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Higher-dimensional self-consistent solution with deformed internal spaces

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We study a system of gravity and free massless scalar fields minimally coupled to gravity in a 7-dimensional background which is a direct product of a 4-dimensional Minkowski space and a 3-dimensional homogeneously deformed three-sphere. Compactification is caused by the vacuum energy of scalar fields. The effective potential as a function of two parameters (scale and deformation) is calculated numerically after dimensional regularization. We find the effective potential decreases rapidly toward negative infinity in both prolate and oblate directions. The classical curvature, however, can balance the quantum effect and yields three extrema. In addition to the round S^3 solution in the quantum-corrected field equations, we find two deformed ones. One of the deformed solutions corresponds to the local minimum of the total potential. The round three-sphere solution, however, corresponds to the local maximum of that. More scalar fields can enlarge the scale of the internal space but not affect the shape. This serves as an example of gauge symmetry breaking by deformation of the internal space in multidimensional theories. The stability of these background solutions is discussed but not established conclusively. A discussion of four different analytic-continuation procedures is presented in one of the appendixes.

I. INTRODUCTION

Several mechanisms have been proposed for higher-dimensional theories to compactify part of the spacetime dimensions (usually spacelike) to make them undetectable at present energies. Some mechanisms are pure classical. One interesting scheme is to introduce elementary gauge fields[1] in higher dimensions. Their nonzero vacuum expectation value in the classical field equations can yield a solution with $M^4\times B^N$ configuration, where M^4 is 4-dimensional Minkowski space and B^N an N-dimensional compact space. Another possibility is to introduce a nonlinear σ model.[2] In the context of supergravity, the vacuum expectation value of antisymmetric tensor fields can also be employed to compactify the 11-dimensional ground state.[3] In this scheme, however, a large negative cosmological constant is induced. As a result, the 4-dimensional spacetime is anti–de Sitter instead of Minkowski. A spin-torsion compactification[4] is proposed as a semiclassical approach to solve the cosmological-constant problem, if an appropriate fermionic bound state can be found. Yet there is another semiclassical scheme which we will discuss here: namely, compactification by the vacuum energy of the matter field.

The scale of the internal space is generally believed to be not too much larger than the Planck length ($\simeq 10^{-33}$ cm). Therefore, quantum effects should be taken into account,[5] although a quantum theory of gravity is still lacking. As was first pointed out by Appelquist and Chodos,[6] the one-loop quantum fluctuation of gravity produces a Casimir effect in the 5-dimensional Kaluza-Klein model and may lead to a collapse of the fifth dimension. However, the properties of vacuum energy depend strongly on the topology and geometry of the manifold. This feature gives the vacuum energy a chance (as demonstrated by Candelas and Weinberg[7]) for certain higher-dimensional backgrounds, to balance the classical Einstein tensor in classical field equations and yield self-consistent solutions. However, one must keep in mind that in order to have a quantum effect comparable to classical terms in the action and without considering effects of quantum gravity, a large number of matter fields (including, perhaps more efficiently, higher-spin fields) should be considered. This model can fix the size of the internal space and provides an opportunity to calculate gauge coupling constants.[8] The calculability of these quantities, however, is limited to odd dimensions and odd-loop quantum fluctuations (with dimensionless regularization[9]). Other cases are plagued by the ambiguity of unknown coupling constants of geometrical counterterms (e.g., $R^{(4+N)/2}$) (Ref. 10). Unfortunately, in odd dimensions one has to surmount the difficult problem of chiral fermions.[11] Nevertheless, this mechanism may still play a role in more realistic models.

In this paper we use a 7-dimensional model to demonstrate that it is possible to have a $M^4\times$Taub background supported by the vacuum energy of a massless scalar field minimally coupled to gravity. Taub space[12] is a 3-dimensional homogeneous but anisotropic space with isometry group SU(2)$\times$U(1). It is, roughly speaking, a deformed three-sphere characterized by two geometric parameters; the scale a and the deformation α (see Sec. II). The reason that we choose to work with the static Taub space is twofold. Firstly, it is the lowest-dimensional model to allow homogeneous deformation which we consider as a slightly more generalized step than the dilatation mode considered by many authors toward a systematic study of the stability of a $M^4\times S^N$ background. Secondly, the Taub space has an isometry group: SU(2)$\times$U(1), which may lead to phenomenologically interesting gauge groups from a 4-dimensional point of view. One may take this "geometric symmetry breaking" as an alternative to the usual Higgs mechanism in grand unification theories (GUT's), except that it occurs at the compactification energy scale which is much higher than the usual GUT symmetry-breaking energy scale. Geometric symmetry breaking[13] is not a new idea, ever since the revival of Kaluza-Klein theories. For example, deforming the spherical background to an ellipsoid to account for gauge symmetry breaking from SO($N+1$) to SO(N) has been considered by Lim.[14] A 6-dimensional gravitation—Maxwell system with a Higgs field and with a nonlinear σ field have been proposed by Sobczyk[15] and Shin,[16] respectively, to account for electoweak symmetry breaking. The low-mass scale (100 GeV) of the vector bosons can be obtained by extremely small deformation of the internal S^2. One of the differences of the present work with the above-mentioned ones is that the geometric deformation here considered is homogeneous which may have less "kinetic energy" than the inhomogeneous ones. Of course, geometric symmetry breaking is relevant only if at least a part of the 4-dimensional gauge group originates in isometries of the internal space.

In this paper we consider only a free massless scalar field minimally coupled to an $M^4\times$Taub background. The exact effective potential can thus be obtained. The effective potential is formally divergent. An analytic continuation which involves a Sommerfeld-Watson transformation is used to render it finite. In this way we obtain an effective potential which is a function of deformation parameter α and internal space volume Ω (see Fig. 4). Taking this potential as the energy source for Einstein equations, we find that in addition to the spherical solution, there are two deformed ones (see Fig. 5). It seems that one deformed solution can be stable; however, it could be an illusion (see Sec. IV). The complete stability analysis will be reported elsewhere. Besides the physical aspects, we find analytic continuation is of its own interest. We have collected four different methods and convinced ourselves that they give the same answer in the common domain. However, each one of them has its advantage and limit. We devote Appendix B to a detailed discussion.

Recently, we found similar work has been done. Okada[17] considered a conformal scalar field in the same background but only for oblate deformation ($\alpha>0$ in our notation). With respect to our work, many technical complexities can be avoided in this case. Page[18] has conducted a minimal scalar-field effective-potential calculation in the small-α expansion which we shall discuss in Appendix C. Shiraishi[19] also discussed small-deformation stability of the effective potential of a Dirac field and a scalar field with an arbitrary positive coupling to gravity in the background of $M^4\times S^3$ and $M^4\times S^7$.

The plan of the remainder of this paper is as follows. In Sec. II we present the calculation of the effective potential. In Sec. III we find self-consistent solutions. In Sec. IV the problem of stability and some physical implication of these solutions are discussed. Technical discussions on sums and regularization are presented in Appendixes A and B. A small-α perturbative calculation of the effective potential is presented in Appendix C.

II. EFFECTIVE POTENTIAL

Consider a scalar field Φ in a 7-dimensional spacetime. The classical action of the system has the form[20]

$$S=\int dV_7\left[\frac{1}{\bar{\kappa}}(\bar{R}-2\bar{\Lambda})+\tfrac{1}{2}\Phi(\bar{\Box}-\xi\bar{R}-m^2)\Phi\right], \quad (2.1)$$

where dV_7 is the volume element and quantities with overbars are in the 7-dimensional spacetime. We assume that this system admits a $M^4\times S^3$ geometry as a solution of the quantum-corrected field equations where M^4 denotes a Minkowski space and S^3 a three-sphere with possible homogeneous deformation. The line element of this background geometry is given by

$$ds^2=\eta_{\mu\nu}dx^\mu dx^\nu+\sum_{a=1}^{3}(l_a\sigma^a)^2\ , \quad (2.2)$$

where x^μ's are coordinates in the Minkowski space; σ^a's form a basis one-form on S^3 satisfying the structure relation $d\sigma^a=\frac{1}{2}\epsilon^a{}_{bc}\sigma^b\wedge\sigma^c$. In the Euler-angle parametrization[21] ($0\le\theta\le\pi$, $0\le\phi$, $\psi\le 2\pi$) the σ^a's are given by

$$\sigma^a=\cos\psi\,d\theta+\sin\psi\sin\theta\,d\phi\ , \quad (2.3a)$$

$$\sigma^b=-\sin\psi\,d\theta+\cos\psi\sin\theta\,d\phi\ , \quad (2.3b)$$

$$\sigma^c=d\psi+\cos\theta\,d\phi\ . \quad (2.3c)$$

The l_a's are principal curvature radii of the homogeneous internal space and in general a function of the external spacetime. The case where all l_a's are equal corresponds to the usual round three-sphere with isometry SU(2)$\times$SU(2)$\simeq$SO(4). The case when two l_a's are equal corresponds to the Taub space with isometry SU(2)$\times$U(1). This is the case we shall discuss in this paper. The most general case with three different l_a's corresponds to the mixmaster space with SU(2) isometry only.

Let $l_1=l_2\neq l_3$. The background geometry depends on two parameters only: the scale a and deformation α defined by

$$a=2l_1\ , \quad (2.4)$$

$$\alpha=(l_1/l_3)^2-1\ . \quad (2.5)$$

The range of α is $-1<\alpha<\infty$ and $\alpha=0$ corresponds to round S^3. In particular, since we are looking for static background, we consider a and α constant. The curvature scalar of the geometry (2.2) can thus be reduced to[22]

$$\bar{R}=\tilde{R}=\frac{4l_1{}^2-l_3{}^2}{2l_1{}^4}=\frac{2(3+4\alpha)}{a^2(1+\alpha)}\ , \quad (2.6)$$

where tilded quantities are in the internal space. The Laplace-Beltrami operator in this geometry can also be simplified into a Minkowski space one and its Taub-space counterpart

$$\bar{\Box}=\Box+\tilde{\Box}\ . \quad (2.7)$$

The scalar field $\Phi(x,y)$, where y denotes internal coordinates, can then be expanded by a complete orthonormal set of eigenfunctions $Y_M(y)$ of the operator $-\tilde{\Box}+\xi\tilde{R}+m^2$ with eigenvalues $\{\lambda_M\}$ as

$$\Phi(x,y)=\sum_M \phi_M(x)Y_M(y) . \tag{2.8}$$

Y_M's are the well-known three-dimensional rotation-group representation functions which are of the form[23]

$$D^J_{KL}(\theta,\phi,\psi)=e^{iL\phi}d^J_{KL}(\theta)e^{iK\psi} , \tag{2.9}$$

where J,K,L take all values of integers and half-integers and $K,L=-J,-J+1,\ldots,J-1,J$. The eigenvalues λ_M are obtained[23] as

$$\lambda_M=\frac{J(J+1)}{l_1{}^2}+\left[\frac{1}{l_3{}^2}-\frac{1}{l_1{}^2}\right]K^2+m^2+\xi\tilde{R} . \tag{2.10}$$

The quantum number L is totally degenerate in the Taub-space case.

On making use of the orthonormality of Y_M and homogeneity of the internal-space geometry, the 7-dimensional scalar action can be reduced to a 4-dimensional one with infinite number of massive scalar fields

$$S^0=\frac{-1}{2}\sum_M\int dV_x\phi_M(x)(-\Box+\lambda_M)\phi(x) . \tag{2.11}$$

The effective action of a real scalar field in vacuum is given formally by

$$\Gamma=-i\ln\int[d\phi]e^{iS^0[\phi]} . \tag{2.12}$$

Since we are dealing with free fields (2.11), the functional integral of (2.12) can be performed exactly and yields

$$\Gamma=-\frac{i}{2}\mathrm{tr}\ln\left[\frac{iH}{\mu^2}\right] , \tag{2.13}$$

where $H=\partial^2S/\partial\phi^2$ and a mass scale μ is introduced to render $[d\phi]$ and Γ dimensionless. The eigenvalues of H can be read off from (2.11) as $k^2+\lambda_M$. The effective action (2.13) then follows:

$$\Gamma=\frac{i\Omega_d\mu^{4-d}}{2(2\pi)^d}\int d^dk\sum_M\ln(k^2+\lambda_M)$$
$$=\frac{\Omega_d\mu^{4-d}}{2(2\pi)^{d/2}}\Gamma\left[\frac{-d}{2}\right]\sum_M(\lambda_M)^{d/2} , \tag{2.14}$$

where the dimension d is a complex variable so that Γ as a function of d can be analytically continued. One can then define the matter effective potential in the Minkowski spacetime by $V=-\Gamma/\Omega_d$. It proves to be convenient to use the following definitions:

$$n=2J+1,\quad q=J-K,\quad \sigma=m^2a^2+\xi\tilde{R}-1 .$$

The eigenvalues can thus be written as

$$\lambda_M=[n^2+\sigma+\alpha(n-1-2q)^2]/a^2 . \tag{2.15}$$

In this paper we shall consider only a massless, minimally coupled ($\xi=0$) scalar field; σ reduces to -1 and the effective potential follows from (2.14) and (2.15):

$$V(a,\alpha,d)=\frac{-\mu^{4-d}\Gamma(-d/2)}{2a^d(4\pi)^{d/2}}\times\sum_{n=1}^{\infty}n\sum_{q=0}^{n-1}[n^2-1+\alpha(n-1-2q)^2]^{d/2} . \tag{2.16}$$

The effective potential as it stands is well defined only for $\mathrm{Re}d<-3$. We shall perform an analytical continuation of V as a function of d to $d=4$, using a Sommerfeld-Watson transformation (see Appendix B for details). Since we use dimensional regularization, the one-loop calculation in an odd-dimensional spacetime is finite.[9] The main result is presented in (2.31) for $\alpha<0$ and in (2.35) for $\alpha>0$. The effective potential (2.36) is plotted in Fig. 4. Detailed calculation is given below.

We first do the sum over q in (2.16) by making use of the Plana sum formula (see Appendix A for details). Since $n=1$ is the zero mode, one can start from $n=2$. One should at the beginning examine the analytic property of the function $\phi(q)$ defined by

$$\phi(q)=[n^2-1+\alpha(n-1-2q)^2]^{d/2} . \tag{2.17}$$

$\phi(q)$ has branch points at

$$q=\frac{n-1}{2}\pm\frac{i}{2}\left[\frac{n^2-1}{\alpha}\right]^{1/2}\quad\text{for }\alpha>0 , \tag{2.18a}$$

$$q=\frac{n-1}{2}\pm\frac{1}{2}\left[\frac{n^2-1}{-\alpha}\right]^{1/2}\quad\text{for }\alpha<0 . \tag{2.18b}$$

We find from (2.18b) that if $-\frac{3}{4}<\alpha<0$, branch points will fall outside of the summation region $(-\frac{1}{2},n-\frac{1}{2})$. Integration path C_1' and C_2' of Fig. 6(b) can be used.

For $-\frac{3}{4}<\alpha\le 0$, we can rewrite the effective potential from relation (A1):

$$V=\frac{-\mu^{4-d}\Gamma(-d/2)}{2a^d(4\pi)^{d/2}}\left[\int_0^1 dy\,F(y)(1+\alpha y^2)^{d/2}+2i(1+\alpha)^{d/2}\int_0^\infty\frac{G(y)dy}{e^{2\pi y}+1}\right]_{d=4} , \tag{2.19}$$

where

$$F(y)=\sum_{n=2}^{\infty}n^2(n^2-A^2)^{d/2} , \tag{2.20}$$

$$G(y)=\sum_{n=2}^{\infty}n\{[(n+iB)^2-E^2]^{d/2}-[(n-iB)^2-E^2]^{d/2}\} , \tag{2.21}$$

$$A=(1+\alpha y^2)^{-1/2},\quad B=\frac{2y\alpha}{1+\alpha},\quad E^2=\frac{4\alpha y^2}{(1+\alpha)^2}+\frac{1}{1+\alpha} . \tag{2.22}$$

Next, we find the infinite sum in $F(y)$ can be converted into an integral in the complex plane by a Sommerfeld-Watson transformation (see Appendix B):

$$\sum_{n=2}^{\infty} n^2(n^2-A^2)^{d/2}=\frac{i}{2}\int_C dz\, z^2(z^2-A^2)^{d/2}\cot\pi z \;, \qquad (2.23)$$

where C is the contour of Fig. 1. Because α is negative and $y\in(0,1)$, we find $1\le A<2$. The analytic continuation consists of two steps. First, one changes the integration path C to C' (see Fig. 1). Equation (2.23) can now be expressed as

$$F(y)=\sin\frac{\pi d}{2}\left[P\int_0^{A(y)} x^2(A^2-x^2)^{d/2}\cot\pi x\, dx + \int_0^{\infty} x^2(A^2+x^2)^{d/2}\coth(\pi x)dx\right] -(A^2-1)^{d/2}\cos\left[\frac{\pi d}{2}\right]. \qquad (2.24)$$

Because $z=1$ is a simple pole on the integration path, the $(0,A)$ integral has to be defined by the principal value. The last term of (2.24) comes from the residue of the $z=1$ pole. The second integral in (2.24) is divergent when the value $d=4$ is taken. Hence further analytic continuation is called for. Rewriting $\coth\pi x$ as $[1+2/(e^{2\pi x}-1)]$ (Ref. 24), one has

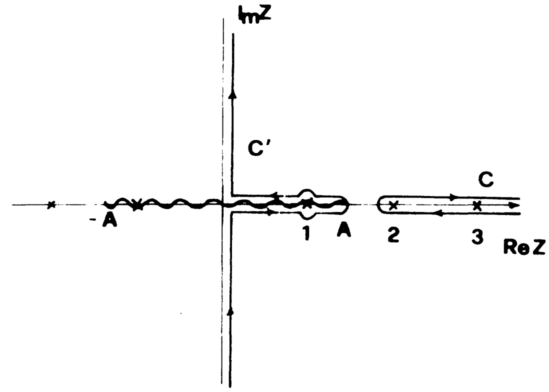

FIG. 1. The n summation in Eq. (2.23) is replaced by a complex integral representation along the contour C. By Cauchy's theorem the contour C can be changed into C' which runs along the cuts associated with the branch points $z=\pm A$. $z=1,2,\ldots$ are simple poles. When this figure is referred to the discussion of Eq. (B14), a change of $A\in(0,1)$ is understood.

$$\int_0^{\infty} dx\, x^2(x^2+A^2)^{d/2}\coth\pi x = \frac{A^{d+3}\Gamma(\frac{3}{2})\Gamma\left[-\frac{d+3}{2}\right]}{2\Gamma\left[\frac{-d}{2}\right]} + 2\int_0^{\infty} dx\frac{x^2(x^2+A^2)^{d/2}}{e^{2\pi x}-1}\;. \qquad (2.25)$$

Substituting (2.25) into (2.24) and taking the $d\to4$ limit, we find

$$F(y)=\lim_{\epsilon\to0}\pi\epsilon\left[\tfrac{1}{2}P\int_0^{A(y)} x^2(A^2-x^2)^2\cot\pi x\, dx+\frac{\Gamma(7)\zeta(7)}{(2\pi)^7}+2A^2\frac{\Gamma(5)\zeta(5)}{(2\pi)^5}+A^4\frac{\Gamma(3)\zeta(3)}{(2\pi)^3}\right]-\lim_{d\to4}\cos\frac{\pi d}{2}(A^2-1)^{d/2}\;, \qquad (2.26)$$

where $\epsilon=d-4$. Note that there is an overall $\Gamma(-d/2)$ factor in (2.19) which gives $-1/\epsilon$ in the limit of $d\to4$. We conclude that (2.26) gives finite contribution to the effective potential except for the last term which will be treated separately.

It remains now to evaluate the function G in (2.19). Because the branch cuts are different, we have to separate the $(0,\infty)$ integral into $(0,u)$ and (u,∞) with $u=\frac{1}{2}[(1+\alpha)/(-\alpha)]^{1/2}$. From (2.22) it is easy to see that when $0\le y\le u$, $E^2\ge0$. Branch cuts of the function $[(z\pm iB)^2-E^2]^{d/2}$ are shown in Fig. 2. One can do a similar Sommerfeld-Watson transformation, and $G(y)$ takes the form

$$2iG(y)=\int_C dz\, z\cot\pi z\{[(z-iB)^2-E^2]^{d/2}-[(z+iB)^2-E^2]^{d/2}\}\;, \qquad (2.27)$$

where C is the contour in Fig. 2. One may observe that contour C can be replaced by contours $(C_1'+C_3')$ and $(C_2'+C_3')$, respectively, for the two terms in (2.27) (this is allowed for $d<-3$). After manipulations similar to that of F, we obtain, for $0\le y\le u$,

$$2iG(y)=\lim_{\epsilon\to0}2\pi\epsilon\left[\int_0^{E(y)} dx\,(E^2-x^2)^2\frac{B\sin(2\pi x)-x\sinh(2\pi B)}{\cosh(2\pi B)-\cos(2\pi x)}+8B\left[\frac{\Gamma(5)\zeta(5)}{(2\pi)^5}+(B^2+E^2)\frac{\Gamma(3)\zeta(3)}{(2\pi)^3}\right] + \int_0^{B(y)} dx\,\coth\pi x[(x-B)^2+E^2]^2 x\right]-\lim_{d\to4}\{[(1+iB)^2-E^2]^{d/2}-[(1-iB)^2-E^2]^{d/2}\}\;. \qquad (2.28)$$

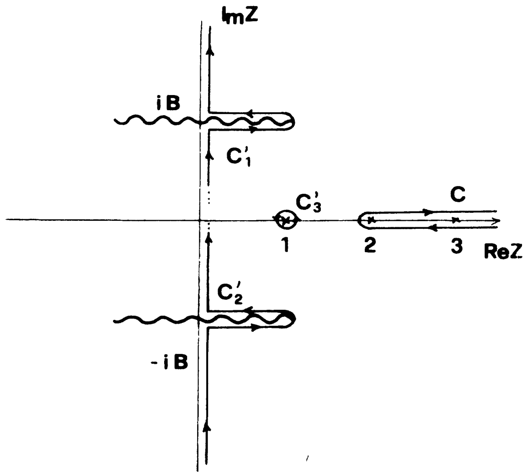

FIG. 2. In Eq. (2.27), the contour C is replaced by $C_1'+C_3'$ for the $[(z-iB)^2-E^2]^{d/2}$ part of the integral and by $C_2'+C_3'$ for the $[(z+iB)^2-E^2]^{d/2}$ part. ($E^2\geq 0$.)

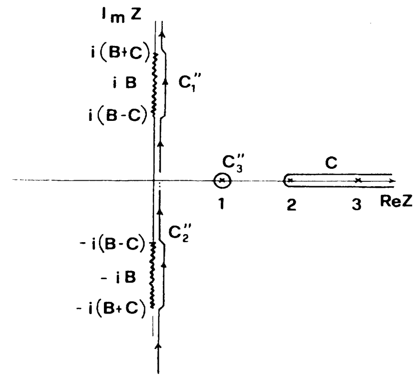

FIG. 3. In the case of $E^2\leq 0$, the two terms on the RHS of Eq. (2.27) have branch points along the imaginary axis. Similar to Fig. 2, the contour C for these two terms is changed to $C_1''+C_3''$ and $C_2''+C_3''$, respectively.

In the case $y\geq u$, (2.22) yields $E^2\leq 0$. The branch cuts associate with (2.27) will be along the imaginary axis (see Fig. 3). We then replace the contour C of Fig. 3 by $(C_1''+C_3'')$ and $(C_2''+C_3'')$ for the two integrals in (2.27), respectively. By carefully keeping track of the phase changes when the integration path is along the cut, we find (2.27) leads to

$$2iG(y)=\lim_{\epsilon\to 0}2\pi\epsilon\left[\tfrac{8}{15}BF^2+8B\left[\frac{\Gamma(5)\zeta(5)}{(2\pi)^5}+(B^2+E^2)\frac{\Gamma(3)\zeta(3)}{(2\pi)^3}\right]\right.$$
$$\left.-\int_0^{F-B}dx\frac{x[(x+B)^2+E^2]^2}{e^{2\pi x}-1}+\int_0^{F+B}dx\frac{x[(x-B)^2+E^2]^2}{e^{2\pi x}-1}\right]$$
$$-\lim_{d\to 4}\{[(1+iB)^2-E^2]^{d/2}-[(1-iB)^2-E^2]^{d/2}\}\ , \qquad (2.29)$$

where $F=(-E^2)^{1/2}$ and $y\geq u$.

Substituting the last terms of (2.26), (2.28), and (2.29) into (2.19) and performing the y integration, we find these terms do give a finite contribution to the effective potential after $d\to 4$ limit is taken. More explicitly, we have

$$\lim_{d\to 4}-\Gamma\left[\frac{-d}{2}\right]\left[\int_0^1 dy\left[1-\frac{1}{A^2}\right]^{d/2}\cos\frac{\pi d}{2}+2i\int_0^\infty\frac{dy}{e^{2\pi y}+1}(1+\alpha)^{d/2}\right.$$
$$\left.\times\{[(1+iB)^2-E^2]^{d/2}-[(1-iB)^2-E^2]^{d/2}\}\right]=\frac{-1}{25}+16\int_0^\infty\frac{dy}{e^{2\pi y}+1}y(4y^2-1)\ln(4y^2+1)$$
$$+2\int_0^\infty\frac{dy}{e^{2\pi y}+1}(16y^4-24y^2+1)\theta\ , \qquad (2.30)$$

where $\tan\theta/2=1/2y$. On using the result of (2.26) to (2.30) in (2.19), we obtain the final expression for the regularized effective potential:

$$V=\frac{1}{32\pi a^4}\left[\tfrac{1}{2}\int_0^1 dy(1+\alpha y^2)^2P\int_0^A x^2(A^2-x^2)^2\cot\pi x\,dx+2(1+\alpha)^2\int_0^u\frac{W_2(y)}{e^{2\pi y}+1}dy\right.$$
$$+2(1+\alpha)^2\int_u^\infty\frac{dy}{e^{2\pi y}+1}\left[\tfrac{8}{15}BF^2-\int_0^{F-B}dx\frac{x[(x+B)^2+E^2]^2}{e^{2\pi x}-1}\right.$$
$$\left.\left.+\int_0^{F+B}dx\frac{x[(x-B)^2+E^2]^2}{e^{2\pi x}-1}\right]+W_1(\theta)\right]\quad(-3/4<\alpha<0)\ , \qquad (2.31)$$

where

$$W_1(\theta)=\frac{\alpha^2}{\pi}\left[\frac{1}{25}-16\int_0^\infty \frac{dy}{e^{2\pi y}+1}y(4y^2-1)\ln(4y^2+1)-2\int_0^\infty \frac{dy}{e^{2\pi y}+1}(16y^4-24y^2+1)\theta\right]+\frac{45\zeta(7)}{8\pi^7}\left[1+\frac{2\alpha}{3}+\frac{\alpha^2}{5}\right]$$
$$+\frac{\zeta(5)}{2\pi^5}(3+2\alpha+\alpha^2)+\frac{\zeta(3)}{\pi^3}\left[\frac{1}{4}+\frac{\alpha}{6}+\frac{7\alpha^2}{60}\right],$$
$$W_2(y)=\int_0^{E(y)} dx(E^2-x^2)^2\frac{B\sin(2\pi x)-x\sinh(2\pi B)}{\cosh(2\pi B)-\cos(2\pi x)}+\int_0^{B(y)} dx\coth\pi x[(x-B)^2+E^2]^2 x .$$

For $\alpha\geq 0$, recall that the branch points (2.18a) force us to take the contour C_1'' and C_2'' of Fig. 6(c) during the q sum (Appendix A). As a result the modified Plana sum formula (A6) gives rise to an extra term inside the large parentheses of (2.19):

$$-(4\alpha)^{d/2}4\sin\frac{\pi d}{2}\sum_{n=2}^{\infty} n\int_p^\infty dy\frac{(y^2-p^2)^{d/2}}{e^{i\pi(n-1)+2\pi y}-1}, \tag{2.32}$$

where $p=\frac{1}{2}[(n^2-1)/\alpha]^{1/2}$. This term is finite in the limit $d=4$ thanks to the exponential factor in the denominator. Other modifications in functions F and G are also needed. In the F part, positive α dictates the range of A: $0<A\leq 1$. No poles will lie on the integration path $(0,A)$. The right-hand side (RHS) of (2.24) is thus replaced by

$$\sin\frac{\pi d}{2}\left[\int_0^{A(y)} x^2(A^2-x^2)^{d/2}\cot\pi x\,dx+\int_0^\infty x^2(A^2+x^2)^{d/2}\cot(\pi x)dx\right]-(1-A^2)^{d/2} . \tag{2.33}$$

Since both B and E^2 are now positive, (2.27) can be regularized by contours shown in Fig. 2. Similar calculation also leads to a change in the definition of θ in (2.30),

$$\tan\frac{\theta'}{2}=-2y , \tag{2.34}$$

the rest of (2.30) remains intact for the case of $\alpha>0$. With these modifications, we find the regularized effective potential for $\alpha\geq 0$ as

$$V=\frac{1}{32\pi a^4}\left[\frac{1}{2}\int_0^1 dy(1+\alpha y^2)^2\int_0^{A(y)} x^2(A^2-x^2)^2\cot\pi x\,dx+2(1+\alpha)^2\int_0^\infty \frac{W_2(y)}{e^{2\pi y}+1}dy\right.$$
$$\left.-32\alpha^2\sum_{n=2}^{\infty} n\int_p^\infty dy\frac{(y^2-p^2)^2}{e^{i\pi(n-1)+2\pi y}-1}+W_1(\theta')\right] . \tag{2.35}$$

All the integrals in (2.31) and (2.35) can be evaluated numerically. It turns out to be convenient for later analysis if we replace the scale a by the internal space volume Ω via the relation

$$\Omega=\frac{2\pi^2a^3}{\sqrt{1+\alpha}} .$$

We can then write the effective potential as

$$V=\frac{Y(\alpha)}{\Omega^{4/3}} . \tag{2.36}$$

The graph of $Y(\alpha)$ is plotted in Fig. 4.

It is interesting to note that even if the topology of the manifold is still S^3, deformation in both prolate ($\alpha<0$) and oblate ($\alpha>0$) directions induce large negative Casimir energy similar to the well-known case of two parallel plates which corresponds to the topology of S^1. The asymptotic behavior of $V(\alpha)$ seems to be independent of the coupling constant ξ (Ref. 25).

We have also compared the present result with that of a small α perturbation calculation (see Appendix C) in Fig. 4. The perturbative calculation agrees with the exact one within 1% up to $|\alpha|=0.14$.

III. SOLUTION OF FIELD EQUATIONS

We have calculated the effective potential of a massless minimally coupled scalar field in a $M^4\times$Taub geometry without asking how to obtain this background in the first place. In this section we shall show that a certain configuration of the background geometry can be sustained by this potential.

Since we consider a vacuum state, the matter Lagrangian consists of only a geometrically dependent effective potential which is generated from quantum fluctuation of matter fields in this background. The action, in general, can be written as

$$S=\int dV_x dV_y\left[\frac{1}{\bar{\kappa}}(\bar{R}-2\bar{\Lambda})-\bar{V}\right], \tag{3.1}$$

where $\bar{V}$ is the 7-dimensional matter effective potential. In particular, we are seeking a vacuum solution with 4-dimensional Poincaré invariance and a static homogeneous internal space, the action can hence be reduced to

$$S=\int dV_x\left[\frac{\Omega}{\kappa}(\tilde{R}-2\bar{\Lambda})-V\right] . \tag{3.2}$$

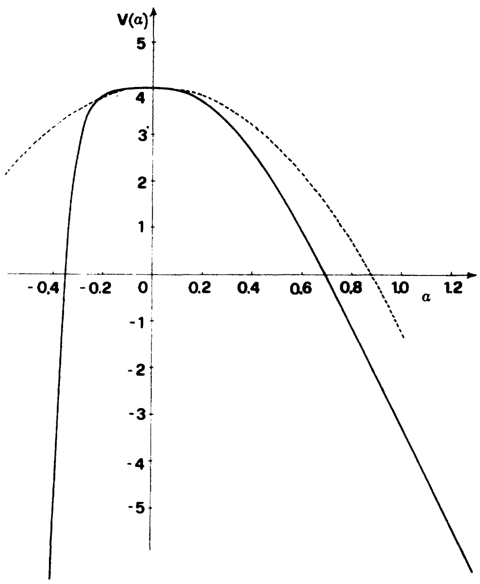

FIG. 4. Effective potential of a massless, minimally coupled scalar field in a $M^4 \times$Taub background. The dotted line denotes the small α perturbative result in Eq. (C6). Deformation parameter α is 0 for a round three-sphere. Here we set the S^3 volume $\Omega=1$. $V(\alpha)$ is plotted in units of 10^{-3}.

There is no kinetic term in the Lagrangian, so the total potential can be read off immediately:

$$V_T=\frac{1}{\bar\kappa}(-\Omega\tilde R+2\Omega\bar\Lambda+\bar\kappa V) \ . \tag{3.3}$$

We can take (3.3) as a potential for a classical system with two dynamical variables: internal space volume Ω and deformation parameter α. $\bar\Lambda$ is a 7-dimensional cosmological constant which will be determined by the value of Ω and α through (3.4b). Static solutions must satisfy the following field equations:

$$\left.\frac{\partial V_T}{\partial\alpha}\right|_{\alpha_0\Omega_0}=\left.\frac{\partial V_T}{\partial\Omega}\right|_{\alpha_0\Omega_0}=0 \ , \tag{3.4a}$$

$$V_T\,|_{\alpha_0\Omega_0}=0 \ . \tag{3.4b}$$

Equations (3.4a) and (3.4b) are the (mn) and $(\mu\nu)$ components of Einstein equations, respectively. [One may also vary the action (3.1) with respect to $g_{\mu\nu}$ to obtain the $(\mu\nu)$ component of the Einstein equations and with respect to (α,Ω) to obtain the (mn) component of the Einstein equations. One then sets $g_{\mu\nu}=\eta_{\mu\nu}$ for a solution of $M^4\times S^3$ and Einstein equations are reduced to Eqs. (3.4).] Equation (3.4b) signifies that cosmological constant in four dimensions is zero. From the general form of the effective potential (2.36), and curvature scalar (2.6), Eqs. (3.3) and (3.4) yield the following set of algebraic equations:

$$-\left[\frac{2\pi^2}{\Omega_0}\right]^{2/3}\frac{3+4\alpha_0}{3(1+\alpha_0)^{4/3}}+\bar\Lambda-\frac{2\bar\kappa Y}{3\Omega_0{}^{7/3}}=0 \ , \tag{3.5a}$$

$$\frac{(2\pi^2)^{2/3}}{3}\Omega_0{}^{1/3}\frac{4\alpha_0}{(1+\alpha_0)^{7/3}}+\frac{\bar\kappa Y'}{2\Omega_0{}^{4/3}}=0 \ , \tag{3.5b}$$

$$-(2\pi^2)^{2/3}\Omega_0{}^{1/3}\frac{3+4\alpha_0}{(1+\alpha_0)^{4/3}}+\Omega_0\bar\Lambda+\frac{\bar\kappa Y}{2\Omega_0{}^{4/3}}=0 \ , \tag{3.5c}$$

where

$$Y'=\left.\frac{dY}{d\alpha}\right|_{\alpha_0} .$$

From these equations we first determine the solution α_0 through

$$\frac{Y'}{Y}=\frac{-14\alpha_0}{3(3+4\alpha_0)(1+\alpha_0)} \ . \tag{3.6}$$

Then, Ω_0 and $\bar\Lambda_0$ follow accordingly:

$$\Omega_0{}^{5/3}=\frac{7\bar\kappa Y(\alpha_0)(1+\alpha_0)^{4/3}}{4(3+4\alpha_0)(2\pi^2)^{2/3}} \ , \tag{3.7}$$

$$\bar\Lambda_0=\frac{5\bar\kappa Y(\alpha_0)}{4\Omega_0{}^{7/3}} \ . \tag{3.8}$$

From (3.2) we see that the 4-dimensional Newton's constant is given by $\kappa=\bar\kappa/\Omega$ (actually this is true only up to quantum corrections[26]). Numerical calculation of (3.6) gives three solutions:

$$\alpha_0=0,\ -0.059\,41,\ -0.1861 \ . \tag{3.9}$$

The total potential V_T is shown in Fig. 5 where $\bar\Lambda$ and Ω are evaluated through (3.7) and (3.8) with $\alpha_0=-0.059\,41$. For the other two values of α_0, the shape of the potential V_T is similar except that fine-tuning effect of the cosmological constant shifts the value of V_T to zero at $\alpha=0$ and -0.1861, respectively. It may be interesting to note that

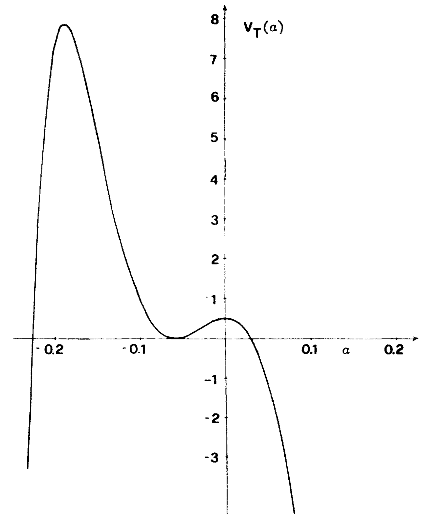

FIG. 5. The total potential in Eq. (3.3). Λ and Ω are determined by Eqs. (3.7) and (3.8) at one of the solutions $\alpha_0=-0.059\,41$. Here we set $2(4\pi^4\Omega)^{1/3}/\bar\kappa=1$.

at $\alpha=0$, $Y'/Y=0$ for any matter fields in this background. This relation follows directly from (2.36) and (3.6). As we have discussed near the end of Sec. II, perturbative calculation of Y gives a very good approximation for $|\alpha| \leq 0.14$. One must be careful, however, in applying the perturbatively computed Y in the analysis leading to the solution of (3.6), because the deviation of Y' thus calculated is already 2.8% for $|\alpha| = 0.005$.

Ω_0 as it stands in (3.7) leads to a scale $a_0 \sim 5\times10^{-3} l_P$ (Planck length) which is beyond the validity regime of semiclassical approximation. One way to justify our result is to consider b scalar fields. The scale a_0 will increase with $\sqrt{b}$; however, the shape (α_0) is preserved.

IV. DISCUSSION

It is important to ask whether the ground-state solution we find in Sec. III is stable. Unfortunately, for vacuum-energy compactification models it is quite involved to calculate the response of the effective potential to an arbitrary metric perturbation. If one considers only the dilatation mode of the internal space and if the effective potential is calculated from conformal fields, one can perform a conformal transformation in $4+N$ dimensions[27] to scale out the spacetime dependence of the internal space. One can then use the static effective potential in the stability discussion. In particular, when the total number of spacetime dimensions is odd, one need not consider the conformal anomaly in the odd-loop effective potential.[9] In our case, there are two complexities. Firstly, we consider massless minimally coupled scalar fields. True, there is still no conformal anomaly; however, at the classical level a scale-dependent mass term appears as is clear from the following.

If

$$g'_{\mu\nu}=\omega^2 g_{\mu\nu},\quad \phi'=\omega^{1-n/2}\phi ,$$

then

$$\sqrt{g'}\phi'\Box\phi=\sqrt{g}\left\{\phi\Box\phi+\phi^2\left[\left(1-\frac{n}{2}\right)\left(\frac{n}{2}-2\right)\frac{\omega_{,\mu}\omega_{,\nu}g^{\mu\nu}}{\omega^2}+\left(1-\frac{n}{2}\right)\frac{\Box\omega}{\omega}\right]\right\}. \quad (4.1)$$

The static massless effective potential could only be used within the limit that the spacetime dependence of the scale ω is small. Secondly, since the deformation cannot be factored out as a conformal factor, the previous conformal transformation scheme cannot give us a new spacetime with static internal space. Therefore the static potential (2.36) should not be used to analyze the stability problem, rather, one needs to consider the effective action with a dynamical background. A perturbative formalism to evaluate the effective potential for Bianchi type-I spacetime has been laid down by Hartle and Hu.[28] One may argue that in the perturbative sense, corrections to the static potential can only affect the kinetic terms (e.g., $\dot\Omega,\dot\alpha$). By assuming that the kinetic term has the "right sign," one can proceed to examine the static potential to see whether our solutions (3.9) are local minima. We would like to point out here that even in this naive stability analysis (i.e., by considering only homogeneous internal space volume and shape perturbation) the abovementioned assumption needs justification. Following the usual small-oscillation analyses of classical mechanics, the stability condition around static solutions reduces to the condition to have only a positive ω^2 solution of the secular equation:

$$\det(V-\omega^2 T)=0 . \quad (4.2)$$

Where the 2×2 matrices V and T denote generalized potential and kinetic terms, respectively, in coordinates Ω and α. It is not difficult to convince oneself that the positive ω^2 solution depends sensitively on the matrix element of T as well as V. Since the quantum correction to the kinetic terms is comparable to the classical part in the vacuum-energy compactification models, to neglect it is highly risky. A detailed study in this direction is now in progress.

The stability consideration we have discussed so far is only a special case to the whole issue. In general, one may classify stability problems into classical and semiclassical ones. In the category of classical stability one can further divide it into linear and nonlinear stability. In the linear case, as in the usual field theory, one introduces the $(4+N)$-dimensional metric perturbations, $h_{\mu\nu}$, $h_{\mu n}$, h_{mn}, and then writes down the perturbed action to the second order to seek possible tachyonic or ghost modes.[29] A particular gauge may be chosen (e.g., a light-cone gauge[30]) or, if possible, a gauge-invariant method[31] can be applied in the analysis. The quantum response due to the gravitational and Yang-Mills wave perturbation has been calculated by Awada and Toms.[32] Taking this into consideration, Candelas and Weinberg[7] reached the conclusion that $M^4\times S^N$ background is stable against gravitational and Yang-Mills perturbation, if fermion fields are introduced and the boson to fermion ratio is less than an N-dependent upper bound. Later on, Gilbert, McClain, and Rubin[33] concluded that a lower bound of this ratio can be obtained by considering a stability condition against a homogeneous dilatation mode of the S^N if this mode is adiabatic. As we have mentioned a general discussion is technically complicated. Many authors[27,34] have studied another particular mode: namely, the homogeneous dilatation in both internal and external spaces. It usually bears the name of cosmological stability due to the homogeneity assumption. Classical linear perturbative stability is the first step in the discussion of stability of a candidate for a ground state in a higher-dimensional system. If it is satisfied one can intuitively think that we find a state at a local minimum of energy. The nonlinear stability of the background geometry has also been explored[35] within the context of cosmology. In addition to classical stability, Witten[36] has demonstrated that a $M^4\times S^1$ background can decay semiclassical (as in tunneling effect) into nothing provided a topology-changing dynamics is allowed. It appears difficult to generalize his argument to other theories admitting Freund-Rubin-type compactification.[37] Recently, some authors[38] suggested that by taking the geometric quantities of the internal space (e.g., radius) as a quantum scalar field, then the shape of the effective potential can dictate the fate of a vacuum-energy compactification ground state. Following these arguments, one may look at Fig. 5 and naively conclude that $\alpha_0=-0.059\,41$ solution is quantum-mechanically unstable. We think the meaning of this approach should be more carefully studied.

If any of the solutions of (3.9) is stable, it is a candidate for the ground state. Consider a general metric perturbation on this background. The $h_{\mu n}$ part of the perturbation in the direction of a Killing vector on S^3 can be identified as gauge field A_μ. The gauge symmetry will be the same as the isometry of the internal space, because there is no other field configuration to break this symmetry. For the solution $\alpha=0$ the gauge symmetry is $SU(2)\times SU(2)$ [or SO(4)] and there are six gauge fields. For $\alpha\neq 0$ solutions gauge symmetry reduces to $SU(2)\times U(1)$ and hence there are four gauge fields only. Using the zero-mode ansatz,[39] we find the mass gained by the two gauge fields is proportional to the deformation $\alpha(a\sqrt{1+\alpha})^{-1}$ (Ref. 40). Symmetry breaking in Kaluza-Klein theories have been considered by many authors.[13–16] However, one has to keep in mind that in our case to preserve the 4-dimensional Poincaré invariance we have to fine-tune the 7-dimensional cosmological constant (3.8). Hence each different solution (α_0,Ω_0) associates a different cosmological constant $\overline{\Lambda}_0$. It is, therefore, meaningless to ponder the transition from one solution to the other as in the case of usual symmetry breaking. The symmetry-broken phase comes together with spontaneous compactification.

If we do have a $M^4\times$Taub background, the gauge coupling constants of SU(2) and U(1) can be determined geometrically.[8] The weak angle at the compactification energy scale hence follows.[41] At a classical level the weak angle is simply the ratio of the "averaged radius" of SU(2) to the radius of U(1):

$$\tan\theta_w=\frac{1}{l_3}\left[\frac{2{l_1}^2+{l_3}^2}{3}\right]^{1/2}=(1+\tfrac{2}{3}\alpha)^{1/2}\,. \qquad (4.3)$$

Since the range of α is $(-1,\infty)$, one finds the range of weak angle in this model is $0.25<\sin^2\theta_w<1$. Note that the common value of the weak angle obtained from various grand unification models (0.3 from $[SU(6)]^4$ and $\frac{3}{8}$ from SU(5), SO(10), SU(16), E_6, and E_8) falls within this range. When quantum correction is included, using Eq. (37) of Ref. 41 and the values of Y at $\alpha=0$, $-0.059\,41$, -0.1864, we find $\sin^2\theta_w=0.5$, 0.5028, 0.5098, respectively. Of course, to be realistic one should include fermions and gravitons. In fact, the contribution of higher-spin field to the effective potential is much greater than that of the scalar.[42,48] Also, more dimensions are required if one wishes to consider more realistic models.[44] Our results holds if the Taub space is a part of the larger internal space. The work presented here serves only as a modest attempt to study the ground state of a higher-dimensional system with a non-maximally-symmetric internal geometry and to investigate possible physical implication of it.

ACKNOWLEDGMENTS

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APPENDIX A: PLANA SUM FORMULA (REF. 45)

In this appendix we outline the derivation of the Plana sum formula and discuss some variation of it. Let $\phi(z)$ be analytic in the region bounded by $(n-\frac{1}{2}\pm i\infty)$ and $(m+\frac{1}{2}\pm i\infty)$ and for $z=a+ib$, $a\in[n-\frac{1}{2},\ m+\frac{1}{2}]$, $\phi(z)e^{-2\pi b}\to 0$ when $b\to+\infty$. The Plana sum formula takes the form[17]

$$\sum_{i=n}^{m}\phi(i)=\int_{n-1/2}^{m+1/2}\phi(x)dx-i\int_0^\infty\frac{dy}{e^{2\pi y}+1}[\phi(n-\tfrac{1}{2}+iy)-\phi(n-\tfrac{1}{2}-iy)-\phi(m+\tfrac{1}{2}+iy)+\phi(m+\tfrac{1}{2}-iy)]\,. \qquad (A1)$$

It can be obtained by adding the following two integrals along the path C_1' and C_2' of Fig. 6(b), respectively:

$$\int_{C_1'}dz\frac{\phi(z)}{e^{-2\pi iz}-1}=\left[\int_{n-1/2+i\infty}^{n-1/2}+\int_{m+1/2}^{m+(1/2)+i\infty}+P\int_{n-1/2}^{m+1/2}\right]\frac{\phi(z)dz}{e^{-2\pi iz}-1}-\pi i\sum_{i=n}^{m}\mathrm{Res}_-(i)\,, \qquad (A2)$$

$$\int_{C_2'}dz\frac{\phi(z)}{e^{2\pi iz}-1}=\left[\int_{n-(1/2)-i\infty}^{n-1/2}+\int_{m+1/2}^{m+1/2-i\infty}+P\int_{n-1/2}^{m+1/2}\right]\frac{\phi(z)dz}{e^{2\pi iz}-1}+\pi i\sum_{i=n}^{m}\mathrm{Res}_+(i)\,, \qquad (A3)$$

where $\mathrm{Res}_\pm(n)$ denotes residues calculated from $\phi(z)/(e^{\pm 2\pi iz}-1)$ and P denotes the Cauchy principal value.

Alternatively, one may also choose the contour as C_1, C_2 in Fig. 6(a) if it proves to be convenient. In this case, the Plana sum formula becomes[46]

$$\sum_{i=n}^{m}\phi(i)=\tfrac{1}{2}[\phi(n)+\phi(m)]+\int_n^m\phi(x)dx-i\int_0^\infty\frac{dy}{e^{2\pi y}+1}[\phi(n-iy)-\phi(n+iy)-\phi(m-iy)+\phi(m+iy)]\,. \qquad (A4)$$

If, however, the function $\phi(z)$ has singular points in the region surrounded by contours, then the above formulas should be modified. Taking the function defined in (2.17), for example,

$$\phi(z)=[k^2-1+\alpha(k-1-2z)^2]^{d/2},\ \ k\in N,\ \ k\geq 2\,, \qquad (A5)$$

it has branch points at

$$z=\frac{k-1}{2}\pm\frac{i}{2}\left[\frac{k^2-1}{\alpha}\right]^{1/2}$$

for $\alpha>0$. If one chooses branch cuts as in Fig. 6(c), paths along the cuts may give a nonzero contribution. Following the same procedure as above we obtain the modified Plana sum formula for the function defined by (A5):

$$\sum_{i=n}^{m}\phi(i)=\int_{n-1/2}^{m+1/2}\phi(x)dx-i\int_0^\infty\frac{dy}{e^{2\pi y}+1}[\phi(n-\tfrac{1}{2}+iy)-\phi(n-\tfrac{1}{2}-iy)-\phi(m+\tfrac{1}{2}+iy)+\phi(m+\tfrac{1}{2}-iy)]$$
$$-4\sin\frac{\pi d}{2}(4\alpha)^{d/2}\int_p^\infty\frac{(y^2-p^2)^{d/2}dy}{e^{i\pi(k-1)+2\pi y}-1}\,, \tag{A6}$$

where

$$p=\frac{1}{2}\left[\frac{n^2-1}{\alpha}\right]^{1/2}.$$

Another version of the Plana sum formula for infinite sums is discussed in Appendix B.

APPENDIX B: ANALYTIC CONTINUATION

In this appendix we shall discuss four different procedures [labeled A to D] of performing analytical continuation. In order to make the exposition more transparent, we introduce the function (B1), which is a typical expression we have encountered in Sec. II, as an example:

$$F(d,A^2)=\Gamma\left[\frac{-d}{2}\right]\sum_{n=1}^{\infty}n^2(n^2-A^2)^{d/2},\quad A^2<1\,. \tag{B1}$$

$F(d,A^2)$ is well defined only for $d<-3$. We shall show how to express the sum in (B1) in integral forms and obtain expressions which make sense also for other values of d. In particular, we are interested in the limit $d\to 4$. The case when $d=0$ and $A^2=1$ will be discussed at the end of this appendix. The generalization of the lower limit of the sum from $n=1$ to $n=m$, $m^2\geq A^2$ will also be discussed along the line.

The first two methods are based on the Laplace transformation. After applying the transformation, the dependence on n is contained in the factor e^{-nt}. Therefore, the sum over n can be performed easily.

1. Method A

This method was introduced by Candelas and Weinberg.[7] For $n>A$ and $D<-1$, we can write $n(n^2-A^2)^{d/2}$ as

$$n(n^2-A^2)^{d/2}=\frac{A\sqrt{\pi}}{\Gamma(-d/2)}\int_0^\infty dt\,e^{-nt}\left[\frac{t}{2A}\right]^{(-d-1)/2}\times I_{(-d-3)/2}(At)\,, \tag{B2}$$

where $I(-d-3)/2$ is a modified Bessel function. Using this relation we obtain, from (B1),

$$F(d,A^2)=\sqrt{\pi}2^{(d+1)/2}\int_0^\infty dt\frac{t^{-d-2}}{(2\sinh t/2)^2}(At)^{(d+3)/2}\times I_{(-d-3)/2}(At)\,. \tag{B3}$$

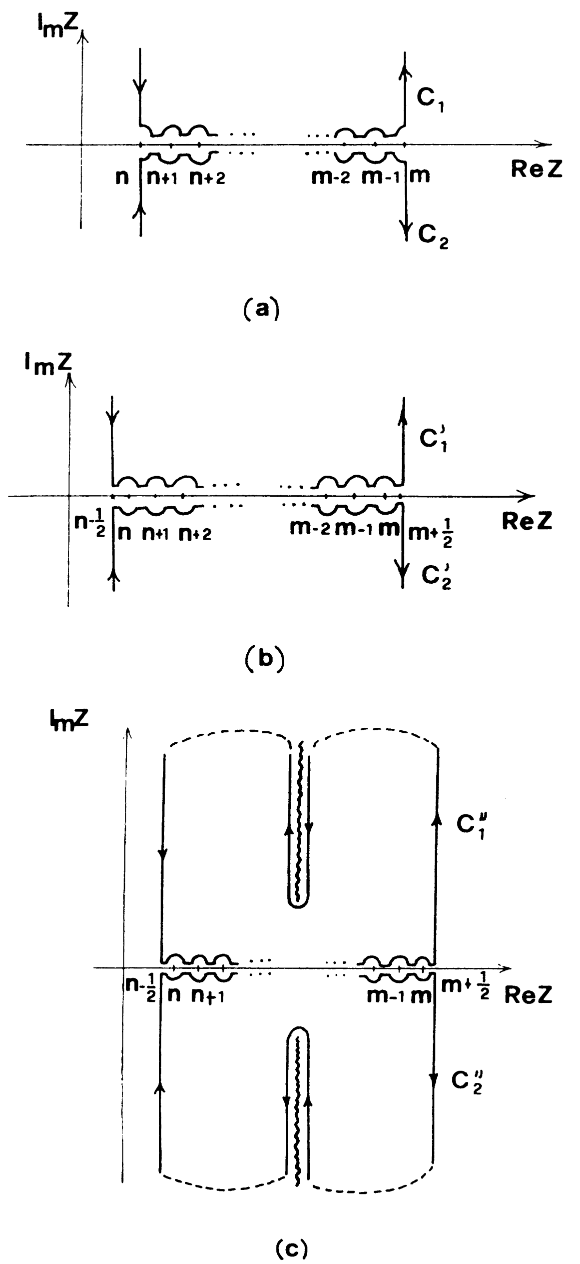

FIG. 6. (a) is used in deriving the Plana sum formula (A4). (b) is used in deriving another version of the Plana sum (A1). When the function is not analytic [(A5) or (2.17)], (c) has to be used in deriving (A6).

Since $I_{n+1/2}(z)$ approaches $e^z/\sqrt{z}$ for large z, the integral in (B3) is convergent in the $t\to\infty$ limit, if $A^2<1$. However, it diverges at the $t\to 0$ limit in the case of $d>-3$, because $I_{-n-1/2}(z)$ approaches $z^{-n-1/2}$ for small z. For even integer d the integrand is an even function of t. Near $t=0$, it behaves like $t^{(-d-5)/2}$ so that for $d<-3$ one can choose the integration contour C along the whole real axis bypass of the origin (see Fig. 1 of Ref. 7). Hence (B4) can be written as

$$F(d,A^2)=\sqrt{\pi}2^{(d-1)/2}\int_C dt\frac{t^{-d-2}}{(2\sinh t/2)^2}(At)^{(d+3)/2}\times I_{(-d-3)/2}(At)\ . \tag{B4}$$

The representation (B4) is well defined also for $d=4$. Using the explicit form of $I_{-7/2}(z)$, one obtains

$$F(4,A^2)=4\int_C\frac{dt}{(2\sinh t/2)^2}\left[\sinh(At)\left[\frac{15A}{t^5}+\frac{A^3}{t^3}\right]-\cosh(At)\left[\frac{15}{t^6}+\frac{6A^2}{t^4}\right]\right]\ . \tag{B5}$$

$F(4,A^2)$ can be evaluated by closing the contour C in the upper (or lower) half-plane and summing over residues of poles at $Z=2\pi pi$ (p integer). The result is

$$F(4,A^2)=8\pi\sum_{p=1}^{\infty}\left[\sin(2\pi pA)\left[\frac{-90A}{(2\pi p)^6}+\frac{9A^3}{(2\pi p)^4}\right]+\cos(2\pi pA)\left[\frac{39A^2}{(2\pi p)^5}-\frac{A^4}{(2\pi p)^3}-\frac{90}{(2\pi p)^7}\right]\right]\ . \tag{B6}$$

In particular, for $A^2=1$ we have

$$F(4,1)=8\pi\left[\frac{39}{(2\pi)^5}\zeta(5)-\frac{1}{(2\pi)^3}\zeta(3)-\frac{90}{(2\pi)^7}\zeta(7)\right]\ . \tag{B7}$$

A problem arises if the integrand of (B2) is not an even function of t. This is the case when we have the sum starting from $n=2$, for example, or d is odd. It is not clear, in this case, how to choose an appropriate integration contour.

2. Method B

This method is also based on the Laplace transformation. It was used by Critchley and Dowker[47] and later by Sarmadi.[43] To employ this method in our case, we have to first calculate $F(4,-A^2)$ and then we will make a "Wick rotation" and obtain the expression for $F(4,A^2)$ which agrees with (B6).

We begin with the relation

$$F(d,-A^2)=\Gamma\left[\frac{-d}{2}\right]\sum_{n=1}^{\infty}[(n^2+A^2)^{1+d/2}-A^2(n^2+A^2)^{d/2}]\ . \tag{B8}$$

From the definition of the Γ function, one has

$$(n^2+A^2)^{\nu}=\frac{1}{\Gamma(-\nu)}\int_0^{\infty}dt\,t^{-\nu-1}e^{-(n^2+A^2)t} \tag{B9}$$

valid for $\nu<0$. On using the property of the θ function,[48]

$$\sum_{n=-\infty}^{\infty}e^{-sn^2}=\left[\frac{\pi}{s}\right]^{1/2}\sum_{n=-\infty}^{\infty}e^{-\pi^2n^2/s}\ , \tag{B10}$$

one can derive the following representation of F:

$$F(d,-A^2)=-\sum_{n=1}^{\infty}\sqrt{\pi}\int_0^{\infty}ds\,e^{-sA^2}e^{-\pi^2n^2/s}\left[A^2s^{(-d-3)/2}+\left[1+\frac{d}{2}\right]s^{(-d-5)/2}\right]-\frac{\sqrt{\pi}}{2}\int_0^{\infty}ds\,e^{-sA^2}\left[A^2s^{(-d-3)/2}+\left[1+\frac{d}{2}\right]s^{(-d-5)/2}\right]\ . \tag{B11}$$

It is here that the minus sign at A^2 is necessary in order to make the integral converge at infinity. One can now rewrite (B11) by an integral representation of the Bessel function K_ν (Ref. 49)

$$K_\nu(xz)=\frac{z^\nu}{2}\int_0^{\infty}\exp\left[-\frac{x}{2}\left[t+\frac{z^2}{t}\right]\right]t^{-\nu-1}dt\ , \tag{B12}$$

and explicit forms of $K_{7/2}$ and $K_{5/2}$. The result is

$$F(4,-A^2)=\frac{\sqrt{\pi}}{4}A^7\Gamma\left[\frac{-7}{2}\right]-8\pi\sum_{p=1}^{\infty}e^{-2\pi pA}\left[\frac{A^4}{(2\pi p)^3}+\frac{9A^3}{(2\pi p)^4}+\frac{39A^2}{(2\pi p)^5}+\frac{90A}{(2\pi p)^6}+\frac{90}{(2\pi p)^7}\right]\ . \tag{B13}$$

We now can do the "Wick rotation" ($A^2\to -A^2$) on (B13) to return to our original expression. Because of the $A\to -A$

symmetry in the original expression, we need to symmetrize the final expression too (we call it a "symmetric prescription"). Finally, one will arrive at the formula (B6) again.

One can also start from the expression (B5) and arrive at (B13) by a symmetrized $A^2 \to -A^2$ transformation and performing the integral (B5). The contour C should be closed in the upper-half plane for terms with e^{itA} factor and in the lower-half plane for terms with e^{-itA}. The residue at zero yields the first term on the RHS of (B13).

This method can be used when the sum (B1) is from $n=m$ ($m^2 > A^2$). Moreover, even if d is odd, in which case method A fails completely, this method can single out the divergent part which will be of the form $\Gamma((-d-1)/2)$ and the regular part can be evaluated numerically.

3. Method C

The third method of performing analytic continuation in d is based on the Sommerfeld-Watson integral.[24] The basic equality is

$$F(d,A^2)=\frac{i}{2}\Gamma\left[\frac{-d}{2}\right]\int_C dz\, z^2(z^2-A^2)^{d/2}\cot(\pi z)\ , \tag{B14}$$

where path C and later C' are close to those shown in Fig. 1. The only difference is that point A lies now in (0,1) internal not in (1,2) as before. The equality follows since the integrand on the RHS of (B14) has simple poles at $z=1,2,3,\ldots$. For $d<-3$, one can change the integration contour of (B14) from C to C' (see Fig. 1) and obtain

$$F(d,A^2)=\Gamma\left[\frac{-d}{2}\right]\sin\left[\frac{\pi d}{2}\right]\left[\int_0^\infty dy\, y^2(y^2+A^2)^{d/2}\coth(\pi y)+\int_0^A dx\, x^2(A^2-x^2)^{d/2}\cot(\pi x)\right]$$

$$+\text{contribution from the "bubble" integral at } A\ . \tag{B15}$$

The first integral on the RHS of (B15) is formally divergent as $d\to 4$. We can use the relation

$$\coth(\pi y)=1+\frac{2}{e^{2\pi y}-1} \tag{B16}$$

to separate out the singular part and proceed to regularize it. The singular part can be written as

$$\int_0^\infty dy\, y^2(y^2+A^2)^{d/2}=\frac{A^{3+d/2}}{2}\frac{\Gamma(\frac{3}{2})\Gamma\left[\frac{-d-3}{2}\right]}{\Gamma\left[\frac{-d}{2}\right]}\ . \tag{B17}$$

This representation gives the regular result in the limit of $d\to 4$ when it is put into (B15). The contribution from the "bubble integral" at point A is also negligible as $d\to 4$. After dropping terms of higher order in $(d-4)$, we have

$$\begin{aligned}F(4,A^2)&=\frac{-\pi}{2}\left[2\int_0^\infty dy\frac{y^6+2y^4A^2+y^2A^4}{e^{2\pi y}-1}+A^7\int_0^1 dx\, x^2(1-x^2)^2\cot(\pi xA)\right]\\&=-\pi\left[720\frac{\zeta(7)}{(2\pi)^7}+48A^2\frac{\zeta(5)}{(2\pi)^5}+2A^4\frac{\zeta(3)}{(2\pi)^3}+\frac{A^7}{2}\int_0^1 dx\, x^2(1-x^2)^2\cot(\pi xA)\right]\ ,\end{aligned} \tag{B18}$$

which is to be compared with (B5). In particular for $A=1$, we have

$$F(4,1)=-\pi\left[720\frac{\zeta(7)}{(2\pi)^7}+48\frac{\zeta(5)}{(2\pi)^5}+2\frac{\zeta(3)}{(2\pi)^3}+\frac{1}{2}\int_0^1 dx\, x^2(1-x^2)^2\cot(\pi x)\right]\ . \tag{B19}$$

Making use of the fact that for a positive integer n,

$$\int_0^1 x^n(1-x)^n\cot\pi x=0\ , \tag{B20}$$

and properties of Bernoulli polynomials $B_n(x)$, (Ref. 49),

$$\int_0^1 B_{2n+1}(x)\cot\pi x\, dx=(-)^{n+1}\frac{2(2n+1)!\zeta(2n+1)}{(2\pi)^{2n+1}}\ , \tag{B21}$$

one recovers the expression given in (B7).

This method still holds if we generalize the lower limit of the sum in (B1) into $n=m$ and $m-1\le A\le m$. If $A<m-1$, the residue contribution from poles such as $m-1$ in the C' contour integral will be finite and the overall $\Gamma(-d/2)$ factor leads to infinity when d is even. When d is odd, this method can also single out the singular part which is just (B17).

4. Method D

The last method of analytic continuation we shall discuss is making use of the infinite Plana sum formula[50]

$$\sum_{n=m}^{\infty} f(n)=\frac{f(m)}{2}+\int_m^{\infty} f(\tau)d\tau +i\int_0^{\infty} dt\frac{f(m+it)-f(m-it)}{e^{2\pi t}-1} \tag{B22}$$

which is valid if (1) $f(z)$ is regular for $\mathrm{Re}z \geq m$, (2)

$$\lim_{t\to\pm\infty} e^{-2\pi|t|}f(\tau+it)=0$$

uniformly for $m \leq \tau < \infty$, and (3)

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty} e^{-2\pi|t|}\,|f(\tau+it)|\,dt=0 .$$

Using (B22) for our function (B1), we have

$$F(d,A^2)=\Gamma\left[\frac{-d}{2}\right]\left[\tfrac{1}{2}(1-A^2)^{d/2}+\int_1^{\infty} t^2(t^2-A^2)^{d/2}dt +i\int_0^{\infty} dt\frac{(1+it)^2[(1+it)^2-A^2]^{d/2}-(1-it)^2[(1-it)^2-A^2]^{d/2}}{e^{2\pi t}-1}\right]. \tag{B23}$$

One must expand the RHS of (B23) in powers of $(d-4)$. The leading term which is potentially problematic because $\lim_{d\to4}\Gamma(-d/2)\sim -1/(d-4)$, turns out to be zero. The linear term in large parentheses combined with the Γ factor gives a finite result. The only formally divergent term (as $d\to4$) on the RHS of (B23) is the $(1,\infty)$ integral. It can be regularized by using the representation

$$\int_1^{\infty} t^2(t^2-A^2)^{d/2}dt=\int_1^{A} t^2(t^2-A^2)^{d/2}dt +\frac{A^{3+d}}{2}\frac{\Gamma\left[\frac{-d-3}{2}\right]\Gamma\left[\frac{d+2}{2}\right]}{\Gamma\left[\frac{-1}{2}\right]} \tag{B24}$$

which is clearly finite as $d\to4$.

The advantage of this method is that it singles out the divergence [essentially the $\Gamma((-d-3)/2)$ factor] quickly in the odd d case. It is useful when one is interested in the renormalization aspect of certain theories. The evaluation of the finite part seems to be more involved than that in other methods. This method can also be used when the summation is from $n=m$ if $m>A$.

For some values of d and A, $F(d,A^2)$ may give a divergent expression. For example, if $d\to0$ and $A\to1$ then F diverges as $\ln(1-A)$ in all four methods. In Appendix C we will encounter this case in the perturbative representation of the effective potential (C3). Fortunately, that expression is multiplied by some polynomials of $(1-A)$ and perturbative expression turns out to be finite [at least up to $O(\alpha^2)$]. Observing that in the minimally coupled case $1-A^2=m^2a^2$, so the $A^2\to1$ limit is a consequence of the massless limit that we are considering. The logarithmic divergence can hence be viewed as an infrared divergence.

APPENDIX C: PERTURBATIVE CALCULATION

If the deformation is small, one can calculate the effective potential (2.26) in power series of the deformation parameter α. The calculation of mode sums in this way is much simper than the exact calculation presented in Sec. II. It may hence serve as a useful test to verify the exact calculation. In this appendix we present the perturbative evaluation of the effective potential based on Ref. 51.

In the case of small α, the internal Taub-space curvature scalar can be written as

$$\tilde{R}=\frac{2(3+4\alpha)}{a^2(1+\alpha)}=\frac{2}{a^2}(3+\alpha-\alpha^2)+O(\alpha^3) . \tag{C1}$$

The eigenvalues of the Laplace-Beltrami operator (2.15) can be expanded up to $O(\alpha^3)$ as

$$\lambda_M=[n^2+\rho+\alpha(4k^2+2\xi)-2\xi\alpha^2]/a^2 , \tag{C2}$$

where $\rho=6\xi-1+m^2a^2$.

The effective potential hence follows from (2.12) and (C2):

$$V=-\frac{\mu^{4-d}\Gamma(-d/2)}{2a^d(4\pi)^{d/2}}\sum_{n=1}^{\infty}\Bigg[n^2(n^2+\rho)^{d/2}+\frac{\alpha d}{6}[n^2(n^2+\rho)^{d/2}+n^2(n^2+\rho)^{(d/2)-1}(6\xi-\rho-1)]-\alpha^2 d\xi n^2(n^2+\rho)^{(d/2)-1} +\frac{d}{60}\left[\frac{d}{2}-1\right]\alpha^2\{3n^2(n^2+\rho)^{d/2}-(6\rho+10-20\xi)n^2(n^2+\rho)^{(d/2)-1} +[3\rho^2+10\rho+7-20\xi(\rho+1)+60\xi^2]n^2(n^2+\rho)^{(d/2)-2}\}+O(\alpha^3)\Bigg] . \tag{C3}$$

To evaluate the effective potential (C3), one may use the analytic continuation scheme of Ref. 7 which is also discussed in method A of our Appendix B. In the case of minimal coupling it appears useful to keep a nonzero mass throughout the calculation to avoid an infrared divergence in the last term of $O(\alpha^2)$ of (C3). Massless limit can be taken at the very end. Alternatively, one can sum from $n=2$ at the expense that the analytic continuation method of Ref. 7 is not applicable and an other method should be employed (see Appendix B).

Let

$$S(\nu)=\Gamma\left[\frac{-\nu}{2}\right]\sum_{n=2}^{\infty} n^2(n^2-1)^{\nu/2} .$$

Similar to Ref. 7, we obtain

$$\underset{\nu\to 4}{S(\nu)}=4\left[\frac{-90}{(2\pi)^6}\zeta(7)+\frac{39}{(2\pi)^4}\zeta(5)-\frac{1}{(2\pi)^2}\zeta(3)\right], \qquad (C4)$$

$$\underset{\nu\to 2}{S(\nu)}=-2\left[\frac{12}{(2\pi)^4}\zeta(5)-\frac{5}{(2\pi)^2}\zeta(3)\right]. \qquad (C5)$$

On using (C3) to (C5), we find that for a massless, minimally coupled scalar field the effective potential is

$$V=\frac{4.038\times 10^{-3}}{\Omega^{4/3}}[1-1.300\alpha^2+O(\alpha^3)] \qquad (C6)$$

and similarly, for a massless conformal scalar field ($\xi=\frac{5}{24}, \rho=\frac{1}{4}$),

$$V=\frac{3.812\times 10^{-4}}{\Omega^{4/3}}[1-5.310\alpha^2+O(\alpha^3)] , \qquad (C7)$$

where $\Omega=2\pi^2 a^3/\sqrt{1+\alpha}$ is the volume of the Taub space.

Page[18] has done a similar calculation on a $M^4\times$(deformed S^3). S^3 is viewed here as a U(1) bundle over a CP^1 manifold. His perturbative result agrees with ours.

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