

High-Order Nonlinear Diffusion

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A number of important physical processes, such as the flow of a surface-tension dominated thin liquid and the diffusion of dopant in semiconductors are governed by the fourth-order nonlinear diffusion equation $u_t + (u^n u_{xxx})_x = 0$ ($n > 0$). The analysis of such equations is doubly complex due to the nonlinearity and the high order. Here we present some of the more immediate and simple results for this equation and our development parallels known results for the classical nonlinear diffusion equation $u_t = (u^n u_x)_x$. In particular we examine in some detail asymptotic solutions for small n (namely $0 < n \ll 1$), simple waiting-time solutions and the similarity source solution. The similarity source solution for $n = 1$ has a particularly simple closed form which may be readily generalized to a nonlinear diffusion equation of arbitrary high order. This simple exact similarity solution appears not to have been noted previously and together with the well-known solution for the classical equation means that the general nonlinear equation $u_t = (-1)^m D(u^n D^{2m+1} u)$, with $D = \partial/\partial x$, admits a simple exact similarity solution either for $m = 0$ and all values of n or for $n = 1$ and all values of m . Details for small- n solutions and waiting-time solutions for this general equation are also briefly noted.

1. Introduction

THE second-order nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(u^n \frac{\partial u}{\partial x} \right), \tag{1.1}$$

which models physical processes such as the percolation of gas through porous media for $n \geq 1$ (see (Muskat, 1937)) and the flow of thin liquid films spreading under gravity for $n = 3$ (see (Buckmaster, 1977)) has received much recent attention. A number of solutions of this equation have been found and it is by now reasonably well understood (see, for example, (Vázquez, 1987; Peletier, 1981; Aronson, 1983)). One of the most important differences between the solutions of (1.1) and the solutions of the linear diffusion equation ((1.1) with $n = 0$) is the existence of fronts behind which the solution is non-zero and in front of which the solution is zero. This is illustrated by the similarity solution of (1.1) obtained by Zel'dovich and Kompaneets (1950), this solution being

$$u(x, t) = \begin{cases} \frac{1}{t^k} \left(C - \frac{n\xi^2}{2(n+2)} \right)^{1/n}, & \xi^2 \leq \frac{2}{n} (n+2)C, \\ 0, & \xi^2 \geq \frac{2}{n} (n+2)C, \end{cases} \tag{1.2}$$

where $\xi = x/t^k$, $k = (n+2)^{-1}$ and C is a constant. This similarity solution is important since it is the long-time limit of any initial distribution of u , with C related to the initial integral of u .

Equation (1.1) also possesses so-called waiting-time solutions in which the interface does not move immediately, but an initial redistribution occurs behind the front before the front begins to move. Aronson (1970), Knerr (1977) and Kamen (1980) showed that an exact solution of (1.1) with this waiting-time property is

$$u(x, t) = \begin{cases} \left(\frac{n}{2(n+2)} \frac{x^2}{(t_0 - t)} \right)^{1/n}, & x \geq 0, \\ 0, & x \leq 0. \end{cases} \quad (1.3)$$

This solution becomes unbounded as t tends to t_0 and ceases to be valid for $t > t_0$. While exhibiting the existence of waiting-time solutions, (1.3) provides no information about the behaviour of general solutions after the front starts to move. By using a phase-plane analysis, Lacey, Ockendon & Tayler (1982) constructed similarity solutions which have waiting-time behaviour and which are valid when the front starts to move. However, these similarity solutions cannot be expressed in simple form. The solution (1.3) provides the initial behaviour of the interface of a waiting-time solution and (1.2) provides the long-term behaviour after the front has started to move (see (Knerr, 1977)).

The solution (1.3) 'blows up' as t tends to t_0 and it cannot be continued in any sense for $t > t_0$. Blow-up is related to the fact that the initial datum grows 'too much' at infinity and has nothing to do with the waiting-time property for general solutions of (1.1). A comprehensive study of the blow-up question can be found in (Benilan, Crandall & Pierre, 1984). The solution (1.3) is, in fact, essentially the formal limit of (1.2) as the constant C tends to zero. As well as being a separable solution, (1.3) is also a similarity solution, being a function of $x/(t_0 - t)^{1/2}$, although it is not the same type of similarity solution as (1.2) with the constant C non-zero. Another second-order nonlinear diffusion equation which has a simple blow-up solution is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(\alpha e^{\beta u} \frac{\partial u}{\partial x} \right),$$

where α and β are appropriate constants ($\alpha > 0$, $\beta < 0$). This equation admits the solution

$$u(x, t) = \frac{1}{\beta} \log \left\{ \frac{(x - x_0)^2 + C}{2\alpha(t_0 - t)} \right\},$$

where x_0 , t_0 and C are all constants, and this solution, which follows immediately from the assumption $u(x, t) = f(x) + g(t)$, blows up for $t = t_0$ and also for $x = x_0$ if $C = 0$. We observe that as u tends to zero this diffusion equation approaches the classical linear equation and therefore it cannot have fronts propagating at finite velocity.

Instead of finding exact solutions of (1.1) for arbitrary n , Kath & Cohen (1982) find asymptotic solutions for $0 < n \ll 1$. This has the advantage of enabling

the solution for any initial condition to be found. By defining a new dependent variable and a new time by

$$v = u^n, \quad t = nt^*, \quad (1.4)$$

and expanding v in the perturbation series

$$v = v_0(x, t^*) + nv_1(x, t^*) + \dots, \quad (1.5)$$

the solution of (1.1) is found to be determined to first order in n by the solutions of

$$v_{0x^*} = v_{0x}^2, \quad v_{1t^*} = 2v_{0x}v_{1x} + v_0v_{0xx}. \quad (1.6)$$

These equations are nonlinear hyperbolic equations and so in general shocks, corresponding to jumps in v_x , will form in the solutions for v_0 and v_1 . At a shock, (1.6) is not a valid approximation to (1.1). Kath & Cohen (1982) show how these shocks can be smoothed out using corner layers.

Depending on the nature of the singularity at the front of the initial distribution of u , the solution of (1.1) for $0 < n \ll 1$ exhibits waiting-time behaviour. It is shown by Kath & Cohen (1982) that if the initial condition has

$$v_0 \sim k(x - x_0)^\alpha, \quad (1.7)$$

$\alpha > 0$ at the front x_0 , then the solution exhibits waiting-time behaviour for $\alpha \geq 2$. For $0 < \alpha < 2$ the front moves immediately, which is in agreement with the results of Knerr (1977) and references therein. This immediate movement of the front was shown to be due to the immediate formation of a shock at the front for $\alpha < 2$.

In the present work, the fourth-order nonlinear diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(u^n \frac{\partial^3 u}{\partial x^3} \right) = 0 \quad (1.8)$$

is studied. This equation arises in the flow of a surface-tension dominated thin liquid film, where $n = 3$ (see, for example, (Greenspan, 1978; Greenspan & McCay, 1981; Hocking, 1981; Lacey, 1982)) and the diffusion of dopant in semiconductors (see, for example, (King, 1986; Tayler, 1987)). At least in some respects the properties and solutions of this equation parallel those of the second-order equation (1.1). In particular, waiting-time and similarity solutions corresponding to (1.3) and (1.2) respectively can be found. Solutions corresponding to (1.3) give the initial behaviour of the interface for certain initial data, and solutions corresponding to (1.2) are expected to give the long-term behaviour. However, due to the higher order of the equation, only the similarity solution for $n = 1$ has a simple closed form. In the following section, small- n asymptotic solutions of (1.8) are found. In contrast to the solutions of (1.1) for $0 < n \ll 1$, these solutions are found to have infinite waiting time, this being independent of the nature of the singularity at the front of the initial distribution of u , provided that this initial distribution satisfies certain conditions. In the subsequent two sections waiting-time solutions and similarity solutions of (1.8) are discussed, while in the final section of the paper some comments relating to the general

equation

$$\frac{\partial u}{\partial t} = (-1)^m \frac{\partial}{\partial x} \left(u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \right) \quad (1.9)$$

are made. King (1986) includes a model which involves equation (1.9) with $m = 2$ and $n = 3$. Here, in particular, we are able to give a simple closed-form expression for the similarity-source solution of (1.9) for the special case of $n = 1$. We remark that the factor $(-1)^m$ in equation (1.9) is necessary to ensure sensible physical behaviour of its solutions. For example, if this factor is not included, the similarity-source solution of this equation, arising from an initial (positive) delta function, can be totally negative, which is of course absurd. In addition (1.9) without the factor $(-1)^m$ is, for odd m , a backward parabolic equation (for $u > 0$) whose initial-value problems are not expected to be well-posed.

2. Solution for small n

In the present section, an asymptotic solution of (1.8) for small n is found. For $0 < n \ll 1$, the diffusion coefficient u^n is near 1 for u away from zero and drops rapidly to zero as u tends to zero. The limit $n \rightarrow 0$ is then a singular limit as u^n cannot be uniformly expanded as a series in n which is uniformly valid for all u . As in (Kath & Cohen, 1982), to overcome this we set

$$v = u^{1/n}, \quad t = \left(\frac{1}{3}n\right)^3 \tau, \quad (2.1)$$

so that equation (1.8) becomes

$$\begin{aligned} v_\tau + \left(1 - \frac{n}{3}\right) \left(1 - \frac{2n}{3}\right) v_x^4 + n \left(1 - \frac{n}{3}\right) \left(2 - \frac{n}{3}\right) v v_x^2 v_{xx} \\ + \frac{n^2}{3} \left(1 - \frac{n}{3}\right) v^2 v_{xx}^2 + \frac{n^2}{9} \left(4 - \frac{n}{3}\right) v^2 v_x v_{xxx} + \frac{n^3}{27} v^3 v_{xxxx} = 0. \end{aligned} \quad (2.2)$$

The solution is a function of the fast time-scale τ due to the rapid variation of v from 1 when u is away from zero, to 0 when u tends to zero. This equation is solved by using the perturbation series

$$v(x, \tau) = v_0(x, \tau) + n v_1(x, \tau) + n^2 v_2(x, \tau) + \dots \quad (2.3)$$

The solution for u is found by inverting (2.1) to obtain

$$u = v^{3/n} = v_0^{3/n} e^{3v_1/v_0} \left[1 + 3n \left(\frac{v_2}{v_0} - \frac{1}{2} \frac{v_1^2}{v_0^2} \right) + \dots \right]. \quad (2.4)$$

We note that to evaluate the solution for u to $O(1)$, the solution for v must be found to $O(n)$.

Substituting the perturbation expansion (2.3) into (2.2), we find that at $O(1)$,

$$v_{0\tau} + v_{0x}^4 = 0. \quad (2.5)$$

This is a first-order nonlinear hyperbolic equation, which can be solved by

standard methods (see (Whitham, 1974: § 2.13)) to yield

$$v_0(x, \tau) = 3f'^4(\zeta)\tau + f(\zeta), \quad x = 4f'^3(\zeta)\tau + \zeta, \quad (2.6)$$

where $v_0 = f(\zeta)$, $x = \zeta$ when $\tau = 0$.

At $O(n)$, (2.2) and (2.3) give the equation

$$v_{1\tau} + 4v_{0\zeta}^3 v_{1\zeta} = -v_0 v_{0\zeta}^2 v_{0\zeta\zeta} + v_{0\zeta}^4, \quad (2.7)$$

which can be solved by the method of characteristics to give

$$\left. \begin{aligned} v_1(x, \tau) &= \frac{3}{4}f'^4(\zeta)\tau - \frac{1}{12} \left[f(\zeta) - \frac{1}{4} \frac{f'^2(\zeta)}{f''(\zeta)} \right] \\ &\quad \times \log(1 + 12f'^2(\zeta)f''(\zeta)\tau) + f_1(\zeta), \\ x &= 4f'^3(\zeta)\tau + \zeta, \end{aligned} \right\} \quad (2.8)$$

where $v_1 = f_1(\zeta)$, $x = \zeta$ when $\tau = 0$.

The initial value of v_0 , namely f , will in general be some positive function which is non-zero only in some finite region. Such a function is sketched in Fig. 1. If $f' \neq 0$ at the fronts of the initial distribution, then it can be seen from (2.6) that there will be a gap between the inward sloping characteristics at the fronts and the characteristics of zero slope outside the region where f is non-zero. This gap is filled by expansion fans originating at the fronts. Let the value of f and its derivative be f_r and f'_r at the right-hand front, f_l and f'_l at the left-hand front and let x_r and x_l be the initial positions of the right- and left-hand fronts respectively. Then from (2.5) it can be found that the solution in the expansion fans originating at the fronts is

$$v_0 = 3\tau \left(\frac{x_r - x}{4\tau} \right)^{\frac{1}{3}}, \quad f'_r{}^3 \leq \frac{x - x_r}{4\tau} \leq 0 \quad (2.9a)$$

for the right-hand front, and

$$v_0 = 3\tau \left(\frac{x - x_l}{4\tau} \right)^{\frac{1}{3}}, \quad 0 \leq \frac{x - x_l}{4\tau} \leq f'_l{}^3 \quad (2.9b)$$

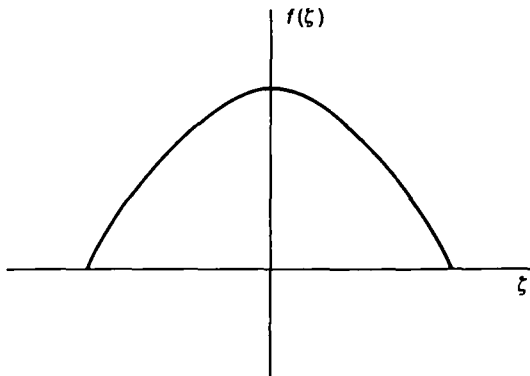


FIG. 1. Initial condition for v_0

for the left-hand front. It can be seen that these expansion fans exist only if $f'_r \neq 0$ and $f'_l \neq 0$. These expansion fans smooth out the initial discontinuity in f' .

From (2.7) and (2.9a,b), it can be found that the solution for v_1 is

$$v_1 = \frac{3}{4} \tau \left(\frac{x_r - x}{4\tau} \right)^{\frac{1}{3}}, \quad f_r'^3 \leq \frac{x - x_r}{4\tau} \leq 0 \quad (2.10a)$$

in the right-hand expansion fan, and

$$v_1 = \frac{3}{4} \tau \left(\frac{x - x_l}{4\tau} \right)^{\frac{1}{3}}, \quad 0 \leq \frac{x - x_l}{4\tau} \leq f_l'^3 \quad (2.10b)$$

in the left-hand expansion fan.

One of the most important features of (1.1) is that it possesses waiting-time solutions. The existence of such solutions for (1.8) when $0 < n \ll 1$ will now be discussed. Since the solution for v_0 has inward propagating characteristics at the initial fronts with expansion fans linking these characteristics to the characteristics with zero slope in the region where v_0 is initially zero, v_0 remains non-zero only in its initial domain unless outward propagating shocks form. Hence the solution exhibits waiting-time behaviour unless outward propagating shocks form at the fronts when $\tau = 0$. To investigate this possibility, we consider equation (2.5) in its conservation form

$$\frac{\partial}{\partial \tau} (v_{0x}) + \frac{\partial}{\partial x} (v_{0x}^4) = 0. \quad (2.11)$$

Since the conserved density is v_{0x} , shocks occur as discontinuities in v_{0x} with v_0 remaining continuous. This conservation equation gives the shock velocity U as

$$U = [v_{0x}^4]/[v_{0x}] = v_{01x}^3 + v_{01x}^2 v_{02x} + v_{01x} v_{02x}^2 + v_{02x}^3, \quad (2.12)$$

where $[]$ denotes a jump in the quantity and inferior 1 and 2 indicate values ahead of and behind the shock respectively. If a shock initially forms at or near a front of the initial distribution of v_0 , it can be seen from (2.12) that the shock initially travels inwards, away from the front, since $v_{0x} < 0$ at the forward front and $v_{0x} > 0$ at the rear front. Hence even if shocks form immediately at the fronts, the solution still exhibits waiting-time behaviour. This is in contrast to the solution of (1.1) for small n , for which it was shown by Kath & Cohen (1982) that if the fronts are such that shocks form immediately there, then these shocks propagate outwards and the solution will not have waiting-time behaviour. The waiting time is in fact infinite and the solution will not move outside its initial domain. This can be seen from the expression (2.12) for the shock velocity; for suppose that a shock moves to the right of the initial domain of v_0 . Then $v_{01x} = 0$ and, since v_0 is continuous, $v_{02x} < 0$, so that $U < 0$. Hence a shock cannot propagate to the right of the initial domain of v_0 . Similarly, a shock cannot propagate to the left of the initial domain of v_0 . In summary then, solutions of (1.8) for $0 < n \ll 1$ exhibit infinite waiting-time behaviour whenever the initial distribution of u is non-zero only in some finite region and $f' \neq 0$ at the fronts.

At a shock, the values of v_{0x} jump discontinuously, so that (2.5) is not a valid approximation of (2.2) there. The solution in the neighbourhood of a shock can

be smoothed out by incorporating a corner layer, as was done by Kath & Cohen (1982) for (1.1) when $0 < n \ll 1$. The evaluation of the corner-layer solution will not be attempted here as it has no major effect on the outer solution, unlike the situation with a boundary-layer problem, where the boundary-layer solution needs to be evaluated to complete the outer solution.

The solution given by (2.4), (2.6), (2.8), (2.9) and (2.10) is not expected to be valid for large time. As t tends to infinity, we expect that any solution of (1.8) will asymptotically approach a similarity solution, which is shown in Section 4 to involve a function of $x/t^{(n+4)^{-1}}$. On the other hand, the solutions of (2.5) and (2.7) approach simple wave solutions, which are functions of x/t . Kath & Cohen (1982) also found that their asymptotic solution of (1.1) for $0 < n \ll 1$ is not valid for large times.

3. Exact waiting-time solution

As stated in the introduction, (1.1) possesses waiting-time solutions of the form (1.3). Equation (1.8) also has such solutions which can be found for all n , except $n = 2$ and $n = 4$, in which cases a simple solution appears not to exist. It may be easily shown that (1.8) has separable solutions of the form

$$u(x, t) = \begin{cases} \left(\frac{x^4}{\lambda n(t_0 + t)} \right)^{1/n}, & x \geq 0, \\ 0, & x \leq 0, \end{cases} \quad (3.1)$$

where t_0 and λ are constants such that

$$\lambda = \frac{4}{n} \left(\frac{4}{n} - 1 \right) \left(\frac{4}{n} - 2 \right) \left(\frac{4}{n} + 1 \right) \quad (3.2)$$

for $n \neq 2, 4$. The solution (3.1) is valid for those values of t for which $\lambda(t_0 + t) > 0$. For $n = 2$ and $n = 4$, separable solutions of the form $u = X(x)T(t)$ respectively yield

$$(X^2 X''')' = -\lambda X, \quad (X^4 X''')' = -\lambda X, \quad (3.3)$$

where λ is the separation constant; the authors are unable to identify simple solutions of these equations.

The separable solutions (3.1) for $2 < n < 4$ have the same behaviour as the corresponding solutions (1.3) for the second-order equation (1.1), in that they become infinite in finite time and remain non-zero only within their initial domain. These separable solutions are postulated to represent the initial behaviour of a waiting-time solution before the front starts to move. The separable solutions for $n < 2$ and $n > 4$ have the opposite behaviour in that the solutions for these cases are defined for large t and decay uniformly as t tends to infinity.

Taking the initial conditions

$$f(\xi) = \begin{cases} \xi^{4/3}, & \xi \geq 0, \\ 0, & \xi \leq 0, \end{cases} \quad (3.4)$$

and $f_1(\xi) = 0$ in (2.6) and (2.8), it can be shown that the solution (2.4) agrees with (3.1) for $0 < n \ll 1$ to $O(\tau)$.

4. Similarity source solution

The similarity source solution of (1.8) has the initial condition

$$u(x, 0) = u_0 \delta(x), \quad (4.1)$$

where u_0 is the constant strength of the source and $\delta(x)$ is the usual Dirac delta function. The appropriate similarity solution of (1.8) takes the form

$$u(x, t) = \frac{1}{t^k} \phi\left(\frac{x}{t^k}\right), \quad (4.2)$$

where here $k = (n + 4)^{-1}$. With $\xi = x/t^k$ we may readily deduce from (1.8) and (4.2) that

$$(\phi^n \phi''')' = k(\xi \phi)', \quad (4.3)$$

where primes here denote differentiation with respect to ξ . Clearly equation (4.3) integrates immediately to yield

$$\phi^n \phi''' - k \xi \phi = C_1, \quad (4.4)$$

where C_1 is a constant. Since $\phi(\xi)$ is expected to be even we have $\phi'(0) = \phi'''(0) = 0$ and therefore this constant of integration is zero. Alternatively for $n > 0$ and since we look for a solution such that $\phi(\xi)$ vanishes for $\xi \geq \xi_1$ (for some ξ_1) we may again conclude that C_1 must be zero. Thus the similarity source solution of (1.8) hinges on solving

$$\phi^{n-1} \phi''' = k \xi, \quad (4.5)$$

subject to $\phi(\xi)$ even, and

$$\phi(\xi_1) = \phi'(\xi_1) = 0 \quad (4.6)$$

for some ξ_1 determined from the initial condition (4.1) in the form

$$\int_{-\xi_1}^{\xi_1} \phi(\xi) d\xi = u_0. \quad (4.7)$$

We note that the condition $\phi'(\xi_1) = 0$ is included to make the solution as smooth as possible at $\xi = \pm \xi_1$. If we assume that the solution has a singularity $(1 - \eta^2)^\beta$ at the front, then $\beta = 3/n$ and this condition is meaningful only for $n < 3$. For $n > 3$ the condition $\phi'(\xi_1) = \infty$ may be the appropriate additional constraint, although the authors have no conclusive evidence to support this claim.

In fact, ξ_1 is simply a scale factor and it is not difficult to show that the problem (4.5) to (4.7) can be alternatively formulated as follows. In terms of Ψ and η defined by

$$\phi(\xi) = \xi_1^{4/n} \Psi(\eta), \quad \eta = \xi/\xi_1, \quad (4.8)$$

we have for $n > 0$,

$$\Psi^{n-1}\Psi'' = k\eta, \quad (4.9)$$

where $\Psi(\eta)$ is even and

$$\Psi(1) = \Psi'(1) = 0, \quad \xi_1^{(n+4)/n} \int_0^1 \Psi(\eta) d\eta = \frac{u_0}{2}; \quad (4.10)$$

of course primes in (4.9) and (4.10) refer to differentiation with respect to η .

Equations (4.5) or (4.9) appear in general not to admit simple closed-form solutions. However, for the special values $n = 0, 1, 2$ we have the following results. The case in which $n = 0$ yields the linear problem (from equation (4.5)),

$$\phi''' - \frac{1}{4}\xi\phi = 0, \quad (4.11)$$

which may be solved in the usual way using the Fourier cosine transform and noting that in this case ξ_1 is infinite. Omitting the details the final result is

$$\phi(\xi) = \frac{u_0}{\pi} \int_0^\infty e^{-w\xi} \cos w\xi dw = \frac{u_0}{4\pi} \sum_{j=0}^\infty \frac{(-1)^j \Gamma(\frac{1}{2}j + \frac{1}{4})}{(2j)!} \xi^{2j}. \quad (4.12)$$

This solution changes sign infinitely many times and moreover, by differentiating the Fourier transform, it is not difficult to show that it satisfies,

$$\int_0^\infty \xi\phi(\xi) d\xi > 0, \quad \int_0^\infty \xi^2\phi(\xi) d\xi = 0, \quad \int_0^\infty \xi^3\phi(\xi) d\xi < 0,$$

so that there is a delicate balance between positive and negative parts. These properties can also be deduced directly from (4.11) and further from (Bernis, 1987b) it follows that for $0 < n < 1$ equation (4.5) has a solution satisfying (4.6) which changes sign infinitely many times (where ϕ^n is to be interpreted as $|\phi|^n$ if ϕ is negative). The change of sign of solutions to certain parabolic equations is discussed further in (Bernis, 1987a). The existence of seemingly non-physical solutions is counter-intuitive and noteworthy.

For the case when $n = 1$, equation (4.5) or (4.9) may be integrated in a trivial manner to give

$$\phi(\xi) = (\xi_1^2 - \xi^2)^2/120, \quad (4.13)$$

where $\xi_1 = (225u_0/2)^{1/3}$. The generalization of this simple solution, appropriate to (1.9), is given in the final section of the paper. Although for $n = 2$ we are unable to give the source solution explicitly we observe that (4.5) or (4.9) admits a simple first integral, namely

$$2\phi\phi'' - \phi'^2 + \frac{1}{8}(\xi_1^2 - \xi^2) = 0, \quad (4.14)$$

where the constant of integration is determined from (4.6).

For $\frac{2}{3} < n < 3$ we may generate approximate solutions of (4.5) and (4.6) by assuming Frobenius-type series expansions. For n outside this range the singularity at the front is not of the assumed form (that is, it may not be algebraic).

in the terminology of (4.9) and (4.10) we look for an even solution of the form

$$\Psi(\eta) = \sum_{j=0}^{\infty} a_j (1 - \eta^2)^{\beta+j}, \quad (4.15)$$

where β and the coefficients a_j ($j \geq 0$) are to be determined. In a routine manner we find from (4.9) that $\beta = 3/n$ and a_0 is determined from

$$a_0^2 8\beta(\beta - 1)(\beta - 2) = -k, \quad (4.16)$$

which for $n > 0$ gives rise to positive a_0 if and only if $1 < \beta < 2$, that is $\frac{3}{2} < n < 3$. If this is the case, formulae for higher coefficients are found to be

$$\left. \begin{aligned} a_1 &= \frac{(6-n)a_0}{4n(3-n)}, \\ a_2 &= \frac{(6-n)(2n^3 - 13n^2 + 69n + 54)a_0}{32n^2(3-n)(2n^2 - 9n + 27)}, \end{aligned} \right\} \quad (4.17)$$

and in principle the procedure may be continued. We observe the special case in which $n = 6$ ($\beta = \frac{1}{2}$) has the 'exact' solution

$$\Psi(\eta) = a_0(1 - \eta^2)^{\frac{1}{2}}, \quad (4.18)$$

which is non-physical because (4.16) gives $a_0^6 = -\frac{1}{30}$ and therefore a_0 is not real.

Although we are unable to generate any further exact solutions of (4.5) or (4.9), for completeness we note that since these equations remain invariant under a simple stretching one-parameter group of transformations, we may therefore reduce the third-order equation to one of second order. With the following transformations:

$$\Psi(\eta) = \eta^{4n} v(\log \eta), \quad z = \log \eta, \quad p = \frac{dv}{dz}, \quad (4.19)$$

we may show that (4.9) becomes

$$p \frac{d^2 p}{dv^2} + \left(\frac{dp}{dv} \right)^2 + 3(\lambda - 1) \frac{dp}{dv} + \lambda(\lambda - 1)(\lambda - 2) \frac{v}{p} = \frac{k}{pv^{n-1}} - (3\lambda^2 - 6\lambda + 2), \quad (4.20)$$

where $k = (n + 4)^{-1}$ and $\lambda = 4/n$. Unfortunately, the authors have not been able to exploit this reduction of order, even for special values of n .

In summary, for the similarity source solution we have presented exact solutions for $n = 0$ and $n = 1$, while for $\frac{3}{2} < n < 3$ we have shown that Frobenius-type series expansions can, at least in principle, be found. We have not demonstrated that such series necessarily converge, nor for other values of n have we established the existence of the similarity source solution. Indeed we have indicated some uncertainty as to exactly which conditions are appropriate at the

front. The principle that the solution be as smooth as possible at the front appears to be the natural and logical assumption. However, as already stated, we do not claim that this is necessarily the case for all values of n .

5. Some extensions and additional comments

In this section we make one or two additional comments relating to the nonlinear diffusion equation (1.9) which is of arbitrary high order. Equation (1.9) can be solved in the limit $0 < n \ll 1$ by using the asymptotic expansion (2.3), where v and τ are now given by

$$v = u^{n/(2m+1)}, \quad t = (n/(2m+1))^{2m+1} \tau. \quad (5.1)$$

Substituting the expansion (2.3) into (1.9), we find that at $O(1)$, v_0 is the solution of

$$v_{0\tau} = (-1)^m v_{0x}^{2m+2}. \quad (5.2)$$

This equation can be solved by standard methods (see (Whitham, 1974: §2.13)) to yield

$$\left. \begin{aligned} v_0(x, \tau) &= (-1)^{m+1} (2m+1) f'^{2(m+1)}(\xi) \tau + f(\xi), \\ x &= (-1)^{m+1} 2(m+1) f'^{2m+1}(\xi) \tau + \xi, \end{aligned} \right\} \quad (5.3)$$

where $v_0 = f(\xi)$, $x = \xi$ when $\tau = 0$. To find u to $O(1)$, the solution for v_1 must be found (see (2.4)). However, to determine the relationship between waiting-time solutions and the nature of the singularity at the fronts of the initial distribution of u , only the solution for v_0 needs to be found. This relationship is again determined by the direction of propagation of any shocks which form at the initial fronts.

If m is odd and $f' \neq 0$ at its fronts, it can be seen from (5.3) that the characteristics at the fronts are inward propagating and so expansion fans will form at the fronts, as was the case in Section 2 for $m = 1$. From (5.2), it can be found that the solution for v_0 in the expansion fans is

$$v_0 = (2m+1) \tau \left(\frac{x_r - x}{2(m+1)\tau} \right)^{2(m+1)/(2m+1)}, \quad 2(m+1) f_r'^{2m+1} \leq \frac{x - x_r}{\tau} \leq 0 \quad (5.4)$$

at the right-hand front, and

$$v_0 = (2m+1) \tau \left(\frac{x - x_l}{2(m+1)\tau} \right)^{2(m+1)/(2m+1)}, \quad 0 \leq \frac{x - x_l}{\tau} \leq 2(m+1) f_l'^{2m+1} \quad (5.5)$$

at the left-hand front, where f_r' , f_l' , x_r and x_l are as in Section 2. These expansion fans smooth out the initial discontinuity in f' .

In conservation form, (5.2) is

$$\frac{\partial}{\partial \tau} (v_{0x}) + \frac{\partial}{\partial x} ((-1)^{m+1} v_{0x}^{2m+2}) = 0, \quad (5.6)$$

so that the shock velocity U is given by

$$U = (-1)^{m+1} [v_{0x}^{2(m+1)}] / [v_{0x}]. \quad (5.7)$$

For m odd, it can be seen that if a shock forms immediately at a front, then the initial shock velocity is inwards. Hence the same situation occurs as in Section 2 and infinite waiting-time behaviour will occur whenever the initial distribution of v_0 is non-zero only in some finite region and satisfies certain conditions. For m even, the initial velocity of any shock forming immediately at the front is outwards and any waiting-time behaviour depends on the exact nature of the singularity at the front, as for the $m = 0$ case considered by Kath & Cohen (1982).

The time at which a shock first forms is given by

$$\tau_c = \min_{\xi} \left(\frac{(-1)^m}{2(m+1)(2m+1)f'^{2m}(\xi)f''(\xi)} \right), \quad (5.8)$$

if $(-1)^m f''(\xi)$ is positive. Let us examine the possibility of a shock forming at the fronts of f . For m even, if it is assumed that $f(\xi) \sim \beta(\xi - \xi_0)^\alpha$ at the front ξ_0 , where α and β are constants, then we find that if $\alpha < 2(m+1)/(2m+1)$ then $\tau_c = 0$, if $\alpha = 2(m+1)/(2m+1)$ then τ_c is finite, and if $\alpha > 2(m+1)/(2m+1)$ then $\tau_c = \infty$. Hence if $\alpha < 2(m+1)/(2m+1)$, a shock forms immediately at the front and there is no waiting-time behaviour. However, if $\alpha \geq 2(m+1)/(2m+1)$, the solution shows waiting-time behaviour and the front will only start to move when it is overtaken by a shock.

We may readily confirm that the simple separable waiting-time solutions of (1.9) have the form

$$u(x, t) = \begin{cases} \left(\frac{x^{2(m+1)}}{\lambda n [t_0 + (-1)^{m+1} t]} \right)^{1/n}, & x \geq 0, \\ 0, & x \leq 0, \end{cases} \quad (5.9)$$

where again t_0 and λ are constants such that

$$\lambda = \frac{2(m+1)}{n} \left(\frac{2(m+1)}{n} - 1 \right) \left(\frac{2(m+1)}{n} - 2 \right) \dots \left(\frac{2(m+1)}{n} - 2m \right) \left(\frac{2(m+1)}{n} + 1 \right), \quad (5.10)$$

and clearly (5.9) is only applicable to those values of m and n for which λ is non-zero.

The similarity source solution of equation (1.9) takes the form (4.2) with $k = (n + 2m + 2)^{-1}$. We may deduce that

$$(\phi^n \phi^{(2m+1)})' = (-1)^{m+1} k (\xi \phi)', \quad (5.11)$$

and as before we look for an even solution such that $\phi(\xi)$ is zero for $\xi \geq \xi_1$ and assuming the source solution to be as smooth as possible at ξ_1 we postulate that

$$\phi(\xi_1) = \phi'(\xi_1) = \phi''(\xi_1) = \dots = \phi^{(m)}(\xi_1) = 0. \quad (5.12)$$

Thus on integrating (5.11) and using (5.12) we have

$$\phi^{n-1} \phi^{(2m+1)} = (-1)^{m+1} k \xi \quad (5.13)$$

For $n = 0$ we have

$$\phi(\xi) = \frac{u_0}{\pi} \int_0^\infty e^{-w^{2(m+1)}} \cos w \xi \, dw = \frac{u_0}{2(m+1)\pi} \sum_{j=0}^\infty \frac{(-1)^j \Gamma\left(\frac{2j+1}{2m+2}\right)}{(2j)!} \xi^{2j}, \quad (5.14)$$

and for m both odd and even and $m \geq 1$ this solution changes sign infinitely many times. For $n = 1$ we may verify that we obtain

$$\phi(\xi) = (\xi_1^2 - \xi^2)^{m+1} / (2m+3)!, \quad (5.15)$$

and ξ_1 is given by

$$\xi_1^{2m+3} = \frac{(2m+3)! \Gamma(m+\frac{5}{2})}{\pi^{\frac{1}{2}}(m+1)!} u_0. \quad (5.16)$$

It is worthwhile noting that in this notation the similarity source solution (1.2) for the classical nonlinear diffusion equation (1.1) is given by

$$\phi(\xi) = \left(\frac{n}{2(n+2)}\right)^{1/n} (\xi_1^2 - \xi^2)^{1/n}, \quad (5.17)$$

where ξ_1 is given by

$$\xi_1^{1+2/n} = \left(\frac{2(n+2)}{n}\right)^{1/n} \frac{\Gamma\left(\frac{1}{n} + \frac{3}{2}\right) u_0}{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{n} + 1\right)}, \quad (5.18)$$

and of course these results coincide with (5.15) and (5.16) for the case when $m = 0$ and $n = 1$. Thus, the general equation (1.9) admits a simple similarity source solution for the two special cases of either $m = 0$ and all values of n or $n = 1$ and all values of m .

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