Tôhoku Math. Journ.
35 (1983), 607-639.

HIGHER DIMENSIONAL ANALOGUES OF PERIODIC CONTINUED FRACTIONS AND CUSP SINGULARITIES

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(Received March 24, 1983)

Introduction. There is a well-known relationship between periodic continued fractions and 2-dimensional cusp singularities. (See, for instance, [10], [11].) Let $\pi: U \to V$ be the minimal resolution of a 2-dimensional cusp singularity (V, p). Then the exceptional set $X = \pi^{-1}(p)$ is either a cycle of s rational curves with self-intersection numbers $a_1, a_2, \cdots, a_s \leqq -2$ at least one of which is strictly smaller than $-2 (s \geqq 2)$, or a rational curve with a node and with a self-intersection number $a < 0$. Then we can associate to it the periodic continued fraction

$$\begin{aligned}\omega &= [[\overline{-a_1, -a_2, \cdots, -a_s}]] \\ &= (-a_1) - \underline{1}\rceil\overline{(-a_2)} - \cdots - \underline{1}\rceil\overline{(-a_s)} - \underline{1}\rceil\overline{(-a_1)} - \cdots ,\end{aligned}$$

or

$$\omega = [[\overline{-a+2}]] = (-a+2) - \underline{1}\rceil\overline{(-a+2)} - \underline{1}\rceil\overline{(-a+2)} \cdots .$$

Conversely, we can construct a 2-dimensional cusp singularity and its resolution as above, from a periodic continued fraction ω first by constructing a convex cone in $\boldsymbol{R}^2$ and then applying the theory of torus embeddings. (See Remark in §4.) Moreover, the dual graph of X can be thought of as a subdivision of a circle S^1, with $a_1, a_2, \cdots, a_s$ attached to s vertices as weights in this order.

In this paper, we generalize the above relationship to higher dimensions and construct higher dimensional cusp singularities from suitable analogues of periodic continued fractions. The well-known Hilbert modular cusp singularities are special cases of the cusp singularities we obtain.

Nakamura [15] found a duality for 2-dimensional cusp singularities. Our higher dimensional cusp singularities also have a duality among themselves generalizing that of Nakamura.

First in Section 1, we show that certain cusp singularities are obtained from suitable cones in $\boldsymbol{R}^r$ with actions of subgroups of $GL(r, \boldsymbol{Z})$, by means of torus embeddings. In Section 2, we study some properties and analytic invariants of such singularities. Especially, they are in general not Cohen-Macaulay but a part of them are quasi-Buchsbaum singularities. (Recently, Ishida [13] showed that all of them are Buchsbaum singularities.)

In Section 4, we show how to obtain these cusp singularities as above explicitly, when $r=3$. Our method is to consider the analogue of the weighted subdivision of S^1 as above in one higher dimension. Namely, we consider a triangulation $\triangle$ of a compact topological surface T, on each edge of which a pair of integers is attached. If it satisfies suitable conditions, then we can construct from it a pair (C, Γ) of a cone C in $\boldsymbol{R}^3$ and a subgroup Γ of $GL(3, \boldsymbol{Z})$ as in Section 1, in a manner similar to the case $r=2$. The corresponding cusp singularity has a resolution whose exceptional set consists of rational surfaces, crossing each other along rational curves and points, in such a way that the "dual graph" agrees with the given triangulation $\triangle$ of T. In the case of 3-dimensional Hilbert modular cusp singularities, the corresponding compact topological surfaces T as above are 2-dimensional real tori. Conversely, in Section 3, we see that when T is a 2-dimensional real torus, the corresponding singularity is a 3-mensional Hilbert modular cusp singularity. Some examples of them can be found in Thomas and Vasquez [19]. Besides 3-dimensional Hilbert modular cusp singularities we give an example with non-orientable T and those with orientable T of genus $g(T)>1$, at the end of this paper.

Our method in Section 4 has an obvious generalization in higher dimensions, but for simplicity we restrict our consideration in Section 4 to 3-dimensional cusp singularities.

The author would like to thank Professors T. Oda and I. Satake and Dr. M.-N. Ishida of Tôhoku University for their useful advice and encouragement. The author also would like to thank Dr. Kimio Watanabe and Y. Koyama who pointed out the possibility that the above singularities are not necessary Cohen-Macaulay but Buchsbaum singularities.

A part of the results in this paper was announced in [20].

1. Cones and singularities. Let $N \simeq \boldsymbol{Z}^r$ and $N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R} \simeq \boldsymbol{R}^r$. Let $\pi: N_{\boldsymbol{R}} \backslash \{0\} \rightarrow S^{r-1}$ be the natural projection onto a sphere $S^{r-1} = (N_{\boldsymbol{R}} \backslash \{0\})/\boldsymbol{R}_{>0}$. Then $\operatorname{Aut}(N) = GL(N)$ acts on S^{r-1} through π. Let $\mathscr{S}$ be the set of equivalence classes of pairs (C, Γ) of a cone C in $N_{\boldsymbol{R}}$ and a subgroup Γ of $GL(N)$ satisfying the following conditions: C is open, nondegenerate (i.e., $\bar{C} \cap (\overline{-C}) = \{0\}$), convex and Γ-invariant. Moreover, the induced action of Γ on $D:=\pi(C)=C/\boldsymbol{R}_{>0}$ is properly discontinuous and fixed point free with the compact quotient D/Γ. Here we say two pairs (C, Γ) and (C', Γ') are equivalent, if there exists an element g

of $GL(N)$ such that $g_R(C) = C'$ and that $g\Gamma g^{-1} = \Gamma'$, where g_R is the image of g in $GL(N_R)$.

Let us denote by $SL(N)$ the special linear group, which is the subgroup of $GL(N)$ consisting of the elements of determinant 1. If Γ is contained in $SL(N)$ for (C, Γ) in $\mathscr{S}$, then D/Γ is orientable. When $r = 3$, the genus $g(D/\Gamma)$ of the orientable surface D/Γ is greater than 0.

Let $M = N^*$ be the $\boldsymbol{Z}$-module dual to N with the canonical pairing $\langle\ ,\ \rangle: M \times N \to \boldsymbol{Z}$. For (C, Γ) in $\mathscr{S}$, let C^* be the dual cone of C in M_R, i.e.,

$$C^*: = \operatorname{Int}\{m \in M_R \mid \langle m, n\rangle > 0 \text{ for all } n \in C\}$$
$$= \{m \in M_R \mid \langle m, n\rangle > 0 \text{ for all } n \in \bar{C}\backslash\{0\}\} ,$$

where Int denotes the interior. Then C^* is also a nondegenerate open convex cone with the canonical action of Γ satisfying $\langle g(m), g(n)\rangle = \langle m, n\rangle$ for any element g of Γ. Let Θ (resp. Θ^*) be the convex hull of $C \cap N$ (resp. $C^* \cap M$). We define the *support function* $h: \bar{C}^* \to \boldsymbol{R}_{>0}$ of Θ by

$$h(x) = \inf\{\langle x, y\rangle \mid \text{ for all } y \in C \cap N\} .$$

Then $h(x)$ is continuous, upper convex, i.e.,

$$h(x + x') \geqq h(x) + h(x') \quad \text{for} \quad x, x' \in \bar{C}^* ,$$

$h(M \cap C^*) = \boldsymbol{Z}_{>0}$ and $\Theta = \{y \in C \mid \langle x, y\rangle \geqq h(x) \text{ for all } x \in \bar{C}^*\}$. We define the polar Θ° of Θ to be

$$\Theta^\circ = \{x \in \bar{C}^* \mid h(x) \geqq 1\} = \{x \in \bar{C}^* \mid \langle x, y\rangle \geqq 1 \text{ for all } y \in \Theta\} .$$

Then clearly Θ° is convex and contains Θ^*. Moreover, we have the duality $(\Theta^\circ)^\circ = \Theta$.

In the following, we show that the boundary $\partial\Theta$ of Θ has a natural Γ-invariant polyhedral decomposition by compact convex polyhedra. Our argument is similar to that of Ehlers [24] in the Hilbert modular case. For each point t of D, there exists a point x of C such that $\bar{\Theta} \cap \pi^{-1}(t) = \boldsymbol{R}_{\geqq 1} \cdot x$ (: $= \{s \cdot x \mid s \geqq 1\}$). Then there sxists a point m of M_R such that $H(m)$ (: $= \{y \in N_R \mid \langle m, y\rangle = 1\}$) is a support hyperplane of Θ containing x, i.e., $\langle m, x\rangle = 1$ and $\langle m, y\rangle \geqq 1$ for all $y \in \Theta$. Let $\Theta(m): = H(m) \cap \bar{\Theta}$ and call it a "face" of $\bar{\Theta}$. We note that D is the union of the images under π of the "faces" of $\bar{\Theta}$.

LEMMA 1.1. *For any support hyperplane $H(m)$ of Θ, the intersection $H(m) \cap \bar{C}$ is compact. Equivalently, m belongs to C^*.*

PROOF. Suppose that $H(m) \cap \bar{C}$ is not compact. Then $\bar{\Theta}$ contains the set $\{y_0 + sy_1 \mid s \geqq 0\}$ for some point y_0 of $\Theta(m)$ and y_1 of ∂C. Let

$$\phi(x) = \int_{C^*} e^{-\langle x^*, x\rangle} dx^*$$

be the characteristic function of the cone C defined by Vinberg [21]. Then $\phi(x) \neq 0$ for all points x of C and

$$\phi(y_0 + sy_1) = \int_{C^*} e^{-\langle x^*, y_0\rangle} \cdot e^{-s\langle x^*, y_1\rangle} dx^*$$

goes to 0 as s goes to infinity, since $\langle x^*, y_1\rangle > 0$ for all x^* in C^*. However, $\inf\{\phi(x) | x \in \Theta\} > 0$, since D/Γ is compact and $\phi(x)$ is Γ-invariant and continuous. Hence we have a contradiction.

Since $\pi(H(m) \cap \bar{C})$ contains D and $H(m) \cap \bar{C}$ is compact, we have $\pi(H(m) \cap \bar{C}) = \bar{D}$. Then $\langle m, x\rangle > 0$ for any point x of $\bar{C} \backslash \{0\}$, since $\boldsymbol{R}_{>0} \cdot x \cap H(m) \neq \emptyset$. Hence m belongs to C^*. q.e.d.

For each support hyperplane $H(m)$ of Θ, we have $H(m) \cap C \cap N \neq \emptyset$. For otherwise, there exists a positive real number s smaller than 1 such that $H^-(sm) \cap C \cap N = \emptyset$, where $H^-(sm) = \{y \in N_{\boldsymbol{R}} | \langle sm, y\rangle \leqq 1\}$. Then $\bar{\Theta}$ must be contained in $H^+(sm) := \{y \in N_{\boldsymbol{R}} | \langle sm, y\rangle \geqq 1\}$, a contradiction.

LEMMA 1.2. *Each "face" $\Theta(m)$ of $\bar{\Theta}$ is equal to the convex hull of the finite set $H(m) \cap C \cap N$. In particular, $\Theta(m)$ is a compact convex polyhedron.*

PROOF. Let $\Theta'(m)$ be the convex hull of $H(m) \cap C \cap N$. It is clear that $\Theta(m) \supset \Theta'(m)$ and $((H^-(m) \cap C) \backslash \Theta'(m)) \cap N = \emptyset$. By Lemma 1.1, $H^-(sm) \cap \bar{C}$ is compact for all positive real numbers s. Hence there exists a positive real number s smaller than 1 so that the convex set $C_1 := \{y_1 + u(y_2 - y_1) | u \geqq 0,\ y_1 \in \Theta'(m) \text{ and } y_2 \in H(sm) \cap \bar{C}\}$ satisfies $(C \backslash C_1) \cap N = \emptyset$. Then C_1 contains Θ and hence $\Theta'(m) = C_1 \cap H(m) \supset \Theta(m)$. Moreover, $H(m) \cap C \cap N$ is a finite set by Lemma 1.1. Hence $\Theta(m) = \Theta'(m)$ is a compact convex polyhedron. q.e.d.

Here we note that if $\Theta(m_1) \cap \Theta(m_2) \neq \emptyset$, then $\Theta(m_1) \cap \Theta(m_2) = \Theta((m_1 + m_2)/2)$, namely, the intersection of two "faces" of $\bar{\Theta}$ is also a "face". Since D/Γ is compact, ϕ is Γ-invariant and $\phi(x) > \phi(tx)$ for $t > 1$ and $x \in C$, the restriction $\phi_{|\Theta}$ of ϕ to Θ is bounded above. Hence $\bar{\Theta}$ is contained in C, since $\phi(x)$ goes to infinity as x approaches the boundary ∂C, by [1, Chapter II, Proposition 1.3] and [21, Chapter I, Proposition 3]. Thus the restriction $\phi_{|\partial\Theta}$ of ϕ to $\partial\Theta$ is a one-to-one and continuous map onto D.

LEMMA 1.3. *$\partial\Theta$ is the union of the $(r-1)$ dimensional "faces" of $\bar{\Theta}$.*

PROOF. For any point t of D, let x be the point of $\partial\Theta$ with $\pi(x) = t$ and let $H(m_0)$ be a support hyperplane of Θ containing x. Assume that

$d := \dim \Theta(m_0) < r-1$. Let $H^*(\Theta(m_0)) := \{x \in M_R \mid \langle x, y\rangle = 1 \text{ for all } y \in \Theta(m_0)\}$. Then $H(m)$ is a support hyperplane containing $\Theta(m_0)$ for any point m of $H^*(\Theta(m_0)) \cap \Theta^\circ$. By Lemma 1.1, $H^*(\Theta(m_0)) \cap \Theta^\circ$ is contained in C^*. Then $H^*(\Theta(m_0)) \cap (C^* \backslash \Theta^\circ) \neq \emptyset$, since $H^*(\Theta(m_0)) \cap \Theta^\circ$ is closed. Take a point x_0 of $H^*(\Theta(m_0)) \cap (C^* \backslash \Theta^\circ)$. Then $H^+(x_0) := \{y \in N_R \mid \langle x_0, y\rangle \geqq 1\}$ contains $\Theta(m_0)$, the closure of $C \backslash H^+(x_0)$ is compact and $\Theta \backslash H^+(x_0) \neq \emptyset$. Hence $(C \backslash H^+(x_0)) \cap N$ is a finite and nonempty set. So we can find a support hyperplane $H(m_1)$ of Θ such that $H(m_1) \cap N \supsetneqq H(m_0) \cap N$. Clearly $\Theta(m_1) \supset \Theta(m_0)$ and $\dim \Theta(m_1) > d$. Repeating this process several times, we have a "face" $\Theta(m)$ containing x with $\dim \Theta(m) = r-1$. q.e.d.

Let vol be the volume on N_R normalized so that $\mathrm{vol}(S) = 1/r!$ for simplices $S = \{a_1 n_1 + a_2 n_2 + \cdots + a_r n_r \mid 0 \leqq a_1, a_2, \cdots, a_r, \sum_{j=1}^r a_j \leqq 1\}$ for any Z-basis $\{n_1, n_2, \cdots, n_r\}$ of N. Since this volume is $GL(N)$-invariant, it induces a volume on C/Γ, which we also denote by the same symbol vol. Since D/Γ is compact, $\mathrm{vol}((C\backslash\Theta)/\Gamma)$ is finite. On the other hand, for each $(r-1)$-dimensional "face" α of $\bar{\Theta}$, the volume $\mathrm{vol}([\alpha])$ of $[\alpha] := \{ty \mid 0 \leqq t \leqq 1 \text{ and } y \in \alpha\}$ is not smaller than $1/r!$, since $\alpha \cap N$ contains at least r points which are linearly independent in N_R. Moreover, $\Gamma_\alpha(:= \{\gamma \in \Gamma \mid \gamma \cdot \alpha = \alpha\}) = \{\mathrm{id}\}$ and $\{\gamma \in \Gamma \mid \gamma \cdot \alpha \cap \alpha \neq \emptyset\}$ is finite set, since α is a compact convex polyhedron and the action of Γ on D is fixed point free. Hence the number of Γ-equivalence classes of "faces" of $\bar{\Theta}$ is finite and the number of "faces" of $\bar{\Theta}$ containing x is also finite for any point x of $\partial\Theta$. Thus we have:

LEMMA 1.4. *The boundary $\partial\Theta$ of Θ has a natural Γ-invariant polyhedral decomposition $\square$ consisting of the "faces" of $\bar{\Theta}$.*

The boundary $\partial\Theta^\circ$ of Θ° has a natural Γ-invariant polyhedral decomposition $\square^\circ$ dual to $\square$ in the following manner: To each d-dimensional "face" α of $\bar{\Theta}$ corresponds the $(r-d-1)$-dimensional "face" $H(y_1) \cap H(y_2) \cap \cdots \cap H(y_s) \cap \partial\Theta^\circ$ of $(\bar{\Theta}^\circ)$ if $\alpha = \Theta(m)$ is the convex hull of $\{y_1, y_2, \cdots, y_s\} = H(m) \cap C \cap N$. The boundary $\partial\Theta^*$ of Θ^* also has a natural Γ-invariant polyhedral decomposition $\square^*$ by the following Lemma 1.6. But Θ^* does not agree with Θ° and hence $\square^*$ may not be dual to $\square$ in general. The following subclass will turn out later to have nicer properties:

DEFINITION 1.5. $\mathscr{S}_0 = \{(C, \Gamma) \in \mathscr{S} \mid \Theta^\circ = \Theta^*\}$.

The polyhedral decompositions $\square$, $\square^\circ$ and $\square^*$ induce Γ-invariant cell divisions of D, D^* and D^* under the homeomorphisms $\pi_{|\partial\Theta}: \partial\Theta \xrightarrow{\sim} D$, $\pi_{|\partial\Theta^\circ}: \partial\Theta^\circ \xrightarrow{\sim} D^*$ and $\pi_{|\partial\Theta^*}: \partial\Theta^* \xrightarrow{\sim} D^*$, respectively, where $D^* = C^*/R_{>0}$. We also denote them by the same symbols $\square$, $\square^\circ$ and $\square^*$.

LEMMA 1.6. *If* (C, Γ) *is in* $\mathscr{S}$ *(resp.* $\mathscr{S}_0$*), then so is* (C^*, Γ).

PROOF. Since $\square$ and $\square^o$ are Γ-invariant and dual to each other as cell divisions and $\Gamma_\alpha = \{\mathrm{id}\}$ for each cell α of $\square$, we have Γ_{α^o} $(:= \{\gamma \in \Gamma \mid \gamma \cdot \alpha^o = \alpha^o\}) = \{\mathrm{id}\}$ for each cell α^o of $\square^o$. Hence the action of Γ on D^* is properly discontinuous and fixed point free. Also D^*/Γ is compact, since so is D/Γ. Moreover, $(\Theta^*)^o = (\Theta^o)^o = \Theta$, if $\Theta^* = \Theta^o$. q.e.d.

In the following, we use the notations of Oda [16]. For (C, Γ) in $\mathscr{S}$ let $\square$ as above be the cell division of $D = C/\boldsymbol{R}_{>0}$ induced by the boundary $\partial\Theta$ of the convex hull Θ of $N \cap C$. Let $\sigma(\alpha) = \boldsymbol{R}_{\geqq 0} \cdot \alpha$ be the closure of the cone $\pi^{-1}(\alpha)$ in $N_{\boldsymbol{R}}$ for each cell α of $\square$. They by Lemma 1.2 and 1.4, we have a Γ-invariant r.p.p. decomposition (N, Σ) with $\Sigma = \{\sigma(\alpha) \mid \text{for all cells } \alpha \text{ of } \square\} \cup \{\{0\}\}$. Clearly $|\Sigma| \backslash \{0\} = C$, where $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. We have a Γ-invariant map ord: $T_N \operatorname{emb}(\Sigma) \to \operatorname{Mc}(N, \Sigma)$ with the commutative diagram:

$$\begin{array}{ccc} T_N \operatorname{emb}(\Sigma) & \xrightarrow{\text{ord}} & \operatorname{Mc}(N, \Sigma) := T_N \operatorname{emb}(N, \Sigma)/CT_N \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ (\boldsymbol{C}^*)^r \simeq T_N & \xrightarrow{-\log|\ |} & N_{\boldsymbol{R}} \simeq \boldsymbol{R}^r, \end{array}$$

where CT_N is the compact real torus $\operatorname{Hom}_{\mathrm{gr}}(M, U(1)) \simeq U(1)^r$, and $\operatorname{Mc}(N, \Sigma)$ is the "manifold with coners" associated to the r.p.p. decomposition Σ (cf. [16, Chap. II]). Let $\hat{C}$ be the interior of the closure of C in $\operatorname{Mc}(N, \Sigma)$, and $\tilde{U} := \operatorname{ord}^{-1}(\hat{C})$. Then $\tilde{U}$ contains $\tilde{X} := T_N \operatorname{emb}(\Sigma) \backslash T_N$, and Γ acts on $\tilde{U}$ properly discontinuously and without fixed points, since so it does on $\hat{C}$. Let $U := \tilde{U}/\Gamma$ and $X := \tilde{X}/\Gamma$. Then X is a compact analytic subset in U, since the cell division of D is a Γ-invariant "dual graph" of $\tilde{X}$ and D/Γ is compact.

PROPOSITION 1.7. *Let* U *and* X *be as above. Then* X *is contractible, i.e., there exists a normal isolated singularity* (V, p) *with a holomorphic map* $\Pi: U \to V$ *which maps* X *to a point* p *and whose restriction to* $U \backslash X$ *gives an isomorphism* $U \backslash X \simeq V \backslash \{p\}$.

PROOF. If Γ is not contained in $SL(N)$, then we have an exact sequence $1 \to \Gamma' \to \Gamma \to \boldsymbol{Z}/2\boldsymbol{Z} \to 0$, with Γ' contained in $SL(N)$. In this case, U is a quotient space of $\tilde{U}/\Gamma'$ by the cyclic group Γ/Γ' of order 2. Hence, we may assume that Γ is contained in $SL(N)$ without loss of generality. The characteristic function $\phi(x)$ in the proof of Lemma 1.1 is a Γ-invariant convex function on C, and can be extended to a continuous function on $\hat{C}$ vanishing on $\hat{C} \backslash C$. (See [1, Chap. II, §1 and Chap. III, §2]). Therefore the Γ-invariant continuous function $\phi \circ \operatorname{ord}$ on $\tilde{U}$ induces a continuous function $\hat{\phi}$ on U vanishing on X. Then $\hat{\phi}$ is strictly subharmonic, since $\phi(x)$ is convex on C. Hence X is contractible in U by [9]. q.e.d.

DEFINITION 1.8. We denote by $\mathrm{Cusp}(C, \Gamma)$ the singularity (V, p) in the above proposition for each (C, Γ) in $\mathscr{S}$, and we let $\mathscr{T}$ (resp. $\mathscr{T}_0$) $= \{\mathrm{Cusp}(C, \Gamma) \mid (C, \Gamma) \in \mathscr{S}\}$ (resp. $\mathscr{S}_0$)$\}$.

PROPOSITION 1.9. *The correspondence* $\mathrm{Cusp}(C, \Gamma) \mapsto \mathrm{Cusp}(C^*, \Gamma)$ *is a duality in* $\mathscr{T}$ *and in* $\mathscr{S}_0$.

PROOF. Let (C, Γ) be in $\mathscr{S}$. Since C is convex, the interior $\mathrm{Int}(\bar{C})$ of the closure of C is equal to C. Then $(C^*)^* = C$, e.g., by [17, Theorem 4.1]. Hence the proposition follows from Lemma 1.6. q.e.d.

REMARK. When $r = 2$, we have $\mathscr{S} = \mathscr{S}_0$ and $\mathscr{T} = \mathscr{T}_0 = \{$cusp singularities of dimension 2$\}$. Moreover, the above duality agrees with Nakamura's duality [15].

2. Some properties and analytic invariants of the singularities in $\mathscr{T}_o$.

DEFINITION 2.1 (Watanabe [22]). An isolated singularity (V, p) is *purely elliptic*, if the plurigenera defined by

$$\delta_m := \dim H^0(V \setminus \{p\}, (K_V)^{\otimes m})/L^{2/m} ,$$

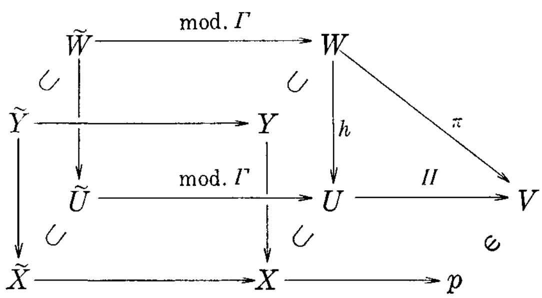
satisfies $\delta_m = 1$ for all positive integers m, where K_V is the canonical sheaf of V and

$$L^{2/m} := \{\omega \in H^0(V \setminus \{p\}, (K_V)^{\otimes m}) \mid \omega \text{ is } L^{2/m}\text{-integrable}\} .$$

PROPOSITION 2.2. *Let* (C, Γ) *be in* $\mathscr{S}$. *If* Γ *is contained in* $SL(N)$, *then* $(V, p) = \mathrm{Cusp}(C, \Gamma)$ *is purely elliptic. If* Γ *is not contained in* $SL(N)$, $\delta_m = 1$ *or* 0 *according as* m *is even or odd.*

PROOF. First assume that $\Gamma \subset SL(N)$. Take a global coordinate $(z_1, z_2, \cdots, z_r)$ of $T_N \simeq (\boldsymbol{C}^*)^r$ (i.e., $z_j = e(m_j)$ for a $\boldsymbol{Z}$-basis $\{m_1, m_2, \cdots, m_r\}$ of M). Then the r-form $\tilde{\omega} = (dz_1/z_1) \wedge (dz_2/z_2) \wedge \cdots \wedge (dz_r/z_r)$ on T_N is Γ-invariant. Hence it induces a nowhere vanishing holomorphic r-form ω on $V \setminus \{p\} = U \setminus X$. Since $K_{\tilde{U}} = \mathscr{O}_{\tilde{U}}(-\tilde{X})$, we have $K_U = \mathscr{O}_U(-X)$ by Oda [16, Proposition 6.6], where $K_{\tilde{U}}$ and K_U denote the canonical sheaves of $\tilde{U}$ and U, respectively. Hence $\delta_m = 1$ for all positive integers m, since ω^m is not $L^{2/m}$-integrable, but $f \cdot \omega^m$ is $L^{2/m}$-integrable for any holomorphic function f on V vanishing at p. When $\Gamma \not\subset SL(N)$, $\tilde{\omega}^m$ is Γ-invariant if and only if m is even. q.e.d.

In the following, we only consider the singularities in $\mathscr{T}_o$, which seem to have relatively nicer properties, and some analytic invariants of which can be calculated. Recall that for (C, Γ) in $\mathscr{S}_o$, (N, Σ) is the r.p.p. decomposition consisting of the cones $\sigma(\alpha)$ joining 0 and cells α in $\square$, where $\square$ is the natural cell division of the boundary $\partial\Theta$ of the convex hull Θ of $N \cap C$. We obtained in Section 1 an open neighborhood $\tilde{U}$ of $\tilde{X} = T_N \operatorname{emb}(\Sigma) \backslash T_N$ in $T_N \operatorname{emb}(\Sigma)$ and a holomorphic map $\Pi: U = \tilde{U}/\Gamma \to V$ with $\Pi^{-1}(p) = X = \tilde{X}/\Gamma$ and $(V, p) = \operatorname{Cusp}(C, \Gamma)$. Let (N, Λ) be a Γ-invariant subdivision of (N, Σ) consisting of nonsingular cones. (As for the existence, we refer the reader to [23] and [25].) Let $\tilde{h}: T_N \operatorname{emb}(\Lambda) \to T_N \operatorname{emb}(\Sigma)$ be the map induced by the identity map of N and let $\tilde{W} = \tilde{h}^{-1}(\tilde{U})$. Then $\tilde{h}$ induces a holomorphic map $h: W := \tilde{W}/\Gamma \to U$. Here W is a complex manifold and $Y := h^{-1}(X)$ is a divisor on W with only normal crossings as singularities. Thus $\pi = \Pi \cdot h: W \to V$ is a resolution of the singularity $(V, p) = \operatorname{Cusp}(C, \Gamma)$ with the exceptional set $Y = \pi^{-1}(p)$.



THEOREM 2.3. *Let* $\operatorname{Cusp}(C, \Gamma)$ *be in* $\mathscr{T}_o$. *In the above notations, we have*

$$R^i\pi_*\mathcal{O}_W = \begin{cases} \mathcal{O}_V & i = 0 \\ H^i(D/\Gamma, \boldsymbol{C}) & i > 0\,, \end{cases}$$

where the i-th cohomology $\boldsymbol{C}$*-vector space* $H^i(D/\Gamma, \boldsymbol{C})$ *of* D/Γ *is regarded as an* $\mathcal{O}_V$*-module through the residue map* $\mathcal{O}_V \to \boldsymbol{C}(p) = \mathcal{O}_V/\mathfrak{m}$. *Here* $\mathcal{O}_W$ *and* $\mathcal{O}_V$ *are the analytic structure sheaves of* W *and* V, *respectively, and* $\mathfrak{m}$ *is the maximal ideal of the stalk* $\mathcal{O}_{V,p}$ *of* $\mathcal{O}_V$ *at* p. *Moreover,*

$$R^i\pi_*\mathcal{O}_W(-Y) = \begin{cases} \mathfrak{m} & i = 0 \\ 0 & i > 0\,. \end{cases}$$

COROLLARY 2.4. *Suppose* Γ *is in* $SL(N)$. *A singularity* $(V, p) =$ $\operatorname{Cusp}(C, \Gamma)$ *in* $\mathscr{T}_o$, *is not a Cohen-Macaulay but a quasi-Buchbaum singularity. Namely, if* $\omega_V^{\cdot}$ *is the normalized dualizing complex of* V, *then the cohomology sheaves* $H^j(\omega_V^{\cdot})$ *are* $\boldsymbol{C}(p)$*-vector spaces for all* $j \neq -\dim V$.

REMARK. Freitag [6] noted that the Hilbert modular cusps are not Cohen-Macaulay. Ishida [13] could recently show that any $\mathrm{Cusp}(C, \Gamma)$ in $\mathscr{T}$, even if Γ is not contained in $SL(N)$ and even if (C, Γ) is not in $\mathscr{T}_0$, is actually a Buchsbaum singularity, i.e., the truncation $\tau_{-r}(\omega_V^{\cdot})$ of the complex $\omega_V^{\cdot}$ itself, not just its cohomology sheaves, is a complex of $C(p)$-vector spaces.

For the proof of the above theorem, we need some lemmas. First, we show that the above map $\Pi: U \to V$ is a "rational resolution".

LEMMA 2.5. *The singularities of U are rational* i.e.,

$$R^i h_* \mathcal{O}_W = \begin{cases} \mathcal{O}_U & i = 0 \\ 0 & i > 0 . \end{cases}$$

Moreover, the Grauert-Riemenschneider type theorem holds, i.e.,

$$R^i h_* \mathcal{O}_W(-Y) = \begin{cases} \mathcal{O}_U(-X) & i = 0 \\ 0 & i > 0 . \end{cases}$$

In particular, we have

$$R^i \pi_* \mathcal{O}_W = R^i \Pi_* \mathcal{O}_U \qquad i \geqq 0$$
$$R^i \pi_* \mathcal{O}_W(-Y) = R^i \Pi_* \mathcal{O}_U(-X) \qquad i \geqq 0 .$$

PROOF. Since $\tilde{h}$ is the pull-back of h by the unramified covering $\tilde{U} \to U$, it suffices to show the corresponding assertions for $\tilde{h}$. But $\tilde{U}$ is an open subset of a torus embedding, hence $\tilde{U}$ has only rational singularities and the Grauert-Riemenschneider vanishing theorem holds by Kempf [14]. The last assertion follows from the Leray spectral sequence. q.e.d.

Let $M_k := \{m \in C^* \cap M \mid h(m) = k\} = M \cap k(\partial\Theta^*)$ for each positive integer k, where h is the support function of C defined in the previous section. The character $e(m)$ determines a nonzero element $g(m)$ of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-k\tilde{X}))$ for each m in M_k. (For the definition of the character, see [16].) As we saw in Section 1, there exists a Γ-invariant duality between the natural cell divisions $\square$ and $\square^*$ of $\partial\Theta$ and $\partial\Theta^*$, since (C, Γ) belongs to $\mathscr{S}_0$. For each cell α of $\square$, we denote by α^* the cell of $\square^*$ dual to α. Let $X_\alpha^\circ = \mathrm{orb}(\boldsymbol{R}_{\geqq 0} \cdot \alpha)$ and $X_\alpha = \bar{X}_\alpha^\circ$ for each cell α of $\square$. We note that each X_α is also a torus embedding with respect to the algebraic torus $X_\alpha^\circ \simeq (C^*)^{r-\dim\alpha-1}$, and $\tilde{X}$ is the union of X_α°, hence of X_α, with α running through $\square$.

LEMMA 2.6. *Let α be a cell of $\square$. For each m in M_k, $g(m)$ vanishes on $\tilde{X}$ if and only if m does not belong to $M \cap k\alpha^*$, and $\{g(m) \mid m \in M \cap k\alpha^*\}$*

is a C-basis of $H^0(X_\alpha, \mathcal{O}_{X_\alpha}(-k\tilde{X}))$. *Moreover, we have* $H^i(X_\alpha, \mathcal{O}_{X_\alpha}(-k\tilde{X}))=0$ *for all positive integers* i *and for all nonnegative integers* k.

PROOF. For each element m of M_k, $g(m)$ is a nonzero element of $H^0(X_\alpha, \mathcal{O}_{X_\alpha}(-k\tilde{X}))$ if and only if $\langle m, \alpha\rangle = k$, or equivalently, m belongs to $k\alpha^*$. Since X_α° is isomorphic to $(C^*)^j$ with $1 \leqq j \leqq r-1$ or to a point, $g(m)$ with $m \in M \cap k\alpha^*$ are linearly independent on X_α° and generate $H^0(X_\alpha, \mathcal{O}_{X_\alpha}(-k\tilde{X}))$. Since X_α is the union of X_β° for all β with α as a face and $g(m)$ with $m \in M \cap k\beta^*$ does not vanish on X_β°, $\mathcal{O}_{X_\alpha}(-k\tilde{X})$ is generated by grobal sections. Then by Demazure [5] and Kempf [14], $H^i(X_\alpha, \mathcal{O}_{X_\alpha}(-k\tilde{X})) = 0$ for all positive integers i. q.e.d.

We say $\square/\Gamma$ is *fine* if $\{\gamma \in \Gamma \mid \gamma(\alpha) \cap \beta \neq \emptyset\} = \{1\}$ for any two cells α and β of $\square$ with $\alpha \cap \beta \neq \emptyset$.

PROPOSITION 2.7. $H^i(X, \mathcal{O}_X) \simeq H^i(D/\Gamma, C)$ *and* $H^i(X, \mathcal{O}_X(-kX)) = 0$ *for all positive integers* i *and* k.

PROOF. First, assume that $\square/\Gamma$, which is the dual graph of X, is orientable and fine. Let Σ_j be the set of all j-dimensional cells of $\square/\Gamma$, and for each cell α of Σ_j let X_α be the corresponding $(r-j-1)$-dimensional analytic subset of X, i.e., $X_\alpha = p(X_{\hat{\alpha}})$ for a cell $\hat{\alpha}$ of $\square$ representing α, where $p: \tilde{X} \to X$ is the natural quotient map by Γ. Fix an orientation of $\square/\Gamma$. Let $Q_\alpha^\beta: \mathcal{O}_{X_\alpha} \to \mathcal{O}_{X_\beta}$ be the restriction maps for cells β and their faces α of $\square$. Then by Ishida [12, §3], we have the exact sequence

$$0 \to \mathcal{O}_X \to \bigoplus_{\alpha_0 \in \Sigma_0} \mathcal{O}_{X_{\alpha_0}} \to \bigoplus_{\alpha_1 \in \Sigma_1} \mathcal{O}_{X_{\alpha_1}} \to \cdots \to \bigoplus_{\alpha_{r-1} \in \Sigma_{r-1}} \mathcal{O}_{\alpha_{r-1}} \to 0 .$$

Here the coboundary map $\delta: \bigoplus_{\alpha \in \Sigma_d} \mathcal{O}_{X_\alpha} \to \bigoplus_{\beta \in \Sigma_{d+1}} \mathcal{O}_{X_\beta}$ is defined by $\delta(f_\alpha) = \sum_{\beta \in \Sigma_{d+1}} \operatorname{sign}(\alpha, \beta) \cdot Q_\alpha^\beta(f_\alpha)$. By tensoring this with $\mathcal{O}_U(-kX)$ for a nonnegative integer k, we also have the exact sequence

$$0 \to \mathcal{O}_X(-kX) \to \bigoplus_{\alpha_0 \in \Sigma_0} \mathcal{O}_{X_{\alpha_0}}(-kX) \to \bigoplus_{\alpha_1 \in \Sigma_1} \mathcal{O}_{X_{\alpha_1}}(-kX) \to \cdots$$
$$\to \bigoplus_{\alpha_{r-1} \in \Sigma_{r-1}} \mathcal{O}_{\alpha_{r-1}} \to 0 .$$

This induces a spectral sequence

$$E_1^{p,q} = \bigoplus_{\alpha \in \Sigma_p} H^q(X_\alpha, \mathcal{O}_{X_\alpha}(-kX)) \Rightarrow H^{p+q}(X, \mathcal{O}_X(-kX)) .$$

But by Lemma 2.6, $E_1^{p,q} = 0$ for all $p \geqq 0$ and $q > 0$. Hence we have $H^i(X, \mathcal{O}_X(-kX)) = H^i(K^\cdot)$ for the complex $K^\cdot$ with $K^d = E_1^{d,0} = \bigoplus_{\alpha \in \Sigma_d} H^0(X_\alpha, \mathcal{O}_{X_\alpha}(-kX))$. Note that we have a complex

$$(1) \qquad 0 \to H^0(X, \mathcal{O}_X(-kX)) \xrightarrow{\delta} K^0 \xrightarrow{\delta} K^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} K^{r-1} \to 0 .$$

When $k=0$, we have $K^d = \bigoplus_{\alpha\in\Sigma_d} C_\alpha$ and $\delta(1_\alpha) = \sum_{\beta\in\Sigma_{d+1}} \mathrm{sign}(\alpha,\beta)\cdot 1_\beta$, since $H^0(X_\alpha, \mathcal{O}_{X_\alpha}) \simeq \boldsymbol{C}$ for any α. Hence we have $H^i(K^\cdot) = H^i(D/\Gamma, \boldsymbol{C})$, since the dual graph $\square/\Gamma$ of X is a cell division of D/Γ. Next we consider the case $k>0$. For each element $\bar{m}$ of M_k/Γ, let $\bar{g}(\bar{m})$ be the section of $\mathcal{O}_X(-kX)$ induced by $\sum_{m\in\bar{m}} g(m)$, which is a finite sum on each point of $\tilde{X}$, by Lemma 2.6. For each cell α of Σ_d, let $\alpha^*(k) := (M\cap k\bar{\alpha}^*)/\Gamma$, where $\bar{\alpha}^*$ is the union of the cells of $\partial\Theta^*$ mapped mod Γ to the cell α^* dual to α. Then by Lemma 2.6, $\{\bar{g}(\bar{m})_{|X_\alpha} \mid \bar{m}\in\alpha^*(k)\}$ is a $\boldsymbol{C}$-basis of $H^0(X_\alpha, \mathcal{O}_{X_\alpha}(-kX))$ for each cell α of $\square/\Gamma$. Moreover, clearly

$$\delta(\bar{g}(\bar{m})_{|X_\alpha}) = \sum_{\beta\in\Sigma_{\dim\alpha+1}} \mathrm{sign}(\alpha,\beta)\cdot\bar{g}(\bar{m})_{|X_\beta}\,.$$

For each element $\bar{m}$ of $(M\cap k(\partial\Theta^*))/\Gamma$, there exists a unique minimal dimensional cell β^* of $\partial\Theta^*/\Gamma$ among the cells α^* with $\bar{m}\in\alpha^*(k)$. Then for each cell α of $\square/\Gamma$, we have $\bar{m}\in\alpha^*(k)$ if and only if β^* is a face of α^*, or equivalently α is a face of the cell β dual to β^*. Since β is contractible, the sequence

$$(1_{\bar{m}})\qquad 0\to\boldsymbol{C}\to\bigoplus_{\alpha_0\in\Sigma_0(\beta)}\boldsymbol{C}\cdot\bar{g}(\bar{m})_{|X_{\alpha_0}}\xrightarrow{\delta'}\bigoplus_{\alpha_1\in\Sigma_1(\beta)}\boldsymbol{C}\cdot\bar{g}(\bar{m})_{|X_{\alpha_1}}\xrightarrow{\delta'}\cdots$$
$$\xrightarrow{\delta'}\boldsymbol{C}\cdot\bar{g}(\bar{m})_{|X_\beta}\to 0$$

is exact, where $\Sigma_d(\beta)$ is the set of all d-dimensional cells of $\square/\Gamma$ which are faces of β and

$$\delta'(\bar{g}(\bar{m})_{|X_{\alpha_d}}) = \sum_{\alpha_{d+1}\in\Sigma_{d+1}(\beta)} \mathrm{sign}(\alpha_d,\alpha_{d+1})\bar{g}(\bar{m})_{|X_{\alpha_{d+1}}}\,.$$

The complex (1) isomorphic to the direct sum of $(1_{\bar{m}})$'s and hence is exact. Thus we have $H^i(X, \mathcal{O}_X(-kX)) = H^i(K^\cdot) = 0$ for all positive integers i.

In the general case, take a finite unramified Galois covering $\psi\colon U'\to U$ of U with the covering transformation group $G = \Gamma/\Gamma'$ for some subgroup Γ' of Γ such that the dual graph of $X' := \psi^{-1}(X)$ is orientable and fine. Then by the above, it suffices to show that $H^j(X', \mathcal{O}(-kX'))^G = H^j(X, \mathcal{O}(-kX))$ for all nonnegative integers j and k. By Grothendieck [7, Corollary 3 to Theorem 5.3.1], we have the spectral sequence:

$$E_2^{p,q} = H^p(G, H^q(X', \mathcal{O}_{X'}(-kX'))) \Rightarrow H^{p+q}(X, \psi_*^G(\mathcal{O}_{X'}(-kX')))\,.$$

Since G is finite and $H^q(X', \mathcal{O}_{X'}(-kX'))$ are $\boldsymbol{C}$-vector spaces, we have $H^p(G, H^q(X', \mathcal{O}_{X'}(-kX'))) = 0$ for all positive integers p. On the other hand, ψ is unramified and $\psi^*\mathcal{O}_X(-kX) = \mathcal{O}_{X'}(-kX')$. Hence we have $\psi_*^G\mathcal{O}_{X'}(-kX') = \mathcal{O}_X(-kX)$. Thus $H^j(X', \mathcal{O}_{X'}(-kX'))^G = H^j(X, \mathcal{O}_X(-kX))$.

q.e.d.

PROOF OF THEOREM 2.3. By the comparison theorem in [8] and Bănica-Stănăşilă [2], we have

$$(R^i\Pi_*\mathcal{O}_U)^\wedge = \text{proj}\lim_k H^i(U, \mathcal{O}_U/\mathcal{O}_U(-kX))$$
$$(R^i\Pi_*\mathcal{O}_U(-X))^\wedge = \text{proj}\lim_k H^i(U, \mathcal{O}_U(-X)/\mathcal{O}_U(-kX)),$$

where $^\wedge$ denotes the $\mathfrak{m}$-adic completion. In view of the exact sequences

$$0 \to \mathcal{O}_X(-kX) \to \mathcal{O}_U/\mathcal{O}_U(-(k+1)X) \to \mathcal{O}_U/\mathcal{O}_U(-kX) \to 0$$
$$0 \to \mathcal{O}_X(-kX) \to \mathcal{O}_U(-X)/\mathcal{O}_U(-(k+1)X) \to \mathcal{O}_U(-X)/\mathcal{O}_U(-kX) \to 0,$$

we are done by Proposition 2.7 and Lemma 2.5. q.e.d.

PROOF OF COROLLARY 2.4. Let $\omega_V^\cdot$ and $\omega_W^\cdot$ be the normalized dualizing complexes of V and W, respectively, so that they are complexes with nonzero terms between degrees $-r$ through 0, where $r = \dim V = \dim W$. (See, for instance, Schenzel [18].) We see as in the proof of Proposition 2.2 that when $\Gamma \subset SL(N)$, the canonical divisor of W is $-Y$. Since W is nonsingular, $\omega_W^\cdot = \mathcal{O}_W(-Y)[r]$. By the duality theorem (see Bănica and Stănăşilă [2]), we have

$$\boldsymbol{R}\pi_*\mathcal{O}_W = \boldsymbol{R}\pi_*\boldsymbol{R}\,\mathrm{Hom}_{\mathcal{O}_W}(\omega_W^\cdot, \omega_W^\cdot) = \boldsymbol{R}\,\mathrm{Hom}_{\mathcal{O}_V}(\boldsymbol{R}\pi_*\omega_W^\cdot, \omega_V^\cdot).$$

On the other hand, $\boldsymbol{R}\pi_*\omega_W = \boldsymbol{R}\pi_*\mathcal{O}_W(-Y)[r] = \mathfrak{m}[r]$ by Theorem 2.3. Hence

$$\boldsymbol{R}\pi_*\mathcal{O}_W = \boldsymbol{R}\,\mathrm{Hom}_{\mathcal{O}_V}(\mathfrak{m}[r], \omega_V^\cdot) = \boldsymbol{R}\,\mathrm{Hom}_{\mathcal{O}_V}(\mathfrak{m}, \omega_V^\cdot[-r]).$$

By the long exact sequence arising from the short exact sequence $0 \to \mathfrak{m} \to \mathcal{O}_V \to \boldsymbol{C} \to 0$, we get the isomorphisms $R^i\pi_*\mathcal{O}_W = \mathrm{Ext}^i_{\mathcal{O}_V}(\mathcal{O}_V, \omega_V^\cdot[-r])$ for $i \leqq r-2$ and the exact sequence

$$0 \to \mathrm{Ext}^{r-1}_{\mathcal{O}_V}(\mathcal{O}_V, \omega_V^\cdot[-r]) \to R^{r-1}\pi_*\mathcal{O}_W \to \boldsymbol{C} \to \mathrm{Ext}^r_{\mathcal{O}_V}(\mathcal{O}_V, \omega_V^\cdot[-r]) \to 0,$$

since $R^j\pi_*\mathcal{O}_W = 0$ for $j \geqq r$ and $\mathrm{Ext}^i_{\mathcal{O}_V}(\boldsymbol{C}, \omega_V^\cdot[-r]) = 0$ for $i \neq r$ and $= \boldsymbol{C}$ for $i = r$, by [18]. On the other hand, $\mathrm{Ext}^i_{\mathcal{O}_V}(\mathcal{O}_V, \omega_V^\cdot[-r]) = H^i(\omega_V^\cdot[-r])$. Hence, in view of Theorem 2.3, we see that $H^j(\omega_V^\cdot[-r])$ for $j \neq 0$ are $\boldsymbol{C}$-vector spaces. q.e.d.

Next we define the length and the principal degree of a singularity in $\mathscr{T}_0$, which may be regarded as a generalization of those of Nakamura [15], and consider the relation between them and the embedding dimension.

DEFINITION 2.8. The length $\mathrm{length}(V)$ of $(V, p) = \mathrm{Cusp}(C, \Gamma)$ in $\mathscr{T}_0$ is the number of the Γ-equivalence classes of $N \cap \partial\Theta$.

DEFINITION 2.9. The principal degree $\mathrm{Deg}(V)$ of (V, p) is $\dim_{\boldsymbol{C}} H^0(X, \mathcal{O}_X(-X))$.

Here, we note that $\mathrm{Deg}(V) = \dim_C H^0(Y, \mathcal{O}_Y(-Y))$ by Lemma 2.5 and that $\mathrm{Deg}(V) = -Y^2$, when $r = \dim V = 2$. Moreover, as we will see in Corollary 2.15, the length and the principal degree are analytic invariants. Let $\bar{M}_k = \{\Gamma\text{-equivalence classes of } M_k = M \cap k(\partial\Theta^*)\}$. Recall that for each element $\bar{m}$ of $\bar{M}_k$, $\bar{g}(\bar{m})$ is a section of $\mathcal{O}_X(-kX)$ induced by $\sum_{m \in \bar{m}} g(m)$.

PROPOSITION 2.10. *$\{\bar{g}(\bar{m}) \mid \bar{m} \in \bar{M}_k\}$ is a C-basis of $H^0(X, \mathcal{O}_X(-kX))$. Especially* $\mathrm{Deg}(V) = \mathrm{length}(V^*)$.

PROOF. We have a homomorphism

$$\bigoplus_{\bar{m} \in \bar{M}_k} C \cdot \bar{m} \xrightarrow{\bar{g}} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-k\tilde{X}))^\Gamma \simeq H^0(X, \mathcal{O}_X(-kX)) ,$$

where the first arrow sends $\bar{m}$ to $\bar{g}(\bar{m})$. The map $\bar{g}$ is injective by Lemma 2.6. For any element s of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-k\tilde{X}))$, we have $s = \sum_{m \in M_k} a_m g(m)$ with $a_m \in C$, by Lemma 2.6. Since s is Γ-invariant, $\{a_m \mid m \in \bar{m}\}$ is constant for any $\bar{m}$ of $\bar{M}_k$. Thus we have $s = \sum_{\bar{m} \in \bar{M}_k} a_{\bar{m}} \cdot \bar{g}(\bar{m})$. Hence $\bar{g}$ is surjective. q.e.d.

We now study the embedding dimension.

LEMMA 2.11. *$\sum_{m \in C^* \cap M} |e(m)|$ converges on $\tilde{W}$.*

PROOF. Take a maximal dimensional cone $\sigma = \boldsymbol{R}_{\geq 0} n_1 + \boldsymbol{R}_{\geq 0} n_2 + \cdots + \boldsymbol{R}_{\geq 0} n_r$ of Λ, let $\{m_1, m_2, \cdots, m_r\}$ be the $\boldsymbol{Z}$-basis of M dual to $\{n_1, n_2, \cdots, n_r\}$ and let $z_j = e(m_j)$ for $j = 1$ through r. Take a point $z = (z_1, z_2, \cdots, z_r)$ of $\tilde{W} \backslash \tilde{Y}$ and let $s = s_1 n_1 + s_2 n_2 + \cdots + s_r n_r \in C$ be the image of z under ord: $T_N \operatorname{emb}(\Lambda) \to \operatorname{Mc}(N, \Lambda)$. Then for any element $m = b_1 m_1 + b_2 m_2 + \cdots + b_r m_r$ of $C^* \cap M$, we have $|e(m)| = |z_1^{b_1} \cdot z_2^{b_2} \cdot \cdots \cdot z_r^{b_r}| = (e^{-s_1})^{b_1} \cdot (e^{-s_2})^{b_2} \cdot \cdots \cdot (e^{-s_r})^{b_r} = e^{-\langle m, s\rangle} < 1$, and $b_1, b_2, \cdots, b_r > 0$, since C^* is contained in the dual cone $\sigma^* = \boldsymbol{R}_{\geq 0} m_1 + \boldsymbol{R}_{\geq 0} m_2 + \cdots + \boldsymbol{R}_{\geq 0} m_r$ of σ. Hence $\sum_{m \in C^* \cap M} |e(m)|$ converges on $U(\sigma, s) := \mathrm{ord}^{-1}(\{t_1 n_1 + t_2 n_2 + \cdots + t_r n_r \mid t_j > s_j$ for $j = 1$ through $r\}) \subset \{(w_1, w_2, \cdots, w_r) \in T_N \operatorname{emb}(\{\text{faces of } \sigma\}) \mid |w_j| < |z_j|$ for $j = 1$ through $r\}$. Since $\tilde{W}$ coincides with the union $\bigcup_{\sigma, s} U(\sigma, s)$ of the open sets $U(\sigma, s)$ of $T_N \operatorname{emb}(\Lambda)$ with σ running through all maximal dimensional cones of Λ and with s running through all points of C, $\sum_{m \in C^* \cap M} |e(m)|$ converges on $\tilde{W}$. q.e.d.

By this lemma, $\sum_{m \in \bar{m}} e(m)$ converges to a holomorphic function on $\tilde{W}$, for any $\bar{m}$ of $(C^* \cap M)/\Gamma$. Clearly it is Γ-invariant and vanishes on $\tilde{Y}$. Thus we have a holomorphic function $f(\bar{m})$ on W vanishing on Y. Since C^* is convex, $M \cap C^*$ is a semi-group and $e(m + m') = e(m) \cdot e(m')$ for any m and m' of $M \cap C^*$. Let $C\{M \cap C^*\}$ be the set consisting of the series $\sum_{m\in M\cap C^*} a_m e(m)$ converging on some neighborhood of $\tilde{Y}$ in $\tilde{W}$. Each element of the subset of Γ-invariants $C\{M\cap C^*\}^\Gamma$ of $C\{M\cap C^*\}$ induces a holomorphic function on some neighborhood of p in V, vanishing at p.

LEMMA 2.12. *We have a canonical isomorphism* $C\{M\cap C^*\}^\Gamma \simeq \mathfrak{m}$, *where* $\mathfrak{m}$ *is the maximal ideal of* $\mathcal{O}_V$ *at* p.

PROOF. Clearly, we have an injective homomorphism $C\{M\cap C^*\}^\Gamma \hookrightarrow \mathfrak{m}$. Let $f\in\mathfrak{m}$, and let $\tilde{f}$ be the holomorphic function on some neighborhood of $\tilde{Y}$ in $\tilde{W}$, induced by f. For any r-dimensional nonsingular cone σ of Λ, $\tilde{f}$ is expressed as a convergent series

$$\sum_{m\in M\cap(\sigma^*\setminus\{0\})} a_m e(m) \qquad (a_m\in C)\ ,$$

on some neighborhood of $\mathrm{orb}(\sigma)$. Clearly, $C^*\setminus\{0\}$ is equal to the intersection $\bigcap\sigma^*$ of the dual cones of all r-dimensional nonsingular cones σ of Λ. Hence $\tilde{f}$ is expressed as a series

$$\sum_{m\in M\cap C^*} a_m e(m) \qquad (a_m\in C)\ .$$

It is contained in $C\{M\cap C^*\}^\Gamma$, since $\tilde{f}$ is Γ-invariant. q.e.d.

REMARK. $C\{M\cap C^*\}^\Gamma$ is contained in the ring $C[[(M\cap C^*)\cup\{0\}]]^\Gamma$ of the Γ-invariant formal power series. Cohn [3, Theorem 3.10] showed that $C[[M\cap C^*]]^\Gamma = C[[\cdots, f(\bar{m}), \cdots]]^\Gamma$ with $\bar{m}$ running through $\bar{M}_1$, when $r=2$ and $\mathrm{length}(V)\geqq 3$. Moreover, he also showed in [4] that $C[[M\cap C^*]]^\Gamma$ are finitely generated for 3-dimensional Hilbert modular cusp singularities and found generators for them, in some special cases.

Recall that $M_j=\{m\in M\cap C^* \mid h(m)=j\}$, where h is the support function of C, defined at the beginning of Section 1. Here, we note that $M_1=\partial\Theta^\circ\cap M$ and $\bigcup_{j\in Z_{>0}} M_j = M\cap C^*$. Then we have a filtration $\{\mathfrak{m}_j\}_{j=1,2,\dots}$ in $\mathfrak{m}$ by $\mathfrak{m}_j = C\{\bigcup_{k\geqq j} M_k\}^\Gamma$. Clearly, the j-th power ideal $\mathfrak{m}^j$ is contained in $\mathfrak{m}_j$.

THEOREM 2.13. $\mathrm{Emb.dim}(V)\geqq\mathrm{Deg}(V)$ *for a cusp singularity* (V, p) *in* $\mathcal{T}_0$. *Moreover, the equality holds if* $r=3$ *and the cell division* $\square^*/\Gamma$ *of the compact topological surface* D^*/Γ *induced by* $\partial\Theta^*$, *is a fine cell division.*

PROOF. The canonical map $\mathfrak{m}/\mathfrak{m}^2\to H^0(X, \mathcal{O}_X(-X))$ is surjective, by Proposition 2.10, since it sends $f(\bar{m})+\mathfrak{m}^2$ to $\bar{g}(\bar{m})$ for any $\bar{m}$ of $\bar{M}_1$. Thus the first assertion of the theorem is proved. As for the second assertion, it is sufficient to show that $\mathfrak{m}^2=\mathfrak{m}_2$, since $\dim_C \mathfrak{m}/\mathfrak{m}_2 = \sharp(M_1/\Gamma) = \mathrm{Deg}(V)$. Clearly, $\mathfrak{m}_j/\mathfrak{m}_{j+1}$ is generated by $\{f(\bar{m})+\mathfrak{m}_{j+1} \mid \bar{m}\in\bar{M}_j = M_j/\Gamma\}$. For each

$\bar{m}$ of $\bar{M}_j$, let m be a representative of $\bar{m}$ and let α^* be the 2-dimensional cell of $\partial\Theta^*$ which contains $(1/j)m$. We can find three elements m_1, m_2 and m_3 of $\alpha^* \cap M$ which form a $\boldsymbol{Z}$-basis of M such that the semi-group $\boldsymbol{Z}_{\geq 0}m_1 + \boldsymbol{Z}_{\geq 0}m_2 + \boldsymbol{Z}_{\geq 0}m_3$ generated by them contains m (See the proof $(2) \Rightarrow (3)$ of Proposition 4.5.). Since $h(m_1) = h(m_2) = h(m_3) = 1$ and $h(m) = j$, we can choose $l_1, l_2, \cdots, l_j$ from $\{m_1, m_2, m_3\}$ so that $l_1 + l_2 + \cdots + l_j = m$. Next for any j elements $\gamma_1, \gamma_2, \cdots, \gamma_j$ of Γ, we see that $\gamma_1 l_1 + \gamma_2 l_2 + \cdots + \gamma_j l_j \in M_j$ if and only if $\gamma_1 l_1, \gamma_2 l_2, \cdots, \gamma_j l_j$ are on one and the same cell of $\partial\Theta^*$. When $\square^*/\Gamma$ is a fine cell division, the latter condition is equivalent to $\gamma_1 = \gamma_2 = \cdots = \gamma_j$. Then $f(\bar{m}) - f(\bar{l}_1)\cdot f(\bar{l}_2)\cdots f(\bar{l}_j) \in \mathfrak{m}_{j+1}$, where $\bar{l}_i \in \bar{M}_1$ is the image of $l_i \in M_1$. Then $(\mathfrak{m}/\mathfrak{m}_2)^{\otimes j} \to \mathfrak{m}_j/\mathfrak{m}_{j+1}$ is surjective and $\mathfrak{m}$ is generated by $f(\bar{m})$ with $\bar{m} \in \bar{M}_1$. Hence $\mathfrak{m}_2 = \mathfrak{m}^2$. q.e.d.

For $(V, p) = \mathrm{Cusp}(C, \Gamma)$ in $\mathscr{T}$, let $\square^\dagger$ be the cell division of $D = C/\boldsymbol{R}_{>0}$ induced by that of the boundary $\partial(\Theta^*)^\circ$ of the polar $(\Theta^*)^\circ$ of the convex hull Θ^* of $M \cap C^*$. Let $(U^\dagger, X^\dagger) \to (V, p)$ be the rational resolution of V obtained from the r.p.p. decomposition $\Sigma^\dagger$ corresponding to the cell division of $\partial(\Theta^*)^\circ$ as in Section 1 and Lemma 2.5 for U, X, Σ. For each vertex v of $\square^\dagger/\Gamma$, we assign a positive integer $l(v)$ as follows: Take a representative $\bar{v}$ of v and let u be the vertex of $\partial(\Theta^*)^\circ$ corresponding to it, i.e., $\bar{v}$ is the image of u by $\pi\colon C \to D$. Then there is a positive integer l such that $l\cdot u$ is the primitive element of $\boldsymbol{R}_{\geq 0}\cdot u \cap N$. We then let $l(v) = l$. Clearly it does not depend on the choice of $\bar{v}$, since $\partial(\Theta^*)^\circ$ is Γ-invariant. Let $\bar{X}^\dagger = \sum l(v) X_v^\dagger$, where $X_v^\dagger$ is the irreducible component of $X^\dagger$ corresponding to v. When (V, p) is in $\mathscr{T}_0$, the above $(U^\dagger, X^\dagger)$ agrees with the rational resolution at the beginning of this section and $\bar{X}^\dagger = X^\dagger$. We now study the geometric significance of $(U^\dagger, X^\dagger)$ for (V, p).

Choose an embedding $h\colon (V, p) \hookrightarrow (B, 0)$ of V into an open set B of $\boldsymbol{C}^L$ with $h(p) = 0$. For the blowing-up $q\colon (Z, \boldsymbol{P}^{L-1}) \to (B, 0)$ of B at 0, let $\bar{V}$ be the proper transform of $h(V)$ by q. Hence $q\colon \bar{V} \to V$ is the blowing-up of V at p.

THEOREM 2.14. *If the dual graph $\square^\dagger/\Gamma$ of $X^\dagger$ is fine, then $U^\dagger$ is isomorphic to the normalization of the blowing up $\bar{V}$ of V at p, where $\Pi\colon (U^\dagger, X^\dagger) \to (V, p)$ is the above rational resolution.*

PROOF. We see that $\mathscr{O}_{X^\dagger}(-\bar{X}^\dagger)$ is base point free, in the same way as in the proof of Lemma 2.6, when $\square^\dagger/\Gamma$ is fine. We have a holomorphic map $\bar{h}\colon U \to Z$ with $q\cdot \bar{h} = h\cdot \Pi$, as follows: Let $(z_1, z_2, \cdots, z_L)$ be a coordinate $\boldsymbol{C}^L$ and let g_j be the image of $z_{j|V} + \mathfrak{m}^2$ by the canonical map $\mathfrak{m}/\mathfrak{m}^2 \to H^0(X^\dagger, \mathscr{O}_{X^\dagger}(-\bar{X}^\dagger))$, which is surjective. Then for each point

x of $X^\dagger$, $\bar{h}(x)=(g_1(x), g_2(x), \cdots, g_L(x))\in \boldsymbol{P}^{L-1}=q^{-1}(0)\subset \boldsymbol{Z}$. Clearly $\bar{h}(U)=\bar{V}$. When $\square^\dagger/\Gamma$ is fine, $H^0(X_\alpha^\dagger, \mathcal{O}_{X_\alpha^\dagger}(-\bar{X}^\dagger))$ is a linear subspace of $H^0(X^\dagger, \mathcal{O}_{X^\dagger}(-\bar{X}^\dagger))$ and has a basis $\{\bar{g}(\bar{m})\,|\,\bar{m}\in(\Gamma\bar{\alpha}^*\cap M)/\Gamma\}$ for any cell α of $\square^\dagger/\Gamma$, where $\bar{\alpha}^*$ is the cell dual to a respresentative $\bar{\alpha}$ of α and $X_\alpha^\dagger$ is an irreducible subvariety of $X^\dagger$ corresponding to α. The restriction $\bar{h}_{|(X_\alpha^\dagger)^o}$ of $\bar{h}$ to the algebraic torus $(X_\alpha^\dagger)^o$ $(\simeq(\boldsymbol{C}^*)^d)$ of $X_\alpha^\dagger$ is finite, since $d=\dim X_\alpha^\dagger=\dim\alpha^*$. Since X is the finite union $\bigcup_{\alpha\in\square^\dagger/\Gamma}(X_\alpha^\dagger)^o$ of $(X_\alpha^\dagger)^o$ for all cells of $\square^\dagger/\Gamma$, the restriction $\bar{h}_{|X^\dagger}$ of $\bar{h}$ to $X^\dagger$ is also finite. On the other hand, $\bar{h}$ is an isomorphism over $U^\dagger\backslash X^\dagger$. Hence $U^\dagger$ is isomorphic to the normalization of $\bar{V}$. q.e.d.

REMARK. When $\square/\Gamma$ is not fine, the above theorem does not necessarily hold. We give a simple 2-dimensional example. Let (V, p) be a 2-dimensional cusp singularity with a resolution $\Pi\colon(U, X)\to(V, p)$

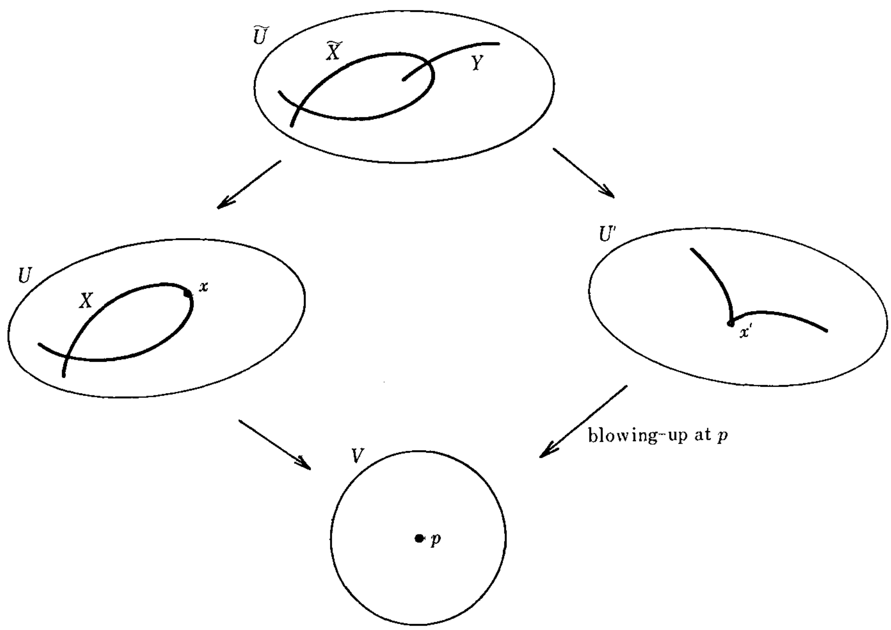


FIGURE 2.1

such that the exceptional set X is a rational curve with a node and with the self-intersection number -1. Then $\dim H^0(X, \mathcal{O}_X(-X))=1$ and X has only one point x, on which all global sections of $\mathcal{O}_X(-X)$ vanish. Let $(\tilde{U}, Y)\to(U, x)$ be the blowing-up of U at x and let $(\tilde{U}, \tilde{X})\to(U', x')$ be the blowing-down of the proper transform $\tilde{X}$ of X in $\tilde{U}$ to a point x'. Then the blowing-up of V at p is isomorphic to U'.

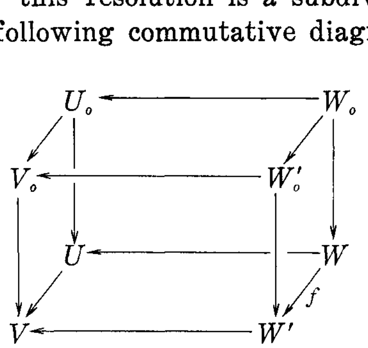
COROLLARY 2.15. *Let* $(V, p) = \mathrm{Cusp}\,(C, \Gamma)$ *and* $(V', p') = \mathrm{Cusp}\,(C', \Gamma')$ *be in* $\mathscr{T}$. *Then* V *and* V' *are analytically isomorphic near* p *and* p', *if and only if* (C, Γ) *and* (C', Γ') *are equivalent in the sense at the beginning of Section* 1.

PROOF. We consider the rational resolutions in Theorem 2.14, but for simplicity, we omit the dagger. Thus let (U, X) and (U', X') be the rational resolutions of (V, p) and (V', p') obtained by the boundaries $\partial(\Theta^*)^\circ$ and $\partial((\Theta')^*)^\circ$ of the polars $(\Theta^*)^\circ$ and $((\Theta')^*)^\circ$ of the convex hulls Θ^* and $(\Theta')^*$ of $M \cap C^*$ and $M \cap (C')^*$, respectively, as in Theorem 2.14. When both cell divisions $\square/\Gamma$ and $\square'/\Gamma'$ of $(C/\boldsymbol{R}_{>0})/\Gamma$ and $(C'/\boldsymbol{R}_{>0})/\Gamma'$ induced by those of $\partial(\Theta^*)^\circ$ and $\partial((\Theta')^*)^\circ$, respectively, are fine, an isomorphism of V and V' near p and p' induces an isomorphism of U and U' near X and X', by Theorem 2.14.

We also have the same assertion in the cases that $\square/\Gamma$ and $\square'/\Gamma'$ are not necessarily fine, as follows. Take a suitable subgroup Γ_0 of Γ of finite index such that the induced cell division $\square/\Gamma_0$ on $(C/\boldsymbol{R}_{>0})/\Gamma_0$ is fine. Then we have the finite covering $U_0 \to U$ induced by the covering map $(C/\boldsymbol{R}_{>0})/\Gamma_0 \to (C/\boldsymbol{R}_{>0})/\Gamma$, where U_0 is the rational resolution of $(V_0, p_0) := \mathrm{Cusp}\,(C, \Gamma_0)$ with the commutative diagram:

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

Take a Γ-invariant subdivision (N, Λ') of (N, Σ') consisting of nonsingular cones, where (N, Σ') is the r.p.p. decomposition of N induced by $\partial((\Theta')^*)^\circ$. If (V, p) and (V', p') are analytically isomorphic near p and p', we have another resolution $W' \to V$ of the singularity (V, p) with nonsingular W' by (N, Λ') and through this isomorphism. The dual graph Δ'/Γ' of the exceptional set of this resolution is a subdivision of $\square'/\Gamma'$ induced by Σ'. We have the following commutative diagram:



Here W (resp. W_o, resp. W_o') is the normalization of the irreducible component of $U \times_V W'$ (resp. $V_o \times_V W$, resp. $V_o \times_V W'$) with the surjective map onto V by the natural projection and $f: W \to W'$ is the map naturally induced by the projection $U \times_V W' \to W'$. The vertical arrows are finite covering maps. Since $U_o \to U$ is unramified, so is $W_o \to W$. Suppose that $W_o' \to W'$ is ramified. The branch locus of W' contains a component Y_o' of the exceptional set Y' of W', since W' is nonsingular and $W_o' \to W'$ is unramified over $W' \setminus Y'$. Clearly, W contains an analytic subvariety Y with $f(Y) = Y_o'$. Then $W_o \to W$ must be ramified at Y, since the restriction of f to $W \setminus f^{-1}(Y')$ is an isomorphism. Hence $W_o' \to W'$ is unramified. The analytic subvariety of W' corresponding to each simplex of Δ'/Γ' is toric and hence simply connected. Therefore, the covering map $W_o' \to W'$ is induced by an unramified covering map $\Delta'/\Gamma_o' \to \Delta'/\Gamma'$ of the dual graph and $(V_o', p_o') := \mathrm{Cusp}(C', \Gamma_o')$ is isomorphic to (V_o, p_o) near p_o and p_o', where Γ_o' is a subgroup of Γ' of finite index. Next take a suitable subgroup Γ_1' of Γ_o' of finite index such that $\square'/\Gamma_1'$ is fine. Then by the same argument, we have a subgroup Γ_1 of Γ_o of finite index such that $(V_1, p_1) := \mathrm{Cusp}(C, \Gamma_1)$ is isomorphic to $(V_1', p_1') := \mathrm{Cusp}(C', \Gamma_1')$. Here $\square/\Gamma_1$ is also fine. Hence by the first consideration, we have an isomorphism $(U_1, X_1) \to (U_1', X_1')$ near X_1 and X_1' between the rational resolutions (U_1, X_1) and (U_1', X_1') of (V_1, p_1) and (V_1', p_1') induced by $\partial(\Theta^*)^o$ and $\partial((\Theta')^*)^o$, respectively. Then we have an isomorphism $(U, X) \simeq (U', X')$ by the following commutative diagram:

$$\begin{array}{ccccc} U_1 & \hookleftarrow & U_1 \setminus X_1 & \longrightarrow & U \setminus X \\ \wr\| & & \wr\| & & \wr\| \\ U_1' & \hookleftarrow & U_1' \setminus X_1' & \longrightarrow & U' \setminus X' . \end{array}$$

The isomorphism $(U, X) \simeq (U', X')$ obtained in the above way, induces an isomorphism $\Gamma \simeq \Gamma'$ and a Γ-equivariant isomorphism $(\tilde{U}, \tilde{X}) \simeq (\tilde{U}', \tilde{X}')$. It is easy to see that (C, Γ) is determined by $\tilde{X}$ and $\mathcal{O}_{\tilde{X}}(-\tilde{X})$, which is dual to the normal sheaf of $\tilde{X}$ in $\tilde{U}$, uniquely up to equivalence. Hence (C, Γ) and (C', Γ') are equivalent. q.e.d.

3. A remark on Hilbert modular cusp singularities. First, we recall Hilbert modular cusp singularities. Let K be a totally positive real algebraic number field of degree r over $\boldsymbol{Q}$. Then we have r distinct embeddings $x \to x^{(i)}$, $i = 1$ through r, of K into $\boldsymbol{R}$. Let N be an additive subgroup of K of rank r, and let Λ be a multiplicative subgroup of U_N^+ of rank $r-1$, where U_N^+ is the group of totally poistive units λ with $\lambda N = N$. Let

$$G(N, \Lambda) := \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & 1 \end{pmatrix} \middle| \lambda \in \Lambda, \mu \in N \right\} .$$

Then $G(N, \Lambda)$ acts properly discontinuously and without fixed points on the product $\mathfrak{H}^r$ of r copies of the upper half plane $\mathfrak{H}$ by

$$(z_1, z_2, \cdots, z_r) \mapsto (\lambda^{(1)} z_1 + \mu^{(1)}, \lambda^{(2)} z_2 + \mu^{(2)}, \cdots, \lambda^{(r)} z_r + \mu^{(r)}) .$$

Then $V(N, \Lambda) = (\mathfrak{H}^r \cup \{\infty\})/G(N, \Lambda)$ is called a Hilbert modular cusp singularity. By the embedding $N \ni x \mapsto (x^{(1)}, x^{(2)}, \cdots, x^{(r)}) \in \boldsymbol{R}^r$, $N(\simeq \boldsymbol{Z}^r)$ is a lattice in $\boldsymbol{R}^r$, i.e., $N_{\boldsymbol{R}} = \boldsymbol{R}^r$. Since $\lambda N = N$ and λ is totally positive for each λ in Λ, Λ is a subgroup of $SL(N)$. We see that $C := (\boldsymbol{R}_{>0})^r$ is Λ-invariant and the induced action of Λ on $D := C/\boldsymbol{R}_{>0}$ is properly discontinuous, fixed point free and with compact quotient by the Dirichlet unit theorem. Hence (C, Λ) is in $\mathscr{S}$ and $\operatorname{Cusp}(C, \Lambda) \simeq V(N, \Lambda)$. In this case, D/Λ is the $(r-1)$-dimensional real torus. Conversely, we have the following, at least in dimension three.

THEOREM 3.1. *Let (C, Γ) be a 3-dimensional pair in $\mathscr{S}$. If $(C/\boldsymbol{R}_{>0})/\Gamma$ is a 2-dimensional real torus, then* Cusp (C, Γ) *is a 3-dimensional Hilbert modular cusp singularity.*

PROOF. Let γ and δ be generators of Γ. We first show that all eigenvalues of γ and δ are real. Suppose not. Then with respect to some basis $\{v_1, v_2, v_3\}$ of $N_{\boldsymbol{R}}$, we can express γ in the form

$$\begin{pmatrix} r^{-2} & 0 & 0 \\ 0 & r\cos\theta & r\sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{pmatrix} \quad \text{with} \quad r \in \boldsymbol{R}_{>0} , \quad 0 < \theta < 2\pi .$$

Since C is open, it contains a point $v = s_1 v_1 + s_2 v_2 + s_3 v_3$ with $s_1 s_2 s_3 \neq 0$. Observing the orbit of v under $\gamma^{\boldsymbol{Z}}$, we see that C contains v_1 or $-v_1$ according as $s_1 > 0$ or $s_1 < 0$, since C is a convex and Γ-invariant cone. But the images $\pi(\pm v_1)$ of $\pm v_1$ by the projection $\pi: N_{\boldsymbol{R}} \backslash \{0\} \to S^2$ are fixed points of γ, a contradiction.

We next show that not all eigenvalues of γ are ± 1. Otherwise, with respect to some basis $\{v_1, v_2, v_3\}$ of $N_{\boldsymbol{Q}}$, we can express γ^2 in the form

$$\text{(i)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \text{(ii)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} .$$

In the first case (i), the orbit $\{(s_1 + k s_2) v_1 + s_2 v_2 + s_3 v_3 \mid k \in \boldsymbol{Z}\}$ of a point $v = s_1 v_1 + s_2 v_2 + s_3 v_3$ of C with $s_2 \neq 0$ under $\gamma^{2\boldsymbol{Z}}$ cannot be contained in any open nondegenerate convex cone of N_R, a contradiction. In the second case (ii), δ^2 is expressed in the form

$$\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \alpha, \beta \in \boldsymbol{Q} ,$$

since Γ is abelian. If $\alpha = q/p$ with $p, q \in \boldsymbol{Z}$, then $\delta^{2p}\gamma^{-2q}$ $(\neq id)$ is expressed in the form

$$\begin{pmatrix} 1 & 0 & \beta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \beta' \neq 0 .$$

But this cannot occur as in the first case. By these considerations, the characteristic polynomial of γ, regarded as an element of $SL(3, \boldsymbol{Z})$ with respect to some $\boldsymbol{Z}$-basis of N, is an irreducible cubic equation over $\boldsymbol{Q}$ with three real roots, or the product of an irreducible quadratic equation with real roots and a linear equation. In both cases, γ has three eigenvectors v_1, v_2 and v_3 with real eigenvalues ξ_1, ξ_2 and ξ_3 different from each other. Let $K = \boldsymbol{Q}(\xi_1, \xi_2, \xi_3)$ be the field generated by ξ_1, ξ_2 and ξ_3 over $\boldsymbol{Q}$. Let $\{n_1, n_2, n_3\}$ be a $\boldsymbol{Z}$-basis of N, and let $v_j = h_{1j}n_1 + h_{2j}n_2 + h_{3j}n_3$ for $j = 1, 2, 3$. We may assume that $h_{ij} \in K$. Then

$$H^{-1}\gamma H = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \quad \text{with} \quad H = (h_{ij}) .$$

Since Γ is abelian,

$$H^{-1}\delta H = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix} \quad \text{with} \quad \eta_j \in K .$$

First, suppose that $\xi_1 = 1$. Then K is a quadratic extension of $\boldsymbol{Q}$. We denote by $\bar{\xi}$, the conjugate of each element ξ of K. Since $v_1 = \gamma v_1$ and $\bar{v}_1 = \gamma \bar{v}_1$, we have $\bar{v}_1 = \varepsilon v_1$ for some $\varepsilon \in K$, where $\bar{v}_1 = {}^t(\bar{h}_{11}, \bar{h}_{21}, \bar{h}_{31})$. Hence $\boldsymbol{R}v_1 \cap N \neq \varnothing$. Then η_1 also must be 1, since $\eta_1 > 0$, $\delta v_1 = \eta_1 v_1$ and $\delta \in SL(N)$. But it contradicts the fact that Γ acts on some open set of S^2 properly discontinuously and without fixed points. Therefore, ξ_1, ξ_2 and ξ_3 are roots of an irreducible cubic equation with integral coefficients. Then we denote by σ_i, the automorphisms of K which send ξ_1 to ξ_i for $i = 1, 2, 3$, respectively. Since $\xi_i {}^t(h_{1i}, h_{2i}, h_{3i}) = \gamma {}^t(h_{1i}, h_{2i}, h_{3i})$ and $\xi_i {}^t(\sigma_i(h_{11}), \sigma_i(h_{21}), \sigma_i(h_{31})) = \gamma {}^t(\sigma_i(h_{11}), \sigma_i(h_{21}), \sigma_i(h_{31}))$, we may assume that $\sigma_i(h_{j1}) = h_{ji}$.

Moreover, we have $\sigma_i(\eta_1) = \eta_i$, since $\sigma_i(\eta_1)v_i = \delta v_i$. Let $H^{-1} = (h'_{ij})$, let $N' = h'_{11}\boldsymbol{Z} + h'_{12}\boldsymbol{Z} + h'_{13}\boldsymbol{Z}$ and let $\Lambda = \xi_1^{\boldsymbol{Z}} \times \eta_1^{\boldsymbol{Z}}$. Then N' is a lattice of rank 3 in K and is Λ-invariant, since $H^{-1}(N) = i(N')$, where $i: N'_{\boldsymbol{R}} \simeq \boldsymbol{R}^3$ is the isomorphism defined by $i(\xi \otimes 1) = (\sigma_1(\xi), \sigma_2(\xi), \sigma_3(\xi))$. Hence we see that $\mathrm{Cusp}(C', \Gamma)$ is isomorphic to $V(N', \Lambda)$, where $C' = \boldsymbol{R}_{>0}v_1 + \boldsymbol{R}_{>0}v_2 + \boldsymbol{R}_{>0}v_3$. Let E be the union of great circles E_1, E_2 and E_3 passing through three pairs of points of $\{\pi(v_1), \pi(v_2), \pi(v_3)\}$. Then Γ acts on $S^2\backslash E$ properly discontinuously and without fixed points and each connected component of $(S^2\backslash E)/\Gamma$ is isomorphic to a 2-dimensional real torus. If $E_j \cap D \neq \emptyset$, then it must be a minor arc of E_j, since $D = C/\boldsymbol{R}_{>0}$ is spherically convex. But it is impossible, since the free abelian group Γ of rank 2 then would act on $E_j \cap D$ properly discontinuously and without fixed points. Hence D agrees with one of the connected components of $S^2\backslash E$, since D/Γ is compact. Therefore, $C = \boldsymbol{R}_{>0}u_1 + \boldsymbol{R}_{>0}u_2 + \boldsymbol{R}_{>0}u_3$ with u_1 (resp. u_2, resp. u_3) $= \pm v_1$ (resp. $\pm v_2$, resp. $\pm v_3$), and hence $\mathrm{Cusp}(C, \Gamma)$ is a Hilbert modular cusp singularity. q.e.d.

COROLLARY 3.2. *If (C, Γ) is in $\mathscr{S}$, then D/Γ cannot be a Klein bottle, where $D = C/\boldsymbol{R}_{>0}$.*

PROOF. Suppose that D/Γ is a Klein bottle. Let $\Gamma' = \Gamma \cap SL(N)$, which is a subgroup of Γ of index 2. Then D/Γ' is a 2-dimensional real torus. We have $C = \boldsymbol{R}_{>0}v_1 + \boldsymbol{R}_{>0}v_2 + \boldsymbol{R}_{>0}v_3$ for some v_1, v_2 and v_3 of $N_{\boldsymbol{R}}$, by the proof of the above theorem. Hence any element γ of Γ not contained in Γ' is expressed in the form

$$\begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & \eta_2 \\ 0 & \eta_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \eta_3 \\ 0 & \eta_1 & 0 \\ \eta_2 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \eta_3 & 0 \\ \eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix},$$

with respect to the basis $\{v_1, v_2, v_3\}$ of $N_{\boldsymbol{R}}$, since C is γ-invariant. Then γ^2 has the double eigenvalue $\eta_2\eta_3$ besides the simple eigenvalue η_1^2. Since the eigenvalues of $\gamma^2 \in SL(N)$ are cubic integers with product 1, we conclude that η_1^2 and $\eta_2\eta_3$ are ± 1. But $\eta_1, \eta_2, \eta_3 > 0$, since $\gamma(C) = C$. Thus $\eta_1 = \eta_2\eta_3 = 1$. Then γ has an eigenvector $v_1 + v_2 + \eta_3 v_3$ or $\eta_3 v_1 + v_2 + v_3$, a contradiction to the fact that $\gamma(\neq I)$ has no fixed point on D. q.e.d.

4. Explicit construction of (C, Γ) in $\mathscr{S}$. In this section, we deal only with the 3-dimensional case for simplicity. We keep the notations from Section 1. Recall that for (C, Γ) in $\mathscr{S}$, we had a Γ-invariant r.p.p. decomposition (N, Δ) consisting of nonsingular cones with $|\Delta|(:= \bigcup_{\sigma \in \Delta} \sigma) = C \cup \{0\}$ and from it a resolution $\tilde{\omega}: (W, Y) \to (V, p)$ of the singularity $(V, p) = \mathrm{Cusp}(C, \Gamma)$. We have a Γ-invariant triangulation $\triangle$ of $D = C/\boldsymbol{R}_{>0}$

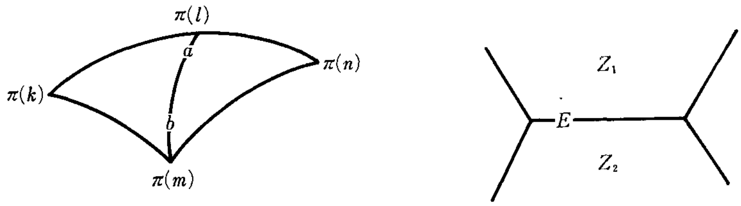


FIGURE 4.1

by the projection $\pi\colon N_R\backslash\{0\} \to S^2$ from (N, Λ). Moreover, we attach integers on both sides of each edge of $\triangle$ in the following way: Let $\mu = \boldsymbol{R}_{\geqq 0}k + \boldsymbol{R}_{\geqq 0}l + \boldsymbol{R}_{\geqq 0}m$ and $\nu = \boldsymbol{R}_{\geqq 0}l + \boldsymbol{R}_{\geqq 0}m + \boldsymbol{R}_{\geqq 0}n$ be 3-dimensional cones of Λ with a common 2-dimensional face $\boldsymbol{R}_{\geqq 0}l + \boldsymbol{R}_{\geqq 0}m$. Since μ and ν are nonsingular, there exist integers a and b such that the equality

$$(*) \qquad n + k + al + bm = 0$$

holds. Then we attach the integers a and b on the sides of $\pi(l)$ and $\pi(m)$, respectively, to the edge $\pi(\boldsymbol{R}_{\geqq 0}l + \boldsymbol{R}_{\geqq 0}m)$ of $\triangle$ as weights (cf. Figure 4.1).

Here, we note that the above integers a an b are equal to the self-intersection numbers $(E_{|Z_2})^2 = Z_1^2 \cdot Z_2$ and $(E_{|Z_1})^2 = Z_1 \cdot Z_2^2$ of the curve $E\colon = \overline{\mathrm{orb}\,(\mu \cap \nu)}$ on the surfaces $Z_2\colon = \overline{\mathrm{orb}\,(\boldsymbol{R}_{\geqq 0}m)}$ and $Z_1\colon = \overline{\mathrm{orb}\,(\boldsymbol{R}_{\geqq 0}l)}$, respectively, e.g., by [16, Proposition 6.7].

Conversely, we can reconstruct $(C, \Gamma) \in \mathscr{S}$ from $\triangle/\Gamma$ and the pair of integers for each edge as follows: Let T be a compact topological surface, $\tilde{T} \to T$ its universal covering and $\Gamma = \pi_1(T)$, the fundamental group of T. Let $\triangle$ be a Γ-invariant triangulation of $\tilde{T}$.

DEFINITION 4.1. A Γ-invariant *N-weighting* of $\triangle$ satisfying the *monodromy condition* at the vertices is a pair (σ, ρ) consisting of a map σ: {all vertices of $\triangle$} $\to N$ and a homomorphism $\rho\colon \Gamma \to GL(N)$ satisfying the following conditions: (i) σ is Γ-equivariant through ρ. (ii) For the three vertices v_1, v_2 and v_3 of each triangle of $\triangle$, their images $\sigma(v_1)$, $\sigma(v_2)$ and $\sigma(v_3)$ form a $\boldsymbol{Z}$-basis of N. (iii) For each vertex v of $\triangle$, if

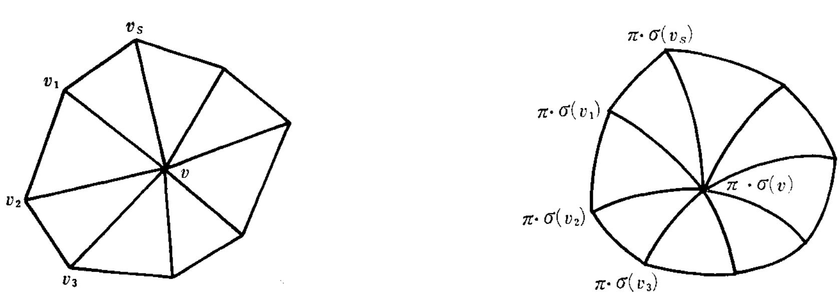


FIGURE 4.2

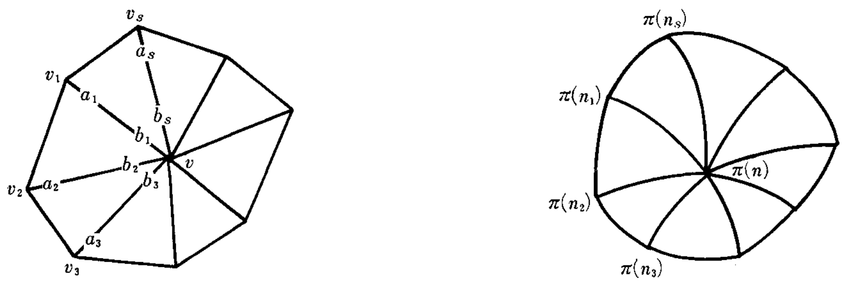


FIGURE 4.3

$v_1, v_2, \cdots, v_s$ are the vertices of its link going around v in this order, then $\pi \cdot \sigma(v_1), \pi \cdot \sigma(v_2), \cdots, \pi \cdot \sigma(v_s)$ also go around $\pi \cdot \sigma(v)$ once in this order (cf. Figure 4.2).

DEFINITION 4.2. A Γ-invariant *double $\boldsymbol{Z}$-weighting* of $\triangle$ satisfying the *monodromy condition* at the vertices is a pair of integers attached to each edge of $\triangle$ with one integer on the side of one vertex and with the other integer on the side of the other vertex satisfying the following conditions: (i) These integers are Γ-invariantly attached. (ii) For each vertex v of $\triangle$, let $v_1, v_2, \cdots, v_s$ be the vertices of its link going around v in this order. Let $\{n, n_1, n_2\}$ be an arbitrary $\boldsymbol{Z}$-basis of N. We then let n_1, n_2 and n be the N-weightings of the vertices v_1, v_2 and v, respectively. Then we can determine the N-weightings $n_3, \cdots, n_s, n_{s+1}$ and n_{s+2} of the vertices $v_3, \cdots, v_s, v_1$ and v_2 in this order by the pair of integers on each edge and by the equality $(*)$. Then we require that $n_{s+1} = n_1$, $n_{s+2} = n_2$ and that $\pi(n_1), \pi(n_2), \cdots, \pi(n_s)$ go around $\pi(n)$ once in this order (cf. Figure 4.3).

Let DZW be the set of all Γ-invariant triangulations $\triangle$ of the universal covering spaces $\tilde{T}$ of compact surfaces T, endowed with Γ-invariant double $\boldsymbol{Z}$-weightings satisfying the monodromy condition at the vertices, where $\Gamma = \pi_1(T)$. For $\triangle$ in DZW, choose a $\boldsymbol{Z}$-basis of N and a triangle of $\triangle$, and attach the three elements of the $\boldsymbol{Z}$-basis to the three vertices of the triangle as N-weights. Then we have an N-weighting of $\triangle$, i.e., a map σ: {all vertices of $\triangle\} \to N$, by the equality $(*)$, since $\tilde{T}$ is simply connected. Moreover, we have a homomorphism $\rho: \Gamma \to GL(N)$, by $\rho(\gamma) \cdot \sigma(v) = \sigma(\gamma \cdot v)$ for any element γ of Γ and vertices v of $\triangle$. Clearly, the pair (σ, ρ) is a Γ-invariant N-weighting of $\triangle$ satisfying the monodromy condition at the vertices. We obtain a Γ-equivariant local homeomorphism $f: \tilde{T} \to S^2$, extending the map $\pi \cdot \sigma$ such that the image of each triangle of $\triangle$ is a spherical triangle. We denote by $C(\mu)$, the cone $\pi^{-1}(f(\mu)) \cup \{0\} = \boldsymbol{R}_{\geq 0} \cdot f(\mu)$ for each simplex μ of $\triangle$. Let

$$\Lambda = \{C(\mu) \mid \text{simplexes } \mu \text{ of } \triangle\} \cup \{\{0\}\},$$

$C = |\Lambda| \setminus \{0\} = \pi^{-1}(f(\tilde{T}))$ and $D = \pi(C) = f(\tilde{T})$. Clearly we have:

PROPOSITION 4.3. *Assume that the following condition* (**) *is satisfied*:

(**) *f is injective, $f(\tilde{T})$ is spherically convex and its closure $\overline{f(\tilde{T})}$ is contained in a hemisphere of S^2.*

Then (N, Λ) is an r.p.p. decomposition of N, C is a Γ-invariant open nondegenerate convex cone, ρ is injective and the action of Γ on D is properly discontinuous and fixed point free. Hence (C, Γ) is in $\mathscr{S}$.

REMARK. A 2-dimensional cusp singularity (V, p) corresponds to a 1-dimensional periodic continued fraction $\omega = [[\overline{b_1, b_2, \cdots, b_s}]]$, where b_j are integers greater than or equal to 2. (See for instance [11].) The former is obtained from the latter in a manner similar to the one above. In this case, $T = S^1$, $\tilde{T}$ is a line and $\triangle$ is a triangulation of $\tilde{T}$, on the vertices $\{v_j\}$ of which the integers $-b_j$ attached periodically, i.e., $-b_j = -b_k$ if $j \equiv k$ (mod. s). Then we have a map σ: {vertices of $\triangle\} \to \boldsymbol{Z}^2$ by the equality: $\sigma(v_{j-1}) + \sigma(v_{j+1}) - b_j\sigma(v_j) = 0$ for all $j \in \boldsymbol{Z}$ and $\sigma(v_0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\sigma(v_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Moreover, we have a matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & b_s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & b_{s-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ -1 & b_1 \end{bmatrix}.$$

Then $(V, p) = \operatorname{Cusp}(C, \Gamma)$, *where* Γ is the cyclic group of infinite order

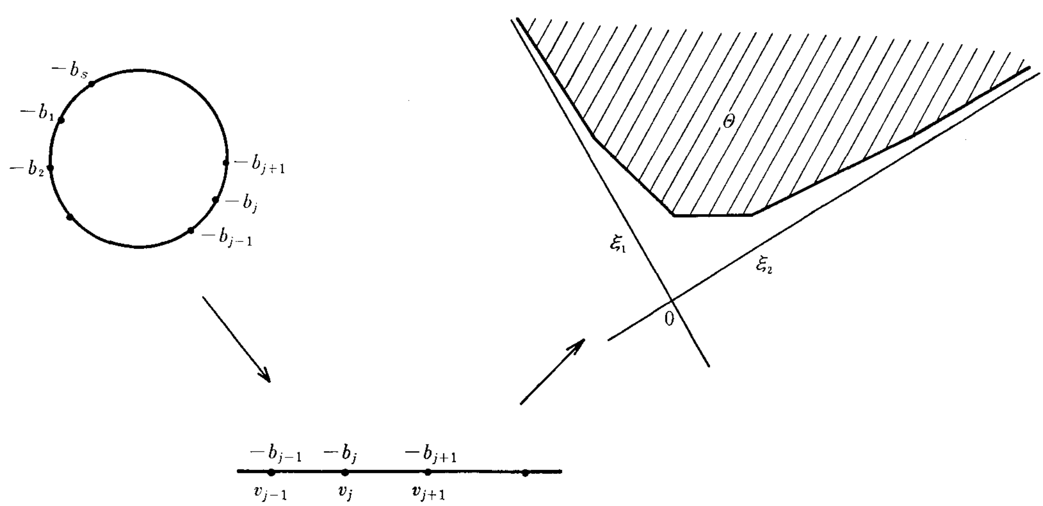


FIGURE 4.4

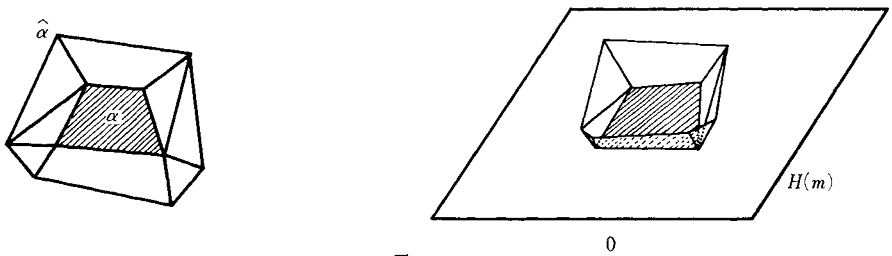


FIGURE 4.5

generated by A, and C is the cone in $\boldsymbol{R}^2$ generated by the two eigenvectors ξ_1 and ξ_2 of A. (See Figure 4.4.)

In the following, we will examine some sufficient condition under which f satisfies the above condition $(**)$. For each nonzero element m of $M_{\boldsymbol{R}}$ we define the affine plane $H(m)$ in $N_{\boldsymbol{R}}$ by $H(m) = \{n \in N_{\boldsymbol{R}} \mid \langle m, n \rangle = 1\}$ as in Section 1.

DEFINITION 4.4. A $\pi_1(T)$-invariant triangulation $\triangle$ of the universal covering space $\tilde{T}$ of a compact topological surface T with a $\pi_1(T)$-invariant double $\boldsymbol{Z}$-weighting satisfying the monodromy condition at the vertices is *strictly locally convex* (resp. *locally convex*) if there exists a $\pi_1(T)$-invariant cell division $\square$ of $\tilde{T}$, of which $\triangle$ is a subdivision and which satisfies the following condition (P) (resp. (P′)): Attaching the three elements of a $\boldsymbol{Z}$-basis of N to the three vertices of a triangle of $\triangle$, we have a $\pi_1(T)$-invariant N-weighting of $\triangle$ satisfying the monodromy condition at the vertices. For each 2-dimensional cell α of $\square$, there exists a unique element m of M (resp. $M_{\boldsymbol{R}}$) such that $\sigma(v)$ is on the plane $H(m)$ (i.e., $\langle m, \sigma(v)\rangle = 1$) for any vertex v of α, and that $\sigma(v)$ is above the plane $H(m)$ (i.e., $\langle m, \sigma(v)\rangle > 1$) for any vertex v of $\hat{\alpha}\backslash\alpha$, where $\hat{\alpha}$ denotes the union of all cells of $\square$, which have common faces with α. (See Figure 4.5.)

In the above definition $\sigma(v)$ need not be on the plane $H(m)$, for a vertex v of $\triangle$ in α if it is not a vertex of $\square$. Clearly "strictly locally convex" implies "locally convex". The local convexity conditions (P) and (P′) in fact imply the "global convexity" as we now see in Theorem 4.5.

THEOREM 4.5. *If a $\pi_1(T)$-invariant triangulation $\triangle$ of the universal covering space $\tilde{T}$ of a compact topological surface T with a $\pi_1(T)$-invariant double $\boldsymbol{Z}$-weighting satisfying the monodromy condition at the vertices is locally convex, then $(**)$ is satisfied,* i.e., *the map $f\colon \tilde{T} \to S^2$ induced by it as after Definition 4.2 is injective, $f(\tilde{T})$ is spherically convex and its closure is contained in a hemisphere of S^2.*

FIGURE 4.6

PROOF. Fix a point s of $\widetilde{T}$ and take a sufficiently small circle S^1 on S^2 with the center $f(s)$. Let W be the union $\bigcup_{\theta \in S^1} W_\theta$ of the longest curves W_θ on $\widetilde{T}$ starting from s whose images by f are on the great circles of S^2 intersecting S^1 at θ. Then W is an open set of $\widetilde{T}$. As we will see shortly in the proof of the sublemma, $f(W_\theta)$ is strictly a minor arc of a great circle for each θ of S^1. Then $f(W_\theta) \cap f(W_{\theta'}) = \{f(s)\}$, if $\theta \neq \theta'$. Hence the restriction $f_{|W}$ of f to W is injective. There is a unique Γ-equivariant continuous locally injective map $\tau\colon \widetilde{T} \to N_R \backslash \{0\}$ such that $f = \pi \cdot \tau$ and that the image $\tau(\lambda)$ of each 2-dimensional cell λ of a cell division $\square$ of $\widetilde{T}$ as in Definition 4.4 is on a plane in N_R. Let α be a 2-dimensional cell of $\square$ containing s, and let m_o be the element of M_R such that $\tau(\alpha)$ is on the plane $H(m_o)$. Then $k(t) := \langle m_o, \tau(t)\rangle$ and $h := k \cdot f_{|W}^{-1}$ are continuous functions on $\widetilde{T}$ and $D = f(W)$, respectively. (See Figure 4.6.)

SUBLEMMA. *$h^{-1}(l)$ is a closed curve for any positive real number l greater than* 1.

PROOF. First we show that the length of $\tau(W_\theta)$ is infinite by any Euclidean metric of N_R for any θ of S^1. Let $\{u_j\}$ be the set of all turning points of $\tau(W_\theta)$ for $\theta \in S^1$ fixed once for all. Namely, u_j is either the image by τ of the point at which W_θ intersects transversally an edge of $\square$ or is a vertex of $\square$ which lies on W_θ. By the local convexity assumption, we easily see that $\{u_j\}$ is an infinite set. Let $v_j = u_j$, when $\tau_{|W}^{-1}(u_j)$ is a vertex of $\square$. When $\tau_{|W}^{-1}(u_j)$ lies on an edge E of $\square$, let v_j be one of the images by τ of two vertices v_j^1 and v_j^2 of E with $\langle m_o, v_j\rangle \leqq \langle m_o, u_j\rangle$. Then $v_j \in \tau W$ and $\{v_j\}$ is an infinite set, since for all vertices v of $\square$, the numbers of edges meeting at v are bounded. Suppose that the length of $\tau(W_\theta)$ is finite and let $l_o = \sup\{k(t) \mid t \in W_\theta\}$. Then clearly $V_o := \tau(\{t \in W \mid k(t) \leqq l_o\})$ must be contained in some compact set K of N_R. Since $k \cdot \tau_{|W}^{-1}(v_j) = \langle m_o, v_j\rangle \leqq \langle m_o, u_j\rangle < l_o$ and $v_j \in N$, $\{v_j\}$ is contained in $K \cap N$, a contradiction to the finiteness of the lattice

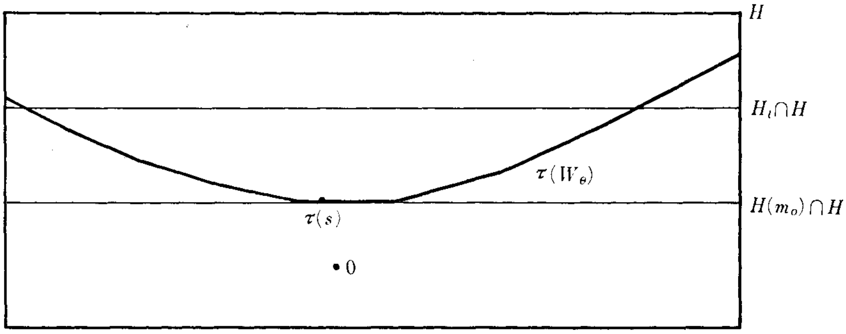


FIGURE 4.7

points $K \cap N$ in the compact set K. Hence the length of $\tau(W_\Theta)$ is infinite. There exists a unique plane H containing $\tau(W_\Theta)$ for each $\Theta \in S^1$. Let H_l be the plane in N_R defined by $\{n \in N_R \mid \langle m_o, n\rangle = l\}$. Then H_l is parallel to $H(m_o)$ and intersects H along a line. By the condition (P′) of Definition 4.3, $\tau(W_\Theta)\backslash I$ is above the line containing I in H, for any line segment I of $\tau(W_\Theta)$. Therefore, $\tau(W_\Theta) \cap H_l$ is exactly one point. Hence we have the map $S^1 \to D$ sending each element Θ of S^1 to $\pi(\tau(W_\Theta) \cap H_l)$. Clearly it is continuous and its image is equal to $h^{-1}(l)$. (See Figure 4.7.) q.e.d.

PROOF OF THEOREM 4.5 CONTINUED. Let $D(l) = \{u \in D \mid h(u) \leqq l\}$. By the above sublemma, $D(l)$ is closed for any real number l greater than 1. Let $\{t_j\}$ be a sequence of points of W, converging to a point of $\tilde{T}$. Since k is a continuous function, $l_o = \max\{k(T_j)\}$ is finite. Then $\{f(t_j)\}$ converges to a point u of $D(l_o) \subset D$. Since $f_{|W}$ is homeomorphic, $t_j = f_{|W}^{-1} \cdot f(t_j)$ converges to $f_{|W}^{-1}(u)$, which is a point of W. Therefore W is closed. Hence $W = \tilde{T}$ and $f\colon \tilde{T} \to D$ is injective. Moreover, D is star-shaped with $f(s)$ as the center, i.e., for any point t of D, the minor arc of the great circle joining $f(s)$ and t is contained in D. However, the choice of s at the beginning of this proof was arbitrary. Hence D is spherically convex. Since $\tau(W \setminus \alpha)$ and $\tau(\alpha)$ are above and on the plane $H(m_o)$, respectively, the closure of $D = \pi \cdot \tau(\tilde{T})$ is contained in the hemisphere which is the image of $H(m_o)$ by the projection π. q.e.d.

By this theorem, each locally convex $\triangle$ in DZW induces a (C, Γ) of $\mathscr{S}$. If $\triangle$ is strictly locally convex, then $\tau(\tilde{T})$ in the proof, coincides with the boundary $\partial\Theta$ of the convex hull Θ of $C \cap N$.

PROPOSITION 4.6. *The following three conditions are equivalent for* (C, Γ) *in* $\mathscr{S}$.

(1) $(C, \Gamma) \in \mathscr{S}_o$.

(2) *(C, Γ) comes from a strictly locally convex $\triangle$ in* DZW.

(3) *(C, Γ) comes from a $\triangle$ in* DZW *satisfying the following conditions*: (i) *The sum of the double weights on each edge of $\triangle$ is not greater than -2.* (ii) *We get a cell division $\square$ by deleting all edges of $\triangle$ which have the sum of the double weights equal to -2.*

In this case, the Euler number $\chi(T) < 0$.

PROOF. (3) $\Rightarrow$ (2). Let (σ, ρ) be the induced Γ-invariant N-weighting of $\triangle$. Let two triples (v_1, v_2, v_3) and (v_2, v_3, v_4) of vertices of $\triangle$ form two adjacent triangles of $\triangle$. Then $\sigma(v_1)$, $\sigma(v_2)$ and $\sigma(v_3)$ lie on a plane $H(m)$ with m belonging to M, since they form a $\boldsymbol{Z}$-basis of N. By the equality $(*)$, $\sigma(v_4)$ is above (resp. on, resp. under) the plane $H(m)$, i.e., $\langle m, \sigma(v_4)\rangle > 1$ (resp. $= 1$, resp. < 1) if and only if the sum of the double weights on the edge incident to both v_2 and v_3, is smaller than (resp. equal to, resp. greater than) -2. From these facts, we see immediately that σ and $\square$ satisfy the condition (P) of Definition 4.4.

(2) $\Rightarrow$ (1). For each 2-dimensional cell α of a cell division $\square$ as in Definition 4.4, $\tau(\alpha)$, which is a face of $\partial\Theta$, is on a plane $H(m)$ with m belonging to M. Thus all vertices of $\partial\Theta^{o}$ are contained in M. Hence $\Theta^{o} = \Theta^{*}$.

(1) $\Rightarrow$ (3). Since $\Theta^{o} = \Theta^{*}$, each face F of $\partial\Theta$ is on a plane $H(m)$ with m belonging to M. Take an arbitrary tiangulation $\triangle_F$ of F with the vertex set $N \cap F$. (See Figure 4.8.) Then the three vertices n_1, n_2 and n_3 of each triangle of $\triangle_F$ form a $\boldsymbol{Z}$-basis of N, since $\langle m, n_1\rangle = 1$ and $n_2 - n_1$ and $n_3 - n_1$ form a basis of the $\boldsymbol{Z}$-submodule $\{n \in N \mid \langle m, n_1\rangle = 0\}$ of N. The manner of the division of each edge of $\partial\Theta$ in the above triangulation is unique. Hence we obtain a Γ-invariant triangulation $\triangle$ of $\partial\Theta$ in the following way: Taking representatives of faces of $\partial\Theta/\Gamma$, triangulate each of them as above, and translate it to other faces of

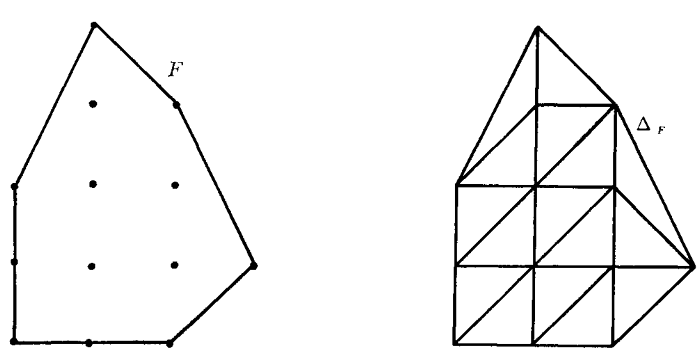


FIGURE 4.8

$\partial\Theta$ by the action of Γ. We have a Γ-invariant double $\boldsymbol{Z}$-weighting of $\triangle$ by the equality $(*)$. Clearly it satisfies the conditions (i) and (ii) of (3), and induces (C, Γ).

To prove the last assertion, we may replace Γ by a subgroup of finite index. Thus without loss of generality, we assume that $\triangle/\Gamma$ gives rise to a triangulation of T. Let s_0, s_1 and s_2 be the numbers of the vertices, edges and faces, respectively, in $\triangle/\Gamma$. Let $v_1, v_2, \cdots, v_{s(v)}$ be the vertices of the link of a vertex v of $\triangle$, and $a_1, a_2, \cdots, a_{s(v)}$ be the weights attached to the sides of $v_1, v_2, \cdots, v_{s(v)}$ of the edges $vv_1, vv_2, \cdots, vv_{s(v)}$, respectively. Then we have the equality: $\sum_{i=1}^{s(v)} a_i = 3(4 - s(v))$ ([16, p. 58]). Since $2s_1 = 3s_2$, we have

$$\begin{aligned} 0 &> \sum_{\text{edges of } \Delta/\Gamma} (2 + \text{the sum of the double weights}) \\ &= \sum_{\text{vertices of } \Delta/\Gamma} \left(\sum_{i=1}^{s(v)} a_i\right) + 2s_1 \\ &= 12s_0 - 3(2s_1) + 2s_1 = 12(s_0 - s_1 + s_2) = 12\chi(T) . \end{aligned} \qquad \text{q.e.d.}$$

5. Examples. (I) Let $\triangle_1$ be an octahedral triangulation of a 2-dimensional sphere S^2. Take a double covering T of S^2 ramified at all six vertices of $\triangle_1$ and let $\triangle_2$ be the triangulation of T induced by $\triangle_1$. Then T is a compact orientable surface of genus 2. Let $\triangle$ be the triangulation of the universal covering space $\tilde{T}$ of T induced by $\triangle_2$, and let $\Gamma = \pi_1(T)$. We have Γ-invariant double $\boldsymbol{Z}$-weightings of $\triangle$, pulling back those of $\triangle_1$ as in Figure 5.1 (i) through (vii), by the map $\tilde{T} \to T \to S^2$. We easily see that they are convex and satisfy the monodromy condition.

(II) Let T be the surface and $\triangle_1$ be its triangulation we obtain from the one in Figure 5.2 by identifying the two edges and the four vertices having the same numbers and the same symbols, respectively. Then T is a non-orientable surface with Euler number $\chi(T) = -2$. Attach -2 (resp. -1) on both sides of the edges of $\triangle_1$ which come from thick (resp. thin) lines of the one in Figure 5.2. Then pulling it back on the universal covering space $\tilde{T}$ of T, we have a convex member in DZW.

(III) Let $\square_1$ be a tetrahedral triangulation of a sphere S^2. Let T be the double covering of S^2 ramified at all vertices and centers of all faces of $\square_1$, and let $\square_2$ be the hexagonal subdivision of T induced by $\square_1$. Then T is a compact topological surface with $g(T) = 3$ and $\square_2$ is self-dual, i.e., the dual graph $\square_2^*$ of $\square_2$ is mapped to $\square_2$ by an isomorphism of T. Let $\triangle_2$ be a triangulation of T we obtain by triangulating each hexagon of $\square_2$ as in Figure 5.3.1. To each edges of $\triangle_2$ which are

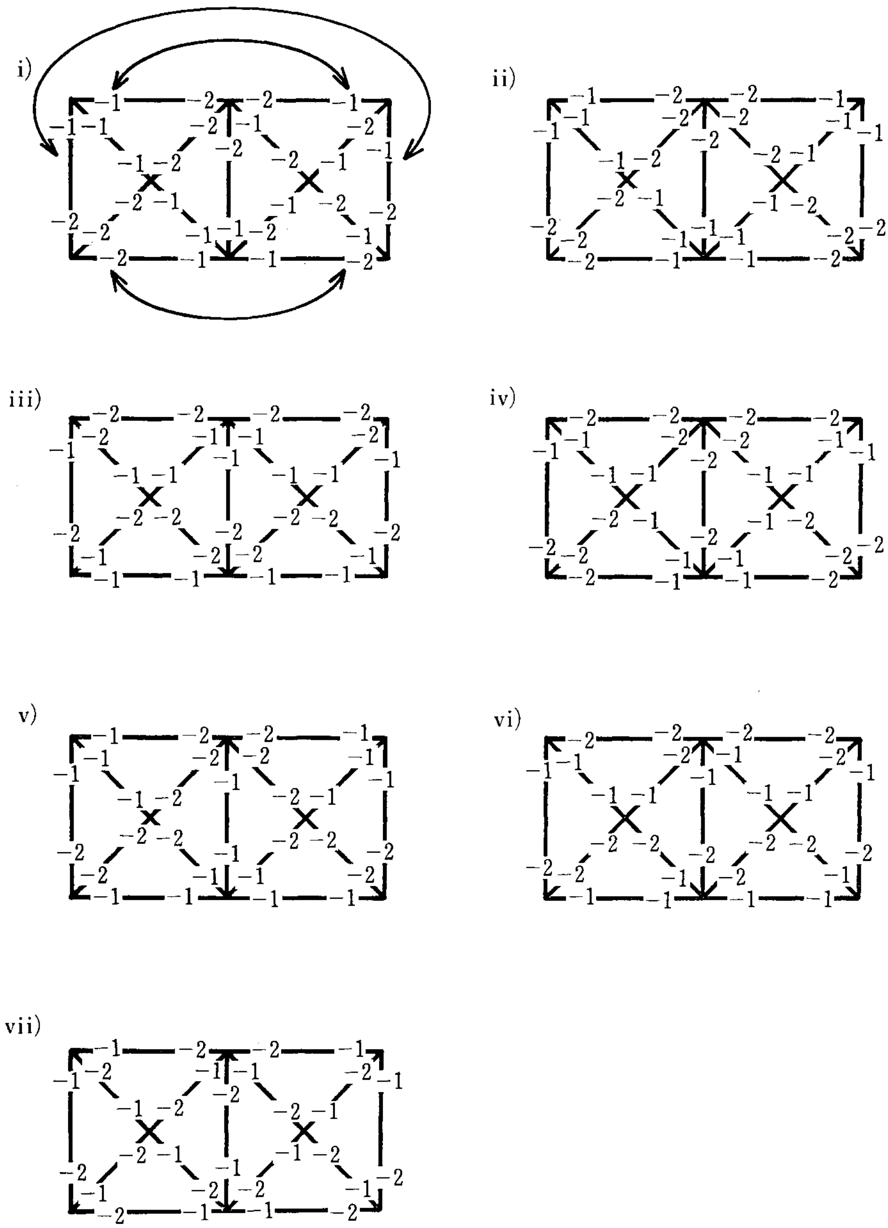


FIGURE 5.1

on the edges of $\square_2$, attach double weights -1 and -3 on the side of the vertex of $\square_2$ and on the opposite side, respectively. Attach -1 on both sides of the other edges of $\triangle_2$. (See Figure 5.3.2.) Pulling them back to the triangulation $\triangle$ on a universal covering space $\tilde{T}$ of T induced by $\triangle_2$, we have a $\pi_1(T)$-invariant double $\boldsymbol{Z}$-weighting on $\triangle$ satisfying the monodromy condition at the vertices. Clearly, it satisfies

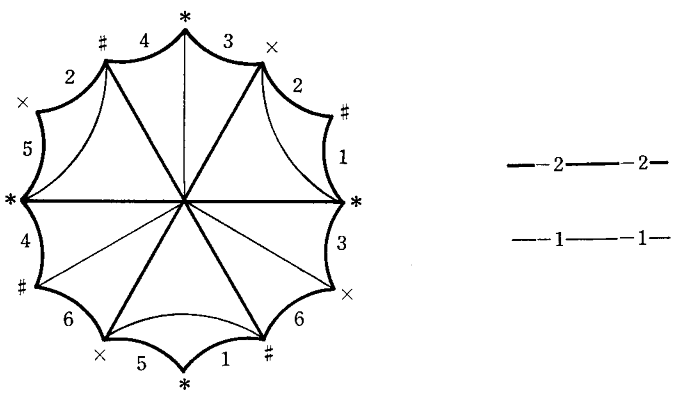


FIGURE 5.2

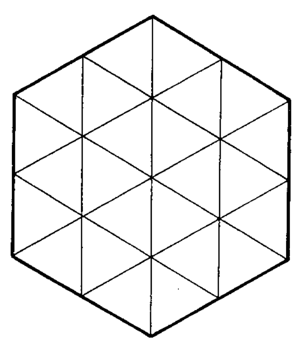


FIGURE 5.3.1

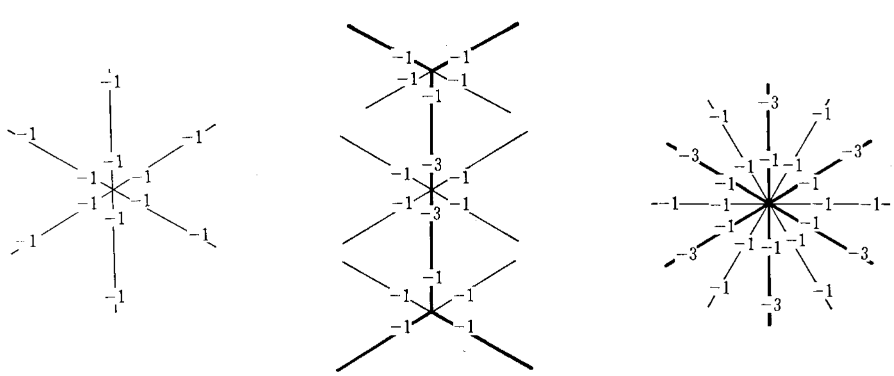


FIGURE 5.3.2

the condition (3) of Proposition 4.6. Let (C, Γ) be an element of $\mathscr{S}_0$ obtained from the above $\triangle$, and let (C^*, Γ) be the dual element of $\mathscr{S}_0$. We see by easy calculations that there is an isomorphism $i: N \to M = \mathrm{Hom}(N, \boldsymbol{Z})$ such that $i_{\boldsymbol{R}}$ sends the cone C onto the dual cone C^* and that $i \cdot \gamma = \gamma \cdot i$ for all γ of Γ. Hence we have $\mathrm{Cusp}(C, \Gamma) \simeq \mathrm{Cusp}(C^*, \Gamma)$.

We list in Table 5.1 the numerical invariants for the singularities we obtain in the above examples.

TABLE 5.1

$\triangle$	T	$\chi(T)$	the length	the principal degree
I (i)	orientable	-2	6	35
(ii)	〃	〃	〃	36
(iii)	〃	〃	〃	29
(iv)	〃	〃	〃	34
(v)	〃	〃	〃	32
(vi)	〃	〃	〃	28
(vii)	〃	〃	〃	40
II	non-orientable	-2	4	46
III	orientable	-3	44	44

REMARK. Let C_1, C_2 and C_3 be the cones arising from the above examples (I) (vii), (II) and (III), respectively. Then the cones C_2 and C_3 are expressed as

$$\{a_1n_1 + a_2n_2 + a_3n_3 \mid a_1^2 - 8(a_2^2 + a_2a_3 + a_3^2) > 0,\ a_1 > 0\}$$

and

$$\{a_1n_1' + a_2n_2' + a_3n_3' \mid a_1^2 - 6(a_2^2 + a_2a_3 + a_3^2) > 0,\ a_1 > 0\}\,,$$

for some $\boldsymbol{Z}$-bases $\{n_1, n_2, n_3\}$ and $\{n_1', n_2', n_3'\}$ of $\boldsymbol{Z}^3$, respectively. Hence these cones are circular. On the other hand, the cone C_1 is not circular. Otherwise, there must be a quadratic form Q on $\boldsymbol{R}^3$ such that $Q(\sigma(gv)) = 1$ for any element g of $\pi_1(T)$ and for a vertex v of a triangulation $\triangle$ of $\tilde{T}$, where σ is an N-weighting of $\triangle$ induced by the double $\boldsymbol{Z}$-weighting of the example (I) (vii). However, by an easy calculation, we can verify that such a quadratic form does not exist.

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