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HIGH SOLITARY WAVES IN WATER: RESULTS OF CALCULATIONS.(U)

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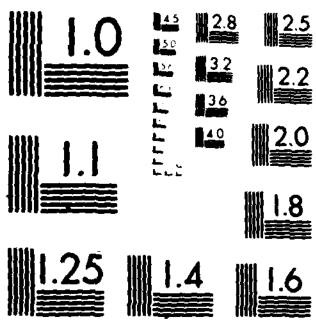

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High Solitary Waves in Water: Results of Calculations

JAMES M. WITTING

Physical Oceanography Branch
Environmental Sciences Division

September 30, 1981

NAVAL RESEARCH LABORATORY
Washington, D.C.

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This report presents results of calculations of certain properties of solitary waves in water as a function of wave strength, with emphasis on the highest and other high waves. A preceding report outlines the method used to arrive at the results presented here. The method involves a Fourier series solution of the flow field at various (numerical) resolutions to arrive at an estimate of limits as the resolution becomes infinitesimal. Values of the properties of intermediate and high waves possess 4 to 6 significant figure accuracies. For some solitary wave parameters, the calculations agree well with those found using independent methods and verify that the total

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20. ABSTRACT (Continued)

energy and other integral properties reach a maximum at strengths less than that of the highest wave. They also verify that the maximum angle of surface inclination can exceed 30° for the not-quite-highest waves. In addition, this report gives solitary wave properties useful in constructing periodic wave solutions by the superimposition of solitary waves. The highest wave in water is higher than previously reported; its amplitude to depth ratio is 0.8332.

CONTENTS

HIGH SOLITARY WAVES IN WATER: RESULTS OF CALCULATIONS

1. INTRODUCTION

From their discovery over a century ago (Russell, 1845) to now (see the review by Miles, 1980), many scientists and mathematicians have studied solitary waves in water. Although a "pure" solitary wave described by the mathematician is only a simplified construct, even a casual observer on a beach usually views at least the long-period waves coming at him as individual (solitary) mounds of water separated by relative quiescence. Because these waves break at some depths, they cannot be described solely in terms of small-amplitude theory.

Solitary waves are a basic ingredient in some descriptions of waves on a beach (Munk, 1949). Modern descriptions of periodic waves can use solitary waves as the basic ingredient (Stiassnie and Peregrine, 1980; Witting, 1981). Many theories describe the properties of solitary waves. Some recent studies pay attention to very high waves and the highest wave. We now have several published reports that give solitary wave properties to several significant figures throughout the entire range from small amplitude to the highest wave. Unhappily, these published reports do not agree with each other to the accuracies claimed.

Two independent sets of calculations agree fairly closely: one involving an expansion about a small parameter and the use of Padé approximants to find the limit of slowly converging or diverging series (Longuet-Higgins and Fenton, 1974), and another involving the solution of an integral equation (Byatt-Smith and Longuet-Higgins, 1976). These investigations indicate that there is a maximum in the wave speed and certain integral properties of solitary waves that occurs short of the wave of maximum amplitude. The maximum in the total energy of a solitary wave leads to important consequences in understanding how solitary waves break as they come up onto a beach (Longuet-Higgins and Fenton, 1974). Longuet-Higgins and Fenton (1974) also report that the amplitude-depth ratio of the highest wave in water is 0.827.

For the highest wave Witting (1975) identifies a singularity in an incomplete mathematical expansion of the fluid field that leads to an amplitude-depth ratio of $3\sqrt{3}/2\pi = 0.82699\ldots$, subject to the validity of the conjecture that the singularity retains its location in computation for the complete solution. The governing parameter distinguishing one wave from another is the falloff rate of amplitude at large distances from the crest. This conjecture must be rejected if the wave speed is a double-valued function of amplitude—the result found by Longuet-Higgins and Fenton (1974) and by Byatt-Smith and Longuet-Higgins (1976).

A series of numerical investigations not dissimilar from those described here also leads to limiting amplitudes of approximately 0.827 (Yamada, 1957; Yamada *et al.*, 1968; Sasaki and Murakami, 1973). Over a range of very high waves that does not include the highest, Sasaki and Murakami (1973) report wave speeds that are up to a percent or so lower than those reported by Longuet-Higgins and Fenton (1974) and Byatt-Smith and Longuet-Higgins (1976). In the NRL Report 8504, Witting and Bergin question the accuracy of these numerical investigations.

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Longuet-Higgins and Fox (1977) show that the highest wave in water, which has a maximum surface angle of inclination of 30°, does not possess the highest angle of inclination. Rather, the almost-highest wave has a maximum angle of inclination of 30.37°. Data presented by Sasaki and Murakami (1973) hint that the maximum angle of inclination is greater than 30°.

In NRL Report 8504, Witting and Bergin describe a refined numerical method which yields about 4-6 place accuracy of the properties of solitary waves of intermediate and high amplitude, up to the highest wave. This paper reports the results of the calculations based on the method of NRL Report 8504.

The results closely match those of Byatt-Smith and Longuet-Higgins (1976) and lead to maximum slopes near 30.37° by extrapolation. They show, however, that the highest wave has an amplitude of 0.8332, considerably above the range of values produced by other theories claiming high accuracy. This limiting amplitude of 0.8332 was reported (without substantiation) by Witting (1975). Details are given here. The calculations include the evaluation of solitary wave amplitudes at their flanks, information necessary to use solitary waves as the material for constructing accurate solutions for long periodic waves (Witting, 1981). The calculations also give profiles and other information about very high waves not previously available.

This report is organized as follows: Section 2 defines various solitary wave parameters and summarizes known relationships among them. Section 3 describes the process by which wave speeds are evaluated for high waves. Section 4 presents the overall results for the entire range of wave amplitudes. Section 5 gives details of the solution for the highest wave in water, and notes striking similarities in these details with the results of Yamada *et al.* (1968), with the exception of the wave speed. Section 6 provides the information necessary to construct the profiles of high and very high waves. It also discusses the limiting maximum surface angle of inclination. Finally, Section 7 summarizes the results and offers some conclusions.

2. SOLITARY WAVE PARAMETERS AND RELATIONSHIPS AMONG THEM

Following Longuet-Higgins and Fenton (1974), let us distinguish one solitary wave from another by the "strength" ω defined by:

$$\omega \equiv 1 - u_c^2/gh \tag{1}$$

where u_c is the speed of the fluid at the crest in wave coordinates, g is the acceleration of gravity, and h is the fluid depth away from the wave. Wave coordinates are those in which the wave profile is stationary.

The nondimensional wave speed, a Froude Number F, is defined by:

$$F^2 \equiv c^2/gh, \tag{2}$$

where c is the speed of the wave relative to still water (alternatively, c is the fluid speed at infinity in wave coordinates). The application of Bernoulli's law along the surface streamline gives the wave amplitude:

$$\alpha \equiv a/h = \frac{1}{2}(F^2 + \omega - 1). \tag{3}$$

At the flanks of the solitary wave the surface elevation tends to the constant h with an exponential decay, so that the elevation above the mean surface level $\eta \propto \exp(-\beta|x|/h)$. The falloff rate β can be found from:

$$\frac{\tan\beta}{\beta} = F^2. \tag{4}$$

Other parameters possessed by the flanks of the solitary wave are a "displacement" x_0 and an "amplitude parameter" B. These are defined by:

$$x_0 \equiv \lim_{\phi \to \infty} \left[\frac{x}{h} - \frac{\phi}{Q} \right] \tag{5}$$

and

$$B \equiv \lim_{|x| \to \infty} \frac{\eta}{h} e^{\beta |x|/h}, \tag{6}$$

where ϕ is the velocity potential, and Q is the volume flux per unit span. In Eqs. (5) and (6) $x = \phi \equiv 0$ at the crest.

Various integral properties of solitary waves have physical interest. Longuet-Higgins (1974) defines them and rederives certain relationships among them. These are the circulation C, the potential energy V, the mass M, the impulse I, the kinetic energy T, and the total energy E. These are defined and some relationships between them are made explicit:

$$C \equiv \frac{1}{g^{1/2} h^{3/2}} \int_{-\infty}^{\infty} \mathbf{u} \cdot ds = 2x_0 F, \tag{7}$$

$$V \equiv \frac{1}{gh^3} \int_{-\infty}^{\infty} \frac{1}{2} g\eta^2 \, dx, \tag{8}$$

$$M \equiv \frac{1}{h^2} \int_{-\infty}^{\infty} \eta dx = 3V/(F^2 - 1), \tag{9}$$

$$I \equiv \frac{1}{g^{1/2} h^{5/2}} \int_{-\infty}^{\infty} \int_0^{h+\eta} u dy dx = FM, \tag{10}$$

$$T \equiv \frac{1}{gh^3} \int_{-\infty}^{\infty} \int_0^{h+\eta} \frac{1}{2} (u^2 + v^2) \, dy dx = \frac{1}{2} F(I - C), \tag{11}$$

and

$$E \equiv T + V. \tag{12}$$

The order in which the integral relationships are written is the order in which they are computed. Note that values of F, x_0, and V are sufficient to determine all of the integral properties. Of course, any three of them are sufficient to determine all of them; the choice of F^2, x_0, and V is made because the numerical techniques developed in NRL Report 8504 determine these three more precisely than any other trio of parameters.

3. THE LIMITING PROCESS FOR HIGH WAVES

As described in NRL Report 8504, calculations are performed at a variety of resolutions, specified by N. There are $N + 1$ grid points at which the free pressure boundary condition is satisfied. For the material presented in this report, $N = 90$, $N = 180$, $N = 360$, $N = 720$, and $N = 1440$ specifies the resolution. Figure 1 shows the results for the calculation of F^2 at the various resolutions. The solid

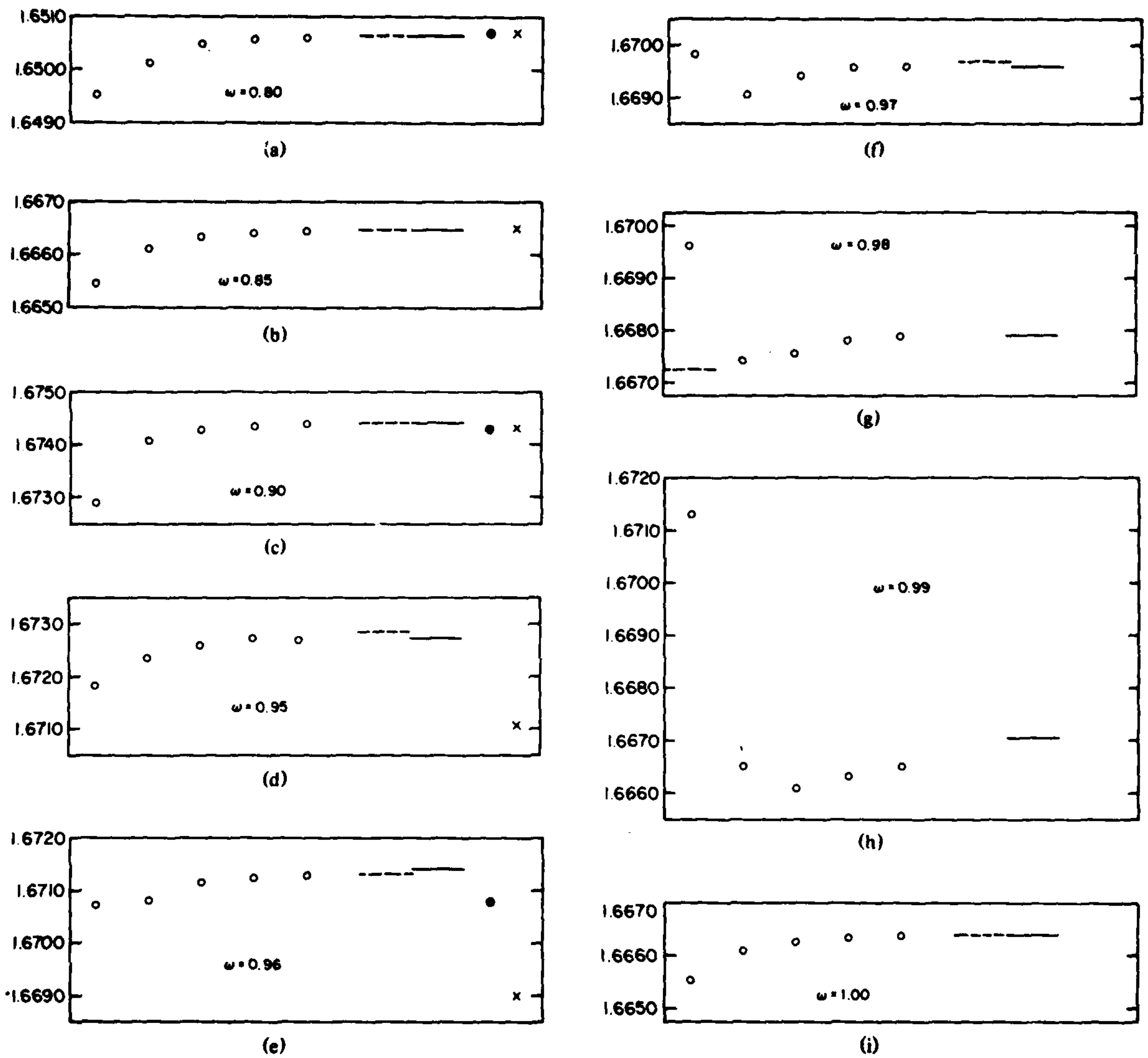

Fig. 1 — The limiting process in the computation of the square of the wave speed, F^2. The open circles are the values found at the resolutions specified by N = 90, 180, 360, 720, and 1440, resolution increasing toward the right. The dashed line is the extrapolation based on the points at N = 180, 360, and 720. The solid line is the extrapolation based on the points at N = 360, 720, and 1440. Each solid circle is the estimate of Byatt-Smith and Longuet-Higgins (1976). Each cross is the estimate of Longuet-Higgins and Fenton (1974). Strengths are: (a) 0.80, (b) 0.85, (c) 0.90, (d) 0.95, (e) 0.96, (f) 0.97, (g) 0.98, (h) 0.99, (i) 1.00.

line is an extrapolation to the estimated limit $N \to \infty$ using the $N = 360$, 720, and 1440 data. The dashed line is an extrapolation to the limit using the $N = 180$, 360, and 720 data. The cases $\omega = 0.80$, 0.85, 0.90, and 1.00 are the best behaved. This behavior is demonstrated by four features: There is an orderly approach to a limit that is obvious to the eye, even including the $N = 90$ estimate. The solid line and the dashed line are aligned, indicating that the two estimated extrapolations are almost identical. The estimates from $N = 1440$ alone are good ones, lying within 0.0001 of the extrapolations. The extrapolations are close to the independent calculations of Longuet-Higgins and Fenton (1974) and with Byatt-Smith and Longuet-Higgins (1976) for $\omega = 0.80$, 0.85, and 0.90.

For the strengths $\omega = 0.95$, 0.96, and 0.97 the extrapolations differ a little from each other, but the data at least for $N \geqslant 180$ appear satisfactory. At $\omega = 0.98$ the extrapolation based on $N = 180$, 360, and 720 is smaller than any of the three data points (this occurs when the curvature is positive). I reject any such estimate. For $\omega = 0.99$ I do not even display an estimate using the $N = 180$, 360, 720 data.

The data shown in Fig. 1 indicate that the estimates of F^2 become less reliable as the strength increases to $\omega = 0.99$, and then hop back to "reliable" for the highest wave $\omega = 1$. These results are consistent with the interpretation that for high waves the singularity at S is the most serious factor limiting accuracy (see NRL Report 8504). This singularity approaches the domain of the computations as $\omega \to 1$. We account for the singularity well at $\omega = 1$, because we know its nature there and account for it explicitly; we do not account for the singularity so well for other high waves.

The behavior displayed by Fig. 1 is characteristic of all of the calculated parameters. The calculations lose some accuracy as the highest wave is approached, but generally revert to very high accuracy for the highest wave.

4. OVERALL RESULTS

Tables 1 to 4 present the overall results for all cases run and are sufficient to determine all of the solitary wave properties appearing on the left side of Eqs. (1 to 12). When the data are well-behaved, as described in the last section, there will be entries in each of columns 2, 3, and 4 of the tables. The extrapolation to $N \to \infty$ using the data from $N = 360$, 720, and 1440 is taken as the best estimate. This is shown in column 2 of each table. Two other estimates come from the $N = 1440$ computation (column 3) and from the extrapolation using the data from $N = 180$, 360, and 720 (column 4). The error estimate in column 5 of each table is quoted as the greater of two comparisons: the difference between the two extrapolations, and one-half the difference between the $N = 360$, 720, and 1440 extrapolation and the $N = 1440$ value itself.

As shown in Fig. 1 for $\omega = 0.98$ and 0.99, the $N = 180$, 360, 720 extrapolation cannot be used. Under these circumstances the entry in column 4 of Tables (1 to 4) is a question mark. If the $N = 360$, 720, 1440 extrapolation is useful, it is quoted, but an error is not. Sometimes the $N = 360$, 720, 1440 extrapolation is also unlikely to be reliable, as in Table 2, $\omega = 0.99$. Columns 2 and 3 receive a question mark. Only the $N = 1440$ value is used as an estimate; the error quoted is the difference between the $N = 720$ and $N = 1440$ values. Although the criteria used to estimate errors are not uniform over the data set, they are as fair as I can manage. For solitary wave strengths that are not too small, the results appear to be very accurate. This includes, in particular, those for the highest wave.

Table 5 lists the estimates of the value of velocity potential at which the surface angle of inclination is a maximum and the angle itself.

Table 1—Calculations of the Square of the Wave Speed, F^2. These are the most direct calculations, based on the iterative procedure described in NRL Report 8504 for each resolution N. In this and the four following tables a dash indicates essentially no error. The text describes the meaning of a question mark.

ω	1440, 720, 360 triad	Departure, 1440 run	Departure, 720, 360, 180 triad	Estimate of F^2	Longuet-Higgins and Fenton (1974)	Byatt-Smith and Longuet-Higgins (1976)
0.10	1.09934	0.00800	0.00029	1.09934 ± 0.00400	1.09891	
0.20	1.19538	0.00244	0.00002	1.19538 ± 0.00122	1.19537	
0.30	1.28846	0.00068	0.00002	1.28846 ± 0.00034	1.28845	
0.40	1.37741	0.00024	0.00001	1.37741 ± 0.00012	1.37741	
0.45	1.41990	0.00016	0.00001	1.41990 ± 0.00008	1.41991	
0.50	1.46080	0.00012	0.00001	1.46080 ± 0.00006	1.46081	
0.55	1.49981	0.00009	–	1.49981 ± 0.00005	1.49982	
0.60	1.53659	0.00006	–	1.53659 ± 0.00003	1.53658	
0.65	1,57071	0.00004	–	1.57071 ± 0.00002	1.57074	
0.70	1.60162	0.00004	–	1.60162 ± 0.00002	1.60167	
0.75	1.62858	0.00003	–	1.62858 ± 0.00002	1.62861	
0.80	1.65063	0.00003	–	1.65063 ± 0.00002	1.6507	1.6507
0.85	1.66646	0.00002	–	1.66646 ± 0.00001	1.6664	
0.90	1.67441	0.00002	–	1.67441 ± 0.00001	1.6742	1.67428
0.95	1.67273	0.00001	0.00001	1.67273 ± 0.00001	1.6711	
0.96	1.67141	0.00014	0.00009	1.67141 ± 0.00009		1.6708
0.97	1.66960	–	0.00008	1.66960 ± 0.00004		
0.98	1.66791	0.00003	?	1.668		
0.99	1.66705	0.00053		1.667		
1.00	1.66640	0.00003	–	1.66640 ± 0.00002	1.654	

Table 2—Calculations of the Potential Energy, V. This parameter is computed by numerical integration over the wave. Integration is stopped at $\sigma = 177°$, and is extended to infinity by analytic integration using the exponential decay of the amplitude and the value of η/h at $\sigma = 177°$.

ω	1440, 720, 360 triad	Departure, 1440 run	Departure 720, 360, 180 triad	Estimate of V	Longuet-Higgins and Fenton (1974)	Byatt-Smith and Longuet-Higgins (1976)
0.10	0.02470	0.00403	0.00131	0.02470 ± 0.00202	0.02484	
0.20	0.07069	0.00135	0.00030	0.07069 ± 0.00068	0.07100	
0.30	0.13025	0.00051	0.00011	0.13025 ± 0.00026	0.13041	
0.40	0.19824	0.00023	0.00005	0.19824 ± 0.00012	0.19833	
0.45	0.23393	0.00016	0.00004	0.23393 ± 0.00008	0.23399	
0.50	0.26986	0.00012	0.00003	0.26986 ± 0.00006	0.26991	
0.55	0.30529	0.00009	0.00002	0.30529 ± 0.00005	0.30532	
0.60	0.33936	0.00006	0.00002	0.33936 ± 0.00003	0.33939	
0.65	0.37115	0.00004	0.00002	0.37115 ± 0.00002	0.37117	
0.70	0.39957	0.00005	0.00001	0.39957 ± 0.00003	0.39956	
0.75	0.42338	0.00004	0.00001	0.42338 ± 0.00002	0.42332	
0.80	0.44123	0.00003	0.00001	0.44123 ± 0.00002	0.4410	0.441
0.85	0.45162	0.00003	0.00001	0.45162 ± 0.00002	0.4510	
0.90	0.45313	0.00003	–	0.45313 ± 0.00002	0.4512	0.453
0.95	0.44546	–	0.00008	0.44546 ± 0.00008	0.4394	
0.96	0.44328	0.00008	0.00011	0.44328 ± 0.00011		0.443
0.97	0.44090	0.00001	0.00011	0.44090 ± 0.00011		
0.98	0.43894	0.00007	?	0.439		
0.99	?	?	?	0.43750 ± 0.00080		
1.00	0.43758	0.00019	0.00022	0.43758 ± 0.00022	0.413	

Table 3—Calculations of the Displacement, x_0. This parameter varies nearly linearly with σ near $\sigma = 180°$. Each estimate of the parameter is a linear extrapolation of $x/h - \phi/Q$ from data at 176° and 178°.

ω	1440, 720, 360 triad	Departure, 1440 run	Departure 720, 360, 180 triad	Estimate of V	Longuet-Higgins and Fenton (1974)	Byatt-Smith and Longuet-Higgins (1976)
0.10	0.3682	0.0512	0.0531	0.3682 ± 0.0531	0.3532	
0.20	0.4819	0.0080	0.0001	0.4819 ± 0.0040	0.4817	
0.30	0.5671	0.0026	0.0017	0.5671 ± 0.0017	0.5667	
0.40	0.6297	0.0026	0.0053	0.6297 ± 0.0053	0.62592	
0.45	?	?	?	0.6475 ± 0.0007	0.64799	
0.50	?	?	?	0.6655 ± 0.0002	0.66583	
0.55	?	?	?	0.6794 ± 0.0002	0.67961	
0.60	?	?	?	0.6894 ± 0.0001	0.68962	
0.65	?	?	?	0.6958 ± 0.0001	0.69593	
0.70	?	?	?	0.6986 ± 0.0001	0.69866	
0.75	?	?	?	0.6980 ± 0.0001	0.69795	
0.80	?	?	?	0.6940 ± 0.0001	0.6935	0.694
0.85	?	?	?	0.6870 ± 0.0001	0.6860	
0.90	?	?	?	0.6774 ± 0.0001	0.6747	0.677
0.95	0.6674	—	0.0001	0.6674 ± 0.0001	0.6603	
0.96	?	?	?	0.6659 ± 0.0002		0.666
0.97	0.6646	—	0.0001	0.6646 ± 0.0001		
0.98	?	?	?	0.6638 ± 0.0004		
0.99	0.6645	0.0009	?	0.664		
1.00	0.6641	0.0002	0.0001	0.6641 ± 0.0001	0.643	

Table 4—Calculations of the Amplitude at the Tail, B. This parameter varies nearly linearly with σ near $\sigma = 180°$. Each estimate of B is a linear extrapolation from data at 176° and 178°.

ω	1440, 720, 360 triad	Departure, 1440 run	Departure, 720, 360, 180 triad	Estimate of B
0.10	0.426	0.203	0.062	0.426 ± 0.102
0.20	0.670	0.079	0.012	0.670 ± 0.040
0.30	0.940	0.051	0.005	0.940 ± 0.026
0.40	1.153	0.030	0.002	1.153 ± 0.015
0.45	1.242	0.025	0.002	1.242 ± 0.013
0.50	1.317	0.022	0.001	1.317 ± 0.011
0.55	1.378	0.018	0.001	1.378 ± 0.009
0.60	1.425	0.016	0.002	1.425 ± 0.008
0.65	1.457	0.016	0.001	1.457 ± 0.008
0.70	1.475	0.012	0.001	1.475 ± 0.006
0.75	1.478	0.011	0.001	1.478 ± 0.006
0.80	1.466	0.009	0.001	1.466 ± 0.005
0.85	1.441	0.009	—	1.441 ± 0.005
0.90	1.405	0.017	—	1.405 ± 0.009
0.95	1.356	—	0.009	1.356 ± 0.009
0.96	1.355	0.003	0.003	1.355 ± 0.003
0.97	1.350	0.003	0.001	1.350 ± 0.002
0.98	1.347	0.003	0.016	1.347 ± 0.016
0.99	1.352	0.019	?	1.35
1.00	1.349	0.004	0.021	1.349 ± 0.021

Table 5—Calculations of the Maximum Angle of Inclination, θ_m, and the Velocity Potential at the Maximum Angle of Inclination, ϕ_m.

ω	ϕ_m/Q	1440, 720, 360 triad	Departure, 1440 run	Departure 720, 360, 180 triad	Estimate of θ_m	Byatt-Smith and Longuet-Higgins (1976)
0.10	2.2	1.157°	0.028°	?	1.157°± 0.014°	1.15°
0.20	1.46	3.150	0.010	—	3.150 ± 0.005	3.15
0.30	1.08	5.590	0.004	0.001	5.590 ± 0.002	5.59
0.40	0.84	8.341	0.001	—	8.341 ± 0.001	8.35
0.45	0.740	9.810	0.001	—	9.810 ± 0.001	
0.50	0.643	11.335	0.001	—	11.335 ± 0.001	11.34
0.55	0.574	12.911	0.001	—	12.911 ± 0.001	
0.60	0.501	14.537	0.001	—	14.537 ± 0.001	14.54
0.65	0.432	16.213	0.001	—	16.213 ± 0.001	
0.70	0.367	17.942	—	—	17.942 ± —	17.95
0.75	0.304	19.726	0.001	0.001	19.726 ± 0.001	
0.80	0.243	21.577	—	—	21.577 ± —	21.60
0.85	0.182	23.508	0.002	0.002	23.508 ± 0.002	
0.90	0.121	25.557	0.001	—	25.557 ± 0.001	25.6
0.95	0.059	27.792	—	—	27.792 ± —	
1.00	0.000				30.000,exactly	30.00, exactly

Figure 2 plots the various integral properties of solitary waves that can be derived from the knowledge of F^2, V, and x_0, using Eqs. (7, 9, 10, 11, and 12). The estimated accuracy of each of the solid circles is much smaller than the size of the circle. For $\omega = 0.98$ and 0.99 there is a little more uncertainty; error bars indicate this uncertainty, based on the uncertainties cited in Tables 1 to 3. The results agree perfectly with those of Byatt-Smith and Longuet-Higgins (1976) where comparisons can be made, i.e., through $\omega = 0.96$. The data fall somewhat above the results of Longuet-Higgins and Fenton (1974) for the very high and highest waves. The feature that the integral properties have a maximum at less than the highest wave is verified by the calculations. That the total energy has such a maximum is particularly important in the fate of waves propagating onto a beach (see Longuet-Higgins and Fenton, 1974).

5. SPECIFICS FOR THE HIGHEST WAVE IN WATER

The previous sections demonstrate a close agreement with results of the completely independent theory of Byatt-Smith and Longuet-Higgins (1976) where they can be compared. No problem is apparent for the calculations of the highest wave, at least by comparison with those of almost highest waves; indeed, the highest wave should be very accurately computed (Fig. 1). It thus makes sense to present detailed results for the highest wave, with confidence that they are probably correct. Table 6 shows these detailed results. The velocity potential is exact. The angle of inclination θ and the fluid speed q are the quantities most directly arising from the calculations.

Yamada *et al.* (1968) present much of the same information. Figure 3 compares the results of the calculations given here with their results for θ and q. Over almost the entire range of σ, the angles of inclination are within 10^{-4} of each other. Also over almost the entire range of σ, the speeds bear the constant ratio of the 2/3 power of the reported wave Froude Numbers (there F is 1.2854; here F is 1.2909). Even though the Froude Numbers differ in the third decimal place, there is an essential sameness about θ and q that merits some discussion.

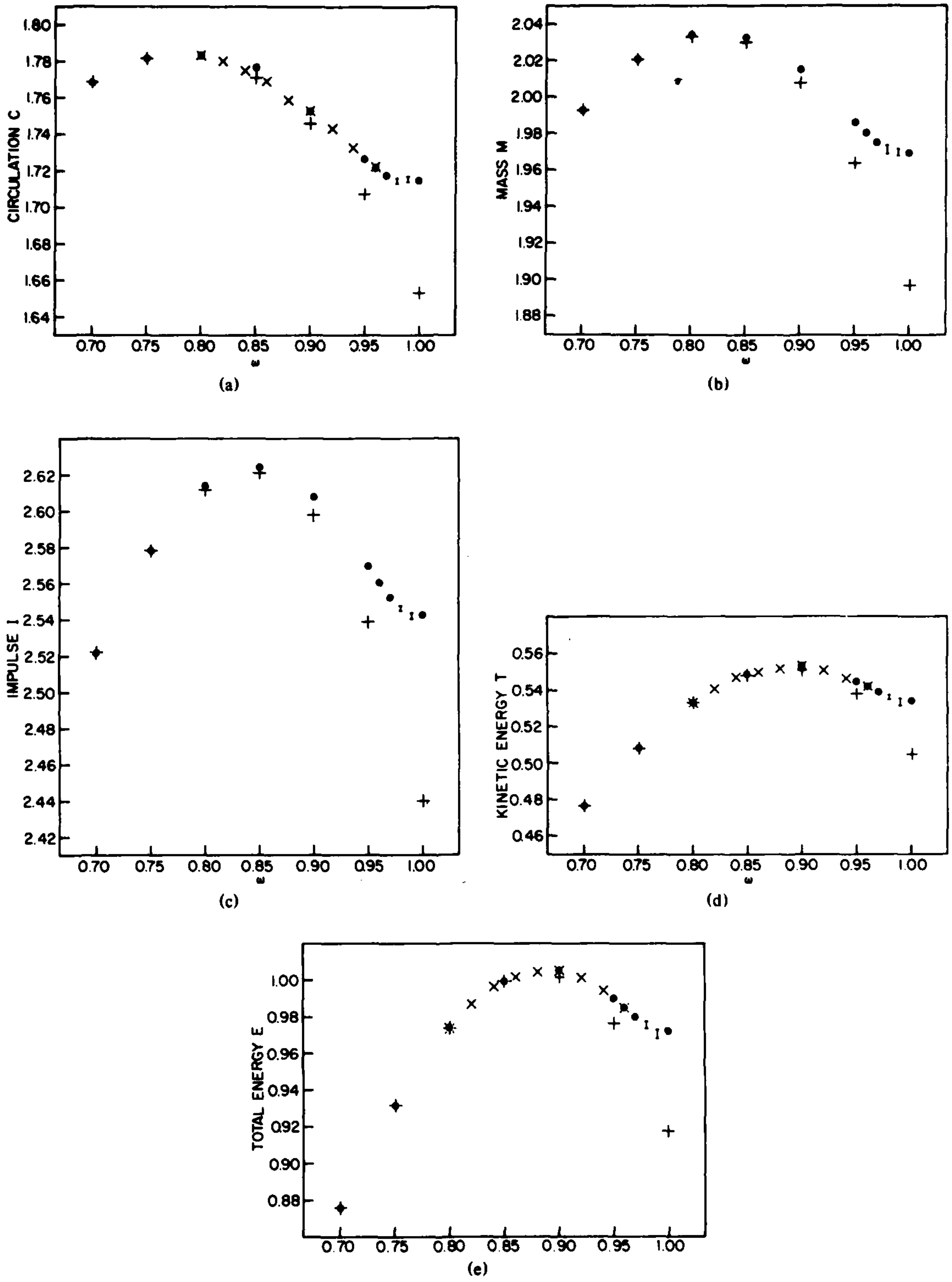

Fig. 2 — Integral properties of high solitary waves. These are (a) the circulation C; (b) the mass M; (c) the impulse I; (d) the kinetic energy T; (e) the total energy E. In each plot, solid points and errors bars are the results of this report, the *x*'s are those of Byatt-Smith and Longuet-Higgins (1976), and the +'s are those of Longuet-Higgins and Fenton (1974).

Table 6—Results of the Calculation of the Highest Solitary Wave in Water. The resolution is N = 1440; no extrapolations have been taken.

σ	ϕ/Q	θ	$q/(Q/h)$	x/h	y/h
0°	–	0.52360	–	–	1.83319
2	0.01111	0.51433	0.21482	0.06725	1.79474
4	0.02223	0.50744	0.27007	0.10706	1.77242
6	0.03335	0.50124	0.30855	0.14067	1.75387
8	0.04448	0.49544	0.33898	0.17084	1.73745
10	0.05563	0.48993	0.36454	0.19875	1.72247
12	0.06679	0.48464	0.38676	0.22499	1.70856
14	0.07797	0.47952	0.40653	0.24997	1.69549
16	0.08917	0.47454	0.42441	0.27393	1.68311
18	0.10041	0.46968	0.44080	0.29705	1.67130
20	0.11168	0.46492	0.45595	0.31948	1.65998
22	0.12298	0.46024	0.47006	0.34132	1.64909
24	0.13432	0.45565	0.48329	0.36266	1.63858
26	0.14570	0.45112	0.49578	0.38356	1.62839
28	0.15713	0.44666	0.50760	0.40408	1.61851
30	0.16860	0.44224	0.51884	0.42427	1.60890
32	0.18013	0.43788	0.52957	0.44416	1.59953
34	0.19172	0.43355	0.53984	0.46381	1.59038
36	0.20337	0.42927	0.54968	0.48324	1.58144
38	0.21509	0.42501	0.55917	0.50247	1.57268
40	0.22688	0.42079	0.56830	0.52154	1.56410
42	0.23874	0.41660	0.57712	0.54046	1.55568
44	0.25068	0.41242	0.58566	0.55926	1.54741
46	0.26271	0.40827	0.59394	0.57796	1.53927
48	0.27483	0.40414	0.60197	0.59657	1.53127
50	0.28704	0.40002	0.60978	0.61512	1.52338
52	0.29935	0.39591	0.61738	0.63362	1.51561
54	0.31176	0.39181	0.62480	0.65208	1.50793
56	0.32429	0.38773	0.63203	0.67051	1.50036
58	0.33693	0.38364	0.63909	0.68895	1.49288
60	0.34970	0.37957	0.64601	0.70738	1.48548
62	0.36259	0.37549	0.65277	0.72584	1.47816
64	0.37563	0.37141	0.65940	0 74434	1.47091
66	0.38880	0.36734	0.66590	0.76288	1.46373
68	0.40213	0.36326	0.67228	0.78148	1.45662
70	0.41561	0.35917	0.67855	0.80015	1.44956
72	0.42926	0.35507	0.68472	0.81891	1.44256
74	0.44308	0.35097	0.69077	0.83777	1.43562
76	0.45709	0.34685	0.69675	0.85674	1.42871
78	0.47128	0.34273	0.70262	0.87584	1.42186
80	0.48568	0.33858	0.70843	0.89508	1.41504
82	0.50030	0.33442	0.71415	0.91447	1.40825
84	0.51513	0.33025	0.71980	0.93403	1.40150
86	0.53020	0.32605	0.72538	0.95377	1.39478
88	0.54552	0.32183	0.73091	0.97371	1.38808
90	0.56110	0.31758	0.73636	0.99387	1.38141

Table 6 (Continued) — Results of the Calculation of the Highest Solitary Wave in Water. The resolution is N = 1440; no extrapolations have been taken.

σ	ϕ/Q	θ	$q/(Q/h)$	x/h	y/h
92	0.57695	0.31331	0.74176	1.01426	1.37476
94	0.59310	0.30901	0.74711	1.03491	1.36812
96	0.60954	0.30469	0.75242	1.05582	1.36149
98	0.62631	0.30032	0.75768	1.07702	1.35487
100	0.64342	0.29593	0.76290	1.09853	1.34826
102	0.66089	0.29149	0.76807	1.12037	1.34166
104	0.67874	0.28702	0.77322	1.14257	1.33505
106	0.69699	0.28250	0.77833	1.16515	1.32844
108	0.71567	0.27794	0.78341	1.18814	1.32183
110	0.73481	0.27333	0.78848	1.21157	1.31520
112	0.75443	0.26867	0.79351	1.23547	1.30856
114	0.77456	0.26396	0.79853	1.25987	1.30190
116	0.79524	0.25919	0.80353	1.28481	1.29523
118	0.81651	0.25436	0.80852	1.31032	1.28853
120	0.83840	0.24946	0.81350	1.33647	1.28180
122	0.86097	0.24450	0.81847	1.36328	1.27504
124	0.88426	0.23946	0.82344	1.39082	1.26824
126	0.90832	0.23435	0.82841	1.41915	1.26140
128	0.93323	0.22915	0.83338	1.44832	1.25452
130	0.95904	0.22387	0.83836	1.47841	1.24758
132	0.98584	0.21850	0.84335	1.50950	1.24059
134	1.01370	0.21302	0.84835	1.54169	1.23354
136	1.04274	0.20744	0.85338	1.57506	1.22642
138	1.07307	0.20175	0.85842	1.60975	1.21922
140	1.10480	0.19593	0.86350	1.64588	1.21194
142	1.13809	0.18999	0.86860	1.68361	1.20457
144	1.17312	0.18390	0.87376	1.72311	1.19709
146	1.21008	0.17766	0.87895	1.76460	1.18951
148	1.24921	0.17126	0.88419	1.80832	1.18181
150	1.29080	0.16467	0.88949	1.85456	1.17397
152	1.33520	0.15789	0.89487	1.90367	1.16598
154	1.38282	0.15090	0.90032	1.95609	1.15782
156	1.43418	0.14366	0.90587	2.01234	1.14948
158	1.48995	0.13616	0.91152	2.07311	1.14092
160	1.55096	0.12836	0.91729	2.13926	1.13212
162	1.61835	0.12022	0.92321	2.21191	1.12305
164	1.69360	0.11169	0.92930	2.29261	1.11365
166	1.77886	0.10270	0.93559	2.38351	1.10387
168	1.87720	0.09317	0.94213	2.48775	1.09364
170	1.99345	0.08297	0.94899	2.61021	1.08283
172	2.13565	0.07193	0.95626	2.75902	1.07130
174	2.31891	0.05976	0.96409	2.94944	1.05877
176	2.57712	0.04592	0.97277	3.21565	1.04476
178	3.01844	0.02914	0.98300	3.66646	1.02809
180°	∞	—	1.00000	∞	1.00000

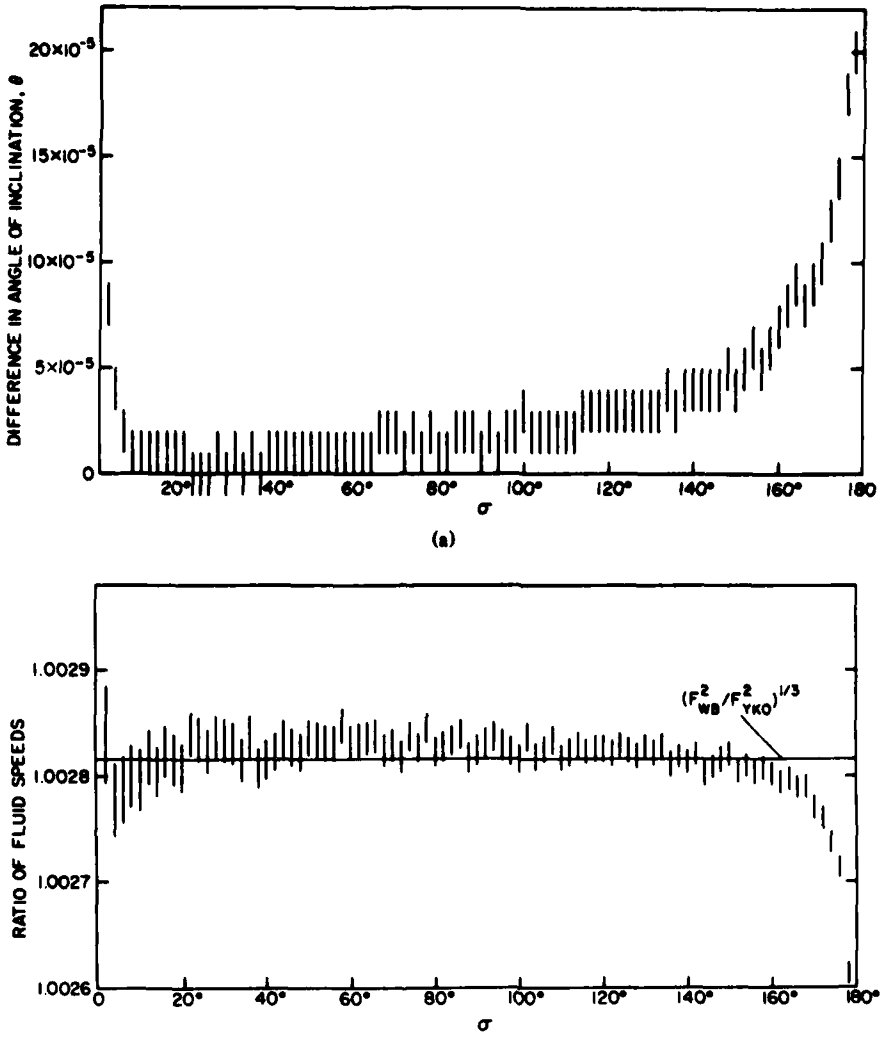

Fig. 3 — Comparison of the basic parameters of the highest solitary wave in water as computed here with those computed by Yamada *et al.* (1968): (a) θ from these calculations minus θ from Yamada *et al.*; (b) $q/(Q/h)$ from Yamada *et al.* divided by $q/(Q/h)$ from these calculations.

Grant (1973) has examined the nature of the singularity at the crest of the highest irrotational progressive waves. Let $z' \equiv x + iy'$ and $w' = \phi + i\psi'$ in some nondimensionalization. For the highest waves one can write, e.g., the function dz'/dw' is as follows:

$$dz'/dw' = \frac{a_0}{(iw')^{1/3}} [1 + a_{11} (iw')^{\mu_1} + a_{21} (iw')^{2\mu_1} + a_{31} (iw')^{3\mu_1} + \dots \qquad (13)$$

$$+ a_{12} (iw')^{\mu_2} + a_{22} (iw')^{\mu_2+\mu_1} + a_{32} (iw')^{\mu_2 + 2\mu_1} + \dots$$

$$+ \text{more}],$$

where w' vanishes at the crest. The coefficients a_{11}, a_{12}, a_{13}, a_{1n} are set to satisfy boundary conditions other than the free pressure boundary condition; all the other coefficients are determined by the free surface boundary condition. The first two exponents are: $\mu_1 = 0.80268$ and $\mu_2 = 2.9066$. In the non-dimensionalization used for both these calculations and those of Yamada *et al.* (1968), i.e., $h = u = 1$, Bernoulli's Law determines $a_0 = (2/3\ F^2)^{1/3}$. If the only difference between the calculations of Yamada *et al.* and ours lies in a difference in identifying F^2, consequently, a_0, then at least near the crest it is not surprising that the angles of inclination are the same, for these do not contain a_0, and that the ratios of q are in the ratios of a_0^{-1}. This is precisely what Fig. 3 shows, not only near the crest, but throughout the entire wave.

The expansion of Eq. (13) can be used to describe the solitary wave near the crest. Applying Bernoulli's Law along the free surface one can go through some cumbersome algebra to calculate the first few coefficients:

$$
\begin{aligned}
&a_{11} = \text{arbitrary} \\
&a_{21} = 0.29042\ a_{11}^2 \\
&a_{31} = -\ 0.15402\ a_{11}^3 \\
&a_{41} = 0.01030\ a_{11}^4 \qquad a_{12} = \text{arbitrary} \\
&a_{51} = 0.14979\ a_{11}^5 \qquad a_{22} = 0.78934\ a_{12}\, a_{21}.
\end{aligned} \tag{14}
$$

The data in Table 6 out to at least $\sigma = 10°$ (more for some entrees) are consistent to five decimal places with the assignment $a_{11} = 0.361$; $a_{12} = 0$. This gives confidence that the calculations, not only of θ and q, but also of x/h and y/h, are accurate. Moreover, we may conclude that a_{11} lies near 0.361, and that a_{12} is small.

6. PROFILES OF HIGH WAVES

Figure 4 shows the difference between the surface elevation of each point of a wave to that of the corresponding point of the highest wave. The display spreads the information sufficiently in elevation η/h that one can construct accurate profiles by using the figure in conjunction with Table 6. The pattern is consistent with that found by other investigators. The integral under the difference curve is positive for every case, indicating that all of the waves displayed have values of M greater than that of the highest wave (see Fig. 2b).

Figure 5 plots the maximum slope of the very highest waves, along with the values given by Sasaki and Murakami (1973). It is probably justifiable to plot these on the same figure, even though the wave speeds given by them are significantly smaller (at a given amplitude) than are calculated here. Their numerical technique is similar to that of Yamada *et al.* (1968), and we have seen that the computed angles in Yamada *et al.* are accurate.

Unfortunately, the calculations with $\omega = 0.99$ are not too accurate, as shown by the error bar. The two curves drawn make use of the highest accurately computed maximum slopes. The data points alone suggest that the maximum slope exceeds 30°, the result proved by Longuet-Higgins and Fox (1977). Indeed, both the straight line and quadratic extrapolations intersect $\omega = 1.00$ very close to the value of 30.37° derived by Longuet-Higgins and Fox (1977).

7. SUMMARY AND CONCLUSIONS

The numerical techniques described in NRL Report 8504 are used to compute properties of solitary waves that are very accurate for intermediate and high waves, a little less accurate for very high waves, and appear again very accurate for the highest wave. Many results of previous work are verified by these calculations (some of them not for the first time). Specifically, they verify the following conclusions: There is a maximum in the integral properties of solitary waves, including the total energy. The maximum surface inclination as extrapolated from maximum slopes of very high waves does

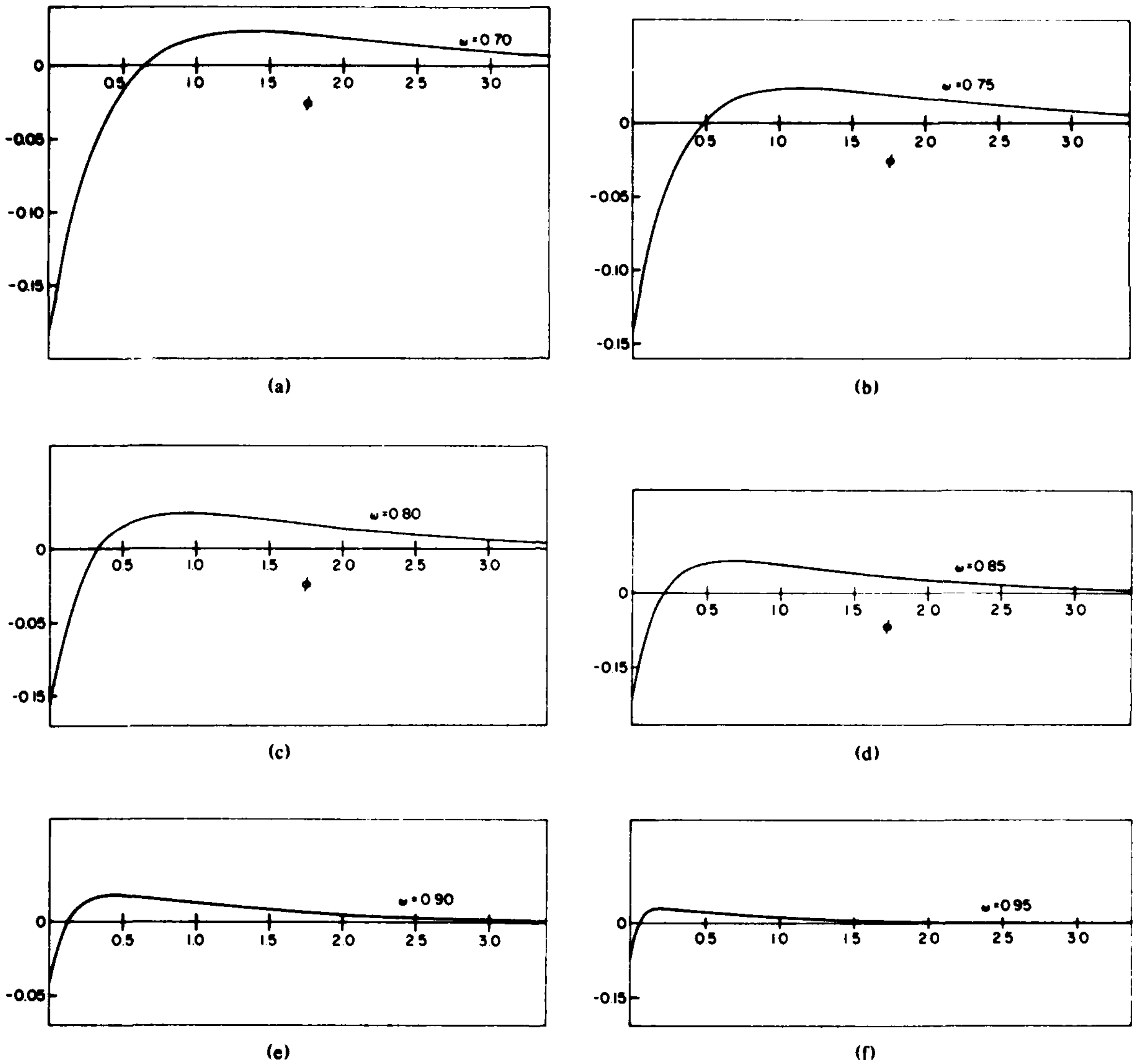

Fig. 4 — Profiles of high solitary waves. The plots are the difference $(\eta/h)_\omega$ minus $(\eta/h)_{1.00}$: (a) $\omega = 0.70$, (b) $\omega = 0.75$, (c) $\omega = 0.80$, (d) $\omega = 0.85$, (e) $\omega = 0.90$, (f) $\omega = 0.95$.

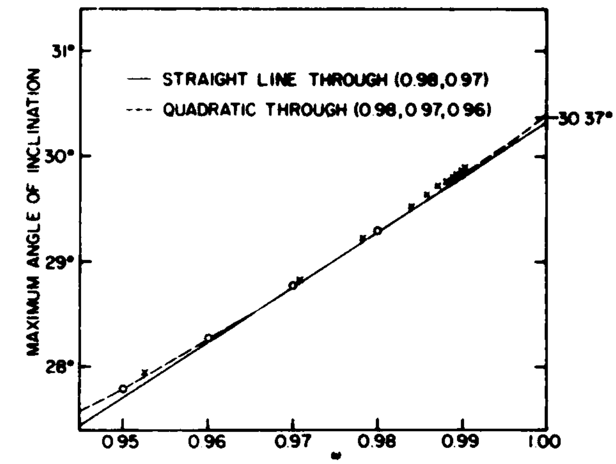

Fig. 5 — The maximum surface angle of inclination for very high solitary waves. The open circles and error bar are from these calculations. The *x*'s are from Sasaki and Murakami (1973). The value of 30.37° is derived by Longuet-Higgins and Fox (1977). The solid curve is a straight line matching the open circles at $\omega = 0.98$ and $\omega = 0.97$. The dashed curve is a quadratic fit to the open circles at $\omega = 0.98$, 0.97, and 0.96.

exceed 30° and lies at or near 30.37°. The calculations of Byatt-Smith and Longuet-Higgins (1976) are accurate to their stated precision, possibly excepting their highest reported wave speeds, which might be a bit low. The calculations of Yamada *et al.* (1968) bear a very close resemblence in the structure of θ and q, differences being dominated by a different assignment of the wave speed and amplitude.

Certain new parameters are calculated for high waves. In particular, knowledge of B permits going beyond the zeroth order of solutions of the properties of long periodic water waves from corresponding solitary wave properties described by Stiassnie and Peregrine (1980), to the first order solutions described by Witting (1981), which see for a definition of the ordering. In addition, enough information is presented to plot accurate profiles of very high waves.

The calculations of the speed and amplitude of the highest wave do not agree well with those of previous work. Figure 6 shows this graphically. Only with Byatt-Smith and Longuet-Higgins (1976) is there exceptionally close agreement, and the comparison there can be made only through $\omega = 0.96$. All other reported highest waves shown lie in the vicinity of $F^2 = 1.654$; a/h = 0.827. I believe that criticisms of each of the other calculations can be made. The other numerical investigations (Yamada, 1957; Yamada *et al.* 1968; Sasaki and Murakami 1973) employ a much coarser resolution than that used here. They also rely to some extent on the use of the surface pressure as an indicator of how error-free are the results. Finally, there may be problems in handling their singularity at $\sigma = 180°$. These questions are addressed in NRL Report 8504, and from that discussion it should be evident that uncertainties in these other numerical calculations leave them suspect. The work of Longuet-Higgins and Fenton (1974) uses Padé approximants to estimate the limit of slowly-converging or diverging series. As Byatt-Smith and Longuet-Higgins (1976) show, for the high waves where a comparison with their integral method can be made, the Padé approximant method can give results for which the error is a few in the last place cited. Longuet-Higgins and Fenton (1974) cite only three decimal places for their limiting amplitude, which is about 0.006 away from that given here. Finally, the "exact" value given by Witting (1975) rests on a conjecture that a mathematical singularity identified by an incomplete solution is in the same location as in the complete solution. This is impossible if the wave speed is a double-valued function of amplitude, which we know from numerous studies to be true. Consequently, the result of Witting (1975) must be rejected.

There are clear uncertainties associated with the detailed accuracies in previous reports of precise values of the height and speed of highest wave in water. I ascribe their mutual agreement to coincidence. The calculations reported here suggest that the correct value of the amplitude of the highest wave in water is 0.8332.

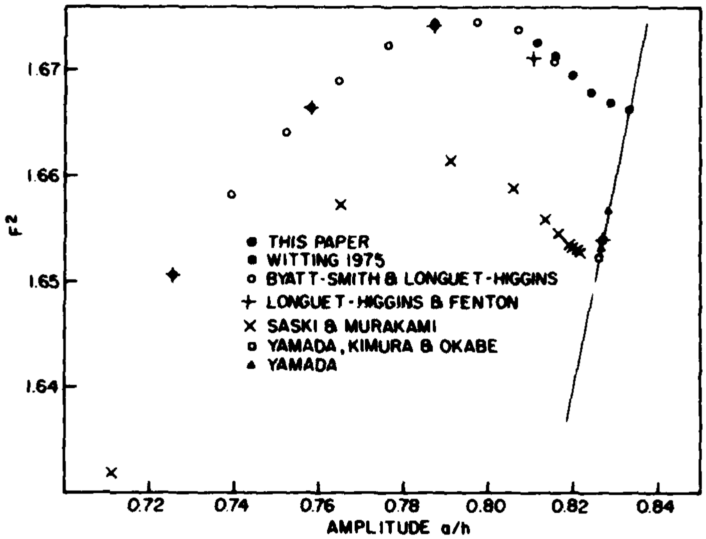

Fig. 6 — The relationship between wave speed and amplitude for high solitary waves. Solid circles are the results of the calculations given here. The open circles are from Byatt-Smith and Longuet-Higgins (1976). The +'s are from Longuet-Higgins and Fenton (1974). The *x*'s are from Sasaki and Murakami (1973). The two open triangles indicate two estimates given by Yamada (1957). The open square is from Yamada *et al.* (1968). The solid square is from Witting (1975).

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