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Mathematics is the art of giving the same name to different things. – Henri Poincaré. (This was in response to 'Poetry is the art of giving different names to the same thing.')[1]

Abstract
Since the first studies dedicated to the mechanics of deformable bodies (by Euler, D'Alembert, Lagrange) the principle of virtual work (or virtual velocities) has been used to provide firm guidance to the formulation of novel theories. Gabrio Piola dedicated his scientific life to formulating a continuum theory in order to encompass a large class of deformation phenomena and was the first author to consider continua with non-local internal interactions and, as a particular case, higher-gradient continua. More recent followers of Piola (Mindlin, Sedov and then Richard Toupin) recognized the principle of virtual work (and its particular case, the principle of least action) as the (only!) firm foundation of continuum mechanics. Mindlin and Toupin managed to formulate a conceptual frame for continuum mechanics which is able to effectively model the complex behaviour of so-called architectured, advanced, multiscale or microstructured (meta)materials. Other postulation schemes, in contrast, do not seem able to be equally efficient. The present work aims to provide a historical and theoretical overview of the subject. Some research perspectives concerning this theoretical approach are outlined in the final section.

Keywords
Principle of virtual work; Principle of virtual velocities; Generalized Continua; Higher Gradient Continua; Gabrio Piola;

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1. Historical perspective as a guide for future researches

The research of the first sources of higher-gradient continua has its own scholarly interest. However, it can also be motivated by a more cogent aim: the search for the most effective tools for conceiving, finding and developing novel theories or models in physics and, in particular, in mechanics. Gabrio Piola (see [1–3]) spent all his scientific activity and his intellectual efforts in proving that the principle of virtual work (or its particular case, the principle of least action) is the most effective conceptual tool for use by a scientist who wants to create new models able to successfully predict the observed experimental evidence and to forecast the existence of unknown phenomena in Continuum Mechanics.

We share Piola's ideas and want to support his point of view by examining the historical evolution of the theory of higher-gradient continua since its first formulation by him. We will show how and why these models were abandoned (or considered logically inconsistent) by those scholars who refused to accept the Lagrangian postulation of mechanics and we will see that they, instead, could be successfully developed only by those scientists (e.g. Rivlin [4], Green and Naghdi [5–8], Pipkin [9, 10], Mindlin [11–16], Toupin [17], Casal [18–21], Germain [22–24]) *who could manage to accept the use of the powerful abstract concepts given to us by the genius of Lagrange* (in this sentence we are paraphrasing Piola [1, 3]). Exactly as happened with Piola (see his preface of the 1848 work in [1]) we are surprised that the principles of virtual work and least action, even though they have been fully supported by undisputed scientific authorities (for instance by D'Alembert, Lagrange, Hamilton, Landau [25, 26], Feynman [27, 28], Sedov [29]), still need to be advocated. It seems necessary to reaffirm, at least to the advantage of the community of specialists in continuum mechanics, that the continuum models which are needed when describing microscopically strongly inhomogeneous systems (see e.g. [30–33]) must consider internal work functionals (see Germain [24], Salençon [34]) involving second (and higher) gradients of virtual displacements. We want to stress that such a statement can already be found in the work by Piola and we will see why it has sometimes been overlooked: we claim that in the theory of higher-gradient continua one can observe the processes of erasure, loss or removal and rediscovery of scientific knowledge which happened to many other scientific theories (see [35, 36]).

1.1. Gabrio Piola advocates the importance of variational principles in mechanics

We reproduce here some excerptions from the work published in 1848 by Piola and translated in [1].

Piola advocates the use of variational principles in mechanics: he claims that this way of thinking has been proven by Lagrange to be the most effective. To support this statement he uses a simile by establishing a parallel between the theory of differential curves and rational mechanics: he also explicitly states that the synthetic analysis allowed by variational methods greatly reduces the possibility of being misled.

> Since Lagrange managed to reduce all the questions of Rational Mechanics to the Calculus of Variations, the decision to insist on avoiding its use is similar to wishing to behave as those who, being involved in the researches in higher geometry, instead of flying to use the formulas taken from integral and differential calculus, stubbornly persist in using, in a pedestrian way, the synthetic methods. Proceeding in this way one does not manage to get many results and one highly risks being wrong. It is instead convenient to persuade oneself that the greater is the part in which the demonstrations are based on simple reasoning, the more they are likely to be wrong, as the intuitive grasp of our reason is very limited and we are very easily misled as soon as the elements of the question increase to a great number and are interconnected in a complex way.[2]

Piola then proceeds by answering a usual objection of those opposing variational principles. Indeed the opposers of variational methods, in every historical period, always use the same argument: they cannot understand the 'intrinsic evidence' of the consequences to which one can arrive by using the variational principles. They usually ask 'why are you using this expression for the action functional?' or 'why are you using this particular virtual work functional?' They also add: 'Lagrangian postulation is abstract and too mathematical and cannot be justified on physical grounds'. The opposers of Lagrangian methods declare that they need to grasp the physical content of every statement in the theories they use. They refuse to accept a unique basic principle (least action or virtual work principles) and calculate from it all relevant logical consequences and, after this mathematical process, to check if the consequences are acceptable from a phenomenological point of view. In our opinion there is, in this kind of criticism, a certain degree of epistemological misunderstanding. If one thinks to the principles of a theory as to its 'true' foundation, it is legitimate to ask for such things as the 'intrinsic evidence' of the principles themselves. If, on the other hand, one judges the correctness of a given set of postulates just by their capability to generate a theory which accurately describes and foresees observable phenomena, it turns out that there is no point in investigating such questions as the 'truth content' of a principle besides the relationship between its logical consequences and experimental evidence, other questions being purely metaphysical. In our opinion, supported by the classic works by Popper (see e.g. [37]), this last approach is the wiser one.

It seems to us that the opposers of Piola's Lagrangian postulation of continuum mechanics (through all ages from 1824 until now) show the same attitude beautifully (and ironically) described by Galileo Galilei in the following excerption from the Assayer (*Il Saggiatore*) (see [38, 39]).

> [. . .]Your Excellency must consider that, for somebody who wants to prove a statement which, if not false, is at least very dubious, it is really advantageous the possibility of using arguments which are probable, conjectures, examples, comparisons and even sophisms, and then of fortifying and entrenching himself by means of influential texts and the authority of other philosophers, rhetors and historians: while the genuine appeal to the severity of the geometrical demonstrations is too dangerous a challenge for those who are not able to handle them correctly; indeed exactly as ex parte rei there is no alternative between the true and the false, also in the necessary demonstrations either one can indubitably conclude or one inexcusably paralogizes, and there is not any other possibility to keep oneself standing by means of limitations, distinctions, words twists and other [logical] whirligigs, and it is instead necessary, in few words and at the first assault, to stay 'either Caesar or nothing'. This geometrical strictness will lead me, shortly and with less tedium for Your Illustrious Excellency, to be able to disentangle from the following demonstrations; which I will call optical or geometrical more with the aim of seconding Mr. Sarsi rather than because I can really find in them, except for the used figures, some perspective or geometry.[3]

They declare that it is preferable to accept many different principles 'on physical grounds', hoping that the whole set of assumptions will not reveal itself to be logically inconsistent. In other words: instead of accepting only one, clearly formulated, principle, they prefer to accept many (and one by one!) principles, too often risking being *deceived*; they prefer to be obliged to find the intrinsic ('physical'?) evidence of many and different principles instead of using a logically correct procedure leading to clear results from a well-specified assumption; in conclusion, they prefer to indefinitely multiply the number of not-well-grounded assumptions. Let us leave Piola to express his ideas again.

> We need to use powerful methods which, representing the simultaneous and compendious expression of many principles, are able to act by simultaneously gathering the power of all of them and are not using each of them separately and one by one, as usually happens in the logical reasoning: [we need] methods which, once reduced to well-determined and immutable processes, do not allow us to be deceived.
>
> Of course, even when it is using this kind of tools our reason still keeps its rights, as it is able to recognize as true their fundaments and correct their applications, although our reason it is not allowed, most of the time, to reach the intrinsic evidence relatively to the consequences to which it managed to arrive.[4]

Piola declares then that variational principles are one of the most *poderosi (formidable, mighty)* conceptual tools to be used in mechanics.

> It is in this way that in our search for the truth we manage to accomplish those great explorations in which direct reasoning is absolutely insufficient, while it becomes advantageous again when, having reached certain destinations, we want to extend the benefits of the obtained knowledge. One of the most formidable among the indicated tools for mechanicians is precisely the calculus of variations. However, I deeply feel that all the present work is also very far from exhausting the fecundity of the Lagrangian methods: I believe I can assure that with these same methods one can conquer all the various parts of mathematical physics. In the previous Memoir we have already seen the panoply of results which can be deduced by means of its use and we treated the many theories which could be connected with the various parts which constitute it.[5]

Piola knew very well how bitter was the opposition to the Lagrangian principles and methods in mechanics and could forecast that future opposers also would try to confute his reasoning. Therefore he concludes his *Memoir* published in 1848 with the following words, whose correctness we strongly believe in.

> I have another work ready, which is not short and which will continue the present one, and I strongly desire to be able to produce ulterior factual evidence of the stated assertion: but whenever I am able to conclude my work and no matter how successful my own efforts will be I am strongly persuaded that time will prove right the words with which I started this Memoir.[6]

1.2. Piola's foundation of continuum mechanics

Gabrio Piola intended to generalize to the theory of deformable bodies the methods introduced by Lagrange for studying mechanical systems. Referring to [3] for an accurate textual analysis of his contribution we resume his ideas here. It has to be recognized that they are topical even nowadays.[7]

Piola considers a deformable body and chooses a reference (Lagrangian) configuration $\mathcal{C}^*$ for this body. The placement function χ maps every material particle X belonging to $\mathcal{C}^*$ to the position occupied in the actual configuration by X. Denoting by $\bar{X}$ another material particle, the actual distance between the considered particles $(X,\bar{X})$ is given by

$$\rho(X,\bar{X}) = \|\chi(\bar{X}) - \chi(X)\|\,. \tag{1}$$

Piola assumes that the internal virtual work corresponding to a virtual displacement $\delta\chi$, that is, the virtual work relative to the 'internal' interactions between the material particles constituting the body, can be given by the double integral

$$\int_{C^*}\int_{C^*}\frac{1}{2}K(X,\bar{X},\rho)\,\delta\rho \tag{2}$$

where the variation $\delta\rho$ corresponds to the variation $\delta\chi$, the scalar quantity K is introduced as the *intensity* of the force (see page 309 in [1], or page 147 in the original work by Piola [40]) exerted by the particle $\bar{X}$ on the particle X and the factor $\frac{1}{2}$ is present as the action–reaction principle holds. The quantity K is assumed to depend on $\bar{X}, X$ and ρ and is manifestly measured in Nm^{-6} (SI units). In number 72 starting on page 150 in [40] (translated completely in [1]), Piola discusses the physical meaning of this scalar quantity and consequently establishes the properties to be verified by the constitutive equations which have to be assigned to it. He refrains moreover from any effort to obtain for it an expression in terms of microscopic quantities relative to the interactions between the microscopic particles which he believes constitute the deformable body considered. On the contrary, he limits himself to requiring that K is objective by assuming that it depends, among all possible Eulerian quantities, only on ρ: this is an assumption which, in the sequel of his considerations, will have some important consequences. Moreover, he argues that if one wants to deal with continua more general than fluids (for a discussion of this point one can have a look at the recent paper [2]) then it may depend (in a symmetric way) also on the Lagrangian coordinates of both $\bar{X}$ and X: therefore he assumes that

$$K(\bar{X},X,\rho) = K(X,\bar{X},\rho).$$

Piola formulates the principle of virtual work (or virtual velocities, as it is called by Lagrange and Piola himself) for a deformable body as follows, where we use more modern notation than in his original formulation:

$$(\forall\delta\chi)\ \left(\int\left[(b_m(X)-a(X))\,\delta\chi(X)+\left(\int_{C^*}\Lambda(X,\bar{X},\rho)\delta\rho^2\,d\bar{X}\right)\right]dX+\delta W(\delta\chi,\partial\mathcal{C}^*)=0\right) \tag{3}$$

Here the following definition has been conveniently introduced (in order to avoid dealing with the variations of expressions involving square roots):

$$\Lambda = \frac{1}{4}\frac{K}{\rho} \tag{4}$$

by means of which it will be possible to introduce the quantity $\Lambda\delta\rho^2$ instead of the quantity $\frac{1}{2}K\delta\rho$ in the integral (2). In (3), moreover, $\partial\mathcal{C}^*$ is the set of boundary material particles in the reference configuration $\mathcal{C}^*$, $b_m(X)$ is the externally applied (volumic) mass force density, $a(X)$ is the acceleration of material point X, and $\delta W(\delta\chi,\partial\mathcal{C}^*)$ is the work expended on the virtual displacement $\delta\chi$ because of the interactions active through the boundary $\partial\mathcal{C}^*$ plus (possibly) the first variations of the equations expressing the applied constraints on that boundary multiplied by the corresponding Lagrange multipliers.

The particular form of the principle of virtual work presented in (3) was reformulated many years later in an interesting effort to find an effective way to study the onset and growth of cracks in continuous bodies (for more details see [3]).

Some attention is required regarding the choice of the external interactions functional used, $\delta W(\delta\chi,\partial\mathcal{C}^*)$ (this point is carefully discussed in [2], [3] and [41]), as it is clear that a given class of bodies, as characterized by a class of internal work functionals, can only sustain certain classes of external interactions. The fact that, when trying to consider non-appropriate external interactions acting on a given class of bodies, one gets some seeming paradoxes (e.g. diverging displacements or deformation energies) should be simply considered as the manifestation of an inconsistency intrinsic in the introduced model, of the same kind, methodologically speaking, as the choice of an inconsistent set of postulates in a purely mathematical theory. Indeed, when the model for a given class of phenomena has to be chosen one has to determine, simultaneously, the most appropriate functional for both the external and internal work functionals.

As modern calculation tools are available to us, the presentation of the theory could, nowadays, stop here. To solve one 'exercise' of mechanics one has simply to introduce the appropriate finite element scheme for solving the variational problem (3). Piola instead had to write a strong form of it. He then proceeds as described in the following subsection.

1.3. Piola's higher-gradient continua

Piola never considers the particular case of linearized deformation measures (which is indeed physically rather unnatural). He characterizes the class of continuum models (see [3]) for which the state of deformation in the neighbourhood of one material point can be described by means of the Green deformation measure and of all its derivatives with respect to Lagrangian referential coordinates.

The method is simple: one has to expand in a Taylor series the variation $\delta\rho^2$ (of course by introducing suitable regularity assumptions about the function $\Lambda(X,\bar{X},\rho)$) and to replace the obtained development in (3). Using modern notation starting from

$$\chi_i(\bar{X})-\chi_i(X)=\sum_{N=1}^{\infty}\frac{1}{N!}\left(\frac{\partial^N\chi_i(X)}{\partial X^{i_1}\dots\partial X^{i_N}}(\bar{X}^{i_1}-X^{i_1})\dots(\bar{X}^{i_N}-X^{i_N})\right)$$

one gets an expression for the Taylor expansion with respect to the variable $\bar{X}$ of centre X for the function

$$\rho^2(\bar{X},X)=\left(\chi^i(\bar{X})-\chi^i(X)\right)\left(\chi_i(\bar{X})-\chi_i(X)\right).$$

He estimates and explicitly writes first, second and third derivatives of ρ^2 with respect to the variable $\bar{X}$. This is what we will do in the sequel, repeating his algebraic procedure with the only difference consisting in the use of Levi-Civita tensor notation. What Piola manages to recognize (also with a courageous conjecture; see [3]) is that in the expression of virtual work all the quantities which undergo infinitesimal variation (which are naturally to be chosen as 'measures of deformation') are indeed either components of the deformation measure C or components of one of its gradients.

Using modern notation we have that

$$\rho^2(\bar{X},X)=\sum_{N=1}^{\infty}\frac{1}{N!}\left.\frac{\partial^N\rho^2(\bar{X},X)}{\partial\bar{X}^{i_1}\dots\partial\bar{X}^{i_N}}\right|_{X=\bar{X}}(\bar{X}^{i_1}-X^{i_1})\dots(\bar{X}^{i_N}-X^{i_N})$$

$$=:\sum_{N=1}^{\infty}\frac{1}{N!}L_{i_1\dots i_N}(\bar{X})(\bar{X}^{i_1}-X^{i_1})\dots(\bar{X}^{i_N}-X^{i_N}) \tag{5}$$

and therefore that

$$\delta\rho^2(\bar{X},X)=\sum_{N=1}^{\infty}\frac{1}{N!}\left(\delta L_{i_1\dots i_N}(\bar{X})\right)(\bar{X}^{i_1}-X^{i_1})\dots(\bar{X}^{i_N}-X^{i_N}).$$

As a consequence

$$\int_{\mathcal{B}}\Lambda(X,\bar{X},\rho)\delta\rho^2(\bar{X},X)\mu(\bar{X})\,d\bar{X}=\sum_{N=1}^{\infty}\frac{1}{N!}\left(T^{i_1\dots i_N}_{\cdot}(X)\delta L_{i_1\dots i_N}(X)\right) \tag{6}$$

where we introduced the tensors

$$T^{i_1\dots i_N}_{\cdot}(X):=\left(\int_{\mathcal{B}}\Lambda(X,\bar{X},\rho)\left((\bar{X}^{i_1}-X^{i_1})\dots(\bar{X}^{i_N}-X^{i_N})\right)\,d\bar{X}\right).$$

Now the results from Appendix E in [3] imply that the tensor having components $L_{i_1\dots i_n}$ can be represented as a linear combination of the tensors $C(X),\dots,\nabla^{n-2}C(X)$ so that equation (6) becomes, by a simple re-arrangement,

$$\int_{\mathcal{B}}\left((b_m(X)-a(X))\,\delta\chi(X)+\sum_{N=1}^{\infty}\langle\nabla^N\delta C(X)|S_{N\cdot}(X)\rangle\right)dX+\delta W(\partial\mathcal{B})=0 \tag{7}$$

where S_N is an Nth-order contravariant totally symmetric tensor[8] and the symbol $\langle | \rangle$ denotes the total saturation (an inner product!) of a pair of totally symmetric contravariant and covariant tensors.

The very elegant, precise and incontrovertible reasoning which we have drafted using modern Levi-Civita tensor calculus immediately leads Gabrio Piola (and ourselves) to introduce and define a class of continuous bodies, which we can call 'Piola's continua' and which includes that constituted by so-called Cauchy continua.

1.4. Sedov: A late disciple of Piola[9]

In [42, 43], the reader will find a deep, erudite and original presentation of field theories. We will present here some annotated excerptions from these two highly underestimated works.

In the first of them one can find the following program.

> Our analysis will show that the Lagrange variational equation for material continua and physical fields can be employed as a basis for all physical models not only of reversible phenomena, but in cases of irreversible phenomena as well. The variational equation has made it possible to unify and synthesize on a common basis various phenomenological and statistical methods of the theory of irreversible processes in thermodynamics and mechanics.

Here Sedov dares to attack a deeply rooted fake belief too much diffused in the literature.

> The present paper contains a description, analysis, and elaboration of the general method which makes it possible to obtain complicated closed systems of equations and complicated supplementary boundary and other conditions for models of media with internal degrees of freedom from the **minimum number of physical assumptions**. The additional boundary and other conditions just mentioned are a means of rendering specific ('concretizing') individual models and particular formulations of problems. The basic variational equation which we propose to investigate and which constitutes the foundation of the present treatise is a **simple and natural generalization of the variational principle of Lagrange**.[10] In many highly important cases it coincides completely with the familiar applications and formulations of this principle.

It is not clear if Sedov was consciously aware of Piola's contribution, however, his way of considering the postulation scheme of physical theories is exactly Piola's. Note that he claims (and in this he responds to many and absolutely unjustified criticisms to variational methods) that variational principles supply the correct boundary conditions to use in physical field models. This concept is still considered controversial in some engineering and mathematical circles. Being a true follower of Piola, Sedov states then the following.

> Some authors hold the view that the mechanics of movable continuous material media can be constructed by means of a single Cartesian coordinate system without significantly limiting generality. This supposition. which is reflected in certain texts and conveyed to students in all sincerity by their teachers, is incorrect and hinders proper understanding of mechanics and its problems. Confusion is bred, on the one hand, by the fact that the mechanics of deformable bodies is usually concerned with linear problems in which one can assume that the observer's system coincides with the comoving system. On the other hand, it is encouraged by the fact that the metric of the comoving Lagrangian coordinate system in the theory of liquids and gases is manifested only by way of density. At the same time it is often forgotten that even though all substantive characteristics such as velocity, acceleration, strain rate tensor, etc., are introduced by way of the observer's coordinate system, the notion of the comoving coordinate system is still essentially involved.

Sedov considers in his action functionals some potentials which can depend on higher gradients of kinematical fields and concludes that

> The presence of such gradients in the expression for the internal energy makes it necessary to reconsider our concepts concerning the equations of motion and processes, boundary and initial values, interaction mechanisms, conditions at discontinuities, and many other matters.

He justifies this need because

> As we know, there is a need in modern physics and mechanics for the construction, analysis and utilization of new models of bodies with complicated properties.

and because

> In elaborating modem theory of complicated macroscopic models of media and fields it is important to bear in mind that even in Newtonian mechanics the description of phenomena with significant involvement of internal degrees of freedom on the basis of only the principal equation of Newtonian mechanics $F = \text{ma}$ is impossible.

This statement should be read carefully and elaborated on by many modern authors, who should also recall that

In recent years, a great deal of papers have been published which involve complicated equations of motion and state for physical media. However, in many cases such papers have a particular character, sometimes they are connected essentially only with simplest particular problems and with empirical formal approach. Because of difficulties, in essence, a large number of these papers poorly or in no way coordinate with universal or specific physical approaches. At best, only certain general restrictions imposed by the second law of thermodynamics on formally ad hoc introduced relations are considered.

Sedov's main scientific contributions in the works [42, 43] can be summarized by the following two statements:

In the following I will dwell on the highly general variational method of constructing models of continuous media, on the method which is based on and uses fundamental physical ideas and the knowledge of modern theoretical constructions. I mean the rational treatments of the field theory and physical media in theoretical mechanics and in relativity theory, in thermodynamics and statistical physics.

and

In order to describe body and surface interactions inside or on the boundary of the body one must use generalized concepts of energy fluxes and of stress tensors of a higher order.

1.5. Truesdell's difficulties with the principle of virtual works

In his work 'Essays in the history of mechanics' (see [44]), Truesdell shows he has misunderstood the ideas of Lagrange and consequently those expressed by Piola, Mindlin and Toupin. Postponing to a further paper the detailed description of the position expressed by Truesdell in this context and its consequences for the development of mechanics we limit ourselves here to quoting some of his statements (pp. 173–175).

Almost as much nonsense has been written about the *Méchanique Analitique* as about the *Principia* and the *Two New Sciences*. Although LAGRANGE'S book is far easier to read with understanding than is NEWTON'S or GALILEO'S, still it does not seem to be easy enough for most historians of science to penetrate the contents. There are few errors, few novelties, and many routine manipulations in it. While it contains interesting historical parentheses, the presentation of mechanics is strictly algebraic, with no explanation of concepts, no illustrations either by diagrams or by developed examples, and no attempt to justify any limit process by rigorous mathematics. It does not enter at all a number of the fields opened by NEWTON, and it leaves unmentioned most of the deeper and harder problems of mechanics solved by the Basel geometers in the century preceding it. In particular, it does not include the general principle of moment of momentum. It could not do so, because the principle of virtual work does not yield the principle of moment of momentum until the nature of the contact forces is made somewhat explicit. CAUCHY had still to be born and to create the general concept of stress, by which all theories of space-filling bodies are united.

It is very strange that Truesdell, being the historian who claimed to have rediscovered the contributions to mechanical science due to Piola does not recognizes that Piola has proven that the principle of virtual work implies the balance of moment of momentum (the statement sometimes called Piola's theorem). Clearly Truesdell overestimates the role of Cauchy in the process of founding continuum mechanics. As clearly shown for example by Piola, Mindlin, Toupin, Sedov and Germain, the role of the concept of Cauchy stress is, although very important, not at all as crucial as believed by Truesdell and Truesdellians.

On pages 245 to 248 of the essays Truesdell continues to show his negative consideration of Lagrange and his contribution to mechanics. We disagree completely, but we will postpone to further works a detailed analysis of the prejudices shown by Truesdell. Only few comments will be inserted while quoting Truesdell.

[…] Neither of these data will surprise a historian of science, for, on the basis of his general knowledge, he will not expect any development of mechanics as a whole along 'Newtonian' lines by NEWTON or anyone else before the middle of the nineteenth century, nor will he expect the general principle of moment of momentum to be stated before its first great application, namely, the theory of general motion of a rigid body, was discovered. He will more likely start by checking the material as presented in the first systematic treatise on analytical dynamics, the *Méchanique Analitique* of LAGRANGE, published in 1788 and thus bisecting the period separating HUYGENS and NEWTON from the textbook writers at the end of the last century.[11] In the *Méchanique Analitique*, the basic law of mechanics is the principle of virtual work, and from it LAGRANGE derives easily the general integrals of energy, momentum, and moment of momentum for a system of mass-points. Since any forces that do no work may be left out of the expression for the virtual work, we may infer that in LAGRANGE'S equations of the form $\dot{H} = L$, the mutual forces do not contribute to L, but Lagrange does not say so. When we search for explanation of a concept in LAGRANGE'S writings, usually we search in vain; in this case all we find is that the forces are 'those that at the same instant act upon each point of the mass m along some given directions, that is, the velocities that each of these forces would impress upon the mass m if they acted separately and equally during the time taken as a unit. However variable may be the action of these forces, nevertheless one can regard it as constant during an instant.' Also he speaks of the accelerating forces as 'tending to given centers'.

Here Truesdell wants to find in the work of Lagrange the exact words which he has in mind. Lagrange gave to this words a completely different meaning, as his treatment is much more general than the most general one which Truesdell accepted conception of:

Beside the looseness of statement, recalling D'ALEMBERT'S mode of expression, we see here the typical evasive vagueness of LAGRANGE. In the statement of the principle of areas LAGRANGE in fact makes us suspect that he does *not* perceive the generality of the integral of moment of momentum, for he adduces only instances where the torque on each body vanishes: 'If the system were not subject to any accelerating force, or if the only forces present all tend to the point we have selected as origin of co-ordinates …' Search of other parts of the book confirms. Datum 3. In the *Méchanique Analitique* there is nothing relevant to Statement 13; as for Statement A, from LAGRANGE'S treatment it may be inferred that forces which do no virtual work do not contribute to the resultant torque, but LAGRANGE gives no evidence of seeing this fact.

Subsequently Truesdell tries to demolish one of the pillars of modern mechanics and, maybe, of modern science.

The historian will consult the *Méchanique Analitique* for a second reason, namely, that it includes in sections at the beginnings of the various parts the first history of mechanics. He who reads that history will obviate the need to consult the secondary works […], because in regard to rational mechanics since GALILEO'S time these do little more than quote, paraphrase, extend, or correct in detail the little sketches by LAGRANGE. LAGRANGE writes that the principle of areas '… seems to have been discovered at the same time by Messrs. EULER, DANIEL BERNOULLI, and the Chevalier D'ARCY, but in different forms. According to the two first, this principle consists in the fact that in the motion of several bodies about a fixed center, the sum of the products of the mass of each body by the velocity of circulation around its center and by the distance from that same center is always independent of the mutual action that the bodies can exert upon each other, and it remains constant so long as there is neither exterior action nor an exterior obstacle…The principle of Mr. D'ARCY…is that the sum of the products of the mass of each body by the area that its radius vector describes about a fixed center is always proportional to the time. It is plain that this principle is a generalization of the beautiful theorem of NEWTON on the areas described in virtue of arbitrary centripetal forces…' While LAGRANGE'S book is a good starting place, experience with it has led me to the following working hypotheses:

(The following conclusions seem to us completely hazardous.)

1. There was little new in the *Méchanique Analitique*; its contents derive from earlier papers of LAGRANGE himself […] or from works of EULER and other predecessors. 2. General principles or concepts of mechanics are misunderstood or neglected by LAGRANGE. 3. LAGRANGE'S histories usually give the right references but misrepresent or slight the contents. When we read LAGRANGE'S sarcastic comment about D'ARCY, '…he even made out of it a kind of metaphysical principle, which he calls the *conservation of action*…, as if vague and arbitrary names were the essence of the laws of nature and could by some secret virtue raise to final causes some simple consequences of the known laws of mechanics.'

In the following statements Truesdell claims that Lagrange's understanding of mechanics is limited. He also implies that Lagrange was a quite 'lazy' boy under the bad influence of D'Alembert.

[Lagrange] states (§13) that the Lagrangian equations hold for 'an infinity of particles subject to any forces proportional to functions of the distances'; the meaning of this statement is not certain, but with any meaning I can conjecture it is generally false. In this paper LAGRANGE stops short of deriving the equations of motion of a rigid body by his method. […] While in §§4–8 LAGRANGE derives the integrals of momentum, moment of momentum, and energy, use of special properties of the potential function tends to conceal their meanings. […]

Turning to Hypothesis 1, we can choose first to follow up LAGRANGE'S own earlier work. Moving slowly backward in his *Œuvres* is a tedious process. The task is lightened by use of a fourth working hypothesis: 4. LAGRANGE'S best ideas in mechanics derive from his earliest period, when he was studying EULER'S papers and had not yet fallen under the personal influence of D'ALEMBERT.

2. State of the art: Higher-gradient continua theory in the language of functional analysis

The pioneering works [13, 14, 45, 46] and especially those authored by Paul Germain [22–24] clarified the role of functional analysis in continuum mechanics.

In [41] and in [47, 48], continuing Germain's line of thought, it has been remarked that the new tools supplied by the theory of distributions developed by Laurent Schwartz (see the fundamental book [49]) are really adapted to frame generalized continuum theories.

Indeed virtual work is clearly to be identified as a linear and continuous functional defined on admissible virtual displacements and, even more evidently, the set of virtual displacements must include, in the great majority of instances, at least the set $\mathcal{D}$ of C^∞ functions having compact support. In some cases a suitable subset of $\mathcal{D}$ is to be considered: this circumstance can be accounted for via the Hahn–Banach prolongation theorem (for a reference see e.g. [49]) but for simplicity will not be treated here.

Once the Fréchet topology is introduced for $\mathcal{D}$ it is possible to introduce the space of the linear and continuous functionals defined on $\mathcal{D}$, that is, the set of compact distributions, as named and introduced by Schwartz. Once restricted to $\mathcal{D}$ a virtual work functional must therefore coincide with a distribution.

Therefore the representation theorems presented in Schwartz [49, pp. 82–104] can be fruitfully used to describe the structure of virtual work functionals. We remark that already in the pioneering works by Piola [1–3, 40], by using a micro–macro identification procedure and an atomistic micro-model with long-range interactions, the most general kind of distribution was recognized to be necessary to encompass all physically admissible virtual work functionals.

2.1. Work functionals

Once we fix a generic subbody SB (i.e. a subset of material particles occupying, in a given configuration, an admissible domain) of a given continuous body B and consider the set $\mathcal{A}(SB)$ of all infinitesimal displacement fields admissible for SB, it is natural to admit that in $\mathcal{A}(SB)$ are included infinitely differentiable functions having compact support included in SB. In other words, we assume that $\mathcal{D}(SB) \subset \mathcal{A}(SB)$.

It is also natural (as done e.g. in [22–24]) to assume that the work expended by the interactions between SB and its external world is a linear and continuous functional (with respect to the Fréchet topology) when restricted to $\mathcal{D}(SB) \subset \mathcal{A}(SB)$.

In other words we accept the following (fundamentally due to D'Alembert and Lagrange).

POSTULATE ON WORK FUNCTIONALS

The work expended by all the interactions relative to a subbody SB are distributions (in the sense of Schwartz) concentrated on $\overline{U(SB)}$, where we denote by $\overline{U(SB)}$ the topological closure (in the sense of the natural topology on $\mathbb{R}^n$) of an open set $U(SB)$ including SB.

It is clear that, once the previous postulate is accepted, theorems and definitions of the theory of distributions (see [49]) become really relevant in continuum mechanics. In particular we know the following.

Theorem 1. Every distribution having compact support K can be represented as the sum of a finite number of derivatives of measures all having their support included in K.

Theorem 2. (A distribution is said to have order smaller than or equal to k if one can represent it as the sum of derivatives with order smaller than or equal to k of measures.) Every distribution having support included in a regular embedded submanifold M can be uniquely decomposed as a finite sum of transversal derivatives of extensions of distributions defined on M.

We can obviously exploit the Schwartz general representation theorems and, by taking into account the aforementioned definitions and theorems, we get that the postulate on work functionals can be rephrased into the following.

POSTULATE ON THE STRUCTURE OF WORK FUNCTIONALS

For every subbody SB, the work of exerted interactions takes the form

$$\mathcal{P}(SB, V) = \int_{\overline{SB}} (\nabla^{N_{SB}} V) \mid dT_{SB}, \quad \forall V \in \mathcal{D}(SB) \tag{8}$$

where N_{SB} is an integer, dT_{SB} is a tensor-valued measure, and the symbol $\mid$ stands for the total saturation of contravariant and covariant indices.

2.2. External and internal work functionals

Given a body B and its subbody SB it is easy to understand that SB can interact with the world external to B and to the remaining part $B - SB$ of B. The interaction between SB and $B - SB$ is internal for B and external for SB. The external world for SB is composed of the union of the external world for B and $B - SB$. Having explained this nomenclature it is obvious what we mean by internal and external work functionals.

When following the approach *à la D'Alembert* one will introduce the following.[12]

POSTULATE ON WORK BALANCE OR D'ALEMBERT PRINCIPLE OF VIRTUAL WORK

For every subbody SB of a given body B and for every test infinitesimal displacement field V the following equality holds:

$$\mathcal{P}^{int}(SB,V)=\mathcal{P}^{ext}(SB,V) \tag{9}$$

where $\mathcal{P}^{int}(SB,V)$ represents the work expended by the interaction of the subbody $B-SB$ with the subbody B and $P^{ext}(SB,V)$ is the work expended by the interaction of the world external to B with the subbody SB.

Indeed the external world interacts with a continuous body B and its subbodies exert internal interactions on each other. When considering a given subbody SB of B a similar distinction can be made and some interactions which must be regarded as internal when referring to B actually become external when referring to SB. We call *internal* and *external* the work expended on any virtual displacement by internal and external interactions respectively: since the works by D'Alembert, inertial forces are included in external interactions.

For a presentation of the ideas inspiring the postulate on work balance, we refer to [22, 23, 34] or to the works [52], [40] (translated in [1]), [53] and [54]. In these works, it is shown that this principle is the most suitable when dealing with more general systems than finite systems of material points: it is for example very effective in continuum mechanics.

Piola, Mindlin and Toupin limited themselves to considering the following class of external interactions.

CONSTITUTIVE ASSUMPTION FOR EXTERNAL WORK

The external interactions exerted on some subbody SB are described by a distribution $\mathcal{P}^{ext}$ made of two parts. The first part corresponds to long-range external interactions exerted on SB. It is assumed that it can be represented by a distribution which is an integrable function with respect to the Lebesgue measure. The inertial power, which D'Alembert included in $\mathcal{P}^{ext}$, is of this type. The second part corresponds to contact actions. It is assumed to be a distribution concentrated on the topological boundary of D.

2.3. Contact interactions and stress states

The theorems recalled in the previous section suggest that the expression for the work of contact interactions usually considered in continuum mechanics, when the classical format due to Cauchy is considered, is very restrictive.

Contact interactions must be mathematically described by introducing the functional which expresses the work they are expending on virtual displacements. The class of subbodies we consider cannot be limited to domains with smooth boundaries. Indeed tetrahedrons have to belong to this class if we want to follow the trail of Cauchy. We admit subbodies with boundaries (or Cauchy dividing surface) which are piecewise regular, with normal fields subjected to jumps on a finite set of regular curves eventually intersecting into wedges. The Cauchy cuts separating subbodies, where we assume that contact interactions are concentrated, can be non-regular in general, and for higher-gradient continua on the edges of Cauchy cuts (i.e. where normals to the faces are jumping) or on wedges of Cauchy cuts (i.e. points where edges are intersecting) new kinds of contact interactions may appear.

In the more general context we must consider a more general expression for internal contact interactions (but presumably not the most general one which can be conceived; see [3]), which is obtained by the logically consistent representation given by the formula (8):[13]

$$\mathfrak{S}(B,U)=\sum_{k=0}^{N-1}\int_{\partial_2 B}\mathsf{F}_k^2\mid\nabla_\perp^k U+\sum_{k=1}^{N-2}\int_{\partial_1 B}\mathsf{F}_k^1\mid\nabla_\perp^k U+\sum_{k=0}^{N-3}\int_{\partial_0 B}\mathsf{F}_k^0\mid\nabla_\perp^k U. \tag{10}$$

The functional $\mathfrak{S}$ therefore characterizes the stress state of the continuum which is then said to be in a *stress state of order N*. The fields $\left(\mathsf{F}_k^2,\mathsf{F}_k^1,\mathsf{F}_k^0\right)$[14] which depend on B and on the material particle are quantities dual to the normal gradients $\nabla_\perp^k U$ of the virtual displacement field and may be called *the contact* $(k+1)$*-forces* (see [4–7, 55, 56]).

The reader must note here that in general the configuration field may take values in a manifold and the velocity field in its tangent bundle, which can be of any tensorial nature. This is the case for Eringen's microstructured continua. This tensorial nature of kinematical fields is irrelevant in the present context and therefore, for the sake of efficiency, we operate as if the kinematics were described by a real-valued function U. Therefore, the tensor $\nabla_{\perp}^{k} U$ and its dual quantities are considered to be of order k, as well as its dual quantities and $\mathsf{F}_{k}^{i} \mid \nabla_{\perp}^{k} U$ denotes the scalar product of the indicated tensors. It is straightforward, by applying our results component-wise, to extend them to the case where U is a tensor, and in particular the classical case where U is a vector. We can conclude that in the presented treatment the micro-structured continua introduced by Eringen are indeed included.

One of the points of the Cauchy approach which are discussed more often (see e.g. [57]) is the assumptions which are needed regarding the dependence of the fields F_{k}^{i} on the (shape of the) subbody SB. We assume that the densities F_{k}^{i} depend in a sufficiently regular way on the position and depend on the considered subbody only in a local way through its shape.[15]

The theory by Cauchy is a particular case of the one described here: indeed if we make the extra assumptions that the stress state is of order one and that the contact surface 1-forces depend on the shape of dividing surfaces only through their normal then we are back to the framework used by Cauchy and our demonstrations and results are identical.

Assuming that the stress state is of order one is indeed a constitutive assumption so deeply rooted in the minds of many authors that it has very often been accepted unconsciously, and we emphasize that Noll's theorem [57] cannot be proven without this assumption.

The generalized contact interactions we previously described are not usually considered in the literature. This point relates more to the history (or maybe even psychology) of science than to science itself. We limit ourselves here to remarking that one can find at least two different reasons for this circumstance. First, this is due to the fact that virtual work is not always the preferred tool for some mechanicians while, on the other hand, it gives the conceptual framework in which generalized contact interactions arise naturally. Secondly, it is a fact that many usual materials are actually modelled by stress states of order one.

Cauchy's proof of the existence of stress tensor is based on the equilibrium of contact forces with a force which is assumed to be absolutely continuous with respect to volume. We also may use a similar assumption (as proposed in [48] and [47]).

HYPOTHESIS OF QUASI-BALANCE OF WORK

For every virtual displacement field U, there exists a constant K_U such that, for every subbody SB included in B,

$$|\mathfrak{S}(SB, U)| \leq K_U \, |SB| \, . \tag{11}$$

Here $|SB|$ denotes the Lebesgue measure of SB.[16]

While inequality (11) could seem a very weak assumption, we emphasize that it rules out some possible stress states, such as for instance those occurring in continua including material surfaces or continua including interfaces with Laplace surface tension.

This inequality may be considered as a basis for a postulation for continuum mechanics when higher order continua are also considered, as proven in [47, 48].

3. Cauchy straightjacket and the consequent conceptual blockages

The polemics between Poisson and Piola also involved Cauchy. We read from [1, p. 261]:

> But is it true that the principle of equal pressure in all directions is intimately linked with the regular distribution of the molecules, so that it can not exist one without the other? (Poisson. Traité de Mécanique. Tome II p. 506). I doubt it very much, and I think that here as well one has gone forward a bit too far into the deductions: and this because the ideas around that quantity which we call the internal pressure of the fluid have not yet completely clarified. And this is a delicate subject, where it is good to make distinctions, nor it is given to hurry in a few words: then I will come back to it in a separate section. Meanwhile, I will observe that also another eminent French geometer Mr. Cauchy openly disagrees with Poisson on this point, having written in his early Exercises of mathematics (Tome III p. 226) «on voit par les détails dans lesquels nous venons d'entrer que, pour obtenir l'égalité de pression en tous sens, dans un systéme des molécules qui se repoussent, on n'a pas besoin d'admettre, comme l'a fait M. Poisson, une distribution particuliére des molécules autour de l'une quelconque d'entre elles.»Which is said without intending to express my views entirely consenting to those considerations thanks to which Mr. Cauchy as well composes the general equations of motion of bodies. I respect his way of seeing, but I keep mine, or rather not mine, but the one connected with the philosophy of the methods of my Schoolmaster, as I have said from the beginning.

Recall that the Piola's schoolmaster is Lagrange and that Piola, while expressing himself against the views of Poisson, seems to be hesitating in starting a controversy against Cauchy.

Piola wants to found continuum mechanics on variational principles. He does not dare to formulate a least action principle as Maupertuis did some time before him (and as Hamilton is doing more or less at the same time). He retreats to a stronghold which he believes can be defended better: the principle of virtual work. The reason is already expressed clearly by Lagrange: it surely also holds for dissipative systems and seems to encompass more general models.

The Cauchy straightjacket consists in the following set of assumptions:

1. on a Cauchy cut separating two subbodies in a continuous body only contact surface density of forces is exerted;
2. contact forces are balanced by volume forces (i.e. forces per unit volume).

Noll [57] has proven that these two assumptions imply the so-called Cauchy postulate: contact surface density of forces depends only on the normal of the Cauchy cuts. The cited apparently general result has made many authors believe that the Cauchy approach can encompass all logically consistent continuum models. This is not the case: indeed contact interaction can be concentrated on lines and points eventually constituting the set of discontinuity for the normal to Cauchy cuts. Moreover, not only contact forces are admissible in general: as clearly stated in many works (among which we want to recall Toupin's works [17, 45, 46] and also [11–15, 22–24]) the possible contact interactions do not reduce simply to forces, in general. In fact, one can have work expended on transverse gradients at Cauchy cuts up to a given differentiation order. This circumstance gives rise to what Green and Rivlin (see [5–8]) call contact *k-forces* and to higher-order hyperstresses (see [45]).

The whole conceptual frame built in [57] collapses when these more general contact interactions must be taken into account. The need for them is not only widely supported by logical arguments (see the already cited works by Laurent Schwartz) but also by the results which are being gathered (since the works by Toupin and Mindlin) about the micro–macro identification processes.

The Cauchy straightjacket produces a particular form of equilibrium (or evolution) equations where the divergence of a tensor appears and the flux of the same tensor on the boundary of the considered region supplies boundary conditions.

If one wants to use Nth-gradient theories the equilibrium equations involve (see [41]) exactly N stress tensors of increasing orders (from the second to the $(N+1)$th in the case of continua where the only kinematical field is placement). The order of each of these tensors is reduced by the application of divergence operators (exactly N times for Nth-order tensors). There is only one bulk equilibrium or evolution equation, while there are as many boundary conditions (on regular surfaces) as many stress tensors are introduced (that is, exactly N). Moreover, suitable boundary conditions are needed on edges and wedges.

This class of mathematical problems and models cannot be incorporated into the so-called general format of mechanics described in [59].

The interested reader will read from (10) the complexity of boundary conditions needed for an Nth-gradient continuum (more details on the representation formulas linking stress tensors and contact interaction generalized forces are given in [41]).

4. Higher-gradient continua as models for microscopically complex systems

Many papers in the literature try to deduce from the structure of microscopic models for mechanical systems the properties of their macroscopic ones. We limit ourselves here to citing a few of them [31, 32].

The common feature which is shared by all systems to which the Cauchy simplified version of continuum mechanics does not apply is clear: these systems show, at the microscopic level, high contrast in geometrical and mechanical properties. This contrast has relevant effects on their macroscopic behaviour and requires (yet to be accounted for), to be accounted for, that higher gradients of displacement or suitable microstructure fields or both must appear in the constitutive equations for deformation energy.

This circumstance seems perfectly clear in the mind of Mindlin, as is shown by his first efforts at getting a micro–macro identification process. Indeed in [60] a first set of lattice interactions is proposed in order to get a third-gradient continuum as a limit model.

Note that the heuristic procedure systematically presented in the literature can be tracked already in the works by Piola. We can call it the Piola micro–macro identification procedure. It will be discussed in further investigations: here we simply describe it briefly.

Piola assumes that there exists a continuous macroscopic placement function which describes the global behaviour of the considered lattice of particles. He assumes that this lattice is equally spaced in all directions and that the distance σ between two close particles is small. Then he assumes that he knows the law of interaction between any couple of particles in the lattice and therefore knows the expression of virtual work for any virtual displacement. Then he calculates the virtual work of the microscopic systems in the presence of a virtual displacement obtained by the variation of the macro-displacement function. The micro-displacement and the micro-virtual displacement are assumed to be estimated by calculating in the points of the lattice the values of macro-fields. This assumption has later been long debated, and it gives a powerful heuristic tool for the micro–macro identification process, and indeed has been systematically used in the literature (see e.g. the results found in [61] for porous systems, and the papers by Cecchi and Rizzi [62] on generalized beams or for Generalized Beam Theory (GBT) by Piccardo et al. [63]).

Following the calculation process by Piola one manages to find an expression of virtual work for macro-models starting from a priori knowledge of the micro-structure of the considered system.

Mindlin uses this process identifying macro-energy in terms of micro-energy and, exactly as done by Piola, in the limiting process assumes that the finite difference converges to a derivative, and higher-order finite differences converge to higher-order derivatives. This point of course deserves careful mathematical attention as this identification is not generally possible: the method of Gamma convergence allows for the establishment of more rigorous results while fixing the limits of applicability of Piola's procedure.

Interesting results in this context, by using the possibilities given by modern computing systems, are found by Forest (see e.g. [64]) where micro-models are three-dimensional continua whose deformation energy depends on the first gradient of displacement only, while obtained macro-models show the dependence of the deformation energy on higher gradients.

5. The original contributions by Mindlin and Toupin

While somebody claimed to have solved the sixth Hilbert problem[17] and somebody else announced this achievement, Mindlin and Toupin, among many others (see e.g. [5–7, 10, 55, 66]) worked to find the most suitable continuum models for describing complex mechanical systems.

They based their efforts on firm conceptual grounds, the principle of virtual work or, in the case of Lagrangian systems, the principle of least action. Why did the followers of Noll and Truesdell not consider such a possibility? This circumstance must be the object of careful investigation, to which future papers will be dedicated.

Mindlin in his masterpiece [13] starts from an expression of the internal power involving up to the third gradient of virtual displacement. Somebody who, with open mind, reads Mindlin's paper immediately after having read Piola [1, 3, 40] cannot avoid remarking on the striking equality of vision and purpose.

The strategy of the investigations presented by Mindlin is clear: he postulates a form for internal work, that is, the work expended by internal interactions on a virtual admissible displacement. He introduces (exactly as done by Piola and later by Germain [22] or in [41, 47, 48]) a set of three stress tensors, dual to the first, second and third gradient of virtual displacement and then, integrating by parts, he finds the set of admissible contact interactions which can be sustained by such a continuum. In this process he closely parallels Piola when he introduces fluids as those continua which, at equilibrium, cannot sustain contact shear forces.

The main mathematical tools used by Mindlin are differential geometry, Levi-Civita absolute calculus and some functional analysis.

His results are some partial differential equations and corresponding boundary conditions which cannot be, without inelegant and sometimes twisted logical contortions, framed in the so-called general solution of Hilbert's sixth problem put forward by Noll (see [59]).

Indeed it is not possible to regard the set of aforementioned partial differential equations plus coherent (i.e. deduced by a well-structured variational principle) boundary conditions as balance equations having the structure envisioned by Truesdell and Noll. The reason for such a statement is clear: there are as many boundary conditions as there are transverse gradients of virtual displacement contributing to the work expended by contact interactions and only one bulk partial differential equation. Therefore, one should postulate the balance of one quantity in the bulk and the balance of many quantities on the boundary of the considered body, a circumstance which is rather difficult to frame logically.

The main contribution of Mindlin is to have recognized such difficulties and to have explicitly described the methods necessary to circumvent them, in a historical period where variational principles were openly despised.

Richard Toupin has witnessed[19] to the first and third authors of the present work, the great influence exerted on his scientific education formation by the textbooks [25, 26] where the principle of least action is considered the basis of all physical theories.

In the papers of Toupin the basic assumption concerns energy (or action): in this aspect his formulation is closer to the one presented by Cosserat and Cosserat (see [53, 54, 67]). He studies continua following the Lagrangian scheme: first he introduces the kinematical fields he considers needed to describe the phenomena to be studied. After he has clearly specified the set of admissible configurations (see [68]) and admissible motions then he postulates an action functional which is minimized by the real motion. The presentation proceeds by calculating the first variation of the introduced action functional (which is a form of the principle of virtual work) from which the evolution equations are deduced via the standard Lagrangian localization process. These evolution equations are composed of bulk conditions (some partial differential equations) and associated boundary conditions.

In the paper [45] a careful description of some pre-existent literature is presented: unfortunately his historical digression stops at the seminal works by Hellinger [69] and Cosserat and Cosserat. It is clear that in the subject he is treating the more ancient works by Piola as more relevant.

Subsequently Toupin imposes the invariance conditions [45, p. 93]:

> Following the COSSERATS, we postulate that the action density L is invariant under the group of Euclidean displacements.

With elegant reasoning Toupin proves a generalization of Piola's theorem (proven in [40] for continua whose kinematics is characterized simply by the placement field) [45, p. 94]:

> Thus, we have established the basic theorem of equivalence between conservation and invariance: linear momentum, angular momentum, and energy are conserved in a perfectly elastic medium with deformable directors if and only if the action density is invariant under the group of Euclidean displacements.

This statement has been misunderstood and misused by many authors: note that the equivalence concerns the structure of action density and does not directly concern the evolution equations deduced from the least action principle.

Subsequently Toupin remarks:

> That the conservation of energy, momentum, and angular momentum are necessary and sufficient conditions for the Euclidean invariance of the action density in Hamilton's principle was emphasized again and again in the COSSERATS' memoir.

He tries to reinforce his logical argument by invoking the principle of authority: a standard necessity when scientific controversies arise (especially in scientifically weak periods: see [36] on this).

Subsequently in Section 7 of [78] the necessary conditions for stationarity of action are recast in the form preferred by the Truesdell school, to show that variational principles produce a more effective and firmer capability of deducing the basic equations governing physical phenomena. On the other hand in Section 8 Toupin continues by studying all measures of strain which can be coherently introduced in the considered theories of microstructured continua and the consequent possible general forms for action density are presented. The particular case of Cosserat continua is recovered in Section 9. Many of the subsequent papers which one can find in the literature of Cosserat continua have rediscovered the results presented there with clarity and economy of thought. At the beginning of Section 10 Toupin 'rediscovers' or 'recovers' the results presented by Piola regarding Nth-gradient continua. Of course he perfectly masters Levi-Civita absolute calculus: the presentation exploits this powerful tool and again a clear economy of thought is fruitfully gained. In Section 10 one reads:

> There follow some results for materials of grade 2. The general features of the theory of non-simple materials are illustrated sufficiently well by materials of grade 2, and the analysis of higher grade materials is only that much more complicated in details.

This statement is rather optimistic as in [41] the analysis of higher-gradient materials is confronted with the additional use of some technical tools from differential geometry: to treat the general case does not simply reduce to the presentation of more complicated details. However, the presented theory of second-gradient materials is very careful and complete and is surpassed only by the later paper by Paul Germain [22].

In Section 11 the technical problem of confronting Cosserat continua and higher-gradient continua is studied: this topic also seems nearly to have been overlooked in the literature, even if it plays a crucial role in many conceptual and numerical aspects of the considered theory. By means of the introduction of suitable Lagrange multipliers one can constrain the Cosserat micro-rotation to coincide with the rotation appearing in the polar decomposition of the displacement field. In this case, formally, one can state that, using the words of Toupin:

> This result shows that the Cosserat media with constrained rotations are a proper subclass of the materials of grade 2.

Section 12 is another small masterpiece: in it one can see how powerful the variational methods are in dealing with problems which otherwise may appear very difficult. The section is concluded with the elegant theorem of initial hyperstress.

> For almost all uniform homogeneous materials of grade 2 there exists a reference configuration free of initial stress, but, except for special materials, such a natural state possesses initial hyperstress. The natural state is unique to within a rigid motion.

The final section, Section 14, deals with the study of boundary layers which can be described by the generalized models introduced, as Section 13, concerning material symmetries (see [70]), reduces to no more than a sketch.

The paper is closed by a warm acknowledgement of Professor Mindlin.

6. Research perspectives

The theory of Piola's continua, as we can call continua in which internal work functionals depend on higher gradients of virtual displacement, needs to be developed. Notwithstanding all the efforts made all his life by Gabrio Piola, and despite the fact that he was the beginner of a strong school of mathematical physics (see [3]), Cauchy's particular kind of continuum has been considered the most general one which can be logically formulated. Some authors who did accept the logical consistency of the more general continua considered by Piola very often also stated that they rather represent a mathematical amusement without any practical applications, except for some very specific phenomenological situations.

The legacy left to us by Piola, and more recently by Mindlin and Toupin, consists in the assessment that such opinions are indeed fallacious and caused a long blockage of the advancement of continuum mechanics.

The study of Piola's continua will produce many interesting results and will allow for the discovery and exploitation of many new phenomena and the design of new advanced metamaterials.

We list here in short some fields where we forecast important developments in the near future.

The modelling of the mechanical behaviour of living tissues and their mechanically induced remodelling. As is discussed in the nowadays wide, dedicated literature [71–73], bone tissues have a complex mechanical behaviour and they must be modelled by means of continuum models incorporating their microstructure by means of suitable additional fields.

Moreover, the biological activity, induced by mechanical actions, of living parts of bone tissues produces a continuous remodelling of them. The process of remodelling takes place by means of the formation of the microstructures constituting the tissue at the many and different length scales which characterize their mechanical behaviour. The introduction of microstructured continua for their mathematical description is therefore unavoidable (see e.g. [74, 75]).

Bone tissues are intrinsically endowed with a multiscale structure and consequently may show exotic behaviour, like internal boundary layer formation for deformation or stress concentration, or like mechanisms of mechanical instabilities.

Such phenomena (as already remarked by Sedov [29, 42, 43]) are most naturally described by higher-gradient models.

The design of metamaterials for biological applications: Bone and tissue reconstruction. When a living tissue is to be reconstructed by the addition of an artificial, although biocompatible and eventually bioresorbable, material, it is desirable that the added material has the closest possible behaviour to the natural living tissue. This principle has been exploited in the investigations presented in [76] where it has been shown how the microstructural response of a hypothetical bioresorbable material may positively influence the remodelling process in a reconstructed bone tissue. This opens an important possible field of research in the theory of the design of metamaterials. The synthesis of specifically tailored metamaterials to be used in tissue reconstruction is becoming a more and more topical research subject, additionally with a view to other applications.

The metamaterials to be designed must exhibit peculiar mechanical properties which have to favour their effective role as a scaffold on which new bone tissue must be formed by a deposit and conglomeration process. Moreover, they must be biocompatible and possibly bioresorbable: in other words, they should not interfere with the physiological remodelling and reconstruction activity of active cells present in bone tissue and they have to be metabolized in a way similar to living tissue.

Their mechanical and biological behaviour must be optimized in order to lead the remodelling process to the formation of the most effective bone tissue possible.

It is undoubtable that generalized continua may supply an effective tool in modelling all aforementioned phenomena and processes.

The description of the mechanical behaviour of fibre reinforcements for composites. It has been widely recognized (see [9, 77, 78]) that having introduced a continuum model for solids comprised of interconnected arrays of fibres it is necessary to account for the bending stiffness of the fibres by means of a suitable dependence of the deformation energy on the second gradient of the displacement field. In general, when the considered system has a complex microstructure standard first-gradient (Cauchy) models are not able to catch all relevant phenomena, including the onset of boundary layers where the gradients of the displacement field may assume high values. The formation of such layers is induced in fibre reinforcements by the inextensibility conditions which induce high gradients of stress in very narrow material regions.

At least in principle, the mechanical behaviour of fibre reinforcements can be described by means of models in which i) each fibre is regarded as a (Euler or Timoshenko) beam and ii) the interaction between fibres is modelled via suitable (eventually elastic or visco-elastic or visco-plastic) constraints. However, such a modelling procedure shows some limits: it requires huge calculating devices even for very simple situations and does not allow for any effective analytical or semi-analytical optimization process. Therefore, at least in the present state of the art, continuum modelling seems required for these complex mechanical systems, and the continua to be introduced have to belong to the class of Piola's continua.

The description of solid–solid, solid–fluid and liquid–gas phase transitions, related to reversible phenomena. Capillarity phenomena are likely the first phenomena for which generalized continua were exploited (see e.g. [2, 19, 79]).

In capillarity internal boundary layers are formed in bodies which experience high gradients of material properties and deformation fields in very narrow material regions. Often the first gradient deformation energy functions are not convex, showing different energetic wells. In this context it is clear that higher gradients of deformation must have an energetic role, rendering the sharp variations of first gradients of kinematical fields energetically expensive. Microscopically, such dependence of the deformation energy is associated with long-range interactions among the material particles constituting the considered continua, as already envisaged by Piola himself and by Toupin.

In phase transition higher-gradient continuum models seem unavoidable. This can be seen in some three-dimensional models concerning complex multi-phase interactions, for instance if one wants to describe the onset of the strong space variation of pressure observed, also in equilibrium conditions, when bubbles or drops are surrounded by the corresponding other phase of the same material. Laplace circumvented this difficulty by introducing *ad hoc* a third, bi-dimensional, phase carrying a surface energy. Laplace's model assumes the possibility to know a priori the localization of the interfacial zones and their constitutive properties. Instead, by means of higher-gradient models it is possible, without any further assumptions added to the choice of deformation energy, to forecast the localization and the varying mechanical properties of interface regions, which are assumed to be three-dimensionally extended.

The formulation of well-posed problems in the theory of damage, crack formation and growth, or in the theory of plasticity. Standard first-gradient continua have been extensively considered in order to model the phenomena of crack formation (see e.g. [80]), of damage growth (see e.g. [81]) or those involving plastic deformations (see e.g. [82–84]). However, many numerical and conceptual problems arise in this context, as very often the considered models are ill-posed or show the possible onset of strongly singular solutions. More and more often (see e.g. [85]) higher-gradient continuum models are being used to formulate regularized, well-posed models, in which the localization of the material regions where damage or plasticity is concentrated is determined by means of the introduced analysis. In other words, higher-gradient continuum theories show in this context the powerful feature of also being able to predict a priori the regions where the considered phenomena originate, and not only the evolution of damage and/or plastic deformation after the formation of damaged or plastic material regions. Note that while the class of considered phenomena shares with phase transitions the common feature of localizing in unknown regions, it still differs from them in its being irreversible (the consideration of phase transitions in the context of elasticity theories can lead to far from trivial problems; the reader is referred to e.g. [86–88]). This feature made the formulation of variational techniques for determining searched-for models more complicated, however, the results presented in [89] show that once unilateral constraints are introduced to the variational scheme very important and useful results may be obtained. Note that very similar considerations can be formulated for the phenomena occurring in crack formation, for the description of which higher-gradient continuum models (also for their capability of accounting for the multiscale structure showed by materials in the vicinity of cracks) seem to be really adapted (see also the recent development of peridynamics, [3] and the references therein).

The design and optimization of multiphysics metamaterials exploiting the coupling of several physical phenomena, the example of pantographic and piezoelectromechanical structures. In [90] a new metamaterial is considered in order to detect damage onset in some newly conceived acoustic shields. The use of higher-gradient continuum models can play a relevant role in modelling the physical behaviour of many complex mechanical systems and structures [91–95]. The models for complex materials and structures may require the introduction of micro-structured continua endowed with additional kinematical fields, also accounting for the activation of internal degrees of freedom. In [96] it is shown how the activation of these microscopic degrees of freedom may be exploited for energy harvesting or dissipation: these kinds of effects have been also considered (e.g. in [97]) in the study of wave propagation in complex mechanical systems, proving once more the importance of the use of higher-order continuum models.

Actually complexity of piezoelectromechanical structures or microscopically strong inhomogeneous mechanical systems cannot be easily studied by means of too detailed models describing the behaviour of every structural subcomponent. Therefore a homogenized or less detailed model seems necessary in order to account for their most relevant features and, in particular, when they need to be tailored to special purposes. Also, when optimization of their physical properties is necessary, simplified models can direct their design, improvement and adaptation to specific purposes.

Notes

1. This quotation was repeated many times by R. Toupin during the symposium in his honour held at the 4th Canadian Conference on Nonlinear Solid Mechanics (CanCNSM2013). Toupin had clearly in mind many ‘poets’ against whom he had had many controversies.
2. The educated Italian reader will appreciate Piola’s original work of prose: ‘Dopo che Lagrange ha ridotto tutte le questioni della Meccanica Razionale al calcolo delle variazioni, volere persistere a farne senza, è un imitare coloro i quali per le ricerche di alta geometria, piuttosto che correre a volo giovandosi di formole prese dal calcolo differenziale e integrale, si ostinano ad andar pedestri col sussidio de’ metodi sintetici. Così procedendo si fa poco, e s’incontra grave pericolo di far male. Conviene persuadersi che le dimostrazioni sempre più ammettono qualche sospetto di errore, quanto maggiore è il tratto nel quale sono appoggiate al semplice ragionamento: chè la portata intuitiva della nostra ragione è assai limitata, e facilmente c’inganniamo appena gli elementi della questione crescono a notabil numero e si complicano fra di loro.’
3. We have given our translation of the following Italian text which reads: ‘[…] e consideri che per uno che voglia persuader cosa, se non falsa, almeno assai dubbiosa, di gran vantaggio è il potersi servire d’argomenti probabili, di conghietture, d’essempi, di verisimili ed anco di sofismi, fortificandosi appresso e ben trincerandosi con testi chiari, con autorità d’altri filosofi, di naturalisti, di rettorici e d’istorici: ma quel ridursi alla severità di geometriche dimostrazioni è troppo pericoloso cimento per chi non le sa ben maneggiare; imperocché, sì come ex parte rei non si dà mezo tra il vero e ’l falso, così nelle dimostrazioni necessarie o indubitabilmente si conclude o inescusabilmente si paralogiza, senza lasciarsi campo di poter con limitazioni, con distinzioni, con istorcimenti di parole o con altre girandole sostenersi più in piede, ma è forza in brevi parole ed al primo assalto restare o Cesare o niente. Questa geometrica strettezza farà ch’io con brevità e con minor tedio di V. S. Illustrissima mi potrò dalle seguenti prove distrigare; le quali io chiamerò ottiche o geometriche più per secondare il Sarsi, che perché io ci ritrovi dentro, dalle figure in poi, molta prospettiva o geometria.’
4. Abbiamo bisogno di metodi potenti i quali essendo come l’espressione simultanea e compendiosa di molti principj, operano col valore di tutti, e non con quello di uno per volta, che è quanto avviene d’ordinario nel ragionamento logico: di metodi che ridotti a processi determinati e immutabili, non ci lasciano forviare. Anche usando mezzi così fatti la nostra ragione mantiene i suoi diritti, in quanto ne riconosce veri i fondamenti, e giuste le applicazioni: sebbene non le sia il più delle volte concesso conseguire un’intrinseca evidenza relativamente alle conseguenze a cui arriva.
5. È per tal modo che nella ricerca della verità facciamo quei grandi viaggi, ai quali il ragionamento diretto è affatto insufficiente, tornandoci esso poi vantaggioso quando, giunti a certe mete, vogliamo estendere il beneficio delle ottenute cognizioni. Uno appunto fra i più poderosi degli indicati mezzi è il calcolo delle variazioni per la meccanica. Eppure io sento profondamente che anche tutto il presente lavoro è ben lungi dall’esaurire la fecondità dei metodi lagrangiani: credo poter assicurare che con questi stessi metodi si percorrono a passi di conquista le varie parti della fisica matematica. Già vedemmo nella precedente Memoria copia di risultati che se ne deducono, e toccammo di molte teoriche che potrebbero rannodarsi alle varie parti di essa.

6. Ho già in pronto altro non breve scritto in continuazione dell'attuale, e nutro desiderio di poter produrre anche ulteriori prove di fatto dell'esposta asserzione: ma qualunque sia per essere il termine a cui riusciranno le mie fatiche, tengo per fermo che il tempo farà ragione alle parole colle quali diedi cominciamento a questa Memoria.
7. The maestro of Piola was Vincenzo Brunacci, who authored the first university textbook in Italian in which the basics of calculus of variations was set in a modern way. Therefore the following rewriting in modern notation of Piola's work is appropriate.
8. The constitutive equations for such tensors must verify the condition of frame invariance. When these tensors are defined in terms of a deformation energy (that is, when the principle of virtual work is obtained as the first variation of a least action principle) the objectivity becomes a restriction on such an energy. The generalization of the results in Steigmann (2003) [98] to the Nth-gradient continua still needs to be found.
9. We are paraphrasing [36, p. 349].
10. Bold is ours: here the spirit of Sedov is very close to that expressed by Piola.
11. This was footnote 7 in the original text: *Méchanique Analitique*, Paris, Veuve Desaint, 1788. The general principle is stated in the second section of the second part; the integrals are obtained in the third section. The history to which we refer is given in the first section. In the second and later editions, as reprinted in LAGRANGE'S Œuvres 11, there are considerable changes in the first and second sections, but they do not materially alter the part of the history to which we refer here, and, while the entire development of the principle of virtual work is recast, it becomes no clearer. In his *Théorie des Fonctions Analitiques*, Paris, Imp. République, An V (1797) = Œuvres 9, LAGRANGE gives what he regards as an improved treatment. In §205 he asserts that mutual forces are central and are analogous to forces of constraint, and in §217 he derives the integral of moment of momentum for a system subject to steady holonomic constraints.
12. This idea is very ancient and some historians [51] believe that it was Archytas of Tarentum who was the true founder of mathematical mechanics by introducing the principle of virtual velocities (the ancient name of the principle of virtual work: the change of name made somebody believe that the principle was different: see the epigraph by Poincaré after the title).
13. The chosen summation bounds may seem restrictive. This is not the case, as one can easily add some extra terms with vanishing densities. The reader will find more technical details in [41].
14. These tensor fields are, by definition, orthogonal to the manifold where they are concentrated. Thus $\mathsf{F}^i_k \mid \nabla^k_\perp U = \mathsf{F}^i_k \mid \nabla^k U$ and in the sequel it will not be necessary to specify that only the orthogonal part of $\nabla^k U$ is involved.
15. This notion is precisely defined in [47, 48].
16. When considering Cauchy continua and rigid virtual velocity fields U, one reduces inequality (11) to the quasi-balance of forces put forward by [58]. As remarked in [48], quasi-balance of forces is not sufficient to obtain a description of a stress state of order two or higher.
17. In [65, p. 29] one can read: 'Indeed, as his sixth problem HILBERT set the construction of a set of axioms, on the model of the axioms of geometry, for "those branches of physics where mathematics now plays a preponderant part; first among them are probability theory and mechanics." Like all of his problems concerning physical applications of mathematics, his proposal for mechanics has received little attention. The possibility that the future may revise the physics of small corpuscles does not reduce the need for axiomatic treatment of the field theories. Physics, like mathematics, may be constructed precisely at several different levels. The interconnection of the different levels, either exactly or by approximation or by addition of new axioms, then furnishes definite mathematical problems. Having reached agreement that we should base the classical field theories on a set of axioms, we must now admit, ruefully, our inability to do so. In our opinion, none of the attempts to form such a system has been successful. Only in very recent years has an adequate set of axioms for pure mechanics, at last, been constructed; it is the work of NOLL' [57].
18. Personal communication during the 4th Canadian Conference on Nonlinear Solid Mechanics (CanCNSM2013), Montreal, Canada.

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