) = \int_0^t \mathbb{E}\left[X_s^2\right] ds.$$

The relation for the variance is also known as the Itô isometry.

Definition 1.2.5. The stochastic process X_t is called an *Itô process* if there exist two Itô integrable stochastic processes Y_t and Z_t, such that

$$X_t = x + \int_0^t Y_s dB_s + \int_0^t Z_s ds. \tag{1.2.5}$$

Note that we assume both processes Y_t and Z_t to be adapted, which leads to the adaptedness of X_t. Moreover, since Z is also Itô integrable, the Itô process has a finite second order moment. In the following theorem we state the Itô formula for an Itô process of the form (1.2.5).

We will now introduce the Itô formula. This formula has a wide range of applications and is the stochastic version of the classical chain rule of differentiation of calculus. It prescribes how a function of a Brownian motion $f(B_t)$, or more generally, a function of an Itô process $f(X_t)$ will be decomposed into the dynamics of the process X_t and the rate of change of $f(x)$ given by its derivatives. The Itô integral is the main ingredient in the stochastic chain rule. Together with the Itô integral, Itô's formula set the foundation for modern stochastic analysis.

Theorem 1.2.6. Assume that $f(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$ and let X_t be an Itô process. Then

$$\begin{aligned} f(t, X_t) = f(0,x) &+ \int_0^t Y_s \frac{\partial f}{\partial x}(s, X_s)\, dB_s \\ &+ \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds + \int_0^t Z_s \frac{\partial f}{\partial x}(s, X_s)\, ds + \frac{1}{2}\int_0^t Y_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)\, ds. \end{aligned}$$

To end this section, we give a result on a representation of martingales in terms of stochastic integrals.

Theorem 1.2.7. (Martingale Representation Theorem) If M_t is an $L^2(\Omega)$, $\mathbb{F}$-adapted martingale, there exists an Itô integrable process X_s such that

$$M_t = M_0 + \int_0^t X_s dB_s.$$

All square integrable martingales with respect to the filtration generated by a Brownian motion can be written as a stochastic integral with respect to that given filtration. Therefore, a consequence of this theorem is that we can define martingales as processes of the form $M_t = M_0 + \int_0^t X_s dB_s$. This will be very convenient since later on, we will manipulate the conditional expectation and martingales to derive option prices in a simple way.

1.2.2 The Discontinuous Case: Pure Jump Lévy Processes

This section is an analogous introduction to the basic concepts and results of stochastic analysis for jump processes instead. We will follow [33] for a quick introduction. A complete and detailed review of this topic can be found in [4]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 1.2.8. A one-dimensional Lévy process is a stochastic process $X = \{X_t, t \geq 0\}$, satisfying the following properties:

1. $X_0 = 0$, $\mathbb{P}-$a.s.,

2. X has *independent increments*, that is, for all $t > 0$ and $h > 0$, the increment $X_{t+h} - X_t$ is independent of X_s for all $s \leq t$.

3. X has *stationary increments*, that is, for all $h > 0$, $X_{t+h} - X_t \overset{d}{=} X_h$.

4. X is *stochastically continuous*, that is for all $\epsilon > 0$ and

$$\lim_{s \to t} \mathbb{P}\left(|X_t - X_s| > \epsilon\right) = 0, \qquad t \geq 0.$$

The jump of X at time t is defined by

$$\Delta X_t \triangleq X_t - X_{t-},$$

and the number of jumps of size $\Delta X_s \in U \subset \mathcal{B}\left(\mathbb{R}_0\right)$, for any $s \in [0, t]$ where $\mathbb{R}_0 \triangleq \mathbb{R} \setminus \{0\}$ is given by

$$N\left(t, U\right) \triangleq \sum_{0 \leq s \leq t} \mathcal{X}_U\left(\Delta X_s\right). \tag{1.2.6}$$

This defines in a natural way a Poisson random measure N on $\mathcal{B}\left(0, \infty\right) \times \mathcal{B}\left(\mathbb{R}_0\right)$, given by

$$(a, b] \times U \longmapsto N\left(b, U\right) - N\left(a, U\right), \qquad 0 < a \leq b, \quad U \in \mathcal{B}\left(\mathbb{R}_0\right).$$

We call this measure, the *jump measure* of X and its differential form is denoted by $N\left(dt, dz\right)$, $t > 0$.

The Lévy measure ℓ of X is defined by

$$\ell\left(U\right) \triangleq \mathbb{E}\left[N\left(1, U\right)\right], \qquad U \in \mathcal{B}\left(\mathbb{R}_0\right).$$

This measure does not need to be finite, but must always satisfy

$$\int_{\mathbb{R}_0} \left(1 \wedge |z|^2\right) \ell\left(dz\right) < \infty.$$

It is possible to have the following

$$\int_{\mathbb{R}_0} \left(1 \wedge |z|\right) \ell\left(dz\right) = \infty,$$

which implies paths of infinite variation.

Now we will present a characterization of Lévy processes through a formula that was first established by Paul Lévy and A. Ya. Khintchine in the 1930s and is known as the Lévy-Khintchine representation formula.

Theorem 1.2.9. (Lévy-Khintchine formula)

(1) Let X be a Lévy process. Then

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{i\Psi(u)}, \qquad u \in \mathbb{R}, \tag{1.2.7}$$

with the characteristic exponent being given by

$$\Psi(u) \triangleq i\alpha u - \frac{1}{2}\sigma^2 u^2 + \int_{|z|<1} \left(e^{iuz} - 1 - iuz\right) \ell(dz) + \int_{|z|\geq 1} \left(e^{iuz} - 1\right) \ell(dz), \tag{1.2.8}$$

where the parameters $\alpha \in \mathbb{R}$ and $\sigma^2 \geq 0$ are constants and $\ell = \ell(dz)$, $z \in \mathbb{R}_0$, is a σ-finite measure on $\mathcal{B}(\mathbb{R}_0)$ satisfying

$$\int_{\mathbb{R}_0} \left(1 \wedge |z|^2\right) \ell(dz) < \infty. \tag{1.2.9}$$

It follows that ℓ is the Lévy measure of X.

(2) Conversely, given the constants $\alpha \in \mathbb{R}$ and $\sigma^2 \geq 0$, and the σ-finite measure $\ell \in \mathcal{B}(\mathbb{R}_0)$, such that (1.2.9) holds, then there exists a process X (unique in law), such that (1.2.7) and (1.2.8) hold. The process X is a Lévy process.

We define the *compensated jump measure* $\tilde{N}$, also referred to as the *compensated Poisson random measure*, by

$$\tilde{N}(dt, dz) \triangleq N(dt, dz) - \ell(dz)\, dt.$$

For any t, let $\mathcal{F}_t$ be the σ-algebra generated by the random variables W_s and $\tilde{N}(s, A)$; $A \in \mathcal{B}(\mathbb{R}_0)$, $s \leq t$ and define the following filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ in the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now, fix $0 < T < \infty$ and let $\mathcal{P}$ denote the smallest σ-algebra with respect to which all mappings $F : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ satisfying (1) and (2) below are measurable:

1. for each $0 \leq t \leq T$, the mapping $(x, \omega) \to F(t, x, \omega)$ is $\mathcal{B}(\mathbb{R}_0) \otimes \mathcal{F}_t$-measurable.
2. For each $x \in \mathbb{R}_0$, $\omega \in \Omega$, the mapping $t \to F(t, x, \omega)$ is left-continuous.

We call $\mathcal{P}$ the predictable σ-algebra. A $\mathcal{P}$-measurable mapping $G : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ is then said to be predictable. Clearly the definition extends naturally to the case where $[0,T]$ is replaced by $\mathbb{R}_+$.

Let $\theta = \theta(t, z)$, $t \geq 0$, $z \in \mathbb{R}_0$, be an $\mathbb{F}$-predictable process, such that

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(t, z)\, \ell(dz)\, dt\right] < \infty \qquad \text{for some } T > 0,$$

the process

$$M(t) \triangleq \int_0^t \int_{\mathbb{R}_0} \theta(s,z)\, \tilde{N}(ds,dz), \qquad 0 \le t \le T, \tag{1.2.10}$$

is a martingale in L^2. Moreover we have the *Itô isometry* given by

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta(t,z)\, \tilde{N}(dt,dz)\right)^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(t,z)\, \ell(dz)\, dt\right]. \tag{1.2.11}$$

A Wiener process is a special case of a Lévy process. In fact, we have the following general representation theorem.

Theorem 1.2.10. (Lévy-Itô decomposition) Let X be a Lévy process. Then $X = X_t$, $t \ge 0$, admits the following integral representation

$$X_t = a_1 t + \sigma W_t + \int_0^T \int_{|z|<1} z\tilde{N}(ds,dz) + \int_0^T \int_{|z|\ge 1} zN(ds,dz), \tag{1.2.12}$$

for some constants $a_1, \sigma \in \mathbb{R}$. Here $W = \{W_t, t \ge 0\}$, is a standard Wiener process.

Finally, in order to end this section, we provide a fundamental result in stochastic calculus for Lévy processes, which is the counterpart of the Itô formula for jump processes.

Theorem 1.2.11. (The one-dimensional Itô formula). Let $X = \{X_t, t \ge 0\}$ be the Lévy process given by

$$X_t = x + \int_0^t \alpha(s)\, ds + \int_0^t \beta(s)\, dW_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\, \tilde{N}(ds,dz),$$

where $\alpha(t), \beta(t)$ and $\gamma(t,z)$ are predictable processes. Consider a function $f : (0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ in $C^{1,2}((0,\infty) \times \mathbb{R})$ and define

$$Y_t \triangleq f(t, X_t), \qquad t \ge 0.$$

Then, the process $Y = (Y_t)_{t\ge 0}$, is also an Itô-Lévy process and its differential form is

$$\begin{aligned} dY_t = {} & \frac{\partial f}{\partial t}(t,X_t)\, dt + \frac{\partial f}{\partial x}(t,X_t)\, \alpha(t)\, dt + \frac{\partial f}{\partial x}(t,X_t)\, \beta(t)\, dW_t \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\, \beta^2(t)\, dt + \int_{\mathbb{R}_0} \Big[f(t, X_t + \gamma(t,z)) \\ & - f(t,X_t) - \frac{\partial f}{\partial x}(t,X_t)\, \gamma(t,z) \Big]\, \ell(dz)\, dt \\ & + \int_{\mathbb{R}_0} \left[f(t, X_{t-} + \gamma(t,z)) - f(t, X_{t-}) \right] \tilde{N}(dt,dz). \end{aligned} \tag{1.2.13}$$

1.3 Stochastic Differential Equations

This subsection aims to provide a basic framework for stochastic differential equations (SDEs) as well as addressing the problem of solving such differential equations. An introductory guide to this topic is found in [36], or alternatively in Chapter 5 of [26].

Starting from the Itô integral (1.2.4) we can define an SDE by the following expression:

$$dX_t = b\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dB_t, \qquad X_0 = x_0 \in \mathbb{R}, \tag{1.3.1}$$

for a Brownian motion B. This equation is an informal version of the corresponding Itô integral equation, given by

$$X_t = x_0 + \int_0^t b\left(s, X_s\right) ds + \int_0^t \sigma\left(s, X_s\right) dB_s. \tag{1.3.2}$$

Several questions arise at this point:

1. Can one obtain a solution to the SDE (1.3.2), i.e. when does a solution to the SDE exist?
2. If a solution to (1.3.2) exists, when is the solution unique?
3. How can one solve an equation such as the one given in (1.3.2)?

In order to answer all these questions, we need to start by introducing the concept of strong solution, given in the following definition.

Definition 1.3.1. (Strong solution) Let $X = \{X_t, t \in [0,T]\}$ be a process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous sample paths, such that:

1. X is adapted to the filtration $\mathbb{F}$,
2. There exists $x \in \mathbb{R}$, such that $\mathbb{P}\left(X_0 = x\right) = 1$,
3. $\mathbb{P}\left(\int_0^t \left[\left|b\left(s, X_s\right)\right| + \sigma^2\left(s, X_s\right)\right] ds < \infty\right) = 1$,
4. $X_t = x + \int_0^t b\left(s, X_s\right) ds + \int_0^t \sigma\left(s, X_s\right) dB_s, \quad \mathbb{P} - a.s.$

Then, X is called a strong solution to (1.3.1), with initial condition $x \in \mathbb{R}$.

Following [25] we will introduce the definition of uniqueness.

Definition 1.3.2. Let $b(t,x)$ and $\sigma(t,x)$ be given. Suppose that, B is a Brownian motion on some $(\Omega, \mathcal{F}, \mathbb{P})$, $X_0 = x_0 \in \mathbb{R}$, $\{\mathcal{F}_t\}$ is the natural filtration generated by B, and X, $\tilde{X}$ are two strong solutions of (1.3.1) relative to B with initial condition x_0, then $\mathbb{P}\left(X_t = \tilde{X}_t;\, 0 \leq t < \infty\right) = 1$. Under these conditions, we say that strong uniqueness holds for the pair (b, σ).

Theorem 1.3.3. (Existence and uniqueness of solutions for stochastic differential equations). Let $T > 0$ and $b(\cdot,\cdot) : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|); \qquad x \in \mathbb{R},\, t \in [0,T], \tag{1.3.3}$$

for some constant C, and such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|, \tag{1.3.4}$$

for all $x, y \in \mathbb{R}$, $t \in [0,T]$ and some constant D. Then, for all $t \in [0,T]$ and $X_0 = x_0 \in \mathbb{R}$, the stochastic differential equation (1.3.1) has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t$ generated by B_s; $s \leq t$ and

$$\mathbb{E}\left[\int_0^T |X_t|^2\, dt\right] < \infty.$$

Proof. The full version of this proof is found in [36]. The proof for the existence of a strong solution is similar to the familiar existence proof for ordinary differential equations based on the technique of Picard iteration, see e.g. Chapter 4 in [22]. ■

The uniqueness of the strong solution follows from the Itô isometry (1.2.11) and the Lipschitz property (1.3.4), answering this way, to the second question posed in the beginning of this subsection. Despite one may prove the existence and uniqueness of a solution to an SDE like the one given in (1.3.1), it may be difficult to explicitly write the actual solution, usually even impossible. Nevertheless, one can derive explicit solutions for simple examples as the ones proposed in the beginning of Section 5 in [36].

1.4 Basics of Mathematical Finance. Option Pricing and Hedging

The goal of this section is to provide the reader with the basic results on option pricing on a simplified framework. We will follow [7] and [11] for this quick introduction, and the reader is referred to Chapter 10 in [11] for a full overview in a more general framework. What we aim to do here, is to derive fair prices of derivative contracts. Furthermore, we shall discuss how one can come up with a strategy, to hedge the risk associated with a position in a derivative contract.

Let us therefore consider a financial market consisting of only two assets: a risk free asset (which can be regarded as a bank account), with price process B, and a stock price process S. Let us start with the formal definition of risk-free asset.

Definition 1.4.1. The price process B is the price of a *risk free* asset, if it has the dynamics

$$dB_t = rB_t dt, \tag{1.4.1}$$

where r is a constant.

A natural interpretation of a riskless asset is that it corresponds to a bank account with short rate interest r. We will assume that the stock dynamics is given by the SDE (1.1.6). The Black-Scholes model consists of two assets with dynamics given by (1.1.6) and (1.4.1), with r, μ and σ being constants. We consider a financial market given by these two assets, and approach the problem of pricing financial derivatives, also known as contingent claims.

Definition 1.4.2. A contingent T-claim is a financial contract that pays the holder a random amount X at time T. The random variable X is square integrable and $\mathcal{F}_T$-adapted, and T is called the exercise time of the contingent claim.

Observe that all contracts with payoff $f(S_T)$, where f is some function of an underlying asset price S, are contingent claims. European call and put options are defined respectively by the following payoffs:

$$f(s) = (s-K)^+ = \max(s-K, 0), \quad \text{(European call option payoff)},$$
$$f(s) = (K-s)^+ = \max(K-s, 0), \quad \text{(European put option payoff)}.$$

In order to give a fair price to this financial products, we start by considering a financial market consisting of a stock (risky investment), a bond (risk-free investment) and a contingent claim. The price process of the stock is modeled by a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

while the price dynamics of the bond takes the form

$$dB_t = rB_t dt; \qquad B_0 = 1.$$

We let the price dynamics of the contingent claim be an adapted stochastic process denoted by $V_t = V(t, S_t)$. We will assume that an investor can form portfolios from three investment alternatives. Let a_t be the number of stocks, b_t the number of bonds and c_t the number of claims in such a portfolio at time t, which are all assumed to be adapted stochastic processes. We call (a, b, c) the portfolio strategy, and the portfolio value at time t is therefore

$$\Pi_t = a_t S_t + b_t B_t + c_t V_t. \tag{1.4.2}$$

The value process Π_t becomes an adapted stochastic process by definition.

Definition 1.4.3. A portfolio strategy (a, b, c) is called *self-financing* if

$$d\Pi_t = a_t dS_t + b_t dB_t + c_t dV_t. \tag{1.4.3}$$

We can think of $d\Pi$ as the change in portfolio value that coincides with the right-hand side of (1.4.3) if the portfolio is self-financing. Recall that the differential form of a stochastic process is just a simplified notation, and the self-financing property is translated into integral form as follows

$$\Pi_t = \Pi_0 + \int_0^t a_s dS_s + \int_0^t b_s dB_s + \int_0^t c_s dV_s.$$

We introduce now the *arbitrage* notion, that is the possibility of earning money from a zero-investment without taking any risk. For instance, if there is a way to short-sell bonds to finance a purchase of stocks and claims which yields a sure profit, the market is pricing the different instruments so that arbitrage is possible. In ideal markets, such opportunities should not exist, simply because investors will see this opportunity and try to exploit it by competing on prices. This would eventually lead to an equilibrium price in a liquid market, where arbitrage is not possible. We now proceed to give a formal mathematical definition to this idea.

Definition 1.4.4. A self-financing portfolio strategy is called an *arbitrage opportunity* if $\Pi_0 \leq 0$, $\Pi_T \geq 0$ and $\mathbb{E}\left[\Pi_T\right] > 0$.

Claim 1. We assume that the price process Π_t is such that there are no arbitrage possibilities on the market consisting of (B_t, S_t, Π_t).

Theorem 1.4.5. (Black-Scholes Equation) Assume that the market is specified by equations (1.1.6) and (1.4.1), and we want to price a contingent claim $f\left(S_T\right)$. Then the only pricing function which is consistent with the absence of arbitrage is the solution to the following boundary value problem in the domain $[0, T] \times \mathbb{R}_+$, given by

$$V\left(t, s\right) + rsV\left(t, s\right) + \frac{1}{2}s^2\sigma^2 V\left(t, s\right)\partial_{ss}V\left(t, s\right) - rV\left(t, s\right) = 0, \tag{1.4.4}$$

$$V\left(T, s\right) = f\left(s\right). \tag{1.4.5}$$

The proof to the previous result revolves over the fact that we explicitly assumed Markovian dynamics for S and B, and these play a key role in obtaining equations (1.4.4) and (1.4.5). A more general result for the fair price of contingent claims can be obtained due to Harrison and Pliska [20], as stated in Theorem 1.1.2. In that case, the fair price of a contingent claim $f\left(S_t\right)$ is given by the formula

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[f\left(S_T\right) \mid \mathcal{F}_t\right], \tag{1.4.6}$$

where the $\mathbb{Q}$-dynamics of S are given by the change of measure established by Girsanov's theorem in [19], resulting in the following SDE for the stock price

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Theorem 1.4.6. (Girsanov Theorem) Let $W^{\mathbb{P}}$ be a standard $\mathbb{P}$-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ and φ be any adapted process, such that is integrable with respect to $W^{\mathbb{P}}$. We define the process L_t as

$$L_t = \exp\left(\int_0^t \varphi_s dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \varphi_s^2 ds\right), \qquad t \in [0, T].$$

Assume that $\mathbb{E}^{\mathbb{P}}[L_T] = 1$ and define a probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, given by

$$L_T = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Then

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \int_0^t \varphi_s ds, \qquad t \in [0, T],$$

is a standard $\mathbb{Q}$-Wiener process.

The following result is due to Richard Feynman and Marc Kač and provides a relationship between the conditional expectation of the risk-neutral pricing formula (1.4.6) and the PDE from equation (1.4.4).

Proposition 1.4.7. (Feynman-Kač) Assume that V is a solution to the boundary problem

$$\partial_t V(t,x) + r\partial_x V(t,x) + \frac{1}{2}\sigma^2(t,x)\,\partial_{xx}^2 V(t,x) = 0,$$
$$V(T,x) = f(x).$$

Assume furthermore that the process $\sigma(t, S_t)\,\partial_x V(t, S_t) \in L^2([0,T] \times \mathbb{R})$. Then one has

$$V(t, S_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} f(S_T) \mid \mathcal{F}_t\right],$$

where S satisfies the SDE in equation (1.1.6).

So far we have shown the tools and main results that lead to the derivation of the pricing equation (1.4.4). A second major interest, once we know how to price financial derivatives, is whether we can build a portfolio such that when the contingent claim matures, its price coincides with the value of such portfolio. This concept is properly introduced in the following definition.

Definition 1.4.8. We say that a T-claim $f(S_T)$ can be *replicated* or *hedged*, if there exists a self-financing portfolio Π, such that

$$\Pi_T = 0, \quad \mathbb{P} - a.s.$$

In this case, we say that Π is a *hedge* against $f(S_T)$. Alternatively, Π is called a *replicating* or *hedging* portfolio. If every contingent claim is replicable, we say that the market is *complete*.

The following result shows the condition needed in order to state that a market is complete.

Proposition 1.4.9. Suppose that a claim $f(S_T)$ can be hedged using the portfolio Π. Then the only portfolio price process which is consistent with the no arbitrage condition, is given by

$$\Pi_t = 0, \quad t \in [0, T].$$

Furthermore if the claim $f(S_T)$ can be hedged using a portfolio $\tilde{\Pi}$ as well, then $\Pi_t = \tilde{\Pi}_t$ $\mathbb{P}$-a.s. for all $t \in [0, T]$.

Note, that the Black-Scholes model is an example of a complete market. See Chapter 8 pp.116 in [11] for a detailed proof of this statement.

1.5 Malliavin Calculus

This mathematical theory was first introduced by Paul Malliavin in [29] as an infinite-dimensional integration by parts technique. The purpose of this calculus was to prove the results about the smoothness of densities of solutions of SDEs driven by Brownian motion. But in 1984, Ocone in [34] obtained an explicit representation of random variables in terms of Itô stochastic integrals and the Malliavin derivative, later to be known as the Clark-Ocone formula. Ocone and Karatzas [35] applied this result to finance, proving that the Clark-Ocone formula could be used to obtain explicit formulae for replicating portfolios of contingent claims in complete markets. Also Alòs used these techniques in [2] to derive approximations to call option prices using an anticipating process, known as expected integrated future variance.

This section will only cover a quick introduction to Malliavin calculus with respect to Brownian motion. This techniques can also be extended to jump processes and they are both indeed used in this thesis, but the jump case has been omitted for sake of briefness in this introduction. The reader is referred to [33] as it is a good introduction to both continuous and discontinuous cases. See also [32] for an alternative introduction to this topic. None of the statements in the following subsection are proved, as the proofs are already in the references provided.

1.5.1 Wiener-Itô Chaos Expansion

Consider a Brownian setup formed by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a one-dimensional Wiener process $W = W_t = W(\omega, t)$, $\omega \in \Omega$, $t \in [0,T]$; $(T > 0)$, such that $W_0 = 0$. We denote the corresponding left- and right-continuous filtration by $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, where $\mathcal{F}_t$ is the σ-algebra generated by $W_s, 0 \leq s \leq t$, augmented by all the $\mathbb{P}$-zero measure events.

Definition 1.5.1. A real function $g : [0,T]^N \to \mathbb{R}$ is called symmetric if

$$g(t_{\sigma_1}, \ldots, t_{\sigma_n}) = g(t_1, \ldots, t_n), \tag{1.5.1}$$

for all permutations $\sigma = (\sigma_1, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$.

We denote by $L^2([0,T]^n)$, the standard space of square integrable Borel real functions on $[0,T]^n$, such that

$$\|g\|^2_{L^2([0,T]^n)} \triangleq \int_{[0,T]^n} g^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n < \infty. \tag{1.5.2}$$

Let $\tilde{L}^2([0,T]^n) \subset L^2([0,T]^n)$ be the space of symmetric square integrable Borel real functions on $[0,T]^n$ and consider the set

$$S_n = \{(t_1, \ldots, t_n) \in [0,T]^n : \quad 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T\}.$$

Now, if $g \in \tilde{L}^2([0,T]^n)$, then $g_{|S_n} \in L^2(S_n)$ and

$$\|g\|^2_{L^2([0,T]^n)} = n! \int_{S_n} g^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = n!\, \|g\|^2_{L^2(S_n)},$$

where $\|\cdot\|_{L^2(S_n)}$ denotes the norm induced by $L^2([0,T]^n)$ on $L^2(S_n)$, the space of square integrable functions on S_n.

If f is a real function on $[0,T]^n$, then its *symmetrization* $\tilde{f}$ is defined by

$$\tilde{f}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \ldots, t_{\sigma_n}),$$

where the sum is taken over all permutations σ of $(1, \ldots, n)$. Note that $\tilde{f} = f$ if, and only if, f is symmetric.

Definition 1.5.2. Let f be a deterministic function defined on S_n, $(n \geq 1)$ such that

$$\|f\|^2_{L^2(S_n)} \triangleq \int_{S_n} f^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n < \infty.$$

Then, we can define the n-fold *iterated Itô integral* as

$$J_n(f) \triangleq \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} f(t_1, \ldots, t_n)\, dW_{t_1} dW_{t_2} \cdots dW_{t_{n-1}} dW_{t_n}. \tag{1.5.3}$$

Thanks to the construction of the Itô integral we have that $J_n(f)$ belongs to $L^2(\mathbb{P})$, that is, the space of square integrable random variables.

Proposition 1.5.3. The following relations hold true:

$$\mathbb{E}\left[J_m(g)\,J_n(h)\right] = \begin{cases} 0 & , \quad n \neq m \\ (g,h)_{L^2(S_n)} & , \quad n = m \end{cases} \qquad (m, n = 1, 2 \ldots), \tag{1.5.4}$$

where

$$(g,h)_{L^2(S_n)} \triangleq \int_{S_n} g(t_1,\ldots,t_n)\,h(t_1,\ldots,t_n)\,dt_1\cdots dt_n$$

is the inner product of $L^2(S_n)$. In particular, we have

$$\|J_n(h)\|_{L^2(\mathbb{P})} = \|h\|_{L^2(S_n)}. \tag{1.5.5}$$

Definition 1.5.4. If $g \in \tilde{L}^2([0,T]^n)$ we define

$$I_n(g) \triangleq \int_{[0,T]^n} g(t_1,\ldots,t_n)\,dW_{t_1}\cdots dW_{t_n} \triangleq n! J_n(g). \tag{1.5.6}$$

We also call n-fold iterated Itô integrals the $I_n(g)$ here above.

Note from (1.5.4) and (1.5.6), that we have

$$\begin{aligned}\|I_n(g)\|^2_{L^2(\mathbb{P})} &= \mathbb{E}\left[I_n^2(g)\right] = \mathbb{E}\left[(n!)^2 J_n^2(g)\right] \\ &= (n!)^2 \|g\|^2_{L^2(S_n)} = n!\,\|g\|^2_{L^2([0,T]^n)},\end{aligned}$$

for all $g \in \tilde{L}^2([0,T]^n)$.

Theorem 1.5.5. Let ξ be an $\mathcal{F}_T$-measurable random variable in $L^2(\mathbb{P})$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \tilde{L}^2([0,T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \tag{1.5.7}$$

where the convergence is in $L^2(\mathbb{P})$. Moreover, we have the isometry

$$\|\xi\|^2_{L^2(\mathbb{P})} = \sum_{n=0}^{\infty} n!\,\|f_n\|^2_{L^2([0,T]^n)}.$$

1.5.2 The Malliavin Derivative

Malliavin Calculus has a huge significance in finance. It has triggered a wide range of applications, such as in numerical methods, stochastic control and insider trading.

Definition 1.5.6. Let $F \in L^2(\mathbb{P})$ be $\mathcal{F}_T$-measurable with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \tilde{L}^2([0,T]^n)$, $n = 1, 2, \dots$.

(i) We say that $F \in \mathbb{D}_{1,2}$, i.e. F is Malliavin differentiable, if

$$\|F\|^2_{D_{1,2,}} \triangleq \sum_{n=1}^{\infty} nn! \, \|f_n\|^2_{L^2([0,T]^n)} < \infty. \tag{1.5.8}$$

(ii) If $F \in \mathbb{D}_{1,2,}$ we define the Malliavin Derivative $D_t F$ of F at time t, as the expansion

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \qquad t \in [0,T], \tag{1.5.9}$$

where $I_{n-1}(f_n(\cdot, t))$ is the $(n-1)$-fold iterated integral of $f_n(t_1, \dots, t_{n-1}, t)$ with respect to the first $n-1$ variables $t_1, \dots, t_{n-1}$ and $t_n = t$ left as a parameter.

1.5.3 Integral Representations

In the following lines, an explicit stochastic representation for random variables in terms of the Malliavin derivative is provided. The central result is the celebrated Clark-Ocone formula.

Theorem 1.5.7. Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. Then

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F \mid \mathcal{F}_t] \, dW_t. \tag{1.5.10}$$

This theorem gives a representation of the random variable F in terms of Itô stochastic integrals.

1.6 Fractional Brownian Motion (fBm)

In order to provide the reader with a quick introduction to the topic, this section summarizes some of the most relevant results found in Chapter 5 of [32]. The proofs for the following results can be found in the reference and therefore will not be included in this section.

Definition 1.6.1. A centered Gaussian process $B = \{B_t, t \geq 0\}$ is called fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ if it has the following covariance function:

$$R_H(t,s) = \mathbb{E}[B_t B_s] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1.6.1}$$

Fractional Brownian motion has the following self-similar property: For any constant $a > 0$, the process $\left\{a^{-H} B_{at}, t \geq 0\right\}$ and $\{B_t, t \geq 0\}$ have the same distribution. This property is an immediate consequence of the fact that the covariance function (1.6.1) is homogeneous of order $2H$.

From (1.6.1) we can deduce the following expression for the variance of the increment of the process in an interval $[s,t]$:

$$\mathbb{E}\left[|B_t - B_s|^2\right] = |t-s|^{2H}.$$

This implies that fBm has *stationary increments*. Also, for all $\epsilon > 0$ and $T > 0$, there exists a nonnegative random variable $G_{\epsilon,T}$, such that $\mathbb{E}\left[|G_{\epsilon,T}|^p\right] < \infty$ for all $p \geq 1$, and

$$|B_t - B_s| \leq G_{\epsilon,T} |t-s|^{H-\epsilon},$$

for all $s, t \in [0,T]$. In other words, the parameter H controls the regularity of all trajectories, which are Hölder continuous of order $H - \epsilon$, for any $\epsilon > 0$.

Remark 1.6.1. For $H = \frac{1}{2}$, the covariance function (1.6.1) can be written as $R_{\frac{1}{2}}(t,s) = t \wedge s$, and the process B is a standard Brownian motion. Hence, in this case the increments of the process in disjoint intervals are independent. However, for $H \neq \frac{1}{2}$, the increments are not independent.

Set $X_n = B_n - B_{n-1}$, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a Gaussian stationary sequence with covariance function

$$\rho_H(n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right).$$

This implies that two increments of the form $B_k - B_{k-1}$ and $B_{k+n} - B_{k+n-1}$ are:

- positively correlated, ($\rho_H(n) > 0$), if $H > \frac{1}{2}$. This implies an aggregation behavior that describes cluster phenomena and the sequence exhibits long range dependence, this is,

$$\sum_{n=1}^{\infty} \rho_H(n) = \infty.$$

- negatively correlated, ($\rho_H(n) < 0$), if $H < \frac{1}{2}$. This is usually observed in sequences that present intermittency and the sequence exhibits short range dependence, this is,

$$\sum_{n=1}^{\infty} |\rho_H(n)| < \infty.$$

We have seen that for $H \neq \frac{1}{2}$ fBm does not have independent increments. The following proposition asserts that it is not a semimartingale.

Proposition 1.6.2. The fBm is not a semimartingale for $H \neq \frac{1}{2}$.

1.6.1 Fractional integrals and derivatives.

In what follows we will recall the basic definitions and properties of fractional calculus.

Let $a, b \in \mathbb{R}$ such that $a < b$. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left- and right-sided fractional integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(x) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, dy, \tag{1.6.2}$$

$$I_{b-}^{\alpha} f(x) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^x (y-x)^{\alpha-1} f(y)\, dy, \tag{1.6.3}$$

respectively. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a,b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$, then the left- and right-sided fractional derivatives are defined by

$$D_{a+}^{\alpha} f(x) \triangleq \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right),$$

$$D_{b-}^{\alpha} f(x) \triangleq \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right),$$

for almost all $x \in (a,b)$. The following inversion formulas are true:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f, \qquad \forall f \in I_{a+}^{\alpha}(L^p),$$

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f, \qquad \forall f \in L^1(a,b).$$

Analogous inversion formulas hold for the operators I_{b-}^{α} and D_{b-}^{α}.

We now show that fractional Brownian motion can be represented as a stochastic integral. This representation of fBm in term of a Wiener process was first proved by Mandelbrot and Van Ness in [30]. Consider,

$$\begin{aligned} X_t^H &= \frac{1}{C(H)} \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dB_s \\ &= \frac{1}{C(H)} \left(\int_{-\infty}^0 \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dB_s + \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \right), \end{aligned}$$

where B_t is a standard Brownian motion and

$$C(H) = \left(\int_{-\infty}^0 \left[(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right]^2 ds + \frac{1}{2H} \right)^{\frac{1}{2}}.$$

One has to notice that X_t^H is a stochastic integral with respect to a standard Brownian motion as in (1.1.3), where the integrand is a square integrable deterministic function. Thus, it must be Gaussian with $\mathbb{E}\left[X_t^H\right] = 0$ and therefore we can characterize X_t^H by its first and second order moments. In order to do so, observe that

$$\mathbb{E}\left[\left|X_t^H - X_s^H\right|^2\right] = \mathbb{E}\left[\left(X_t^H\right)^2\right] - 2\mathbb{E}\left[X_t^H X_s^H\right] + \mathbb{E}\left[\left(X_s^H\right)^2\right]. \tag{1.6.4}$$

Since by hypothesis it is a square integrable process, if we do the change of variable $s = tu$, one obtains the following,

$$\begin{aligned}
\mathbb{E}\left[\left(X_t^H\right)^2\right] &= \frac{1}{C(H)^2}\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right)^2 ds \\
&= \frac{1}{C(H)^2} t^{2H}\int_{\mathbb{R}}\left((1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right)^2 du \\
&= t^{2H}.
\end{aligned}$$

Analogously, one can show that $\mathbb{E}\left[\left|X_t^H - X_s^H\right|^2\right] = |t-s|^{2H}$. Using (1.6.4), we can write the following,

$$\begin{aligned}
\mathbb{E}\left[X_t^H X_s^H\right] &= -\frac{1}{2}\left(\mathbb{E}\left[\left|X_t^H - X_s^H\right|^2\right] - \mathbb{E}\left[\left(X_t^H\right)^2\right] - \mathbb{E}\left[\left(X_s^H\right)^2\right]\right) \\
&= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).
\end{aligned}$$

Therefore, X_t^H is a fBm since we have proved that its mean and variance are the ones from a fractional Brownian motion.

Prior to the introduction of fBm, Paul Lévy in [28] used the Riemann-Liouville integral, defined by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\,dt,$$

where Γ is the Gamma function, f a locally integrable function and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, to define the process today known, as the Riemann-Liouville representation of fBm:

$$B_t^H = \frac{1}{\Gamma(H+1/2)}\int_0^t (t-s)^{H-\frac{1}{2}}\,dB_s, \tag{1.6.5}$$

where the integration is with respect to the Brownian measure dB_s. Despite being an approximation to fBm through a nice and simple representation, this integral turns out to over-emphasize the origin. This is why later Mandelbrot and Van Ness came out with a more accurate representation for fBm, as introduced previously.

1.7 Summary of Papers

The papers in this thesis appear following a chronological order and are listed below, together with a brief description of the study performed in each of them.

Paper I: "Self-exciting multifractional processes"

In Paper I, we propose a new class of stochastic processes where the future state depends directly on all the past states of the process. This processes are often referred to as self-excited multifractional processes. Starting from the Riemann-Liouville representation of a fBm $\left\{B_t^H, t \in [0,T]\right\}$, with Hurst exponent $H \in (0,1)$, we follow [38] and study a continuous time version of the process, found as the solution to the SDE

$$X_t^{h,f} = \int_0^t \exp\left\{-f\left(t, X_s^{h,f}\right)(t-s)\right\}(t-s)^{h\left(t, X_s^{h,f}\right)-\frac{1}{2}}\, dB_s. \tag{1.7.1}$$

Existence and uniqueness of equation (1.7.1) is shown, as well as a study of probabilistic and path properties such as variance and Hölder regularity of the process. An Euler-Maruyama scheme to approximate the process is provided and we show its strong convergence, as well as estimate its rate of convergence.

Paper II: "High order approximations to call option prices in the Heston model"

In Paper II, we provide a new version of the decomposition formula for a call option price from Alòs in 2012, see [1]. We use this new decomposition result, to find sharper estimates of the error term than in previously known approximations. In particular estimates of the form $O\left(\nu^3\left(|\rho| + \nu\right)\right)$ and $O\left(\nu^4\left(1 + |\rho|\,\nu\right)\right)$, where ν and ρ are, respectively, the volatility of volatility and the correlation terms in the Heston model. A higher order approximation of the form $O\left(\nu^6\right)$ is provided for the uncorrelated case, i.e. $\rho = 0$.

Paper III: "Variance and interest rate risk in unit-linked insurance policies"

In Paper III, we provide a risk-neutral pricing formula for unit-linked life insurance policies, as well as, provide a perfect hedging strategy. We characterize the unique pricing measure $\mathbb{Q}$, for a stochastic volatility model written in forward variance and stochastic interest rates. By considering this setup we are able to complete the market and characterize the measure $\mathbb{Q}$ for the particular Heston model written in forward variance and Vasicek model for interest rates. The study concludes with a simulation, where we price unit-linked policies using the Norwegian mortality rates. In addition we compare prices for the classical Black-Scholes model against the one we propose.

Paper IV: "A decomposition formula for fractional Heston jump diffusion models"

In Paper IV, we start from the fractional Heston version of the decomposition formula from Alòs and Yang in 2017, see [3], to extend it under a jump diffusion model with both jumps in the asset prices and volatility. We provide a martingale representation of the integrated future average variance, by means of Malliavin calculus, and deduce an exact general formula to approximate call option prices V. The expression is built from the Black-Scholes formula, plus some correction terms due to the model parameters, and is written as follows.

$$\begin{aligned}
V_t &= BS\left(t, X_t, v_t\right) - \zeta\left(\rho_2, \eta\right) \mathbb{E}_t \left[\int_t^T e^{-rs} \Lambda BS\left(s, X_s, v_s\right) ds \right] \qquad (1.7.2) \\
&+ \frac{\rho_1}{2} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} \Lambda \Gamma BS\left(s, X_s, v_s\right) \sigma_s d\left[W, M^c\right]_s \right] \\
&+ \frac{1}{8} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} \Gamma^2 BS\left(s, X_s, v_s\right) d\left[M^c, M^c\right]_s \right] \\
&+ \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} \Gamma BS\left(s, X_s, v_s\right) \left[v_s\left(T-s\right) \int_0^\infty \Delta_m^2 g\left(s, M_{s-}, Y_{s-}\right) \ell\left(dz\right) \right] ds \right] \\
&+ \mathbb{E}_t \left[\int_t^T \int_0^\infty e^{-r(s-t)} \left[\Delta_x^2 BS\left(s, X_{s-}, v_{s-}\right) + \Delta_y^2 BS\left(s, X_{s-}, v_{s-}\right) \right] \ell(dz) ds \right].
\end{aligned}$$

We provide a first order approximation formula for equation (1.7.2), in terms of $O\left(\nu^2 + \eta^2\right)$, where ν and η are, respectively, the vol-of-vol and the jump diffusion term. The approximation formula is achieved as a recurrent application of the general result, and is written as a Taylor-type of expansion formula, which can be extended to higher orders.

1.8 Further Research

In future research, one could try to extend the results provided in this thesis in several directions as suggested below:

In Paper I, it would be worth trying to investigate how to use this new class of SEMP processes, in order to model the dynamics of financial assets and study their properties. A first suggestion would be to study instantaneous volatility processes given by an SDE of the following type

$$d\sigma_t = -\kappa\left(\sigma_t - \theta\right) + \nu \sqrt{\sigma_t} dX_t^{h,f}.$$

One could study, for instance, if the term structure of skew could easily be fitted, and how to match this with the no-arbitrage market hypothesis.

In Paper II, one could try to extend the approximating results provided and apply them to the computation of the option's greeks, in order to approximate the greeks of an option in terms of the model parameters. By doing this, one would not only faster estimates of the greeks, but would also gain further interpretation of the results in terms of the model parameters.

In Paper III, there is a major dependence between the completeness of the market and a structure correlation between the assets considered. One loses the uniqueness of the equivalent martingale measure $\mathbb{Q}$ when providing a correlation structure between the underlying asset S, the forward variance ξ and the zero-coupon bond P. The study of how to characterize this relationship between the uniqueness of the measure and the correlation is of high interest due to the natural dependence between S and ξ particularly.

In Paper IV, providing a second order approximation formula as an extension of Theorem 14 would be very relevant. In the first order formula, the discontinuous terms are of order $O\left(\nu^2, \eta^2\right)$ and therefore are not part of the approximation but rather of the error term, leading to an analogous version of the continuous approximation formula already developed in [3]. If one were to extend the approximation up to second order, the discontinuous terms would appear in the approximation, leading to a new formula that would give clearer understanding of the effects of jumps in the final price of a call option. The difficulty here, lies in proving the integrability of a numerous amount of terms that come from applying iteratively the general expansion result provided in this thesis.

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Papers

I

Paper I

Self-Exciting Multifractional Processes

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Paper II

High Order Approximations to Call Option Prices in the Heston Model

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II

Paper III

Variance and Interest Rate Risk in Unit-Linked Insurance Policies

David Baños, Marc Lagunas-Merino, Salvador Ortiz-Latorre

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III

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Article

Variance and Interest Rate Risk in Unit-Linked Insurance Policies

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Abstract: One of the risks derived from selling long-term policies that any insurance company has arises from interest rates. In this paper, we consider a general class of stochastic volatility models written in forward variance form. We also deal with stochastic interest rates to obtain the risk-free price for unit-linked life insurance contracts, as well as providing a perfect hedging strategy by completing the market. We conclude with a simulation experiment, where we price unit-linked policies using Norwegian mortality rates. In addition, we compare prices for the classical Black-Scholes model against the Heston stochastic volatility model with a Vasicek interest rate model.

Keywords: unit-linked policies; pure endowment; term insurance; stochastic volatility models; stochastic interest rates

MSC: 60H30; 91G20; 91G30; 91G60

1. Introduction

A unit-linked insurance policy is a product offered by insurance companies. Such contract specifies an event under which the insured of the contract obtains a fixed amount. Typically, the payoff of such contract is the maximum value between some prescribed quantity, the guarantee, and some quantity depending on the performance of a stock or fund. For instance, if G is some positive constant amount, and S is the value of some equity or stock at the time of expiration of the contract, then a unit-linked contract pays

$$H = \max\{G, f(S)\},$$

where f is some suitable function of S. Here, the payoff H is always larger than G, hence being G a minimum guaranteed amount that the insured will receive. Naturally, the price of such contract depends on the age of the insured at the moment of entering the contract and the time of expiration, likewise, it also depends on the event that the insured is alive at the time of expiration.

The risk of such contracts depends on the risk of the financial instruments used to hedge the claim H, and there are many ways to model it. The most classical one is considering the evolution of S under a Black-Scholes model; this is, for instance, the case in Boyle and Schwartz (1977) or Aase and Persson (1994), where the authors derive pricing and reserving formulas for unit-linked contracts in such setting. One can also consider a more general class of models. For example, it is empirically known that the driving volatility of S is, in general, not constant. One could then take a stochastic model for the volatility, as it is done in Wang et al. (2013), where the authors carry out pricing and hedging under stochastic volatility. Since there is more randomness in the model, complete hedging is no longer possible, the authors in Wang et al. (2013) provide the so-called local risk minimizing strategies. Another result that considers both stochastic interest rates and stochastic volatility is van Haastrecht et al. (2009). This paper focuses on the pricing problem.

In this paper, instead, we look at the problem from two different perspectives. On the one hand, we also consider stochastic volatility, as market evidence shows. Nonetheless, there are available instruments in the market for hedging against volatility risk, the so-called forward variance swaps. Such products are contracts on the future performance of the volatility of the stock. In such a way, we want to price unit-linked contracts taking into account that the insurance company can trade these instruments as well. On the other hand, it is known that unit-linked products share similarities with European call options. For example, authors in Boyle and Schwartz (1977) recognize the payoff of unit-linked products as European call options plus some certain amount. However, European call options have very short maturities, typically between the same day of the contract up to two years, while it is not uncommon to have unit-linked insurance contracts that last for up to 40 years. For this reason, there is an inherent risk in the interest rate driving the intrinsic value of money. In this paper, we take such long-term risk into account as well. Our simulations for the particular contracts in Sections 5.1 and 5.2 suggest that the premiums are underpriced in the Black-Scholes model. The insurer should, therefore, be aware that long maturities in unit-linked contracts have a significant impact on their premiums depending on the model chosen. Classically, most of the literature about equity-linked policies assumes deterministic interest rates. Nevertheless, some research on stochastic interest rates has also been carried. For example, in Bacinello and Persson (2002), the authors consider stochastic interest rates under the Heath-Jarrow-Morton framework and study different types of premium payments. In addition, a comparison with the classical Black-Scholes model is offered in Bacinello and Persson (2002). In addition, in Bacinello and Persson (1993), the Vasicek and Cox-Ingersoll-Ross model is considered for the interest rate. In this paper, we consider a general framework including both cases.

While many results in the literature deal with the construction of risk minimizing strategies in incomplete markets, in this paper instead, inspired by Romano and Touzi (1997), we complete the market by allowing for the possibility to trade other instruments that one can find in the market. On the one hand, we introduce zero-coupon bonds to hedge against interest rate risk and, on the other hand, we introduce variance swaps to hedge against volatility risk.

This paper is organized as follows. First, we introduce in Section 2 our insurance and economic framework with the specific models for the money account, stock, and volatility. Then, in Section 3, we complete the market by incorporating zero-coupon bonds and variance swaps in the market. We derive the dynamics of the new instruments used to hedge and apply the risk-neutral theory to price insurance contracts subject to the performance of an equity or fund with stochastic interest and volatility. In Section 4, we take a particular model, the Vasicek model for the interest rate, and a Heston model written in forward variance form. We implement the model and do a comparison study with the classical Black-Scholes model in Section 5, where we generate price surfaces under Norwegian mortality rates and different maturities. We conclude Section 5 with a Monte Carlo simulation of the price distributions.

2. Framework

The two basic elements needed in order to build a financial model robust enough to be able to price unit-linked policies are a financial market and a group of individuals to write insurance on. We consider a finite time horizon $T > 0$ and a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where Ω is the set of all possible states of the world, $\mathcal{A}$ is a σ-algebra of subsets of Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{A})$. We model the information flow at each given time with a filtration $\mathcal{F} = \{\mathcal{F}_t, t \in [0, T]\}$ given by a collection of increasing σ-algebras, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{A}$ for $t \geq s$. We will also assume that $\mathcal{F}_0$ contains all the sets of probability zero and that the filtration is right continuous. We also take $\mathcal{A} = \mathcal{F}_T$. The information flow $\mathcal{F}$ comes from two sources; the financial market and the states of the insured that are relevant in the policy. The market information available at time t will be denoted by $\mathcal{G}_t$ and the information regarding to the state of the insured by $\mathcal{H}_t$. We will assume throughout the paper that the σ-algebras $\mathcal{G}_t$ and $\mathcal{H}_t$ are independent for all t, which implies that the value of the market assets is independent from the health condition of the insured. We also assume that $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, for all t, where $\mathcal{G}_t \vee \mathcal{H}_t$ is the σ-algebra generated by the union of $\mathcal{G}_t$ and $\mathcal{H}_t$. This can be understood as the total amount of information available in the economy at time t that is the information one can get by recording the values of market assets and the health state of the insured from time 0 to time t.

2.1. The Market Model

The market information $\mathcal{G}$ will be modeled using the filtration generated by three independent standard Brownian motions, W_t^0, W_t^1, and W_t^2. These three Brownian motions represent the sources of risk in our model. We will consider a market formed by assets of two different natures. A bank account, considered to be of a riskless nature and stock or bond prices, which are of risky nature.

We start by defining the bank account, whose price process is denoted by $B = \{B_t\}_{t\in[0,T]}$, such that $B_0 = 1$. We will assume that the asset evolves according to the following differential equation:

$$dB_t = r_t B_t dt, \quad t \in [0,T], \tag{1}$$

where r_t is the instantaneous spot rate and it is assumed to have integrable trajectories. Actually, we will assume that this rate evolves under the physical measure $\mathbb{P}$, according to the following stochastic differential equation (SDE):

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t^0, \quad r_0 > 0, \quad t \in [0,T], \tag{2}$$

where $\mu, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are Borel measurable functions such that, for every $t \in [0,T]$ and $x \in \mathbb{R}$,

$$|\mu(t,x)| + |\sigma(t,x)| \le C(1 + |x|),$$

for some positive constant C, and such that for every $t \in [0,T]$ and $x, y \in \mathbb{R}$

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,x)| \le L|x-y|,$$

for some constant $L > 0$. We will also assume there exists $\epsilon > 0$, such that $\sigma(t,x) \ge \epsilon > 0$ for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. These conditions are sufficient to guarantee a unique global strong solution of (2), weaker conditions may be imposed, see, e.g., (Revuz and Yor 1999, Chapter IX, Theorem 3.5).

One of the risky assets will be the stock. We describe the stock price process $S = \{S_t\}_{t\in[0,T]}$ by a general mean-reverting stochastic volatility model. Specifically, we will consider the following SDEs:

$$\frac{dS_t}{S_t} = b(t, S_t)\, dt + a(t, S_t)\, f(\nu_t)\, dW_t^1, \quad S_0 > 0, \tag{3}$$

$$d\nu_t = -\kappa(\nu_t - \bar{\nu})\, dt + h(\nu_t)\, dW_t^2, \quad \nu_0 > 0, \tag{4}$$

for $t \in [0,T]$. Here, a, b are uniformly Lipschitz continuous and bounded functions, such that $a(t,x) > 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$. The function f is assumed to be continuous with linear growth and strictly positive. We assume that h is a non-negative, linear growth, invertible function such that

$$|h(x) - h(y)|^2 \le \ell(|x-y|),$$

for some function ℓ defined on $(0,\infty)$ such that

$$\int_0^\epsilon \frac{dz}{\ell(z)} = \infty, \quad \text{for any } \epsilon > 0.$$

Then, (Revuz and Yor 1999, Chapter IX, Theorem 3.5(ii)) guarantees the existence of a pathwise unique solution of Equation (3). We call ν_t the instantaneous variance.

Due to the fact that neither ν nor r are tradable, our market model is highly incomplete. In the forthcoming section, we will complete the market by introducing financial instruments in order to hedge against the risk derived from instantaneous variance and interest rates.

We introduce the numéraire, with respect to which we will discount our cashflows.

Definition 1. *The (stochastic) discount factor $D_{t,T}$ between two time intervals t and T, $0 \leq t \leq T$, is the amount at time t that is equivalent to one unit of currency payable at time T, and is given by*

$$D_{t,T} = \frac{B_t}{B_T} = \exp\left(-\int_t^T r_s ds\right). \tag{5}$$

2.2. The Insurance Model

In what follows, we introduce our insurance model. More specifically, we want to model the insurance information $\mathcal{H}$ as the one generated by a regular Markov chain $X = \{X_t, t \in [0,T]\}$ with finite state space $\mathcal{S}$ which regulates the states of the insured at each time $t \in [0,T]$. For instance, in an endowment insurance, the state $\mathcal{S} = \{*, \dagger\}$ consists of the two states, $* =$ "alive" and $\dagger =$ "deceased". In a disability insurance, we have three states, $\mathcal{S} = \{*, \diamond, \dagger\}$, where $\diamond$ stands for "disabled". Observe that X is right-continuous with left limits and, in particular, $\mathcal{H}$ satisfies the usual conditions.

We introduce the following processes:

$$I_i^X(t) = \begin{cases} 1, \text{ if } X_t = i, \\ 0, \text{ if } X_t \neq i \end{cases}, \quad i \in \mathcal{S},$$

$$N_{ij}^X(t) = \#\{s \in (0,t) : X_{t^-} = i, X_t = j\}, \quad i,j \in \mathcal{S}, \quad i \neq j.$$

Here, # denotes the counting measure and $X_{t^-} \triangleq \lim_{\substack{u \to t \\ u<t}} X_u$ the left limit of X at the point t. The random variable $I_i^X(t)$ tells us whether the insured is in state i at time t and $N_{ij}^X(t)$ tells us the number of transitions from i to j in the whole period $(0,t)$.

Definition 2 (Stochastic cash flow)**.** *A stochastic cash flow is a stochastic process $A = \{A_t\}_{t \geq 0}$ with almost all sample paths with bounded variation.*

More concretely, we will consider cash flows described by an insurance policy entirely determined by its payout functions. We denote by $a_i(t)$, $i \in S$, the sum of payments from the insurer to the insured up to time t, given that we know that the insured has always been in state i. Moreover, we will denote by $a_{ij}(t)$, $i,j \in S$, $i \neq j$, the payments which are due when the insured switches state from i to j at time t. We always assume that these functions are of bounded variation. The cash flows we will consider are entirely described by the policy functions, defined by an insurance policy. Observe that the policy functions can be stochastic in the case where the payout is linked to a fund modeled by a stochastic process.

Definition 3 (Policy cash flow)**.** *We consider payout functions $a_i(t)$, $i \in S$ and $a_{ij}(t)$, $i,j \in S$, $i \neq j$ for $t \geq 0$ of bounded variation. The (stochastic) cash flow associated with this insurance is defined by*

$$A(t) = \sum_{i \in S} A_i(t) + \sum_{\substack{i,j \in S \\ i \neq j}} A_{ij}(t),$$

where

$$A_i(t) = \int_0^t I_i^X(s) da_i(s), \quad A_{ij}(t) = \int_0^t a_{ij}(s) dN_{ij}^X(s).$$

The quantity A_i corresponds to the accumulated liabilities while the insured is in state i and A_{ij} for the case when the insured switches from i to j.

The value of a stochastic cash flow A at time t will be denoted by $V^+(t, A)$, or simply $V^+(t)$, and is defined as

$$V^+(t, A) = B_T \int_t^\infty \frac{dA(s)}{B_s},$$

where B is the reference discount factor in (1). The stochastic integral is a well-defined pathwise Riemann–Stieltjes integral since A is almost surely of bounded variation and B is almost surely continuous. The quantity $V^+(t, A)$ is stochastic since both B and A are stochastic. The prospective reserve of an insurance policy with cash flow A given the information $\mathcal{F}_t$ is then defined as

$$V^+_{\mathcal{F}}(t, A) = \mathbb{E}^{\mathbb{Q}}[V^+(t, A)|\mathcal{F}_t],$$

where $\mathbb{Q}$ is an equivalent martingale measure.

It turns out, see (Koller 2012, Theorem 4.6.3), that one can find explicit expressions when the policy functions a_i, $i \in S$, a_{ij}, $i, j \in S$, $i \neq j$ and the force of interest are deterministic. Combining the previous result with a conditioning argument allows us to recast the expression for the reserves as the following conditional expectation:

$$V_i^+(t, A) \triangleq \mathbb{E}^{\mathbb{Q}}\left[\sum_{j\in S}\int_t^\infty \frac{B_t}{B_s} p_{ij}(t,s)da_j(s) + \sum_{\substack{j,k\in S\\ k\neq j}}\int_t^\infty \frac{B_t}{B_s} p_{ij}(t,s)\mu_{jk}(s)a_{jk}(s)ds \,\middle|\, \mathcal{G}_t\right], \tag{6}$$

where μ_{ij} are the continuous transition rates associated with the Markov chain X and $p_{ij}(s, t)$ are the transition probabilities from changing from state i at time s to state j at time t.

In this paper, we will focus on the pricing and hedging of unit-linked pure endowment policies with stochastic volatility and stochastic interest rate. Other more general insurances can be reduced to this. For instance, in (6), if a_i is of bounded variation and a.e. differentiable with derivative $\dot{a}_i$, then we can look at

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_s}\dot{a}_i(s)\middle|\mathcal{G}_t\right] \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_s}a_{ij}(s)\middle|\mathcal{G}_t\right]$$

as contracts with payoff $\dot{a}_i(s)$, respectively $a_{ij}(s)$, with maturity $s \geq t$.

3. Pricing and Hedging of the Unit-Linked Life Insurance Contract

The aim of this section is to price and hedge insurance claims linked to the fund S. However, we cannot hedge any contingent claim using a portfolio with S only. In the spirit of Romano and Touzi (1997), we will complete the market, including the possibility to trade products whose underlying are the forward variance and interest rate, which are indeed actively traded in the market.

3.1. Completing the Market Using Variance Swaps and Zero-Coupon Bonds

First, we will introduce a family of equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$ given by

$$\mathbb{Q}(A) = \mathbb{E}[Z_T \mathbf{1}_A], \qquad A \in \mathcal{G}_T, \tag{7}$$

where $Z = \{Z_t, t \in [0, T]\}$ is given by

$$Z_t = \mathcal{E}\left(\sum_{i=0}^2 \int_0^\cdot \gamma_s^i dW_s^i\right)_t, \qquad t \in [0, T],$$

and $\mathcal{G}$-adapted γ^i, for every $i = 0,1,2$, such that $\mathbb{E}[Z_T] = 1$. Here, $\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M,M]_t\right)$ denotes the stochastic exponential for a continuous semimartingale M.

The following processes are Brownian motions under $\mathbb{Q}$

$$W_t^{\mathbb{Q},i} = W_t^i - \int_0^t \gamma_s^i ds, \qquad i = 0,1,2. \tag{8}$$

Note that not all probability measures given in (7) are risk-neutral in our market model. In particular, γ^1 is determined by the fact that S is a tradable asset and takes the form

$$\gamma_t^1 \triangleq \frac{r_t - b(t,S_t)}{a(t,S_t) f(\nu_t)}.$$

All probability measures in (7) fixing γ^1 are valid risk-neutral measures. In particular, choosing $\gamma^0 = \gamma^2 = 0$ is one of them. From now on, we denote by $\mathbb{Q}^0$ this choice, that is,

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}} = \mathcal{E}\left(\int_0^{\cdot} \gamma_s^1 dW_s^1\right)_T. \tag{9}$$

Now, we are in a position to introduce the financial instruments whose valuation will be done under $\mathbb{Q}^0$. One of the most traded assets in interest rate markets are zero-coupon bonds.

Definition 4. *A T-maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $0 \le t \le T$ is denoted by $P_{t,T}$ and by definition $P_{T,T} = 1$, for all T.*

A risk-neutral price of a zero-coupon bond in our framework is given in the following definition.

Definition 5. *The price of a zero-coupon bond, $P_{t,T}$ is given by*

$$P_{t,T} = \mathbb{E}^{\mathbb{Q}^0}[D_{t,T} \mid \mathcal{G}_t] = \mathbb{E}^{\mathbb{Q}^0}\left[\frac{B_t}{B_T} \mid \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}^0}\left[\exp\left\{-\int_t^T r_s ds\right\} \mid \mathcal{G}_t\right], \tag{10}$$

where $\mathbb{Q}^0$ is the equivalent martingale measure given by (9). See (Filipović 2009, Definition 4.1. in Section 4.3.1 and Section 5.1) for definitions.

The next classical result gives a connection between the bond price in (10) and the solution to a linear partial differential equation (PDE), see e.g., Filipović (2009).

Lemma 1. *Assume that, for any $T > 0$, $F_T \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ is a solution to the boundary problem on $[0,T] \times \mathbb{R}$ given by*

$$\partial_t F_T(t,x) + \mu(t,x)\partial_x F_T(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_x^2 F_T(t,x) - xF_T(t,x) = 0,$$
$$F_T(T,x) = 1.$$

Then,

$$M_t \triangleq F_T(t,r_t) e^{-\int_0^t r_u du}, \qquad t \in [0,T]$$

is a local martingale. If in addition either:

(a) $\mathbb{E}^{\mathbb{Q}^0}\left[\int_0^T \left|\partial_x F_T(t,r_t) e^{-\int_0^t r_u du}\sigma(t,r_t)\right|^2 dt\right] < \infty$, or
(b) M is uniformly bounded,

then M is a martingale, and

$$F_T(t, r_t) = \mathbb{E}^{\mathbb{Q}^0}\left[e^{-\int_t^T r_u du} \mid \mathcal{G}_t\right], \qquad t \in [0, T]. \tag{11}$$

The dynamics of the zero-coupon bond P in terms of the function F_T are given by

$$dP_{t,T} = \mathcal{L}_P(F_T)(t, r_t)\, dt + \partial_x F_T(t, r_t)\, \sigma(t, r_t)\, dW_t^0, \tag{12}$$

where $\mathcal{L}_P \triangleq \partial_t + \mu(t, x)\,\partial_x + \frac{1}{2}\sigma^2(t, x)\,\partial_x^2 - x$.

We turn now to the definition of the forward variance process. The forward variance $\xi_{t,u}$, for $0 \le t \le u$, is by definition the conditional expectation of the future instantaneous variance, see, e.g., Ould Aly (2014), that is,

$$\xi_{t,u} \triangleq \mathbb{E}^{\mathbb{Q}^0}[\nu_u \mid \mathcal{G}_t], \quad 0 \le t \le u, \tag{13}$$

where $\mathbb{Q}^0$ is the risk-neutral pricing measure defined in (9). Following Bergomi and Guyon (2012), one can easily rewrite the general stochastic volatility model, given by Equations (3) and (4) in forward variance form. This is achieved by taking conditional expectation of Equation (4), which yields

$$d\mathbb{E}^{\mathbb{Q}^0}[\nu_u \mid \mathcal{G}_t] \quad = \quad -\kappa\left(\mathbb{E}^{\mathbb{Q}^0}[\nu_u \mid \mathcal{G}_t] - \bar{\nu}\right) du, \quad u > t,$$

Solving the previous linear ordinary differential equation (ODE), by integrating on $[t, u]$, we have

$$\xi_{t,u} \quad = \quad \bar{\nu} + e^{-\kappa(u-t)}(\nu_t - \bar{\nu}). \tag{14}$$

There are two things to notice at this point. The first is that, by construction, $\nu_t = \xi_{t,t}$, for every $t \in [0, T]$. The second is that, differentiating the previous equation, we can characterize the dynamics with respect to t for the forward variance as follows:

$$d\xi_{t,u} \quad = \quad e^{-\kappa(u-t)} h(\nu_t)\, dW_t^2. \tag{15}$$

Solving Equation (14) for ν_t yields

$$\nu_t \quad = \quad \bar{\nu} + e^{\kappa(u-t)}(\xi_{t,u} - \bar{\nu}) \triangleq \psi(t, u, \xi_{t,u}).$$

Usually, the dynamics of the forward variance in any forward variance model are given through the following SDE:

$$d\xi_{t,u} \quad = \lambda(t, u, \xi_{t,u})\, dW_t^2. \tag{16}$$

As a consequence of the previous result, in our case, the function λ in Equation (16) is fully characterized by

$$\lambda(t, u, \xi_{t,u}) \quad \triangleq \quad e^{-\kappa(u-t)}(h \circ \psi)(t, u, \xi_{t,u}). \tag{17}$$

Note that any finite-dimensional Markovian stochastic volatility model can be rewritten in forward variance form. Since we will only be interested in the fixed case $u = T$, we will drop the dependence on T for $\xi_{t,T}$ and write instead $\xi_t = \xi_{t,T}$.

We will show how to form a portfolio with a perfect hedge. The financial instruments needed in order to build a riskless portfolio are the underlying asset, a variance swap, and the zero-coupon bond.

From now on, we will assume that the function F_T, solution to the PDE in Lemma 1 is invertible in the space variable, for every $t \in [0,T]$, e.g., this is the case if r_t, $t \in [0,T]$ is given by the Vasicek model. Introduce the notation

$$G_T(t,x) \triangleq \partial_x F_T(t,x), \tag{18}$$

then $\partial_x F_T(t,r_t) = G_T\left(t, F_T^{-1}(t,P_{t,T})\right)$, where $r_t = F_T^{-1}(t,P_{t,T})$.

3.2. Pricing and Hedging in the Completed Market

Let $\Pi = \{\Pi_t\}_{t\in[0,T]}$ be a stochastic process representing the value of a portfolio consisting of a long position on an option with price V_t, where $V_t = V(t,S_t,\xi_t,P_{t,T})$, and respective short positions on Δ_t units of the underlying asset, Σ_t units of a variance swap, and Ψ_t units of a zero-coupon bond. Therefore, we can characterize the process Π as

$$\Pi_t = V(t,S_t,\xi_t,P_{t,T}) - \Delta_t S_t - \Sigma_t\xi_t - \Psi_t P_{t,T}, \quad t \in [0,T]. \tag{19}$$

Definition 6. *We say that the portfolio Π is self-financing if, and only if,*

$$d\Pi_t = dV(t,S_t,\xi_t,P_{t,T}) - \Delta_t dS_t - \Sigma_t d\xi_t - \Psi_t dP_{t,T},$$

for every $t \in [0,T]$.

Definition 7. *We say that the portfolio Π is perfectly hedged, or risk-neutral, if it is self-financing and*

$$\Pi_T = 0.$$

From now on, and throughout the rest of the paper, we will only differentiate between time derivative $\partial_t V$ and space derivatives $\partial_x V, \partial_y V, \partial_z V$, to write the partial derivatives of $V = V(t,x,y,z)$. We will also denote second order spatial partial derivatives of V with respect to S_t, ξ_t, $P_{t,T}$, respectively by $\partial_x^2 V, \partial_y^2 V, \partial_z^2 V$ and the second order crossed derivatives as $\partial_x\partial_y V, \partial_x\partial_z V, \partial_y\partial_z V$. In order to simplify the notation in the following results, we shall define

$$\Xi_T(t,x) \triangleq G_T\left(t, F_T^{-1}(t,x)\right) \cdot \sigma\left(t, F_T^{-1}(t,x)\right),$$

where one should recall that G_T is given in (18).

Theorem 1. *Let Π be a portfolio defined as in (19), and assume $V \in C^{1,2}\left([0,T]\times\mathbb{R}^3\right)$. If Π is a replicating portfolio, then V fulfills*

$$\begin{aligned}\partial_t V + \frac{1}{2}\left(x^2 a(t,x)^2 f(\psi(t,T,y))^2 \partial_x^2 V + \lambda(t,T,y)^2 \partial_y^2 V + \Xi_T^2(t,z)\partial_z^2 V\right)& \\ -r_t\left(V - x\partial_x V - y\partial_y V - z\partial_z V\right) = 0,&\end{aligned} \tag{20}$$

for every $t \in [0,T]$ and

$$V(T,S_T,\xi_T,P_{T,T}) = \max(S_T,G). \tag{21}$$

Proof. See Appendix A.1 in Appendix A. □

From now on, in order to ease the notation, we will define the differential operator in (20) as

$$\mathcal{L}_V \triangleq \partial_t + \frac{1}{2}\left(x^2 a(t,x)^2 f(\psi(t,T,y))^2 \partial_x^2 + \lambda(t,T,y)^2 \partial_y^2 + \Xi_T^2(t,z)\,\partial_z^2\right) \\ - r_t\left(1 - x\partial_x - y\partial_y - z\partial_z\right). \tag{22}$$

We will now prove that the discounted option price is a martingale.

Theorem 2. *Let V be the solution to the PDE given by Equation (20) with terminal condition (21). Then,*

$$B_t^{-1} V(t, S_t, \xi_t, P_{t,T}) = \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} V(T, S_T, \xi_T, P_{T,T}) \mid \mathcal{G}_t\right],$$

where $\mathbb{Q}$ indicates the risk-neutral measure.

Proof. See Appendix A.2 in the Appendix A. □

4. The Vasicek Model and Heston Model Written in Forward Variance

This section is devoted to providing the reader with a particular model. We will assume that the evolution of the short-term rate is given by a Vasicek model and consider a Heston model for the risky asset written in forward variance form.

Let us consider the following SDE for the short-term rate given by the Vasicek model:

$$dr_t = k(\theta - r_t)\,dt + \sigma dW_t^0, \quad r_0 > 0, \quad t \in [0,T], \tag{23}$$

and the Heston model for the risky asset, given by

$$dS_t = \mu_t S_t dt + S_t\sqrt{v_t}dW_t^1, \quad S_0 > 0, \quad t \in [0,T], \tag{24}$$

$$dv_t = -\kappa(v_t - \bar{v})\,dt + \eta\sqrt{v_t}dW_t^2, \quad v_0 > 0, \quad t \in [0,T]. \tag{25}$$

It is well known that the SDE (23) admits the following closed expression:

$$r_T = e^{-k(T-t)}r_t + \theta\left(1 - e^{-k(T-t)}\right) + \sigma\int_t^T e^{-k(T-s)}dW_s^0.$$

Now, we know that r_T, conditional on $\mathcal{G}_t$, is normally distributed with mean and variance

$$\mathbb{E}[r_T \mid \mathcal{G}_t] = e^{-k(T-t)}r_t + \theta\left(1 - e^{-k(T-t)}\right),$$

$$\mathrm{Var}[r_T \mid \mathcal{G}_t] = \frac{\sigma^2}{2k}\left(1 - e^{-2k(T-t)}\right).$$

One can show, see, e.g., Musiela and Rutkowski (2005), that the price of the zero-coupon bond under the dynamics given in (23) is

$$P_{t,T} = A(t,T)\,e^{-B(t,T)r_t},$$

where $B(t,T) \triangleq \frac{1}{k}\left(1-e^{-k(T-t)}\right)$ and $A(t,T) \triangleq \exp\left(\left(\theta-\frac{\sigma^2}{2k^2}\right)(B(t,T)+t-T)-\frac{\sigma^2}{4k}B(t,T)^2\right)$. If we now apply Itô's Lemma to $f(t,r_t) = A(t,T)e^{-B(t,T)r_t}$, we have

$$\begin{aligned} dP_{t,T} &= \partial_t f(t,r_t)\,dt + \partial_r f(t,r_t)\,dr_t + \frac{1}{2}\partial^2_{rr} f(t,r_t)\,d[r,r]_t \\ &= \partial_t P_{t,T}dt - A(t,T)\,B(t,T)\,e^{-B(t,T)r_t}dr_t + \frac{1}{2}A(t,T)\,B(t,T)^2\,e^{-B(t,T)r_t}d[r,r]_t \\ &= \partial_t P_{t,T}dt + P_{t,T}\left(-B(t,T)\,dr_t + \frac{1}{2}B(t,T)^2\,d[r,r]_t\right). \end{aligned}$$

Replacing the term dr_t in the previous equation by its SDE (23), we have

$$\frac{dP_{t,T}}{P_{t,T}} = -\left(B(t,T)\,k(\theta-r_t) - \frac{1}{2}B(t,T)^2\sigma^2\right)dt - \sigma B(t,T)\,dW^0_t. \tag{26}$$

The forward variance in this case has the following dynamics:

$$d\xi_{t,u} = \eta e^{-\kappa(u-t)}\sqrt{\xi_{t,t}}dW^2_t. \tag{27}$$

The Heston model, as any Markovian model, can be rewritten in forward variance form by means of Equations (24) and (27) . The following corollary gives the specific risk-neutral measure for the Vasicek-Heston model that will be useful for simulation purposes in the next section.

Corollary 1. *The risk-neutral measure under the Vasicek-Heston model is given by the measure in* (7) *with*

$$\begin{aligned} \gamma^0_t &= \frac{1}{\Theta(t,\nu_t)}\left[\eta\sqrt{\nu_t}\Big(2(-1+B(t,T)\,k)\,r_t + B(t,T)\left(B(t,T)\,\sigma^2-2k\theta\right)\Big)\right], \\ \gamma^1_t &= \frac{-1}{\Theta(t,\nu_t)}\left[2B(t,T)\,\sigma\eta\,(\mu_t-r_t)\right], \\ \gamma^2_t &= \frac{1}{\Theta(t,\nu_t)}\left[2B(t,T)\,e^{\kappa(T-t)}r_t\sigma\xi_t\right], \end{aligned}$$

where

$$\Theta(t,x) \triangleq 2B(t,T)\,\sigma\eta\sqrt{x}.$$

Proof. We will proceed similarly as in Theorem 2. We have to impose that the discounted price process, $\tilde{S}_t$, the discounted variance swap $\tilde{\xi}_t$, and the discounted zero-coupon bond price $\tilde{P}_{t,T}$ are $\mathbb{Q}$-martingales:

$$\begin{aligned} d\tilde{S}_t &= dB^{-1}_t S_t + B^{-1}_t dS_t \\ &= -r_t B^{-1}_t S_t dt + B^{-1}_t\left[\mu_t S_t dt + S_t\sqrt{\nu_t}dW^1_t\right] \\ &= \tilde{S}_t\left(\left[\mu_t - r_t + \sqrt{\nu_t}\gamma^1_t\right]dt + \sqrt{\nu_t}dW^{\mathbb{Q},1}_t\right), \end{aligned}$$

now the discounted price process $\tilde{S}_t$ is a $\mathbb{Q}-$martingale if and only if

$$\gamma^1_t = \frac{r_t-\mu_t}{\sqrt{\nu_t}}. \tag{28}$$

We do the same for the discounted forward variance, hence we obtain

$$\begin{aligned}
d\tilde{\xi}_t &= dB_t^{-1}\xi_t + B_t^{-1}d\xi_t \\
&= -r_t B_t^{-1}\xi_t dt + B_t^{-1}\eta e^{-\kappa(T-t)}\sqrt{\nu_t}dW_t^2 \\
&= B_t^{-1}\left(-r_t\xi_t dt + \eta e^{-\kappa(T-t)}\sqrt{\nu_t}\left[dW_t^{\mathbb{Q},2} + \gamma_t^2 dt\right]\right) \\
&= B_t^{-1}\left(\left[\eta e^{-\kappa(T-t)}\sqrt{\nu_t}\gamma_t^2 - r_t\xi_t\right]dt + \eta e^{-\kappa(T-t)}\sqrt{\nu_t}dW_t^{\mathbb{Q},2}\right).
\end{aligned}$$

Therefore, the discounted variance swap $\tilde{\xi}_t$, is a $\mathbb{Q}-$martingale if and only if

$$\gamma_t^2 = \frac{\xi_t r_t}{\eta e^{-\kappa(T-t)}\sqrt{\nu_t}}. \tag{29}$$

Finally, we impose that the discounted zero-coupon bond price process is a $\mathbb{Q}$-martingale in an analogous computation,

$$\begin{aligned}
d\tilde{P}_{t,T} &= dB_t^{-1}P_{t,T} + B_t^{-1}dP_{t,T} \\
&= -r_t B_t^{-1}P_{t,T} + B_t^{-1}P_{t,T}\left[-\left(B(t,T)k(\theta - r_t) - \frac{1}{2}B(t,T)^2\sigma^2\right)dt - \sigma B(t,T)dW_t^0\right] \\
&= -\tilde{P}_{t,T}\left(\left(r_t + B(t,T)k(\theta - r_t) - \frac{1}{2}B(t,T)^2\sigma^2\right)dt + \sigma B(t,T)dW_t^0\right) \\
&= -\tilde{P}_{t,T}\left(r_t + \sigma B(t,T)\gamma_t^0 + B(t,T)k(\theta - r_t) - \frac{1}{2}B(t,T)^2\sigma^2\right)dt - \tilde{P}_{t,T}\left(\sigma B(t,T)dW_t^{\mathbb{Q},0}\right),
\end{aligned}$$

therefore the discounted zero-coupon bond is a $\mathbb{Q}$-martingale if and only if

$$\frac{-1}{1 - B(t,T)k}\left(\sigma B(t,T)\gamma_t^0 + B(t,T)k\theta - \frac{1}{2}B(t,T)^2\sigma^2\right) = r_t. \tag{30}$$

The result follows, solving the linear system formed by Equations (28)–(30). □

5. Model Implementation and Examples

In this section, we present an implementation of the Heston model written in forward variance together with a Vasicek model for the interest rates, in order to price numerically a unit-linked product. We will implement a Monte Carlo scheme for simulating prices under this model and compare it against a classical Black-Scholes model. The Heston process will be simulated using a full-truncation scheme Andersen (2007) in the Euler discretization in both models. We first show the discretized versions of the SDE's for each model and the result of the model comparison given some initial conditions.

Let $N \in \mathbb{N}$ be the number of time steps in which the interval $[0, T]$ is equally divided. Then, consider the uniform time grid $t_k \triangleq (kT)/N$, for all $k = 1, \ldots, N$ of length $\Delta t = T/N$. We present the following Euler schemes for each model:

1. Classical Black Scholes

$$S_{t_{k+1}} = S_{t_k}\exp\left(\left(r_0 - \frac{1}{2}\bar{v}^2\right)\Delta t + \bar{v}\sqrt{\Delta t}\left(W_1^{\mathbb{Q}}(t_{k+1}) - W_1^{\mathbb{Q}}(t_k)\right)\right),$$

where the parameters for the simulation are $(S_0, r, \bar{v})$, given by: $S_0 = 100$, $r_0 = 0.01$, $\bar{v} = 0.04$.

2. Vasicek-Heston Model written in forward variance

$$r_{t_{k+1}} = r_{t_k} + \left[k\left(\theta - (r_{t_k})^+\right) + \sigma\gamma_0(t_k)\right]\Delta t + \sigma\sqrt{\Delta t}\left(W_0^{\mathbb{Q}}(t_{k+1}) - W_0^{\mathbb{Q}}(t_k)\right),$$
$$\xi_{t_{k+1}}(t_N) = \xi_{t_k}(t_N) + r_{t_k}\xi_{t_k}(t_N)\Delta t + \eta e^{-\kappa(t_N - t_k)}\sqrt{(\nu_{t_k})^+\Delta t}\left(W_2^{\mathbb{Q}}(t_{k+1}) - W_2^{\mathbb{Q}}(t_k)\right),$$
$$\nu_{t_{k+1}} = \bar{\nu} + e^{\kappa(t_N - t_{k+1})}\left(\xi_{t_{k+1}}(t_N) - \bar{\nu}\right),$$
$$S_{t_{k+1}} = S_{t_k} + r_{t_k}S_{t_k}\Delta t + S_{t_k}\sqrt{(\nu_{t_k})^+\Delta t}\left(W_1^{\mathbb{Q}}(t_{k+1}) - W_1^{\mathbb{Q}}(t_k)\right),$$
$$P(t_{k+1}, t_N) = P(t_k, t_N) + P(t_k, t_N)\left[r_{t_k}\Delta t - \sigma B(t_k, t_N)\sqrt{\Delta t}\left(W_0^{\mathbb{Q}}(t_{k+1}) - W_0^{\mathbb{Q}}(t_k)\right)\right].$$

where the parameters are $(S_0, \mu, \nu_0, \bar{\nu}, \kappa, \eta, r_0, \theta, k, \sigma)$ and were set as $S_0 = 100$, $\mu = 0.015$, $\bar{\nu} = 0.01$, $\nu_0 = 0.04$, $\kappa = 10^{-3}$, $\eta = 0.01$, $\theta = r_0 = 0.01$, $k = 0.3$, and $\sigma = 0.02$.

For simulation purposes, the Monte Carlo scheme was implemented using 5000 simulations. The following graphs in Figure 1 result from the implementation of the previous models with the mentioned initial conditions, and for $T = \{10, 20, 30, 40\}$. As one can see in Figure 1, it seems like the classic Black-Scholes model tends to underprice the risks derived from volatility and interest rate risk. It is worth noting that the difference between prices increases with both time to maturity and the guarantee. When it comes to what source of risk is bigger, it seems like the fundamental risk lies in the driving interest rate.

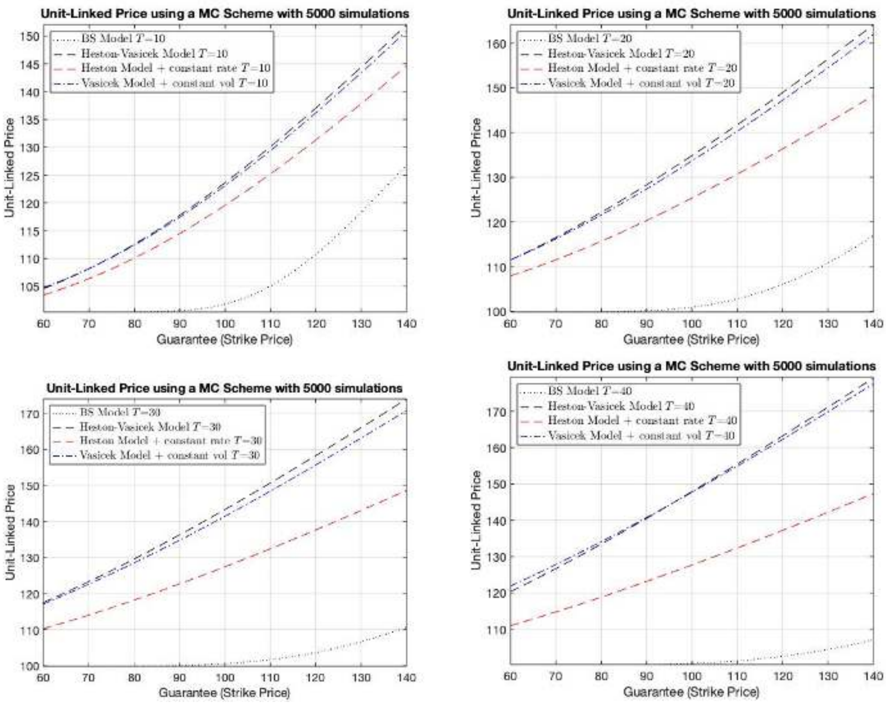

Figure 1. Pricing comparative between the Black-Scholes model, Vasicek-Heston model written in forward variance, Heston model with constant rate r_0, and a Vasicek model with constant volatility ν_0 for $T \in \{10, 20, 30, 40\}$.

As shown in Theorem 2 and Corollary 1, in order to properly price a unit-linked product, it only remains to multiply the value of the derivative priced using the Monte Carlo scheme, times the probability that an x-year old insured survives during the life of the product (T years). To do so, we have used Norwegian mortality from 2018 extracted from Statistics Norway.

As it is usual, mortality among men is higher. We consider, however, the aggregated mortality for simplicity. To model the mortality given in Table 1, we use the Gompertz-Makeham law of mortality which states that the death rate is the sum of an age-dependent component, which increases exponentially with age, and an age-independent component, i.e., $\mu_{*\dagger}(t) = a + be^{ct}$, $t \in [0, T]$. This law of mortality describes the age dynamics of human mortality rather accurately in the age window from about 30 to 80 years of age, which is good enough for our purposes. For this reason, we excluded the very first and last observations from the table. We then find the best fit for $\mu_{*\dagger}$ in the class of functions $\mathcal{C} = \{f(t) = a + be^{ct}, t \in [0, T], a, b, c \in \mathbb{R}\}$. As stated previously, since the stochastic process $X = \{X_t\}_{t\in[0,T]}$, which regulates the states of the insured, is a regular Markov chain, then the survival probability of an x-year old individual during the next T years is

$$_Tp_x = \bar{p}_{**}(x, x+T) = \exp\left(-\int_x^{x+T} \mu_{*\dagger}(\tau)\, d\tau\right).$$

Table 1. Norwegian mortality in 2018, per 100,000 inhabitants. Data from Statistics Norway, table: 05381.

Age	Men	Women	Total
4	50	45	95
9	7	2	9
14	10	3	13
19	26	13	39
24	33	6	39
29	63	24	87
34	72	27	99
39	93	43	136
44	109	68	177
49	156	111	267
54	258	177	435
59	454	310	764
64	737	495	1232
69	1206	824	2030
74	1990	1331	3321
79	3602	2447	6049
84	6626	4628	11254
89	12,469	9053	21,522
≥90	21,909	24,230	46,139

Figure 2 shows the fitted Gompertz–Makeham law based on the mortality data from Table 1.

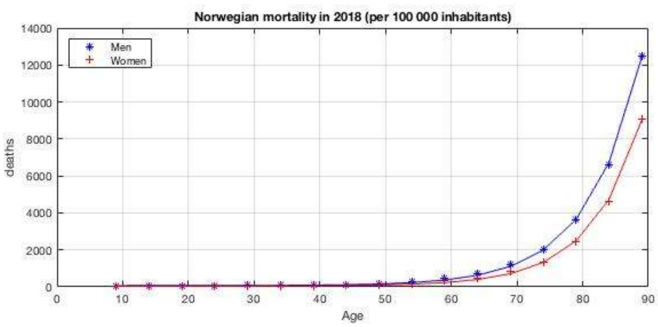

Figure 2. Joint plot of the mortality data given in Table 1, together with the fitted curve using the Gompertz–Makeham law of mortality.

Now, using the Vasicek-Heston model written in forward variance, we can compute a unit-linked price surface in terms of the guarantee, or strike price, and the age of the insured given a terminal time for the product $T > 0$. In particular, the graphs below show the price surfaces for fixed $T = \{10, 20, 30, 40\}$.

From the plots in Figure 3, we can observe that the longer time to maturity is, the lower the unit-linked price is, since the less probable it is that the insured survives. This effect has greater impact on the price than the effect of future volatility, or uncertainty arising from the stochasticity in interest rates. This behavior is easily observed by noting how the price surface collapses to zero as the contract's time to maturity increases, as well as the age of the insured when entering the contract. Hence, we can say that time to maturity has a cancelling effect on price, i.e., on one hand, it increases price as the stock or fund pays longer performance, but, on the other hand, it decreases price due to a lower probability of surviving during the time to maturity of the unit-linked contract.

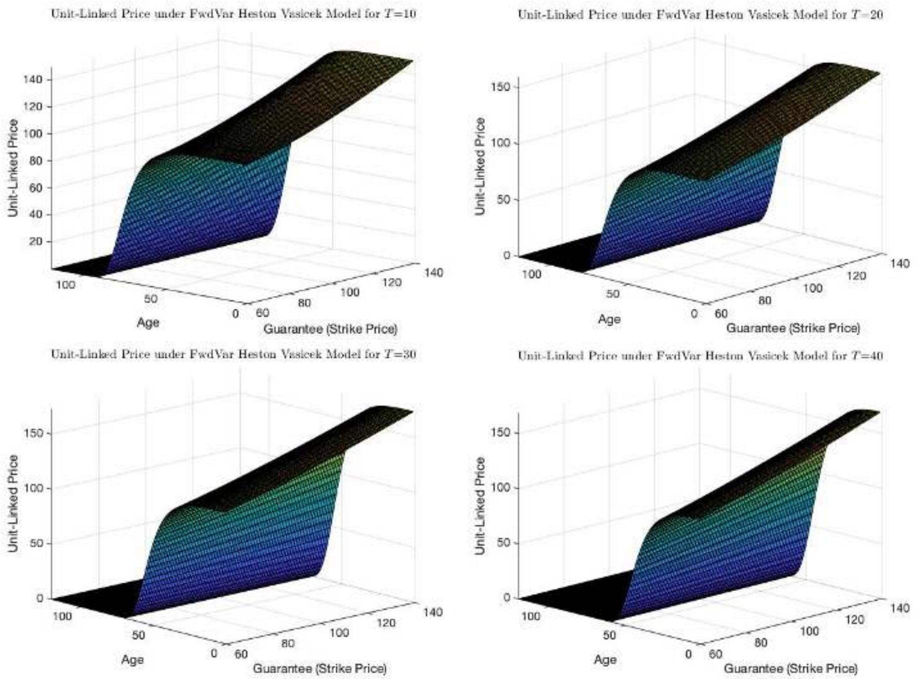

Figure 3. Unit-linked price surfaces under a Vasicek-Heston model written in forward variance for different policy maturities, in terms of the guaranteed amount desired by the insured and his age at the time of acquisition.

The following plots in Figures 4 and 5 are aimed at providing the reader with an overview of the distributional properties of the price process at a constant survival rate equal to one. The first thing that comes to sight is how the variance and time to maturity are directly proportional. In addition, the longer the time to maturity of the unit-linked product is, the more leptokurtic the distribution of the insurance product price is. This is an important thing to take into account in the modeling of prices due to the impact in the hedging of such insurance products.

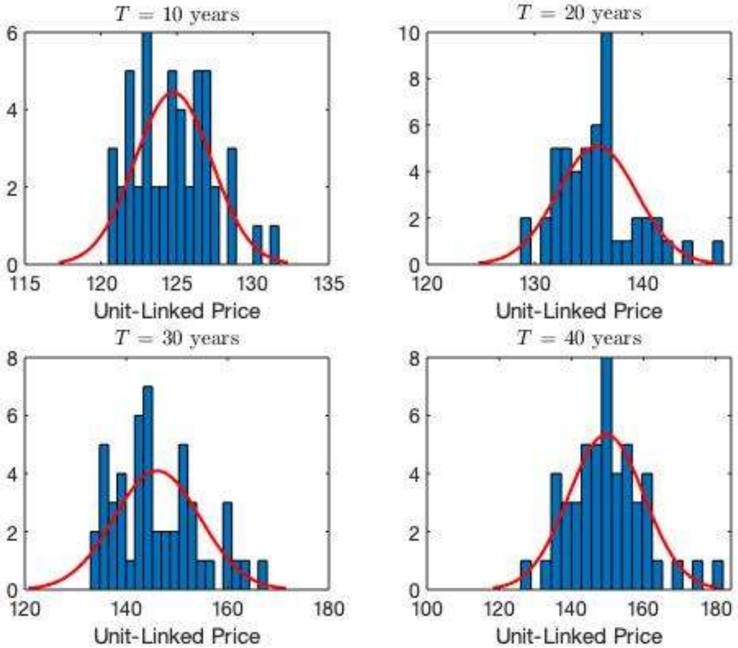

Figure 4. Unit-Linked Price histograms with constant survival rate equal to 1.

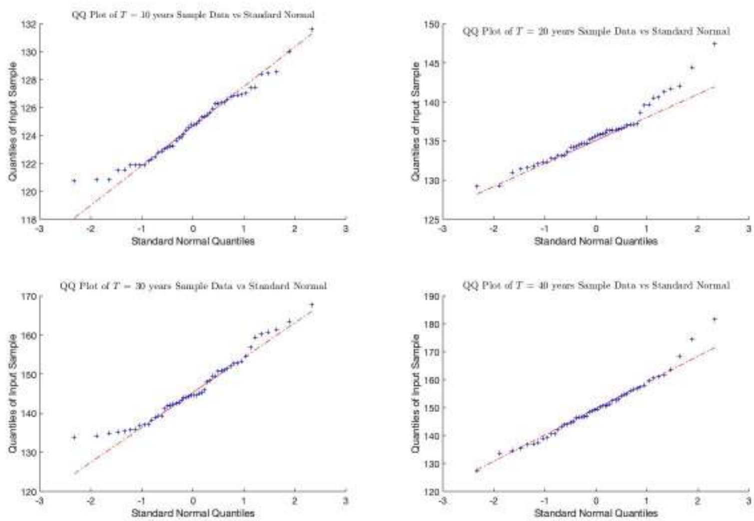

Figure 5. QQ-Plot between the unit-linked price input data and the standard normal distribution for maturities $T = \{10, 20, 30, 40\}$ years.

5.1. Pure Endowment

Consider an endowment for a life aged x with maturity $T > 0$. The policy pays the amount $E_T \triangleq \max\{G_e, S_T\}$ if the insured survives by time T where $G_e > 0$ is a guaranteed amount and S_T is the value of a fund at the expiration time. This policy is entirely determined by the policy function:

$$a_*(t) = \begin{cases} E_T \text{ if } t \geq T \\ 0 \text{ else} \end{cases}.$$

In view of (6) and the above function, the value of this insurance at time t given that the insured is still alive is then given by

$$V_*^+(t, A) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{B_t}{B_s} p_{**}(x+t, x+s) da_*(s) \Big| \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} E_T \Big| \mathcal{G}_t\right] p_{**}(x+t, x+T), \tag{31}$$

The above quantity corresponds to the formula in Theorem 2.

Observe that the payoff of an endowment can be written as

$$\max\{G_e, S_T\} = (G_e - S_T)_+ + G_e,$$

where $(x)_+ \triangleq \max\{x, 0\}$, which corresponds to a call option with strike price G_e plus G_e. In the case that S is modelled by the Black-Scholes model (with constant interest rate), we know that the price at time t of a call option with strike G_e and maturity T is given by

$$BS(t, T, S_t, G_e) \triangleq \Phi(d_1(t, T))S_t - \Phi(d_2(t, T))G_e e^{-r(T-t)},$$

where Φ denotes the distribution function of a standard normally distributed random variable and

$$d_1(t, T) \triangleq \frac{\log(S_t/G_e) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t, T) \triangleq d_1(t, T) - \sigma\sqrt{T-t}.$$

Then, we have that the unit-linked pure endowment under the Black-Scholes model has the price

$$BSE(t, T, S_t, G_e) \triangleq \Phi(d_1(t, T))S_t + G_e e^{-r(T-t)}\Phi(-d_2(t, T)). \tag{32}$$

The single premium at the beginning of this contract under the Black-Scholes model is then

$$\pi_{BS}^0 \triangleq BSE(0, T, S_0, G_e).$$

It is also possible to compute yearly premiums by introducing payment of yearly premiums π_{BS} in the policy function a_*, i.e., $a_*(t) = -\pi_{BS} t$ if $t \in [0, T)$ and $a_*(t) = -\pi_{BS} T + E_T$ if $t \geq T$, then the value of the insurance at any given time $t \geq 0$ with yearly premiums, denoted by V_*^π, becomes

$$-\pi_{BS} \int_t^T e^{-r(s-t)} p_{**}(x+t, x+s) ds + BSE(t, T, S_t, G_e).$$

We choose the premiums in accordance with the equivalence principle, i.e., such that the value today is 0,

$$\pi_{BS} = \frac{BSE(0, T, S_0, G_e)}{\int_0^T e^{-rs} p_{**}(x, x+s) ds}.$$

Under the Vasicek-Heston model instead, the value of policy at time $t \geq 0$ with yearly premiums π_{VH} is

$$V_*^+(t, A) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{B_t}{B_s} p_{**}(x+t, x+s) da_*(s) \Big| \mathcal{G}_t\right]$$
$$= -\pi_{VH} \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_s}\Big|\mathcal{G}_t\right] p_{**}(x+t, x+s)ds + \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T}E_T\Big|\mathcal{G}_t\right] p_{**}(x+t, x+T).$$

A single premium payment π_{VH}^0 corresponds to $V_*^+(0, A)$, i.e.,

$$\pi_{VH}^0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{E_T}{B_T}\right] p_{**}(x, x+T)$$

and the yearly ones correspond to

$$\pi_{VH} = \frac{V_*^+(0, A)}{\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_s}\right] p_{**}(x, x+s)ds}.$$

In Figure 6, we compare the single premiums using the classical Black-Scholes unit-linked model in contrast to the Vasicek-Heston model proposed for different maturities T with parameters $S_0 = 1$, $G_e = 1$, $r = 1\%$ and $\mu = 1.5\%$, $\sigma = 4\%$ for the Black-Scholes model, and $S_0 = 1$, $G_e = 1$, $\mu = 1.5\%$, $\bar{\nu} = 1\%$, $\nu_0 = 4\%$, $\kappa = 10^{-3}$, $\eta = 10^{-2}$, $\theta = r_0 = 1\%$, $k = 0.3$, $\sigma = 2\%$ for the Vasicek-Heston model.

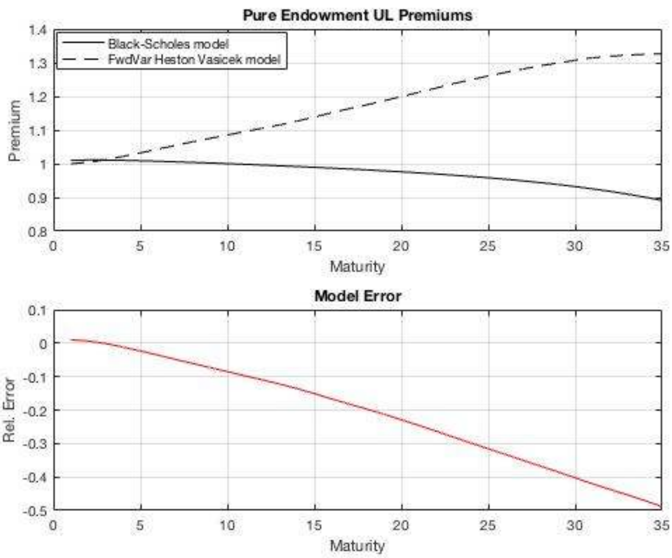

Figure 6. Single premiums for a pure endowment with benefit equal to 1 monetary unit, using the classical Black-Scholes with constant interest $r = 1\%$ and $\mu = 1.5\%$, $\sigma = 4\%$ and a Vasicek-Heston model with parameters $S_0 = 1$, $\mu = 1.5\%$, $\bar{\nu} = 1\%$, $\nu_0 = 4\%$, $\kappa = 10^{-3}$, $\eta = 10^{-2}$, $\theta = r_0 = 1\%$, $k = 0.3$, $\sigma = 2\%$.

5.2. Endowment with Death Benefit

Consider now an endowment for a life aged x with maturity $T > 0$ that pays, in addition, a death benefit in case the insured dies within the period of the contract. That is, the policy pays the amount

$E_T := \max\{G_e, S_T\}$ if the insured survives by time T as before and, in addition, a death benefit of $D_t := \max\{G_d, S_t\}$ if $t \in [0, T)$. This policy is entirely determined by the two policy functions:

$$a_*(t) = \begin{cases} E_T \text{ if } t \geq T \\ 0 \text{ else} \end{cases} \quad , \quad a_{*\dagger}(t) = \begin{cases} D_t \text{ if } t \in [0, T) \\ 0 \text{ else} \end{cases} \quad .$$

In view of (6) and the above functions, the value of this insurance at time t given that the insured is still alive is then given by

$$V_*^+(t, A) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} E_T \Big| \mathcal{G}_t\right] p_{**}(x+t, x+T) + \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_s} D_s \Big| \mathcal{G}_t\right] p_{**}(x+t, x+s)\mu_{*\dagger}(x+s)ds. \quad (33)$$

Following similar arguments as in the case of a pure endowment, by adding the function $a_{*\dagger}$ in the computations, we obtain that the single premiums π_{BS}^0 and π_{VH}^0 for the Black-Scholes model and Vasicek-Heston model, respectively, are given by.

$$\pi_{BS}^0 = BSE(0, T, S_0, G_e) + \int_0^T e^{-rs} BSE(0, s, S_0, G_d) p_{**}(x, x+s)\mu_{*\dagger}(x+s)ds,$$

where the function BSE is given in (32), and

$$\pi_{VH}^0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{E_T}{B_T}\right] p_{**}(x, x+T) + \int_0^T \mathbb{E}^{\mathbb{Q}}\left[\frac{D_s}{B_s}\right] p_{**}(x, x+s)\mu_{*\dagger}(x+s)ds.$$

In Figure 7, we compare the single premiums using the classical Black-Scholes unit-linked model in contrast to the Vasicek-Heston model proposed for different maturities T with parameters $S_0 = 1$, $G_e = G_d = 1$, $r = 1\%$ and $\mu = 1.5\%$, $\sigma = 4\%$ for the Black-Scholes model, and $S_0 = 1$, $G_e = 1$, $\mu = 1.5\%$, $\bar{v} = 1\%$, $v_0 = 4\%$, $\kappa = 10^{-3}$, $\eta = 10^{-2}$, $\theta = r_0 = 1\%$, $k = 0.3$, $\sigma = 2\%$ for the Vasicek-Heston model.

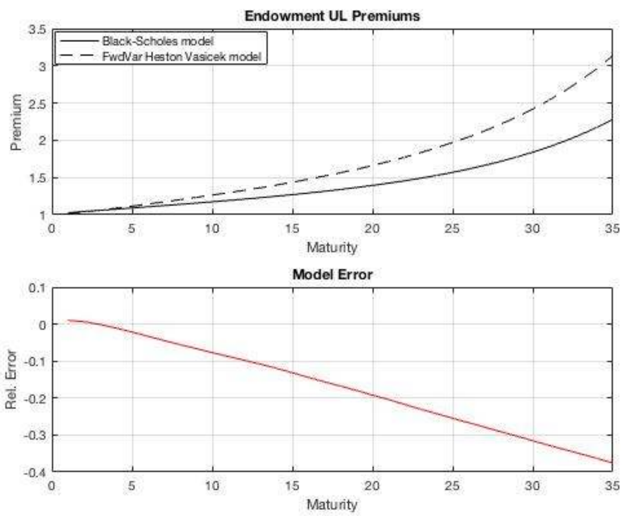

Figure 7. Single premiums for an endowment with benefits equal to 1 monetary unit, using the classical Black-Scholes with constant interest $r = 1\%$ and $\mu = 1.5\%$, $\sigma = 4\%$ and a Vasicek-Heston model with parameters $S_0 = 1$, $\mu = 1.5\%$, $\bar{v} = 1\%$, $v_0 = 4\%$, $\kappa = 10^{-3}$, $\eta = 10^{-2}$, $\theta = r_0 = 1\%$, $k = 0.3$, $\sigma = 2\%$.

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Appendix A. Technical Results

Appendix A.1. Proof of Theorem 1

Proof. It is important to notice that we will use the notation V_t to refer to the process $V\left(t, S_t, \xi_t, P_{t,T}\right)$, and similarly for the partial derivatives. For instance, $\partial_x V_t = \partial_x V\left(t, S_t, \xi_t, P_{t,T}\right)$. By means of Itô's lemma, we are able to write the change in our portfolio $\{V_t\}_{t\in[0,T]}$ as follows:

$$
\begin{aligned}
d\Pi_t &= \partial_t V_t dt + \partial_x V_t dS_t + \partial_y V_t d\xi_t + \partial_z V_t dP_{t,T} \\
&+ \frac{1}{2}\partial_x^2 V_t d\left[S,S\right]_t + \frac{1}{2}\partial_y^2 V_t d\left[\xi,\xi\right]_t + \frac{1}{2}\partial_z^2 V_t d\left[P,P\right]_t \\
&+ \partial_x\partial_y V_t d\left[S,\xi\right]_t + \partial_x\partial_z V_t d\left[S,P\right]_t + \partial_y\partial_z V_t d\left[\xi,P\right]_t \\
&- \Delta_t dS\left(t\right) - \Sigma_t d\xi_t - \Psi_t dP_{t,T} \\
&= \partial_t V_t dt + \left\{\partial_x V_t - \Delta_t\right\} dS_t + \left\{\partial_y V_t - \Sigma_t\right\} d\xi_t + \left\{\partial_z V_t - \Psi_t\right\} dP_{t,T} \\
&+ \frac{1}{2}\partial_x^2 V_t d\left[S,S\right]_t + \frac{1}{2}\partial_y^2 V_t d\left[\xi,\xi\right]_t + \frac{1}{2}\partial_z^2 V_t d\left[P,P\right]_t \\
&+ \partial_x\partial_y V_t d\left[S,\xi\right]_t + \partial_x\partial_z V_t d\left[S,P\right]_t + \partial_y\partial_z V_t d\left[\xi,P\right]_t .
\end{aligned}
$$

Using the dynamics for dS_t, $d\xi_t$, $dP_{t,T}$ and the quadratic covariations, given by

$$
\begin{aligned}
d\left[S,S\right]_t &= S_t^2 a\left(t,S_t\right)^2 f\left(\psi\left(t,T,\xi_t\right)\right)^2 dt, \\
d\left[\xi,\xi\right]_t &= \lambda\left(t,T,\xi_t\right)^2 dt, \\
d\left[P,P\right]_t &= \Xi_T^2\left(t,P_{t,T}\right) dt, \\
d\left[S,\xi\right]_t &= 0, \\
d\left[S,P\right]_t &= 0, \\
d\left[\xi,P\right]_t &= 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
d\Pi_t &= \partial_t V_t dt + \left\{\partial_x V_t - \Delta_t\right\} dS_t + \left\{\partial_y V_t - \Sigma_t\right\} d\xi_t + \left\{\partial_z V_t - \Psi_t\right\} dP_{t,T} \\
&+ \frac{1}{2} S_t^2 a\left(t,S_t\right)^2 f\left(\psi\left(t,T,\xi_t\right)\right)^2 \partial_x^2 V_t dt + \frac{1}{2}\lambda\left(t,T,\xi_t\right)^2 \partial_y^2 V_t dt + \frac{1}{2}\Xi_T^2\left(t,P_{t,T}\right)\partial_z^2 V_t dt \\
&= \left\{\partial_t V_t + \frac{1}{2}\left(S_t^2 a\left(t,S_t\right)^2 f\left(\psi\left(t,T,\xi_t\right)\right)^2 \partial_x^2 V_t + \lambda\left(t,T,\xi_t\right)^2 \partial_y^2 V_t + \Xi_T^2\left(t,P_{t,T}\right)\partial_z^2 V_t\right)\right\} dt \\
&+ \left\{\partial_x V_t - \Delta_t\right\} dS_t + \left\{\partial_y V_t - \Sigma_t\right\} d\xi_t + \left\{\partial_z V_t - \Psi_t\right\} dP_{t,T}.
\end{aligned}
$$

Now, in order to make the portfolio instantaneously risk-free, we must impose that the return on our portfolio equals the risk-free rate r_t, i.e., $d\Pi_t = r_t \Pi_t dt = r_t \left(V_t - \Delta_t S_t - \Sigma_t \xi_t - \Psi_t P_{t,T} \right) dt$, and force the coefficients in front of dS_t, $d\xi_t$ and $dP_{t,T}$ to be zero, i.e.,

$$\begin{aligned} \Delta_t &= \partial_x V_t, \\ \Sigma_t &= \partial_y V_t, \\ \Psi_t &= \partial_z V_t. \end{aligned}$$

This implies that

$$d\Pi_t = \left\{ \partial_t V_t + \frac{1}{2} \left(S_t^2 a\left(t, S_t\right)^2 f\left(\psi\left(t, T, \xi_t\right)\right)^2 \partial_x^2 V_t + \lambda\left(t, T, \xi_t\right)^2 \partial_y^2 V_t + \Xi_T^2\left(t, P_{t,T}\right) \partial_z^2 V_t \right) \right\} dt.$$

Therefore, rearranging the terms in the previous expression and taking into account that we have imposed $\Delta_t = \partial_x V_t$, $\Sigma_t = \partial_y V_t$, $\Psi_t = \partial_z V_t$, we have the PDE for the unit-linked product, ending the proof. □

Appendix A.2. Proof of Theorem 2

Proof. We start by imposing that the discounted price process, $\tilde{S}_t = B_t^{-1} S_t$, the discounted variance swap $\tilde{\xi}_t$, and the discounted zero-coupon bond price $\tilde{P}_{t,T}$ are $\mathbb{Q}-$martingales, where $dB_t = r_t B_t dt$ and $dB_t^{-1} = -r_t B_t^{-1} dt$. To do so, we will also make use of the relationship between the Brownian motions and their $\mathbb{Q}$-measure counterparts, given by (8):

$$\begin{aligned} d\tilde{S}_t &= dB_t^{-1} S_t + B_t^{-1} dS_t \\ &= -r_t B_t^{-1} S_t dt + B_t^{-1} \left[b\left(t, S_t\right) S_t dt + a\left(t, S_t\right) f\left(\psi\left(t, T, \xi_t\right)\right) S_t dW_t^1 \right] \\ &= \tilde{S}_t \left[\left(b\left(t, S_t\right) - r_t \right) dt + a\left(t, S_t\right) f\left(\psi\left(t, T, \xi_t\right)\right) \left[dW_t^{\mathbb{Q},1} + \gamma_t^1 dt \right] \right] \\ &= \tilde{S}_t \left[b\left(t, S_t\right) - r_t + a\left(t, S_t\right) f\left(\psi\left(t, T, \xi_t\right)\right) \gamma_t^1 \right] dt + \tilde{S}_t a\left(t, S_t\right) f\left(\nu_t\right) dW_t^{\mathbb{Q},1}. \end{aligned}$$

Now, the discounted price process $\tilde{S}_t$ is a $\mathbb{Q}-$martingale if, and only if,

$$\gamma_t^1 = \frac{r_t - b\left(t, S_t\right)}{a\left(t, S_t\right) f\left(\psi\left(t, T, \xi_t\right)\right)}. \tag{A1}$$

We do the same for the discounted forward variance process,

$$\begin{aligned} d\tilde{\xi}_t &= dB_t^{-1} \xi_t + B_t^{-1} d\xi_t \\ &= -r_t B_t^{-1} \xi_t dt + B_t^{-1} \lambda\left(t, T, \xi_t\right) dW_t^2 \\ &= B_t^{-1} \left[\lambda\left(t, T, \xi_t\right) \gamma_t^2 - r_t \xi_t \right] dt + B_t^{-1} \lambda\left(t, T, \xi_t\right) dW_t^{\mathbb{Q},2}. \end{aligned}$$

Therefore, the discounted variance swap is a $\mathbb{Q}-$martingale if, and only if,

$$\gamma_t^2 = \frac{r_t \xi_t}{\lambda\left(t, T, \xi_t\right)}. \tag{A2}$$

Finally, we impose that the discounted zero-coupon bond price process is a $\mathbb{Q}$-martingale analogously

$$
\begin{aligned}
d\tilde{P}_{t,T} &= dB_t^{-1}P_{t,T} + B_t^{-1}dP_{t,T} \\
&= -r_t B_t^{-1}P_{t,T}dt \\
&+ B_t^{-1}\Big[\partial_t F_T(t,r_t) + \mu(t,r_t)\,\partial_x F_T(t,r_t) \\
&+ \frac{1}{2}\sigma^2(t,r_t)\,\partial_x^2 F_T(t,r_t) - r_t F_T(t,r_t)\Big]dt + B_t^{-1}\partial_x F_T(t,r_t)\,\sigma(t,r_t)\,dW_t^0 \\
&= B_t^{-1}\Big[\partial_t F_T(t,r_t) + \Big(\mu(t,r_t) + \sigma(t,r_t)\,\gamma_t^0\Big)\partial_x F_T(t,r_t) \\
&+ \frac{1}{2}\sigma^2(t,r_t)\,\partial_x^2 F_T(t,r_t) - r_t\left(F_T(t,r_t) + P_{t,T}\right)\Big]dt \\
&+ B_t^{-1}\Xi_T^2(t,P_{t,T})\,dW_t^{\mathbb{Q},0}.
\end{aligned}
$$

Therefore, the discounted zero-coupon bond is a $\mathbb{Q}$-martingale if, and only if,

$$
\frac{1}{F_T(t,r_t) + P_{t,T}}\left[\partial_t F_T(t,r_t) + \Big(\mu(t,r_t) + \sigma(t,r_t)\,\gamma_t^0\Big)\partial_x F_T(t,r_t) + \frac{1}{2}\Xi_T^2(t,P_{t,T})\right] = r_t. \tag{A3}
$$

Now, we are able to characterize γ^i, for all $i \in \{0,1,2\}$, by solving the linear system given by Equations (A1)–(A3).

Therefore, we will apply Itô's lemma to the discounted price of the option,

$$
d\left[B_t^{-1}V(t,S_t,\xi_t,P_{t,T})\right] = dB_t^{-1}V(t,S_t,\xi_t,P_{t,T}) + B_t^{-1}dV(t,S_t,\xi_t,P_{t,T}).
$$

In order to relax the notation, we will drop the dependencies of V, allowing us to rewrite the previous expression as

$$
\begin{aligned}
d\left[B_t^{-1}V_t\right] &= dB_t^{-1}V_t + B_t^{-1}dV_t \\
&= -r_t B_t^{-1}V_t dt + B_t^{-1}\left[\partial_t V_t dt + \partial_x V_t dS_t + \partial_y V_t d\xi_t + \partial_z V_t dP_{t,T}\right] \\
&+ B_t^{-1}\left[\frac{1}{2}\partial_x^2 V_t d[S,S]_t + \frac{1}{2}\partial_y^2 V_t d[\xi,\xi]_t + \frac{1}{2}\partial_z^2 V_t d[P,P]_t\right] \\
&+ B_t^{-1}\left[\partial_x\partial_y V_t d[S,\xi]_t + \partial_x\partial_z V_t d[S,P]_t + \partial_y\partial_z V_t d[\xi,P]_t\right].
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
d\left[B_t^{-1}V_t\right] &= B_t^{-1}\left(\partial_t V_t - r_t V_t\right)dt \\
&+ B_t^{-1}\left(\partial_x V_t\left[b(t,S_t)S_t dt + S_t a(t,S_t) f(\psi(t,T,\xi_t))\,dW_t^1\right] + \partial_y V_t\left[\lambda(t,T,\xi_t)\,dW_t^2\right]\right) \\
&+ B_t^{-1}\partial_z V_t\left[\mathcal{L}_P(F_T(t,r_t))\,dt + \Xi_T(t,P_{t,T})\,dW_t^0\right] \\
&+ B_t^{-1}\left[\tfrac{1}{2}S_t^2 a(t,S_t)^2 f(\psi(t,T,\xi_t))^2\,\partial_x^2 V_t + \tfrac{1}{2}\lambda(t,T,\xi_t)^2\,\partial_y^2 V_t + \tfrac{1}{2}\Xi_T^2(t,P_{t,T})\,\partial_z^2 V_t\right]dt \\
&= B_t^{-1}\Big[\partial_t V_t - r_t V_t + b(t,S_t)S_t\partial_x V_t + \mathcal{L}_P(F_T(t,r_t))\,\partial_z V_t \\
&+ \tfrac{1}{2}\Big(S_t^2 a(t,S_t)^2 f(\psi(t,T,\xi_t))^2\,\partial_x^2 V_t + \lambda(t,T,\xi_t)^2\,\partial_y^2 V_t + \Xi_T^2(t,P_{t,T})\,\partial_z^2 V_t\Big)\Big]dt \\
&+ B_t^{-1}\left[S_t f(\psi(t,T,\xi_t))\,\partial_x V_t dW_t^1 + \lambda(t,T,\xi_t)\,\partial_y V_t dW_t^2 + \Xi_T(t,P_{t,T})\,\partial_z V_t dW_t^0\right].
\end{aligned}
$$

If we replace the Brownian motions under the $\mathbb{P}$-measure by the ones under the $\mathbb{Q}$-measure given by Equation (8), we can rewrite the previous expression as follows:

$$\begin{aligned}
d\left[B_t^{-1}V_t\right] &= B_t^{-1}\Big[\partial_t V_t - r_t V_t + b(t,S_t)\,S_t\partial_x V_t + \mathcal{L}_P\left(F_T(t,r_t)\right)\partial_z V_t \\
&+\tfrac{1}{2}\left(S_t^2 a(t,S_t)^2 f\left(\psi(t,T,\xi_t)\right)^2\partial_x^2 V_t + \lambda(t,T,\xi_t)^2\partial_y^2 V_t + \Xi_T^2(t,P_{t,T})\,\partial_z^2 V_t\right)\Big]dt \\
&+B_t^{-1}S_t a(t,S_t)\,f\left(\psi(t,T,\xi_t)\right)\partial_x V_t\left[dW_t^{\mathbb{Q},1}+\gamma_t^1 dt\right] \\
&+B_t^{-1}\lambda(t,T,\xi_t)\,\partial_y V_t\left[dW_t^{\mathbb{Q},2}+\gamma_t^2 dt\right] \\
&+B_t^{-1}\Xi_T(t,P_{t,T})\,\partial_z V_t\left[dW_t^{\mathbb{Q},0}+\gamma_t^0 dt\right] \\
&= B_t^{-1}\Big[\partial_t V_t - r_t V_t + S_t\left[b(t,S_t)+\gamma_t^1 a(t,S_t)\,f\left(\psi(t,T,\xi_t)\right)\right]\partial_x V_t \\
&+\gamma_t^2\lambda(t,T,\xi_t)\,\partial_y V_t + \left[\mathcal{L}_P\left(F_T(t,r_t)\right)+\gamma_t^0\Xi_T(t,P_{t,T})\right]\partial_z V_t \\
&+\tfrac{1}{2}\left(S_t^2 a(t,S_t)^2 f\left(\psi(t,T,\xi_t)\right)^2\partial_x^2 V_t + \lambda(t,T,\xi_t)^2\partial_y^2 V_t + \Xi_T^2(t,P_{t,T})\,\partial_z^2 V_t\right)\Big]dt \\
&+B_t^{-1}\left[S_t a(t,S_t)\,f\left(\psi(t,T,\xi_t)\right)\partial_x V_t dW_t^{\mathbb{Q},1} + \lambda(t,T,\xi_t)\,\partial_y V_t dW_t^{\mathbb{Q},2} + \Xi_T(t,P_{t,T}))\partial_z V_t dW_t^{\mathbb{Q},0}\right].
\end{aligned}$$

Applying Equations (A1)–(A3) and reorganizing the terms in the previous equation, we have

$$\begin{aligned}
d\left[B_t^{-1}V_t\right] &= B_t^{-1}\Big[\partial_t V_t + r_t\left(S_t\partial_x V_t + \xi_t\partial_y V_t + P_{t,T}\partial_z V_t - V_t\right) \\
&+\tfrac{1}{2}\left(S_t^2 a(t,S_t)^2 f\left(\psi(t,T,\xi_t)\right)^2\partial_x^2 V_t + \lambda(t,T,\xi_t)^2\partial_y^2 V_t + \Xi_T^2(t,P_{t,T})\,\partial_z^2 V_t\right)\Big]dt \\
&+B_t^{-1}\left[S_t a(t,S_t)\,f\left(\psi(t,T,\xi_t)\right)\partial_x V_t dW_t^{\mathbb{Q},1} + \lambda(t,T,\xi_t)\,\partial_y V_t dW_t^{\mathbb{Q},2} + \Xi_T(t,P_{t,T})\,\partial_z V_t dW_t^{\mathbb{Q},0}\right].
\end{aligned}$$

Now, noticing that the dt term in the previous equation is the differential operator (22) applied to V, we can write the following:

$$\begin{aligned}
d\left[B_t^{-1}V(t,S_t,\xi_t,P_{t,T})\right] &= B_t^{-1}\mathcal{L}_V V(t,S_t,\xi_t,P_{t,T})\,dt \\
&+ B_t^{-1}S_t a(t,S_t)\,f\left(\psi(t,T,\xi_t)\right)\partial_x V(t,S_t,\xi_t,P_{t,T})\,dW_t^{\mathbb{Q},1} \\
&+ B_t^{-1}\lambda(t,u,\xi_t)\,\partial_y V(t,S_t,\xi_t,P_{t,T})\,dW_t^{\mathbb{Q},2} \\
&+ B_t^{-1}\Xi_T(t,P_{t,T})\,\partial_z V(t,S_t,\xi_t,P_{t,T})\,dW_t^{\mathbb{Q},0}.
\end{aligned}$$

Next, integrating on the interval $[s,t]$, with $s \le t$, we can write the previous equation in integral form as

$$\begin{aligned}
B_t^{-1}V(t,S_t,\xi_t,P_{t,T}) = V(s,S_s,\xi_s,P_{s,T}) &+ \int_s^t B_\tau^{-1}\mathcal{L}_V V(\tau,S_\tau,\xi_\tau,P_{\tau,T})\,d\tau \\
&+ \int_s^t B_\tau^{-1}S_\tau a(\tau,S_\tau)\,f\left(\psi(\tau,T,\xi_\tau)\right)\partial_x V(\tau,S_\tau,\xi_\tau,P_{\tau,T})\,dW_\tau^{\mathbb{Q},1} \\
&+ \int_s^t B_\tau^{-1}\lambda(\tau,u,\xi_\tau)\,\partial_y V(\tau,S_\tau,\xi_\tau,P_{\tau,T})\,dW_\tau^{\mathbb{Q},2} \\
&+ \int_s^t B_\tau^{-1}\Xi_T(t,P_{t,T})\,\partial_z V(\tau,S_\tau,\xi_\tau,P_{\tau,T})\,dW_\tau^{\mathbb{Q},0}.
\end{aligned}$$

Taking the conditional expectation with respect to the risk neutral measure, we have that

$$\mathbb{E}^{\mathbb{Q}}\left[B_t^{-1}V(t,S_t,\xi_t,P_{t,T}) \mid \mathcal{G}_s\right] = V_s + \mathbb{E}^{\mathbb{Q}}\left[\int_s^t B_\tau^{-1}\mathcal{L}_V V(\tau,S_\tau,\xi_\tau,P_{\tau,T})\,d\tau \mid \mathcal{G}_s\right].$$

Notice that the previous expression is a martingale if, and only if, $\mathcal{L}_V V(t,S_t,\xi_t,P_{t,T}) \equiv 0$, for all $t \in [0,T]$. □

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Paper IV

A Decomposition Formula for Fractional Heston Jump Diffusion Models

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Appendices

Appendix A

Source Codes

This appendix section aims to provide some of the source codes used in the simulations performed in each of the papers. The programing language chosen was Matlab and the codes are also available in the following repository http://www.mlagunas.com/my-work

A.1 Paper I

The simulation of a self-exciting multifractional gamma process was implemented through the code named `SEMGamma.m` in the following way:

```
function [ D X ] = SEMGamma(N)

%   INPUT Variables
%       N -> number of partitions of the interval [0,T]

%   OUTPUT Variables
%       X -> Values of the process
%       D -> Differences of the process (X(i+1) - X(i))


N = 100;
T = 5;


%Hurst Function definition
%h = @(x) ((1/2)+(1/2)/(1+power(x,2)));
h = @(x) (1/(1+power(x,2)));
alpha = 10;

%hurst function plot
figure();
fplot(h,[-20,20],'k');
title('Hurst Function Graph');
legend('h(x)');
ylabel('Range of h=(0,1)');
grid on


%Initializing Variables
dt = 1/N; %Increment size of the partition
t = 0:dt:T*N;% Time vector
Norm = randn(length(t),1); %N(0,1) simulation
X = repmat(0,length(t),1); % Initial X vector centered at 0
D = repmat(0,length(t),1); % Initial D vector of 0's
H = repmat(h(X(1)),length(t),1); % Initial H vector of H(X(0))
```

```
%Computation of the series X, D, H.

for i=2:length(t)
    Term = 0;
    for j=1:i-1
        Term(j) = exp(-alpha*dt*(i-j))*power(dt*(i-j),h(X(j))-1/2)*Norm(j)*sqrt(dt);
    end
    X(i) = X(1)+sum(Term);
    D(i) = X(i)-X(i-1);
    H(i) = h(X(i));
end



%Small plot of the Process and its variations against the Hurst parameter
figure
my_col = repmat(linspace(0,1,10)',1,3);  % create your own gray colors color map

subplot(2,1,1)
yyaxis right
plot(t/N,X,'k');
ylabel('X_t');
yyaxis left
plot(t/N,D,'color',my_col(7,:));
ylabel('Increments');
title(['SEM-Gamma Process Simulation']);
legend('Process Increments','X_t Process');
xlabel('time (t)');
grid on

subplot(2,1,2)
%yyaxis right
axis([0 t(end)/N 0 1]);
plot(t/N,H,'r');
grid on
title('Hurst Function');
legend('h(x(t))');
xlabel('time (t)');
hold on

end
```

A.2 Paper II

The code to compute a second order approximation is provided below and can be found in the repository under the name of `HestonApprox2nd.m`.

```
%Approximation Heston 2nd Order

function Price = HestonApprox2nd(S0,K,v0,r,tau,kappa,theta,sigma,rho)

kt=kappa.*tau;
D = exp(-kt);

```

```
%We define av0 as the average initial variance
av0= theta + ((v0-theta)./(kappa*tau)).*(1-D);

%We use the average initial volatility instead of initital
%volatility
[BS,temp]= BSeuCall_Approx(S0,K,sqrt(av0),r,tau);

    %The following variable corresponds to rho/2 L[W,M]
    U=0.5*rho...
        .*(sigma./kappa.^2 .*(theta.*kappa.*tau-2*theta+v0+D.*(2*theta-v0)-
             kappa.*tau.*D.*(v0-theta)));

    %The following variable corresponds to 1/8 D[M,M]
    R=0.125.*(sigma./kappa).^2 .*(theta.*tau+(v0-theta)./kappa .*(1-D)...
        -2*theta./kappa.*(1-D)-2*(v0-theta).*tau.*D...
        +theta./(2*kappa).*(1-D.^2)+(v0-theta)./kappa.*(D-D.^2));

    %The following variable corresponds to rho*L[W, rho/2 L[W,M]]
    L_UW=(rho^2 * sigma^2)*(2*(v0 + theta*(kt -3))...
            + D*(theta*(kt^2 + 4*kt + 6) - v0*(kt^2 + 2*kt +2)))/(4*kappa^3);

    Price=BS+ temp(3).*U + temp(5).*R + (1./2).*temp(7).*(U.^2)+ temp(4).*L_UW;

end

function [BS,temp]= BSeuCall_Approx(S0,K,sigma, r, tau)
% Black-Scholes formula:
    d1 = (log(S0./K) +(r+(sigma.^2)/2).*tau)./(sigma.*sqrt(tau));
    d2 = d1- sigma.*sqrt(tau);

    BS = S0.*normcdf(d1,0,1)-K.*exp(-r.*tau).*normcdf(d2,0,1);

    st=sigma.*sqrt(tau);
    st2pi=sigma.*sqrt(2*pi.*tau);

    temp=zeros(7,1);


    temp(3)=(S0*exp(-(d1^2)./2)./st2pi)...
        *(1-d1./st); %DeltaGamma
    temp(5)=(S0*exp(-(d1^2)/2)/st2pi)...
        *((d1^2)/(st^2) - d1/st - 1/(st^2) );%Gamma^2
    temp(4)=(S0*exp(-(d1^2)/2)/st2pi)...
        *((1-d1/st)^2 - 1/(st.^2)); %Delta^2Gamma
    temp(7)=(S0*exp(-(d1^2)/2)/st2pi)...
        *((d1^4)/(st^4)-3*((d1^3)/(st^3))...
        +((d1^2)/(st^2))*(3-6/(st^2))...
        +(d1/st)*(9/(st^2)-1)...
        +(3/(st^2))*(1/(st^2) -1) );  %Delta^2Gamma^2

end
```

A.3 Paper III

The file `HestonFwdVarVasicek.m` provides unit-linked policy prices for different strike prices as a result of a MonteCarlo simulation using a full truncation scheme. It also plots a price surface resulting of the fit for mortality rates.

```
%MonteCarlo Simulation using Full-truncated Scheme

B = @(t,T,k) ((1/k)*(1-exp(-k*(T-t))));


 % SIMULATION PARAMETERS
    T = 10; %Maturity
    N = 1000; %Number of subintervals of [0,T]
    dt = T/N; %Stepsize
    n = 10000; %number of simulations (1.000.000)


%% Classical Heston Model

    %MODEL PARAMETERS

        %VASICEK PARAMETERS

        k = 0.3; %Speed of mean reversion
        theta = 0.01; %Long term mean level
        sigma = 0.02; % Volatility term
        r_0 = 0.01; %spot rate initial value

        %HESTON MODEL in FWD VARIANCE FORM PARAMETERS

            %Instantaneous vol parameters
            kappa = 0.001; %Speed of mean reversion
            nu = 0.01; %Long term mean level
            eta = 0.01; % Vol of vol
            V_0 = 0.04; %Instantaneous vol initial value

            %Price parameters
            mu = 0.015; %drift parameter
            S_0 = 100; %Price initial value

            %Fwd Variance
            fV_0 = nu + exp(-kappa*T)*(V_0-nu);

        %STRIKES
        K = 60:1:140; % Option Strike Prices

    %Correlation Structure
    rho_01 = 0;
    rho_02 = 0;
    rho_12 = 0;
    rho = [[1, rho_01, rho_02];[rho_01,1,rho_12];[rho_02, rho_12,1]];

    corrMat = chol(rho,'lower'); %cholesky factorization
    check = corrMat*corrMat'-rho; %if check = 0 then the matrix is positive
        definite

```

```

    %Simulation of indep. random variables
    indep_rate_BM = randn(n,N); % Brownian Motion simulations for spot rate
    indep_price_BM = randn(n,N); %Brownian Motion simulations for price
    indep_vol_BM = randn(n,N); %Brownian Motion simulations for vol/variance

    %Correlate the random variables
    rate_BM = zeros(n,N); %initialization of correlated BM for rate
    price_BM = zeros(n,N); %initialization of correlated BM for price
    vol_BM = zeros(n,N); %initialization of correlated BM for vol/variance

    for i=1:1:N
        A = [indep_rate_BM(:,i),indep_price_BM(:,i),indep_vol_BM(:,i)]*corrMat
             ';
        rate_BM(:,i)=A(:,1);
        price_BM(:,i) = A(:,2);
        vol_BM(:,i) = A(:,3);
    end


    %Initialize rates/prices/variance/fwdvariance and gamma vectors
    r = repmat(r_0,n,N);
    S = repmat(S_0,n,N);
    V = repmat(V_0,n,N);
    fV = repmat(fV_0,n,N);
    gamma0 = zeros(n,N);
    gamma1 = zeros(n,N);
    gamma2 = zeros(n,N);

    %Compute trajectories
    for i=1:1:n
        for j=1:1:N-1
            b = B(j*dt,T,k);
            gamma0(i,j) = ((1-power(rho_12,2))/(1-power(rho_12,2)-power(rho_01
                 ,2)+2*rho_02*rho_12*rho_01-power(rho_02,2)))...
                *((-1/(sigma*b))*((1-b*k)*r(i,j)+b*k*theta-0.5*power(b,2)*power
                     (sigma,2)))...
                +(1/(1-power(rho_12,2)-power(rho_01,2)+2*rho_02*rho_12*rho_01-
                     power(rho_02,2)))...
                *((rho_02*rho_12-rho_01)*((r(i,j)-mu)/(sqrt(V(i,j))))+(rho_01*
                     rho_12-rho_02)*((r(i,j)*fV(i,j))/(eta*exp(-kappa*(T-j*dt))*
                     sqrt(V(i,j)))));
            gamma2(i,j) = (1/(1-power(rho_12,2)))*...
                (((r(i,j)*fV(i,j))/(eta*exp(-kappa*(T-j*dt))*sqrt(V(i,j))))...
                -rho_12*((r(i,j)-mu)/(sqrt(V(i,j))))...
                +gamma0(i,j)*(rho_12*rho_01-rho_02));
            gamma1(i,j) = ((r(i,j)-mu)/(sqrt(V(i,j))))-rho_01*gamma0(i,j)-
                 rho_12*gamma2(i,j);

            %Non-tradable assets
            r(i,j+1) = r(i,j) + k*(theta-max(r(i,j),0))*dt + sigma*sqrt(dt)*
                 rate_BM(i,j);
            V(i,j+1) = V(i,j)+kappa*(nu-max(V(i,j),0))*dt + eta*sqrt(max(V(i,j)
                 ,0))*sqrt(dt)*vol_BM(i,j);

            %Tradable assets
            S(i,j+1) = S(i,j) + (r(i,j)+sqrt(max(V(i,j),0))*...
```

```
                (gamma1(i,j)+rho_01*gamma0(i,j)+rho_12*gamma2(i,j)))*S(i,j)*dt +...
                S(i,j)*sigma*sqrt(max(V(i,j),0)*dt)*vol_BM(i,j);
            fV(i,j+1) = fV(i,j) + r(i,j)*fV(i,j)*dt...
                + eta*exp(-kappa*(T-j*dt))*sqrt(max(V(i,j),0)*dt)*vol_BM(i,j);

        end
    end
    S_Heston = S;
    V_Heston = V;
    %Compute Expectations

    %Heston_Optprice = max(repmat(S(:,end),1,length(K))-repmat(K,n,1),0);
    Heston_Optprice = max(repmat(S(:,end),1,length(K)),repmat(K,n,1));
    time = [dt:dt:T];
    DF = zeros(n,1);
    for i=1:1:n
        DF(i)=exp(-trapz(time,r(i,:)));
    end
    DF = repmat(DF,1,length(K));
    avg_Heston_Optprice = mean(DF.*Heston_Optprice);



%% Black-Scholes Model

    %Model Parameters
    sigma = sqrt(V_0);
    rate = 0; %rate
    %sigma = Vo;
    %Simulation of Price BM
    BSprice_BM = randn(n,N); %Brownian Motion simulations for price

    %Initialize Price vector
    S_BS = repmat(S_0,n,N);

    %Compute trajectories
    for i=1:1:n
        for j=1:1:N-1
            S_BS(i,j+1) = S_BS(i,j)*exp((rate-0.5*sigma.^2)*dt + sigma*sqrt(dt)*BSprice_BM(i,j));
        end
    end

    %Compute Expectations

    %BS_Optprices = max(repmat(S_BS(:,end),1,length(K))-repmat(K,n,1),0);
    BS_Optprices = max(repmat(S_BS(:,end),1,length(K)),repmat(K,n,1));
    avg_BS_Optprice = mean(BS_Optprices);



%% Insurance: Mortality rates and prices

    mu = @(x) (exp(-9.13275+8.09438*power(10,-2)*x-1.10180*power(10,-5)*power(x,2)));
    lifespan = 120;
```

```
    P = zeros(lifespan,1);

    for i=1:1:lifespan
        P(i) = exp(-integral(mu,i-1,i-1+T));
    end



%% PLOTS

    % PLOT 1
    figure();

    %Upper Plot: Option Prices
        subplot(2,1,1);
        plot(K,avg_BS_Optprice,':k');
        hold on
        plot(K,avg_Heston_Optprice, '--k');
        hold off
        xlabel('Guarantee (Strike Price)');
        ylabel('Unit-Linked Price');
        legend(['BS Model T=' num2str(T)],[' Classic Heston Model T=' num2str(T
             )]);
        title(['Unit-Linked Price using a MC Scheme with ' num2str(n) '
             simulations']);
        axis([K(1) K(end) 80 250]);
        grid on


    %Lower Plot: Price error:
        subplot(2,1,2);
        error1 = rdivide(abs(avg_BS_Optprice - avg_Heston_Optprice),
             avg_BS_Optprice);
        plot(K,error1,'r');
        grid on
        xlabel('Guarantee (Strike Price)');
        ylabel('Error (%)');
        legend('Error Heston vs BS');
        title('Model error');
        axis([K(1) K(end) 0 1]);

    %PLOT 2 (Unit-Linked price surface)

        Price = P*avg_Heston_Optprice;
        figure();
        surf(K,1:1:lifespan,Price);
        xlabel('Guarantee (Strike Price)');
        ylabel('Age');
        zlabel('Unit-Linked Price');
        title([' Unit-Linked Price under Heston Model for T=' num2str(T)]);
```