Higher Chain Formula proved by Combinatorics

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Abstract

We present an elementary combinatorial proof of a formula to express the higher partial derivatives of composite functions in terms of those of factor functions.

1. **Introduction**. Consider $h : x \in X \subset \mathbb{R}^\nu \xrightarrow{f} y \in Y \subset \mathbb{R}^\mu \xrightarrow{g} z \in \mathbb{R}$ where X, Y are open subsets of $\mathbb{R}^\nu, \mathbb{R}^\mu$ respectively and f, g are sufficiently smooth functions. Lots of work have been done to express the higher partial derivatives of the composite function $h = gf$ in terms of those of f, g. Among many important contributions not included in our references, see [1], [2], [3] and [4] for $\mu = \nu = 1$. See [5] for $\nu = 1$ and any μ. See [6] and [7] for $\mu = \nu = 2$. Finally see [8], [9] [10], [11]and [12] for the general case. Here we propose a concise version in §6 with a combinatorial proof modified from [13] extending μ from their one dimension to our many dimensions. This paper is self-contained and should be readable by broad audiences including students studying routine operations of partial differentiation. In addition to many other applications, the corollary of this strategically important formula is required to investigate holomorphic functions on test spaces under coordinate transformations.

2. It is more intuitive to work with variables $x = (x_1, \cdots, x_\nu)$ and $y = (y_1, \cdots, y_\mu)$. For each k in the set J_n of integers $1, 2, \cdots, n$, let t_k denote one of the independent variables $x_1, \cdots, x_\nu$. A *partition* of J_n is a family of pairwise disjoint nonempty subsets of J_n whose union is J_n. Sets in a partition are called *blocks*. A *block function* is to assign a label to each block of a partition. The set of all functions from a partition P of J_n into J_μ is denoted by P_μ. The set of all partitions of J_n is denoted by $\mathbb{P}_n$.

3. **Lemma**.

$$\frac{\partial^n z}{\partial t_1 \cdots \partial t_n} = \sum_{P \in \mathbb{P}_n} \sum_{\lambda \in P_\mu} \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial y_{\lambda(B)}} \right) z \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) y_{\lambda(B)} \right] \right\}.$$

*Dedicated to Galileo Galilei and Giordano Bruno

In fact, it is obviously true for $n = 1$. Inductively, differentiation with respect to t_{n+1} produces terms that are products of factors of the form

$$\left\{ \left(\prod_{B\in P} \frac{\partial}{\partial y_{\lambda(B)}} \right) z \right\} \left\{ \prod_{B\in P, B\neq A\in P} \left[\left(\prod_{b\in B} \frac{\partial}{\partial t_b} \right) y_{\lambda(B)} \right] \right\} \frac{\partial}{\partial t_{n+1}} \left(\prod_{a\in A} \frac{\partial}{\partial t_a} \right) y_{\lambda(A)} \qquad (3.1)$$

and

$$\left\{ \frac{\partial}{\partial t_{n+1}} \left(\prod_{B\in P} \frac{\partial}{\partial y_{\lambda(B)}} \right) z \right\} \left\{ \prod_{B\in P} \left[\left(\prod_{b\in B} \frac{\partial}{\partial t_b} \right) y_{\lambda(B)} \right] \right\}. \qquad (3.2)$$

Eq(3.1) corresponds to a partition of J_{n+1} obtained by adding $n+1$ to block A of P while all other blocks remain to be the same. We do not need to change any label of any block. For Eq(3.2), the family $Q = P \cup \{S_{n+1}\}$ is a partition of J_{n+1} where $S_{n+1} = \{n+1\}$. Define a function $\varphi_i : Q \to J_\mu$ by $\varphi_i(S_{n+1}) = i$ and $\varphi_i(B) = \lambda(B)$ for all $B \in P$. Then Eq(3.2) becomes

$$\begin{aligned} & \left\{ \sum_{i=1}^{\mu} \frac{\partial y_i}{\partial t_{n+1}} \frac{\partial}{\partial y_i} \left(\prod_{B\in P} \frac{\partial}{\partial y_{\lambda(B)}} \right) z \right\} \left\{ \prod_{B\in P} \left[\left(\prod_{b\in B} \frac{\partial}{\partial t_b} \right) y_{\lambda(B)} \right] \right\} \\ = & \left\{ \sum_{i=1}^{\mu} \frac{\partial}{\partial y_i} \left(\prod_{B\in P} \frac{\partial}{\partial y_{\lambda(B)}} \right) z \right\} \left\{ \frac{\partial y_i}{\partial t_{n+1}} \prod_{B\in P} \left[\left(\prod_{b\in B} \frac{\partial}{\partial t_b} \right) y_{\lambda(B)} \right] \right\} \\ = & \sum_{i=1}^{\mu} \left\{ \left(\prod_{A\in Q} \frac{\partial}{\partial y_{\varphi_i(A)}} \right) z \right\} \left\{ \prod_{A\in Q} \left[\left(\prod_{a\in A} \frac{\partial}{\partial t_a} \right) y_{\varphi_i(A)} \right] \right\}. \end{aligned}$$

Clearly this covers all cases for $Q \in \mathbb{P}_{n+1}$ and $\varphi \in Q_\mu$. Hence we finish the proof.

4. For every multi-index $\alpha = (\alpha_1, \cdots, \alpha_\nu) \in \mathbb{N}^\nu$ of *length* ν where $\mathbb{N}$ is the set of all nonnegative integers and for every $x \in \mathbb{R}^\nu$, let

$$|\alpha| = \sum_{j=1}^{\nu} \alpha_j, \quad \alpha! = \prod_{j=1}^{\nu} \alpha_j!, \quad x^\alpha = \prod_{j=1}^{\nu} x_j^{\alpha_j}, \quad \frac{\partial^{|\alpha|} z}{\partial x^\alpha} = \prod_{j=1}^{\nu} \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} z\,.$$

By convention, define $\theta^0 = 1$ regardless whether θ is a number or a differential operator. Suppose that $\{e_j : j \in J_\mu\}$ denotes the standard basis of $\mathbb{R}^\mu$. In particular each e_j is a multi-index of length μ.

5. A multi-index α in $\mathbb{R}^\nu$ is said to be *decomposed* into s *parts* $p_1, \cdots, p_s$ in $\mathbb{N}^\nu$ with *multiplicities* $m_1, \cdots, m_s$ in $\mathbb{N}^\mu$ respectively if the *decomposition equation*

$$\alpha = |m_1|\, p_1 + |m_2|\, p_2 + \cdots + |m_s|\, p_s$$

holds and all parts are different. Note that the parts p's and the multiplicities m's are multi-indexes of order ν, μ respectively. In this case, the *total multiplicity* is defined by

$$m = m_1 + m_2 + \cdots + m_s.$$

The list (s, p, m) is called a *μ-decomposition* or just a decomposition of α. One of many ways to ensure that the parts are all different is to define $\alpha \ll \beta$ if there is $j \in J_\nu$ such that $\alpha_1 = \beta_1, \cdots, \alpha_{j-1} = \beta_{j-1}$ but $\alpha_j < \beta_j$. We may demand the parts of a decomposition (s, p, m) to satisfy the additional condition $0 \ll p_1 \ll p_2 \ll \cdots \ll p_s$.

6. **Higher Chain Formula**.

$$\frac{\partial^{|\alpha|} z}{\partial x^\alpha} = \alpha! \sum_{(s,p,m)\in\mathscr{D}} \frac{\partial^{|m|} z}{\partial y^m} \prod_{k=1}^{s} \frac{1}{m_k!} \left[\frac{1}{p_k!} \frac{\partial^{|p_k|} y}{\partial x^{p_k}} \right]^{m_k}$$

where $\mathscr{D}$ is the set of all decompositions of α. Let $p_k = (p_{k1}, \cdots, p_{k\nu})$, $m_k = (m_{k1}, \cdots, m_{k\mu})$ and $r_i = m_{1i} + \cdots + m_{si}$. Clearly we have $m = (r_1, \cdots, r_\mu)$. The terms on the right hand side in long form become

$$\alpha! \frac{\partial^{r_1+\cdots+r_\mu} z}{\partial y_1^{r_1} \cdots \partial y_\mu^{r_\mu}} \prod_{k=1}^{s} \prod_{i=1}^{\mu} \frac{1}{m_{ki}!} \left[\frac{1}{p_{k1}! \cdots p_{k\nu}!} \frac{\partial^{p_{k1}+\cdots+p_{k\nu}} y_i}{\partial x^{p_{k1}} \cdots \partial x^{p_{k\nu}}} \right]^{m_{ki}}. \tag{6.1}$$

7. We shall prove this formula by interpretation rather than by formal argument. Let us examine the special case when $\nu = 4$, $\mu = 2$ and $\alpha = (39, 50, 9, 0)$. We have independent variables x_1, x_2, x_3, x_4 and dependent variables y_1, y_2. For

$$\begin{aligned} t_1 = \cdots = t_{39} &= x_1 \\ t_{40} = \cdots = t_{89} &= x_2 \\ t_{90} = \cdots = t_{98} &= x_3 \end{aligned}$$

the left hand side of the lemma becomes

$$\frac{\partial^{|\alpha|} z}{\partial x^\alpha} = \frac{\partial^{98} z}{\partial x_1^{39}\ \partial x_2^{50}\ \partial x_3^{9}} = \frac{\partial^{98} z}{\partial t_1 \cdots \partial t_{98}}.$$

8. Consider a partition P of J_{98} whose first 5 blocks are displayed by the following table:

A	B	C	D	E	F	G
1	2	e_2	1,2,3,4	x_1, x_1, x_1, x_1	40,41,42,43,44	x_2, x_2, x_2, x_2, x_2
2	1	e_1	5,6,7,8	x_1, x_1, x_1, x_1	45,46,47,48,49	x_2, x_2, x_2, x_2, x_2
3	1	e_1	9,10,11,12	x_1, x_1, x_1, x_1	50,51,52,63,54	x_2, x_2, x_2, x_2, x_2
4	2	e_2	13,14,15,16	x_1, x_1, x_1, x_1	55,56,57,58,59	x_2, x_2, x_2, x_2, x_2
5	2	e_2	17,18,19,20	x_1, x_1, x_1, x_1	60,61,62,63,64	x_2, x_2, x_2, x_2, x_2

Column-A is the serial number of the blocks. Column-B assigns an integer label to each block in order to define the block function λ, for example $\lambda_3 = 1$. Column-C uses the basis of $\mathbb{R}^\mu$ to label the blocks. Hence the first multiplicity is $m_1 = 2e_1 + 3e_2 = (2, 3)$. Column-D indicates the integers in each block that produce 4 x_1 in column-E. Column-F indicates the integers in each block that produce 5 x_2 in column-G. Since no x_3, x_4 are involved, the first part is $p_1 = (4,5,0,0)$. In this particular example, the other blocks of P and corresponding parts with multiplicities are

A2	B2	C2	D2	E2	F2	G2	H2	J2
7	1	e_1	21,22	x_1, x_1	65,66,67	x_2, x_2, x_2	90	x_3
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
13	2	e_2	33,34	x_1, x_1	83,84,85	x_2, x_2, x_2	96	x_3

and

A3	B3	C3	D3	E3	F3	G3	H3	J3
14	2	e_2	35,$\cdots$,39	$x_1, \cdots, x_1$	56, $\cdots$,59	$x_2, \cdots, x_2$	97,98	x_3, x_3

This partition can be compactly represented by the decomposition equation

$$(39,50,9,0) = |(2,3)|\,(4,5,0,0) + |(3,4)|\,(2,3,1,0) + |(0,1)|\,(5,4,2,0).$$

We set $m_2 = (3,4)$, $p_2 = (2,3,1,0)$, $m_3 = (0,1)$ and $p_3 = (5,4,2,0)$. The total multiplicity is $m = m_1 + m_2 + m_3 = (5,8)$.

9. The block function restricted to column-B offers the differential operator

$$\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_2} = \left(\frac{\partial}{\partial y_1}\right)^2 \left(\frac{\partial}{\partial y_2}\right)^3.$$

Three tables together give

$$\left(\prod_{B\in P}\frac{\partial}{\partial y_{\lambda(B)}}\right) z = \left(\frac{\partial}{\partial y_1}\right)^{2+3+0}\left(\frac{\partial}{\partial y_2}\right)^{3+4+1} z = \frac{\partial^{13} z}{\partial y_1^5 \partial x_2^8} = \frac{\partial^{|m|} z}{\partial y^m}.$$

Next for the first block in table-1, we obtain

$$\left(\prod_{b\in B}\frac{\partial}{\partial t_b}\right) y_{\lambda(B)} = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\right) y_2.$$

All rows in three tables together produce

$$\begin{aligned}
&\prod_{B\in P}\left[\left(\prod_{b\in B}\frac{\partial}{\partial x_b}\right) y_{\lambda(B)}\right] \\
= \;&\left(\frac{\partial^9 y_1}{\partial x_1^4 \partial x_2^5}\right)^2 \left(\frac{\partial^9 y_2}{\partial x_1^4 \partial x_2^5}\right)^3 \cdot \left(\frac{\partial^6 y_1}{\partial x_1^2 \partial x_2^3 \partial x_3}\right)^3 \left(\frac{\partial^6 y_2}{\partial x_1^2 \partial x_2^3 \partial x_3}\right)^4 \\
&\cdot \left(\frac{\partial^{17} y_1}{\partial x_1^5 \partial x_2^4 \partial x_3^2}\right)^0 \left(\frac{\partial^{17} y_2}{\partial x_1^5 \partial x_2^4 \partial x_3^2}\right)^1 \\
= \;&\left[\frac{\partial^{|p_1|} y}{\partial x^{p_1}}\right]^{m_1} \left[\frac{\partial^{|p_2|} y}{\partial x^{p_2}}\right]^{m_2} \left[\frac{\partial^{|p_3|} y}{\partial x^{p_3}}\right]^{m_3} = \prod_{k=1}^{s}\left[\frac{\partial^{|p_k|} y}{\partial x^{p_k}}\right]^{m_k}.
\end{aligned}$$

This is simplified to

$$\frac{\partial^{13} z}{\partial y_1^5 \partial x_2^8} \left(\frac{\partial^9 y_1}{\partial x_1^4 \partial x_2^5}\right)^2 \left(\frac{\partial^9 y_2}{\partial x_1^4 \partial x_2^5}\right)^3 \left(\frac{\partial^6 y_1}{\partial x_1^2 \partial x_2^3 \partial x_3}\right)^3 \left(\frac{\partial^6 y_2}{\partial x_1^2 \partial x_2^3 \partial x_3}\right)^4 \left(\frac{\partial^{17} y_2}{\partial x_1^5 \partial x_2^4 \partial x_3^2}\right). \quad (9.1)$$

10. The columns E, E2 and E3 remain to be the same if we permute the order of integers in each cell of columns D, D2 and D3. Hence the number of different ways to distribute the integers $1, 2, \cdots, 39$ into columns D, D2 and D3 is $39!/(4!^5\ 2!^7\ 5!)$. Taking all variables $x_1, \cdots, x_\mu$ into account, we have

$$\frac{39!}{4!^5\ 2!^7\ 5!}\ \frac{50!}{5^5\ 3!^7\ 4!}\ \frac{9!}{0!^5\ 1!^7\ 2!}.$$

If we permute the rows of each table, it does not alter the columns E, G, E2, G2, J2, E3, G3 and J3. Hence the total number of different ways to distribute all integers in J_{98} is

$$\left(\frac{39!}{4!^5\ 2!^7\ 5!}\ \frac{50!}{5^5\ 3!^7\ 4!}\ \frac{9!}{0!^5\ 1!^7\ 2!}\right)\left(\frac{1}{5!\ 7!\ 1!}\right) = \left(\prod_{j=1}^{\nu} \frac{\alpha_j!}{\prod_{k=1}^{s}(p_{kj}!)^{|m_k|}}\right) \frac{1}{\prod_{k=1}^{s}|m_k|\,!}$$

On the other hand if we interchange rows 2,3 without altering their serial numbers in table-1, we have different block functions with the same effect on the term (9.1). Hence the number of different ways to distribute the integers $1, 2, 3, 4, 5$ is $5!/(2!\,3!\,)$. Because we demand that the parts are all different, the total number of different block functions is

$$\frac{5!}{2!\,3!}\ \frac{7!}{3!\,4!}\ \frac{1!}{0!\,1!} = \prod_{k=1}^{s} \frac{|m_k|!}{\prod_{i=1}^{\mu} m_{ki}!}.$$

Putting everything together, we obtain

$$\begin{aligned}
&\left\{\left(\prod_{j=1}^{\nu} \frac{\alpha_j!}{\prod_{k=1}^{s}(p_{kj}!)^{|m_k|}}\right) \frac{1}{\prod_{k=1}^{s}|m_k|!}\right\} \left(\prod_{k=1}^{s} \frac{|m_k|!}{\prod_{i=1}^{\mu} m_{ki}!}\right) \\
= &\left(\alpha!\ \prod_{j=1}^{\nu}\prod_{k=1}^{s} \frac{1}{(p_{kj}!)^{m_{k1}+\cdots+m_{k\mu}}}\right)\left(\prod_{k=1}^{s}\prod_{i=1}^{\mu}\frac{1}{m_{ki}!}\right) \\
= &\ \alpha! \left(\prod_{j=1}^{\nu}\prod_{k=1}^{s}\prod_{i=1}^{\mu} \frac{1}{(p_{kj}!)^{m_{ki}}}\right)\left(\prod_{k=1}^{s}\prod_{i=1}^{\mu}\frac{1}{m_{ki}!}\right) \\
= &\ \alpha! \prod_{k=1}^{s}\prod_{i=1}^{\mu} \frac{1}{m_{ki}!\prod_{j=1}^{\nu}(p_{kj}!)^{m_{ki}}}
\end{aligned}$$

which is identical to the coefficient of the term (6.1). This completes the proof.

11. Consider an alternative illustrative example to find all possible decompositions. Suppose $\mu = 2$, $\nu = 3$ and $\alpha = (3, 7, 5)$. The maximum among $\alpha_1, \alpha_2, \alpha_3$ is 7. Solve for nonnegative integers α_{ij} from the traditional equations

$$\alpha_i = \alpha_{i1} + 2\alpha_{i2} + 3\alpha_{i3} + 4\alpha_{i4} + 5\alpha_{i5} + 6\alpha_{i6} + 7\alpha_{i7}\,, \qquad \text{for } i = 1, 2, 3.$$

One of many solutions is

$$\begin{cases} 3 &= 1 + 2\cdot 1 + 3\cdot 0 + 4\cdot 0 + 5\cdot 0 + 6\cdot 0 + 7\cdot 0\,, \\ 7 &= 1 + 2\cdot 1 + 3\cdot 0 + 4\cdot 1 + 5\cdot 0 + 6\cdot 0 + 7\cdot 0\,, \\ 5 &= 0 + 2\cdot 0 + 3\cdot 0 + 4\cdot 0 + 5\cdot 1 + 6\cdot 0 + 7\cdot 0\,. \end{cases}$$

The nonzero columns are $(1,1,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,0,1)$. Hence we get an equation

$$(3,7,5) = (1,1,0) + 2(1,1,0) + 4(0,1,0) + 5(0,0,1).$$

Combine the first two terms so that all parts are different,

$$(3,7,5) = 3(1,1,0) + 4(0,1,0) + 5(0,0,1).$$

Arrange the solutions into $0 \ll p_1 \ll p_2 \ll p_3$ where $p_1 = (0,0,1)$, $p_2 = (0,1,0)$ and $p_3 = (1,1,0)$ so that $|m_1| = 5$, $|m_2| = 4$ and $|m_3| = 3$. One of many possible cases is $m_1 = (3,2)$, $m_2 = (0,4)$ and $m_3 = (2,1)$. In this way, we obtain a decomposition $\alpha = |m_1|\, p_1 + |m_2|\, p_2 + |m_3|\, p_3$.

12. **Corollary**. For $n = |\alpha|$, $m \in \mathbb{N}^\mu$ and $\beta \in \mathbb{N}^\nu$, we have

$$\left| \frac{\partial^{|\alpha|} z}{\partial x^\alpha} \right| \le (1+n)^{\mu+\nu+n} A\ (1+B)^n$$

where

$$A = \max_{|m|\le n} \left| \frac{\partial^{|m|} z}{\partial y^m} \right| \quad \text{and} \quad B = \max_{1\le i\le \mu}\ \max_{|\beta|\le n} \left| \frac{\partial^{|\beta|} y_i}{\partial x^\beta} \right|.$$

13. Indeed, from $\sum_{k=1}^s |m_k|\ p_{kj} = \alpha_j$, we obtain

$$\sum_{k=1}^s |m_k|\ |p_k| = \sum_{k=1}^s |m_k|\ \sum_{j=1}^\nu p_{kj} = \sum_{j=1}^\nu \sum_{k=1}^s |m_k|\ p_{kj} = \sum_{j=1}^\nu \alpha_j = |\alpha| = n.$$

In particular, we get $s \le n$, $|m_k| \le n$, $|p_k| \le n$ and

$$|m| = \sum_{i=1}^\mu r_i = \sum_{i=1}^\mu \sum_{k=1}^s m_{ki} = \sum_{k=1}^s \sum_{i=1}^\mu m_{ki} = \sum_{k=1}^s |m_k| \le \sum_{k=1}^s |m_k|\ |p_k| = n.$$

Furthermore, we get

$$\begin{aligned} \left| \frac{\partial^{|\alpha|} z}{\partial x^\alpha} \right| &= \left| \alpha! \sum_{(s,p,m)\in\mathscr{D}} \frac{\partial^{|m|} z}{\partial y^m} \prod_{k=1}^s \frac{1}{m_k!} \left[\frac{1}{p_k!}\ \frac{\partial^{|p_k|} y}{\partial x^{p_k}} \right]^{m_k} \right| \\ &\le \alpha! \sum_{(s,p,m)\in\mathscr{D}} \left| \frac{\partial^{|m|} z}{\partial y^m} \right| \left| \prod_{k=1}^s \left[\frac{\partial^{|p_k|} y}{\partial x^{p_k}} \right]^{m_k} \right| = n! \sum_{(s,p,m)\in\mathscr{D}} A \prod_{k=1}^s \prod_{i=1}^\mu \left| \frac{\partial^{|p_k|} y_i}{\partial x^{p_k}} \right|^{m_{ki}} \\ &\le n! \sum_{(s,p,m)\in\mathscr{D}} A \prod_{k=1}^s \prod_{i=1}^\mu B^{m_{ki}} = n! \sum_{(s,p,m)\in\mathscr{D}} A\ B^{\sum_{k=1}^s \sum_{i=1}^\mu m_{ki}} \\ &\le n! \sum_{(s,p,m)\in\mathscr{D}} A\ B^{|m|} \le n! \sum_{(s,p,m)\in\mathscr{D}} A(1+B)^n. \end{aligned}$$

For $m_k = (m_{k1}, \cdots, m_{k\mu}) \in \mathbb{N}^\mu$ satisfying $|m_k| \leq n$, there are $n+1$ possibilities for each m_{ki} from 0 to n. The totality of m_k is no more than $(1+n)^\mu$. Similarly the totality of $p_k \in \mathbb{N}^\nu$ satisfying $|p_k| \leq n$ is no more than $(1+n)^\nu$. Hence the total number of elements in $\mathscr{D}$ is no more than $n(1+n)^{\mu+\nu}$. The result follows from $n!\, n(1+n)^{\mu+\nu} \leq (1+n)^{\mu+\nu+n}$.

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