High-order gauge-invariant perturbations of a spherical spacetime

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Abstract. We construct a covariant and gauge-invariant framework to deal with arbitrary high-order perturbations of a spherical spacetime. It can be regarded as the generalization to high orders of the Gerlach and Sengupta formalism for first-order nonspherical perturbations. The Regge-Wheeler-Zerilli harmonics are generalized to an arbitrary number of indices and a closed formula is deduced for their products. An iterative procedure is given in order to construct gauge-invariant quantities up to any perturbative order. Focusing on second-order perturbation theory, we explicitly compute the sources for the gauge invariants as well as for the evolution equations.

1. Introduction

Perturbation theory is very useful in many branches of theoretical physics. General Relativity is not an exception, among other reasons because the equations of motion of the full theory are too complicated to be analytically solved. During several years, perturbation theory has been used to analyze the stability of black-hole and cosmological solutions to the Einstein equations. It also can be used to check the stability of the different formulations of equations implemented in numerical codes, since a numerical error can be interpreted as a small departure from the solution that one is considering. Another important application is in the obtention of waveforms produced in certain astrophysical scenarios.

For background spacetimes with spherical symmetry, perturbation theory has provided very good approximations to many situations of physical interest. Already in the early fifties Regge and Wheeler (RW) [1] analyzed the non-spherical perturbations of the Schwarzschild spacetime. As we will see below, the gravitational waves have two kinds of polarity: axial and polar. Regge and Wheeler were able to find a wave equation for the axial degree of freedom. Some years later, Zerilli [2] achieved the same goal for the polar degree of freedom. Moncrief [3] was the first person to study the problem using variables that are invariant under a change of gauge, for the particular case of the Schwarzschild background. Gerlach and Sengupta (GS) [4] generalized all these results to generic spherically symmetric backgrounds. Their formalism possesses many nice features. They performed a 2+2 splitting of the background spacetime and decomposed the perturbations in the Regge-Wheeler-Zerilli (RWZ) basis of harmonics, which absorbs all the dependence on the coordinates of the sphere. The GS formalism is covariant, i.e., it does not require fixing any privilege system of coordinates. In addition, it makes use of gauge-invariant objects to describe the perturbations. The aim of our work is to generalize this formalism in order to have an efficient tool to deal with the problem of high-order perturbations of spherical spacetimes.

There are several reasons that justify the convenience of going beyond first order in perturbation theory. The most obvious motivation is the desire to attain more accuracy in the numerical simulations, as well as to validate regions and quantitative errors for the first-order results. Besides, the non-linearity of the full theory will be reflected in the coupling of first-order modes. Owing to this non-linearity, intrinsic scales may appear in certain problems.

The calculations involved in problems with high-order perturbations are very complicated. This explains why, until very recently, it has been impossible to study these high-order corrections. Nowadays, computer algebra is developed to a point that permits one to face this kind of computations. In this context, another goal of our work is to provide a suitable computer framework to cope with the study of high-order perturbations of spherical spacetimes. This framework is composed by two *Mathematica* packages [5] that essentially consist in the algebraic implementation of the theory presented in this article.

The rest of this article is organized in six sections. Section 2 discusses high-order perturbation theory for a generic background. In Section 3, the GS notation for the spherical background is introduced. Section 4 deals with the tensor spherical harmonics. In particular, the RWZ harmonics are presented and generalized to any rank. A product formula is also obtained for them. In Section 5 the perturbations are decomposed using the RWZ harmonics and a procedure to construct gauge-invariant variables is given. Section 6 studies in detail the second-order case and in Section 7 we summarize the main results and conclude.

2. High-order perturbation theory in General Relativity

We start by considering a family of spacetimes that depend on a dimensionless parameter ε. Each spacetime is composed by a four dimensional manifold $\tilde{\mathcal{M}}(\varepsilon)$, a metric $\tilde{g}_{\mu\nu}(\varepsilon)$ and some matter fields that are abstractly denoted by $\tilde{\Phi}(\varepsilon)$. In particular, the stress-energy tensor $\tilde{T}(\varepsilon)$ is given in terms of these matter fields. We call background manifold the manifold with vanishing ε and denote all the objects defined in it without tildes, that is, $\{\tilde{\mathcal{M}}(0), \tilde{g}_{\mu\nu}(0), \tilde{\Phi}(0)\} \equiv \{\mathcal{M}, g_{\mu\nu}, \Phi\}$. Besides, the background metric $g_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ are supposed to provide a known solution of the Einstein equations.

We then introduce a perturbative hierarchy by expanding all the ε-dependent objects into power series. In order to do this, we assume that the dependence on ε is sufficiently smooth (e. g. C^n if we want to work up to order n):

$$\tilde{g}_{\mu\nu}(\varepsilon) = g_{\mu\nu} + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \, {}^{\{n\}}h_{\mu\nu}, \tag{1}$$

$$\tilde{\Phi}(\varepsilon) = \Phi + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \, {}^{\{n\}}\Phi, \tag{2}$$

$$\tilde{T}_{\mu\nu}(\varepsilon) = T_{\mu\nu} + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \, {}^{\{n\}}T_{\mu\nu}. \tag{3}$$

It is convenient to define a formal perturbative operator Δ, with the properties of a derivative, acting on any ε-dependent tensor $\tilde{\Omega}$,

$$\Delta^n[\Omega] \equiv \left. \frac{d^n\tilde{\Omega}(\varepsilon)}{d\varepsilon^n} \right|_{\varepsilon=0}. \tag{4}$$

In this way, the tensor $\tilde{\Omega}(\varepsilon)$ can be expanded in the following way

$$\tilde{\Omega}(\varepsilon) = \Omega + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \Delta^n[\Omega]. \tag{5}$$

In particular, it is clear from expansion (1) that, by definition, $\Delta[g_{\mu\nu}] \equiv {}^{\{1\}}h_{\mu\nu}$ and $\Delta[{}^{\{n\}}h_{\mu\nu}] \equiv {}^{\{n+1\}}h_{\mu\nu}$. We have been able to deduce closed formulas for the action of the operator Δ^n on all the curvature tensors of interest. Details can be found in reference [6]. For instance, the nth perturbation of the inverse of the metric and of the Christoffel symbols are given by

$$\begin{aligned}
\Delta^n[g^{\mu\nu}] &= \sum_{(k_i)} (-1)^m \frac{n!}{k_1! \ldots k_m!} {}^{\{k_m\}}h^{\mu\alpha} \; {}^{\{k_{m-1}\}}h_\alpha{}^\beta \ldots {}^{\{k_2\}}h_\tau{}^\rho \; {}^{\{k_1\}}h_\rho{}^\nu, &(6)\\
\Delta^n[\Gamma^\alpha{}_{\mu\nu}] &= \sum_{(k_i)} (-1)^{m+1} \frac{n!}{k_1! \ldots k_m!} {}^{\{k_m\}}h^{\alpha\beta} \; {}^{\{k_{m-1}\}}h_{\beta\gamma} \ldots {}^{\{k_2\}}h_{\tau\rho} \; {}^{\{k_1\}}h^\rho{}_{\mu\nu}, &(7)
\end{aligned}$$

where the sums run over the 2^{n-1} sorted partitions of the number n into (k_i) positive integers. This means that we have to consider all the combinations of integers (k_i) such that $k_1 + \ldots + k_m = n$ for all $m \leq n$. It is worth noting that all the derivatives of the metric perturbations in formula (7) are encoded in the three-index perturbation ${}^{\{k_1\}}h^\rho{}_{\mu\nu}$:

$${}^{\{n\}}h^\rho{}_{\mu\nu} \equiv \frac{1}{2} \left({}^{\{n\}}h^\rho{}_{\mu;\nu} + {}^{\{n\}}h^\rho{}_{\nu;\mu} - {}^{\{n\}}h_{\mu\nu}{}^{;\rho} \right). \tag{8}$$

3. Spherical background. Gerlach and Sengupta notation

We will only consider spherically symmetric backgrounds and use the notation introduced by GS [4]. We take the four-dimensional background manifold $\mathcal{M}$ and write it as the product $\mathcal{M}^2 \times S^2$, where $\mathcal{M}^2$ is a two-dimensional Lorentzian manifold with boundary (this boundary will be associated with the center of symmetry) and S^2 the two-sphere. We will use Greek indices for the four dimensional manifold and choose a coordinate system adapted to this decomposition $x^\mu \equiv (x^A, x^a)$. Capital Latin indices denote basis indices on the manifold $\mathcal{M}^2$, $x^A \equiv (x^0, x^1)$, and lowercase Latin letters indicate indices on the two sphere, $x^a \equiv (x^2, x^3)$. We will make all the calculations using this covariant notation, so that the equations presented in this paper are valid in any coordinate system.

Any spherically symmetric metric and stress-energy tensor can be respectively written as

$$\begin{aligned}
g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu &= g_{AB}(x^D)dx^A dx^B + r^2(x^D)\, \gamma_{ab}(x^d) dx^a dx^b,\\
T_{\mu\nu}(x^\lambda)dx^\mu dx^\nu &= t_{AB}(x^D)dx^A dx^B + \frac{1}{2} r^2(x^D)\, Q(x^D)\, \gamma_{ab}(x^d) dx^a dx^b,
\end{aligned}$$

where g_{AB} is the metric of the manifold $\mathcal{M}^2$, γ_{ab} is the round metric of the sphere and r is a scalar defined on $\mathcal{M}^2$. We also define the following notation for the covariant derivatives associated with each metric:

$$g_{\mu\nu;\rho} = 0, \qquad g_{AB|D} = 0, \qquad \gamma_{ab:d} = 0. \tag{9}$$

For future convenience, we introduce the Newman and Penrose basis of vectors on the sphere [7] that are defined by

$$m^a = \frac{1}{\sqrt{2}}(e_\theta{}^a + i e_\phi{}^a), \quad \text{and} \quad \bar{m}^a = \frac{1}{\sqrt{2}}(e_\theta{}^a - i e_\phi{}^a), \tag{10}$$

where $e_\theta{}^a$ and $e_\phi{}^a$ are the unit coordinate basis vectors. These vectors are null, $\gamma_{ab} m^a m^b = \gamma_{ab} \bar{m}^a \bar{m}^b = 0$, and their product can be decomposed as

$$\bar{m}^a m^b = \frac{1}{2}(\gamma^{ab} + i\varepsilon^{ab}). \tag{11}$$

4. Spherical tensor harmonics

4.1. Regge-Wheeler-Zerilli harmonics

The scalar spherical harmonics are defined as the eigenfunctions of the Laplacian operator

$$\gamma^{ab} Y_l^m{}_{:ab} = -l(l+1) Y_l^m, \tag{12}$$

where γ^{ab} is the inverse of the round metric and l a non-negative integer that has a covariant, coordinate-independent, meaning. The other harmonic label m is an integer such that $-l \leq m \leq l$ and its definition relies on the choice of a fixed Z axis

$$\partial_\phi Y_l^m = im Y_l^m. \tag{13}$$

These scalar harmonics have the following behavior under a parity transformation,

$$Y_l^m(\pi - \theta, \pi + \phi) = (-1)^l Y_l^m(\theta, \phi). \tag{14}$$

In order to construct a basis of vectors on the sphere, one can first take the covariant derivative of the scalar harmonics $Y_l^m{}_{:a}$. The basis can be completed with vectors orthogonal to these ones, obtained by means of the Levi-Civita tensor, $X_l^m{}_a \equiv \varepsilon_{ab}\gamma^{bc} Y_l^m{}_{:c}$. In this way, a basis for the vector fields on the sphere is given by $\{Y_l^m{}_{:a}, X_l^m{}_a\}$. These objects have a well-defined parity, inherited from the scalar harmonics

$$Y_l^m{}_{:a}(\pi - \theta, \pi + \phi) = (-1)^l Y_l^m{}_{:a}(\theta, \phi), \tag{15}$$
$$X_l^m{}_a(\pi - \theta, \pi + \phi) = (-1)^{l+1} X_l^m{}_a(\theta, \phi), \tag{16}$$

and are split into two polarity families. Those which change sign as $(-1)^l$ under a parity transformation form the *polar* family and those which change sign as $(-1)^{l+1}$ the *axial* family. There exist other names for the polarity families, for instance Regge and Wheeler used the names *even* and *odd*, instead of polar and axial, respectively.

The basis used by Regge and Wheeler for the rank-two symmetric tensors is given by [1]

$$\{Y_l^m{}_{:ab}, X_l^m{}_{ab} \equiv X_l^m{}_{(a:b)}, Y_l^m \gamma_{ab}\}. \tag{17}$$

(Note that there is a difference in a factor of 2 between $X_l^m{}_{ab}$ and the harmonic used by GS.) The problem of this basis is that, for the particular case $l = 1$, $Y_1^m{}_{:ab}$ and $Y_1^m \gamma_{ab}$ are linearly dependent. This is why Zerilli defined the alternate harmonic

$$Z_l^m{}_{ab} \equiv (Y_l^m{}_{:ab})^{\mathrm{STF}} = Y_l^m{}_{:ab} + \frac{l(l+1)}{2} Y_l^m \gamma_{ab}, \tag{18}$$

where the superscript STF means the symmetric and trace-free part. In this way, the elements of the basis $\{Z_l^m{}_{ab}, X_l^m{}_{(a:b)}, Y_l^m \gamma_{ab}\}$ for the symmetric two-tensors are linearly independent for all the values of the labels l and m. Besides, all these elements have a well-defined parity; $Z_l^m{}_{ab}$ and $Y_l^m \gamma_{ab}$ are polar, whereas $X_l^m{}_{(a:b)}$ is axial.

We will refer to the basis obtained in this section as the RWZ basis of harmonics. This basis proves sufficient if we want to study linear perturbation theory. But in order to do higher-order perturbation theory we have to face two problems. The first one is that we have to know how to expand the product between any pair of these harmonics as a linear combination of other harmonics. The second problem is that, to construct such linear combinations, we will need harmonics with more than two indices. We will solve these two problems by generalizing the RWZ harmonics to any rank and giving a closed formula for the product of any pair of them.

4.2. Generalization of the Regge-Wheeler-Zerilli harmonics

In order to define a basis for the tensors of rank s defined on the sphere, we need a basis with 2^s elements. Two of these elements will be given by the two symmetric traceless tensors on the sphere that are defined in the following way:

$$Z_l^m{}_{a_1...a_s} \equiv (Y_l^m{}_{:a_1...a_s})^{\rm STF}, \tag{19}$$

$$X_l^m{}_{a_1...a_s} \equiv \varepsilon_{(a_1}{}^b Z_l^m{}_{ba_2...a_s)}, \tag{20}$$

valid for $-l \leq m \leq l$ and $1 \leq s \leq l$. In all other cases, these harmonics are defined to be identically zero, except for $s = 0$, when $Z_l^m \equiv Y_l^m$. Since the parity of the scalar harmonics Y_l^m, the round metric γ_{ab} and the Levi-Civita tensor ε_{ab} is respectively $(-1)^l$, $+1$ and -1 and since taking covariant derivatives does not change the parity, the harmonic $Z_l^m{}_{a_1...a_s}$ is polar and $X_l^m{}_{a_1...a_s}$ is axial. The rest of elements for the basis of rank s tensors are given by independent linear combinations of products between γ_{ab} and ε_{ab} with the basis for the tensors of rank $(s-2)$. For instance, the basis of the tensors with three indices is formed by $Z_l^m{}_{abc}$, $X_l^m{}_{abc}$ and six independent combinations of $\gamma_{ab}Z_l^m{}_c$, $\gamma_{ab}X_l^m{}_c$, $\varepsilon_{ab}Z_l^m{}_c$, $\varepsilon_{ab}X_l^m{}_c$ and their index permutation.

4.3. Product formula

The product between two scalar harmonics can be expanded as

$$Y_{l'}^{m'} Y_l^m = \sum_{l''=|l'-l|}^{l'+l} E_{0\,l'\,m'\,l''}^{0l\;m} Y_{l''}^{m+m'}, \tag{21}$$

where we have defined the symbol

$$E_{0\,l'\,m'\,l''}^{0l\;m} \equiv \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2l''+1)}}\, C_{l'\;\,l\;\,l''}^{m'\,m\,m'+m} C_{l'\,l\,l''}^{000}, \tag{22}$$

with $C_{l_1\;l_2\;l}^{m_1\,m_2\,m_1+m_2}$ being the Clebsch-Gordan coefficients.

In order to obtain a similar formula for any pair of the RWZ generalized harmonics, we will introduce the so-called pure-spin harmonics that are very closely related to the Wigner rotation matrices, for which a product formula is well known.

The rank two pure-spin harmonics were defined by Zerilli [8] and we generalize them up to any rank s in the following way,

$$\mathcal{Y}_l^{s,m}{}_{a_1...a_s} \equiv (-1)^s k(l,s)\mathcal{D}_{s,m}^{(l)}(0,\theta,\phi) m_{a_1}...m_{a_s}, \tag{23}$$

$$\mathcal{Y}_l^{-s,m}{}_{a_1...a_s} \equiv k(l,s)\mathcal{D}_{-s,m}^{(l)}(0,\theta,\phi)\bar{m}_{a_1}...\bar{m}_{a_s}, \tag{24}$$

where $\mathcal{D}_{m_1,m_2}^{(l)}$ are the Wigner matrices, that is, the irreducible representations of the rotation group in terms of the Euler angles (α,β,γ). In addition, the normalization factors

$$k(l,s) = \sqrt{\frac{(2l+1)(l+s)!}{2^{s+2}\,\pi\,(l-s)!}}, \tag{25}$$

are chosen in such a way that the relation between the pure-spin harmonics and the generalized RWZ ones is (for $s \geq 1$)

$$Z_l^m{}_{a_1...a_s} = \mathcal{Y}_l^{s,m}{}_{a_1...a_s} + \mathcal{Y}_l^{-s,m}{}_{a_1...a_s}, \tag{26}$$

$$-iX_l^m{}_{a_1...a_s} = \mathcal{Y}_l^{s,m}{}_{a_1...a_s} - \mathcal{Y}_l^{-s,m}{}_{a_1...a_s}. \tag{27}$$

For the case $s = 0$, one has $Z_l^m = \mathcal{Y}_l^{0,m} = Y_l^m$. Because of the properties of the vectors m_a and $\bar{m}_a$, the pure-spin harmonics are symmetric and trace-free tensors.

It is well known that a product between two Wigner matrices, with the same Euler angles (α, β, γ), can be decomposed in the following way

$$\mathcal{D}^{(j_1)}_{m_1' m_1}(\alpha,\beta,\gamma)\, \mathcal{D}^{(j_2)}_{m_2' m_2}(\alpha,\beta,\gamma) = \sum_j C^{m_1\, m_2\, m_1+m_2}_{j_1\; j_2\;\; j} C^{m_1'\, m_2'\, m_1'+m_2'}_{j_1\; j_2\;\; j} \mathcal{D}^{(j)}_{m_1'+m_2', m_1+m_2}(\alpha,\beta,\gamma). \quad (28)$$

Making use of this formula, it is straightforward to find that the product between two pure-spin harmonics with the same sign is given by

$$\mathcal{Y}_{l'}^{\pm s',m'}{}_{a_1...a_{s'}} \mathcal{Y}_l^{\pm s,m}{}_{b_1...b_s} = \sum_{l''=|l-l'|}^{l'+l} E^{\pm s'\, l'\, m'}_{\pm s\;\, l\;\, m\;\, l''} \mathcal{Y}_{l''}^{\pm(s'+s),m'+m}{}_{a_1...a_{s'}b_1...b_s}, \quad (29)$$

with

$$E^{s\;l\;m}_{s'\,l'\,m'\,l''} \equiv \frac{k(l',|s'|)k(l,|s|)}{k(l'',|s+s'|)} C^{m'\,m\,m'+m}_{l'\;\,l\;\;l''} C^{s'\,s\,s'+s}_{l'\;l\;l''}. \quad (30)$$

We also obtain the following decomposition for the product between pure-spin harmonics with different sign (assuming, without loss of generality, that $s' \geq s$),

$$\mathcal{Y}_{l'}^{\mp s',m'}{}_{a_1...a_{s'}} \mathcal{Y}_l^{\pm s,m}{}_{b_1...b_s} = \sum_{l''=|l-l'|}^{l'+l} E^{\mp s'\, l'\, m'}_{\pm s\;\, l\;\, m\;\, l''} \mathcal{Y}_{l''}^{\mp(s'-s),m'+m}{}_{a_{s+1}...a_{s'}} T^{\pm s}{}_{a_1b_1...a_sb_s}, \quad (31)$$

where we have defined the tensors

$$T^s{}_{a_1b_1...a_sb_s} \equiv (-1)^s \bar{m}_{a_1} m_{b_1} ... \bar{m}_{a_s} m_{b_s}, \quad (32)$$

$$T^{-s}{}_{a_1b_1...a_sb_s} \equiv (-1)^s m_{a_1} \bar{m}_{b_1} ... m_{a_s} \bar{m}_{b_s}, \quad (33)$$

that must be expanded using relation (11). By definition $T^0 \equiv 1$.

From formulas (29) and (31) and relations (26)–(27), it is easy to find a similar formula for the product between any pair of RWZ generalized harmonics.

5. Non-spherical perturbations

5.1. Harmonic decomposition of the perturbations

Making use of the RWZ harmonics, we decompose the perturbations of the metric ${}^{\{n\}}h_{\mu\nu}$ and those of the stress-energy tensor ${}^{\{n\}}T_{\mu\nu}$ in the following way:

$${}^{\{n\}}h_{\mu\nu} \equiv \sum_{l,m} \begin{pmatrix} {}^{\{n\}}H^m_l{}_{AB}\, Z^m_l & {}^{\{n\}}H^m_l{}_A\, Z^m_l{}_b + {}^{\{n\}}h^m_l{}_A\, X^m_l{}_b \\ \text{Sym.} & {}^{\{n\}}K^m_l\, r^2 \gamma_{ab} Z^m_l + {}^{\{n\}}G^m_l\, r^2 Z^m_l{}_{ab} + {}^{\{n\}}h^m_l\, X^m_l{}_{ab} \end{pmatrix}, \quad (34)$$

$${}^{\{n\}}T_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{n\}}T^m_l{}_{AB}\, Z^m_l & {}^{\{n\}}T^m_l{}_A\, Z^m_l{}_b + {}^{\{n\}}t^m_l{}_A\, X^m_l{}_b \\ \text{Sym.} & {}^{\{n\}}\tilde{T}^m_l\, r^2 \gamma_{ab} Z^m_l + {}^{\{n\}}T^m_l\, Z^m_l{}_{ab} + {}^{\{n\}}t^m_l\, X^m_l{}_{ab} \end{pmatrix}. \quad (35)$$

Note that the axial (polar) harmonic coefficients are denoted in lowercase (uppercase) letters. This convention will be followed from now on and will be very useful for identifying the polarity of the objects under consideration. These decompositions reproduce those of reference [4] up to the mentioned normalization in the axial tensor $X^m_l{}_{ab}$ and some changes in notation.

5.2. Gauge freedom

In dealing with perturbation theory, one always has to face the problem of how perturbed and unperturbed tensors can be compared. Since they are defined in different manifolds and there is no preferred structure that one could use as a reference, an arbitrary mapping has to be given. Different choices of this mapping are related by gauge transformations. The nth order gauge transformation of any tensor is given by [9]

$$\overline{\Delta^n[\Omega]} = \sum_{m=1}^{n} \frac{n!}{(n-m)!} \sum_{(K_m)} \frac{1}{2^{k_2}...(m!)^{k_m} k_1!...k_m!} \mathcal{L}^{k_1}_{\{1\}\xi}...\mathcal{L}^{k_m}_{\{m\}\xi} \Delta^{n-m}[\Omega], \tag{36}$$

where Ω is any background tensor field, $(K_m) = \{(k_1, ..., k_m) \in \mathbb{N} / \sum_{i=1}^{m} ik_i = m\}$ and the overbar quantity denotes the perturbation of the tensor Ω defined in another gauge. In particular if we apply that formula to the metric tensor up to second order the result is

$$\overline{{}^{\{1\}}h_{\mu\nu}} - {}^{\{1\}}h_{\mu\nu} = \mathcal{L}_{\{1\}\xi} g_{\mu\nu}, \tag{37}$$

$$\overline{{}^{\{2\}}h_{\mu\nu}} - {}^{\{2\}}h_{\mu\nu} = \left(\mathcal{L}_{\{2\}\xi} + \mathcal{L}^2_{\{1\}\xi}\right) g_{\mu\nu} + 2\mathcal{L}_{\{1\}\xi}\, {}^{\{1\}}h_{\mu\nu}. \tag{38}$$

The most important consequence of these formulas is that the gauge freedom is encoded in a vector ${}^{\{n\}}\xi$ for each perturbative order. Making the harmonic decomposition of these vectors

$${}^{\{n\}}\xi_\mu = \sum_{l,m} \left({}^{\{n\}}\Phi^m_l{}_A\, Z^m_l, r^2\, {}^{\{n\}}\Phi^m_l\, Z^m_l{}_a + r^2\, {}^{\{n\}}\xi^m_l\, X^m_l{}_a \right), \tag{39}$$

we realize that there are one axial and three polar gauge degrees of freedom per order.

In order to extract this non-physical freedom from our perturbations, we can take two approaches. Following RW [1] we can fix the gauge by choosing

$${}^{\{n\}}H^m_l{}_A = 0, \qquad {}^{\{n\}}G^m_l = 0 \qquad {}^{\{n\}}h^m_l = 0. \tag{40}$$

We have proven [6] that this gauge, the so-called RW gauge, can be imposed up to any perturbative order and, because of the properties of the spherical harmonics, leads to a full metric $\tilde{g}_{\mu\nu}$ such that

$$\tilde{g}_{Ab:c}\, g^{bc} = 0, \qquad \tilde{g}_{ab} = \tilde{K} g_{ab}, \tag{41}$$

where $\tilde{K}$ is a generic function that depends on all of the four coordinates of the background manifold.

As an alternative we can instead follow Moncrief [3] and, making linear combinations of the harmonic coefficients appearing in equations (34) and (35), construct gauge-invariant quantities. These gauge invariants are conceptually different from those defined in reference [10], since they are tied to the RW gauge (40). We note that this gauge is almost rigid, in the sense that it exhausts all the gauge freedom except for the cases $l = 0, 1$ that have to be treated in a different way. We have succeeded in constructing a procedure to generalize these gauge-invariant quantities for any tensor up to the desired order in perturbation theory [11]. In order to explain this procedure, let us define a tensor ${}^{\{n\}}\mathcal{K}_{\mu\nu}$ that obeys the RW gauge (40) and, therefore, is decomposed in harmonics as

$${}^{\{n\}}\mathcal{K}_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{n\}}\mathcal{K}^m_l{}_{AB}\, Z^m_l & {}^{\{n\}}\kappa^m_l{}_A\, X^m_l{}_b \\ \text{Sym.} & {}^{\{n\}}\mathcal{K}^m_l\, r^2 \gamma_{ab} Z^m_l \end{pmatrix}. \tag{42}$$

The gauge invariants will be the harmonic coefficients of this tensor in terms of the coefficients of the decomposition (34) [and (35) for the stress-energy tensor]. In order to relate them, one just has to take a generic unrestricted metric perturbation ${}^{\{n\}}h_{\mu\nu}$ decomposed as (34) and apply a gauge transformation to take it to the RW form ${}^{\{n\}}\mathcal{K}_{\mu\nu}$. For example, at first order, from relation (37) we have that

$$ {}^{\{1\}}\mathcal{K}_{\mu\nu} = {}^{\{1\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{1\}}p}\, g_{\mu\nu}, \tag{43} $$

where ${}^{\{1\}}p$ is the first-order gauge vector that we have to determine and that is decomposed as

$$ {}^{\{1\}}p \equiv \sum_{l,m} \left({}^{\{1\}}P_l^m{}_A \; Z_l^m, r^2\, {}^{\{1\}}P_l^m \; Z_l^m{}_a + r^2\, {}^{\{1\}}q_l^m \; X_l^m{}_a \right). \tag{44} $$

Therefore, from (42) and (43) we have that these harmonic coefficients are given by

$$ {}^{\{1\}}P_l^m{}_A = \frac{r^2}{2}\, {}^{\{1\}}G_l^m{}_{|A} - {}^{\{1\}}H_l^m{}_A, \qquad {}^{\{1\}}P_l^m = -\frac{1}{2}\, {}^{\{1\}}G_l^m, \qquad {}^{\{1\}}q_l^m = -\frac{1}{2r^2}\, {}^{\{1\}}h_l^m, \tag{45} $$

and hence the first-order gauge invariants defined by GS are

$$ \begin{aligned} {}^{\{1\}}\mathcal{K}_l^m{}_{AB} &= {}^{\{1\}}H_l^m{}_{AB} + {}^{\{1\}}P_l^m{}_{A|B} + {}^{\{1\}}P_l^m{}_{B|A}, &(46)\\ {}^{\{1\}}\mathcal{K}_l^m &= {}^{\{1\}}K_l^m + 2v^A\, {}^{\{1\}}P_l^m{}_A - l(l+1)\, {}^{\{1\}}P_l^m, &(47)\\ {}^{\{1\}}\kappa_l^m{}_A &= {}^{\{1\}}h_l^m{}_A + r^2\, {}^{\{1\}}q_l^m{}_{|A}. &(48) \end{aligned} $$

In order to obtain the nth order gauge invariants, it is easy to see from (36) that the equation we have to solve is the same as at first order (43) but with a source term ${}^{\{n\}}\mathcal{H}_{\mu\nu}$,

$$ {}^{\{n\}}\mathcal{K}_{\mu\nu} = {}^{\{n\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{n\}}p}\, g_{\mu\nu} + {}^{\{n\}}\mathcal{H}_{\mu\nu}. \tag{49} $$

This hierarchy of equations can be solved iteratively because, as can be seen in relation (36), ${}^{\{n\}}\mathcal{H}_{\mu\nu}$ depends on perturbations of lower order ${}^{\{m\}}h_{\mu\nu}$ $(m \le n)$ and on the gauge vectors ${}^{\{m\}}p$ which are supposed to have been determined in terms of ${}^{\{m\}}h_{\mu\nu}$. As usual, we can decompose ${}^{\{n\}}\mathcal{H}_{\mu\nu}$ in spherical harmonics:

$$ {}^{\{n\}}\mathcal{H}_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{n\}}\mathcal{H}_l^m{}_{AB}\; Z_l^m & {}^{\{n\}}\mathcal{H}_l^m{}_A\; Z_l^m{}_b + {}^{\{n\}}\check{h}_l^m{}_A\; X_l^m{}_b \\ \text{Sym.} & {}^{\{n\}}\tilde{\mathcal{H}}_l^m\; r^2\gamma_{ab} Z_l^m + {}^{\{n\}}\mathcal{H}_l^m\; Z_l^m{}_{ab} + {}^{\{n\}}\check{h}_l^m\; X_l^m{}_{ab} \end{pmatrix}. \tag{50} $$

In this way, the nth order metric invariants are

$$ \begin{aligned} {}^{\{n\}}\mathcal{K}_l^m{}_{AB} &= {}^{\{n\}}H_l^m{}_{AB} + {}^{\{n\}}P_l^m{}_{A|B} + {}^{\{n\}}P_l^m{}_{B|A} + {}^{\{n\}}\mathcal{H}_l^m{}_{AB}, &(51)\\ {}^{\{n\}}\mathcal{K}_l^m &= {}^{\{n\}}K_l^m + 2v^A\, {}^{\{n\}}P_l^m{}_A - l(l+1)\, {}^{\{n\}}P_l^m + {}^{\{n\}}\tilde{\mathcal{H}}_l^m, &(52)\\ {}^{\{n\}}\kappa_l^m{}_A &= {}^{\{n\}}h_l^m{}_A + r^2\, {}^{\{n\}}q_l^m{}_{|A} + {}^{\{n\}}\check{h}_l^m{}_A, &(53) \end{aligned} $$

where the harmonic coefficients of the vector ${}^{\{n\}}p$ are given by

$$ \begin{aligned} {}^{\{n\}}P_l^m{}_A &= \frac{r^2}{2}\left({}^{\{n\}}G_l^m + \frac{1}{r^2}\, {}^{\{n\}}\mathcal{H}_l^m \right)_{|A} - {}^{\{n\}}H_l^m{}_A - {}^{\{n\}}\mathcal{H}_l^m{}_A, &(54)\\ {}^{\{n\}}P_l^m &= -\frac{1}{2}\left({}^{\{n\}}G_l^m + \frac{1}{r^2}\, {}^{\{n\}}\mathcal{H}_l^m \right), &(55)\\ {}^{\{n\}}q_l^m &= -\frac{1}{2r^2}\left({}^{\{n\}}h_l^m + {}^{\{n\}}\check{h}_l^m \right). &(56) \end{aligned} $$

In order to find the gauge invariants of another tensor field, in particular of the stress-energy tensor, it suffices to apply a gauge transformation parameterized by the vectors $\{ {}^{\{1\}}p, ..., {}^{\{n\}}p\}$.

6. Second-order

6.1. Second-order gauge invariants

The second-order metric gauge invariants are explicitly given by equations (51)–(53) with the source term

$$ {}^{\{2\}}\mathcal{H}_{\mu\nu} \equiv \mathcal{L}^2_{\{1\}p} g_{\mu\nu} + 2\mathcal{L}_{\{1\}p}\, {}^{\{1\}}h_{\mu\nu}. \tag{57} $$

The matter invariants are encoded in the tensor ${}^{\{2\}}\Psi_{\mu\nu}$, that is decomposed in harmonics as

$$ {}^{\{2\}}\Psi_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{2\}}\Psi^m_l{}_{AB}\, Z^m_l & {}^{\{2\}}\Psi^m_l{}_A\, Z^m_l{}_b + {}^{\{2\}}\psi^m_l{}_A\, X^m_l{}_b \\ \text{Sym.} & {}^{\{2\}}\tilde\Psi^m_l\, r^2\gamma_{ab} Z^m_l + {}^{\{2\}}\Psi^m_l\, Z^m_l{}_{ab} + {}^{\{2\}}\psi^m_l\, X^m_l{}_{ab} \end{pmatrix}, \tag{58} $$

and is defined applying a gauge transformation to the second-order stress-energy tensor ${}^{\{2\}}T_{\mu\nu}$ parameterized by the vectors $\{{}^{\{1\}}p, {}^{\{2\}}p\}$:

$$ {}^{\{2\}}\Psi_{\mu\nu} = {}^{\{2\}}T_{\mu\nu} + \mathcal{L}_{\{2\}p} T_{\mu\nu} + \mathcal{L}^2_{\{1\}p} T_{\mu\nu} + 2\mathcal{L}_{\{1\}p}\, {}^{\{1\}}T_{\mu\nu}. \tag{59} $$

6.2. Second-order Einstein equations

Making use of the gauge-invariant variables defined in the previous section, the second-order Einstein equations can be schematically written as

$$ E_{AB}[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar l,\hat l}\sum_{\bar m,\hat m} {}^{(\epsilon)}S^{\bar m \hat m m}_{\bar l\ \hat l\ l}{}_{AB} = 8\pi\, {}^{\{2\}}\Psi^m_l{}_{AB}, \tag{60} $$

$$ E_{A}[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar l,\hat l}\sum_{\bar m,\hat m} {}^{(\epsilon)}S^{\bar m \hat m m}_{\bar l\ \hat l\ l}{}_{A} = 8\pi\, {}^{\{2\}}\Psi^m_l{}_{A}, \tag{61} $$

$$ \tilde E[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar l,\hat l}\sum_{\bar m,\hat m} {}^{(\epsilon)}\tilde S^{\bar m \hat m m}_{\bar l\ \hat l\ l} = 8\pi\, {}^{\{2\}}\tilde\Psi^m_l, \tag{62} $$

$$ E[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar l,\hat l}\sum_{\bar m,\hat m} {}^{(\epsilon)}S^{\bar m \hat m m}_{\bar l\ \hat l\ l} = 8\pi\, {}^{\{2\}}\Psi^m_l, \tag{63} $$

$$ O_{A}[{}^{\{2\}}\kappa^m_l] - i\sum_{\bar l,\hat l}\sum_{\bar m,\hat m} {}^{(-\epsilon)}S^{\bar m \hat m m}_{\bar l\ \hat l\ l}{}_{A} = 8\pi\, {}^{\{2\}}\psi^m_l{}_{A}, \tag{64} $$

$$ O[{}^{\{2\}}\kappa^m_l] - i\sum_{\bar l,\hat l}\sum_{\bar m,\hat m} {}^{(-\epsilon)}S^{\bar m \hat m m}_{\bar l\ \hat l\ l} = 8\pi\, {}^{\{2\}}\psi^m_l, \tag{65} $$

where $(\bar l, \bar m)$ and $(\hat l, \hat m)$ are the harmonic labels corresponding to the first-order modes that couple giving rise to a second-order mode with labels (l, m). That is why the sums run over all non-negative integers with the usual restrictions $|\bar l - \hat l| \le l \le \bar l + \hat l$ and $\bar m + \hat m = m$. E and O are linear differential operators, in fact they are the same operators that appear at first order, but they now act on second-order quantities. As can be seen recalling our notation of capital or lowercase letters for polar or axial objects, respectively, E acts on the polar invariants $\{{}^{\{2\}}\mathcal{K}^m_l{}_{AB}, {}^{\{2\}}\mathcal{K}\}$, whereas O only acts on the axial invariant $\kappa^m_l{}_A$. We have explicitly calculated and simplified the sources S that are quadratic in first-order perturbations and depend on the sign $\epsilon \equiv (-1)^{\bar l + \hat l - l}$. There exist two kind of sources: ${}^{(+)}S$ and ${}^{(-)}S$. On the one hand, ${}^{(+)}S$ sources are composed by *polar* $\times$ *polar* and *axial* $\times$ *axial* terms with real coefficients. On the other hand, ${}^{(-)}S$ sources contain *polar* $\times$ *axial* mixed terms with purely imaginary coefficients. The sign ϵ flips when any of the l labels changes, so all equations have generically both type of sources. As a consequence, some of the equations share the sources. In particular, equations (61) and (64) alternate their sources. The same thing happens with the pair (63) and (65).

7. Conclusions

In order to face the problem of high-order perturbation theory in a spherical background, we have generalized to higher orders the well-known GS formalism for non-spherical first-order perturbations of a spherical spacetime. This formalism is considered to be optimal for the perturbative study of a number of astrophysical scenarios of interest. The generalization put forward here will make it even more powerful, leading to more precise results and allowing to describe interactions between different modes. In doing this generalization a number of results have been obtained.

Without restricting ourselves to any background, we have achieved closed formulas to calculate the perturbation of all the curvature tensors of interest at any order. Because of their differential character, these formulas turn out to be combinatorial, what makes them very effective from the point of view of an algebraic implementation.

We have generalized the so-called RWZ and pure-spin harmonics to any rank. We also have provided a formula that expands the product between any pair of them as a linear combination of harmonics. This formula is essential to complete the formalism.

In addition, we have studied the gauge-freedom problem, showing that the RW gauge can be imposed up to any order in perturbation theory and obtaining an iterative procedure that, in general, allows one to construct the gauge-invariant quantities that are tied to this gauge.

For the particular case of second order, we have explicitly computed and simplified the gauge invariants for spherical backgrounds, as well as the Einstein equations, that govern the evolution of these invariant objects. The second-order equations are essentially the same as the first-order equations, but they also include complicated quadratic sources in first-order perturbations. We have disentangled the structure of these sources and simplified them to a manageable form.

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