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# HIGHER ORDER AVERAGING THEORY FOR FINDING PERIODIC SOLUTIONS VIA BROUWER DEGREE

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Abstract. In this paper we deal with nonlinear differential systems of the form

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 0, 1, \cdots, k$, and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, $T$–periodic in the first variable, being $D$ an open subset of $\mathbb{R}^n$, and $\varepsilon$ a small parameter. For such differential systems, which do not need to be of class $C^1$, under convenient assumptions we extend the averaging theory for computing their periodic solutions to $k$–th order in $\varepsilon$. Some applications are also performed.

## 1. Introduction and statement of the main results

The method of averaging is a classical and matured tool that allows to study the dynamics of the nonlinear *differential systems* under periodic forcing. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [11]. Important practical and theoretical contributions to the averaging theory were made in the 1930's by Bogoliubov and Krylov [2], and in 1945 by Bogoliubov [1]. In 2004, Buica and Llibre [5] extended the averaging theory for studying periodic orbits to *continuous differential systems* using the Brouwer degree. Recently a version of averaging theory for studying periodic orbits of *discontinuous differential systems* has been provided by Llibre, Novaes and Teixeira in [19]. We refer to the book of Sanders, Verhulst and Murdock [21] for a general introduction to this subject.

All these previous works develop the averaging theory usually up to first order in a small parameter $\varepsilon$, and at most up to third order. In a recent work of Giné, Grau and Llibre [12] the averaging theory for computing periodic solutions was developed to an arbitrary order in $\varepsilon$ for differential equations of one variable. The goal of this paper is to extend the averaging theory for computing periodic solutions to an arbitrary order in $\varepsilon$ for continuous differential equations in $n$ variables. Thus, the main theorem stated in this paper extends the results of Buica and Llibre [5] to an arbitrary order in a small parameter $\varepsilon$.

Here we are interested in studying the existence of periodic orbits of general differential systems expressed by

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{1}$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 1, 2, \cdots, k$, and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, $T$–periodic in the first variable, being $D$ an open subset of $\mathbb{R}^n$.

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In order to state our main result we introduce some notation. Let $x = (x_1, \ldots, x_n) \in D$ and let $y_j = (y_{j1}, \ldots, y_{jn}) \in \mathbb{R}^n$ for $j = 1, \ldots, l$. We denote by $\dfrac{\partial^L}{\partial x^L} F_m(s, x)$ the symmetric $L$–multilinear map which is applied to a "product" of $L$ vectors of $\mathbb{R}^n$, which we denote as $\bigodot_{j=1}^{l} y_j^{b_j} \in \mathbb{R}^{nL}$ where $L = b_1 + b_2 + \cdots + b_l$ and $y_j^{b_j} = (y_j, \ldots y_j) \in \mathbb{R}^{nb_j}$. The definition of the $L$–multilinear map is

$$
\begin{aligned}
&\frac{\partial^L}{\partial x^L} F_m(s,x) \bigodot_{j=1}^{l} y_j^{b_j} = \\
&\qquad \sum_{i_1,\ldots,i_L=1}^{n} \frac{\partial^L F_j(s,x)}{\partial x_{i_1} \cdots \partial x_{i_L}} y_{1i_1} \cdots y_{1i_{b_1}} y_{2i_{b_1+1}} \cdots y_{2i_{b_1+b_2}} y_{li_{b_1+\cdots+b_{l-1}+1}} \cdots y_{li_{b_1+\cdots+b_l}}.
\end{aligned}
$$

We define $f_0, f_i : D \to \mathbb{R}^n$ for $i = 1, 2, \ldots, k$ as

$$
f_0(z) = \int_0^T F_0(t,z)dt, \quad f_i(z) = \frac{y_i(T,z)}{i!}, \tag{2}
$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k-1$, are defined recurrently by the following integral equations:

$$
y_i(t,z) = i! \int_0^t \left( F_i(s,z) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2!2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L}{\partial z^L} F_{i-l}(s,z) \bigodot_{j=1}^{l} y_j(s,z)^{b_j} \right) ds, \tag{3}
$$

where, $S_l$ is the set of all $l$-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

We observe that for $F_0 \equiv 0$,

$$
y_1(t,z) = \int_0^t F_1(s,z)ds
$$

as usual (see for instance [5]).

Our main results are the following.

**Theorem A** ($k$–th order averaging theorem for computing periodic solutions)**.** *Suppose that $F_0 \equiv 0$. In addition, for the functions of the differential system* (1) *we assume also the following conditions.*

(i) *$F_i(t, \cdot) \in C^{k-i}$ for all $t \in \mathbb{R}$, for $i = 1, 2, \cdots, k$, and $R$ and $F_k$ are locally Lipschitz with respect to $x$.*
(ii) *Assume that $f_i \equiv 0$ for $i = 1, 2, \ldots, r-1$ and $f_r \not\equiv 0$ with $r \in \{1, 2, \ldots, k\}$ (here by definition $f_0(z)$ as the zero constant function). Moreover, suppose that for $a \in D$ with $f_r(a) = 0$, there exists a neighborhood $V \subset D$ of $a$ such that $f_r(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$, and that the Brouwer degree $d_B(f_r(z), V, a) \neq 0$.*

*Then, for $|\varepsilon| > 0$ sufficiently small, there exists a $T$–periodic solution $x(\cdot, \varepsilon)$ of system* (1) *such that $x(0, \varepsilon) \to a$ when $\varepsilon \to 0$.*

**Remark 1.** *If $F_0 \not\equiv 0$ we observe that*

$$
y_1(t,z) = \int_0^t F_1(s,z) + \frac{\partial}{\partial x} F_0(s,z) y_1(s,z) ds. \tag{4}
$$

*So the integral equation* (4) *is equivalent to the following Initial Value Problem:*

$$\dot{u}(t) = F_1(t,u) + \frac{\partial}{\partial x}F_0(t,z)u \quad and \quad u(0) = 0, \tag{5}$$

*i.e,* $y_1(t,z) = u(t)$*. Moreover, each* $y_i(t,z)$ *is obtained similarly from a differential initial value problem.*

**Theorem B** ($k$–th order averaging theorem for computing periodic solutions)**.** *Suppose that* $F_0 \equiv 0$*. In addition, for the functions of the differential system* (1) *we assume the following conditions.*

(j) *There exists an open subset* $W$ *of* $D$ *such that for any* $z \in \overline{W}$*, there exists* $d_z > 0$ *such that,* $B(z,d_z) \subset D$*, and*

$$||F_0||_{G_z} < \frac{d_z}{T},$$

*where* $||F_0||_{G_z} = \sup\{||F(t,x)|| : (t,x) \in G_z\}$ *and* $G_z = [0,T] \times B(z,d_z)$*.*

(jj) $F_i(t,\cdot) \in C^{k-i}$ *for all* $t \in \mathbb{R}$*, for* $i = 0,1,\cdots,k$*, and* $R$ *and* $F_k$ *are locally Lipschitz with respect to* $x$*.*

(jjj) *Assume that* $f_i \equiv 0$ *for* $i = 0,1,\ldots,r-1$ *and* $f_r \not\equiv 0$ *with* $r \in \{0,1,\ldots,k\}$*. Moreover, suppose that for* $a \in W$ *with* $f_r(a) = 0$*, there exists a neighborhood* $V \subset W$ *of* $a$ *such that* $f_r(z) \neq 0$ *for all* $z \in \overline{V} \setminus \{a\}$*, and that the Brouwer degree* $d_B(f_r(z),V,a) \neq 0$*.*

*Then, for* $|\varepsilon| > 0$ *sufficiently small, there exists a* $T$*–periodic solution* $x(\cdot,\varepsilon)$ *of system* (1) *such that* $x(0,\varepsilon) \to a$ *when* $\varepsilon \to 0$*.*

**Remark 2.** *Instead hypothesis (j), we could assume a more general hypothesis:*

(j') *Let* $x(\cdot,z,\varepsilon)$ *be a solution of system* (1) *such that* $x(0,z,\varepsilon) = z$*. Assume that for each* $z \in \overline{V}$*, there exists* $\varepsilon_1 > 0$ *such that if* $\varepsilon \in [-\varepsilon_1,\varepsilon_1]$ *then* $x(\cdot,z,\varepsilon)$ *is defined in* $[0,T]$*.*

*For instance, if there exists an open neighborhood* $U$ *of* $\overline{V}$ *such that the solutions* $x_0(\cdot,z)$ *of the unperturbed system* $\dot{x}(t) = F_0(t,x)$ *such that* $x_0(0,z) = z \in U$ *are* $T$*–periodic. Then hypothesis (j') holds.*

Theorems A and B are proved in section 2.

See the appendix for additional information on the Brouwer degree $d_B$.

The functions $f_k(z)$ defined in (2) are given explicitly in section 3 for $k = 1,2,3,4,5$.

## 2. Proofs of Theorems A and B

For proving Theorem A we need the following lemma.

**Lemma 1** (Fundamental Lemma)**.** *Let* $x(\cdot,z,\varepsilon) : [0,t_z) \to \mathbb{R}^n$ *be a solution of* (1) *with* $x(0,z,\varepsilon) = z$*, then*

$$x(t,z,\varepsilon) = z + \int_0^t F_0(t,z)ds + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t,z)}{i!} + \varepsilon^{k+1}\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds + \varepsilon^{k+1}\mathcal{O}(1),$$

*where* $y_i(t,z)$ *for* $i = 1,2,\ldots,k$ *are defined in* (3)*.*

In the proof of Lemma 1 we use the *Faá di Bruno's Formula* (see [14]), about the $l^{th}$ derivative of a composite function.

**Faá di Bruno's Formula** *If* $g$ *and* $f$ *are functions with a sufficient number of derivatives, then*

$$\frac{d^l}{dt^l}g(f(t)) = \sum_{S_l} \frac{l!}{b_1!\, b_2!2!^{b_2} \cdots b_l! l!^{b_l}} g^{(L)}(f(t)) \bigodot_{j=1}^{l} f^{(j)}(t)^{b_j},$$

*where $S_l$ is the set of all $l$–tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ which are solutions of the equation $b_1 + 2b_2 + \cdots + lb_l = l$ and $L = b_1 + b_2 + \cdots + b_l$.*

*Proof of Lemma 1.* Clearly,

$$x(t,z,\varepsilon) = z + \sum_{i=0}^{k} \varepsilon^i \int_0^t F_i(s,x(s,z,\varepsilon))ds + \varepsilon^{k+1}\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds. \tag{6}$$

The Taylor expansion of $F_i(t,x(t,z,\varepsilon))$ around $\varepsilon = 0$, for $i = 0,1,\ldots,k-1$, is given by

$$F_i(t,x(t,z,\varepsilon)) = F_i(t,x(t,z,0)) + \sum_{l=1}^{k-i} \frac{\varepsilon^l}{l!}\left(\frac{\partial^l}{\partial\varepsilon^l}F_i(t,x(t,z,\varepsilon))\right)\Bigg|_{\varepsilon=0} + \varepsilon^{k-i+1}\mathcal{O}(1). \tag{7}$$

The Faá di Bruno's formula allows to compute the $l$–derivatives of $F_i(t,x(t,z,\varepsilon))$ in $\varepsilon$, for $i = 0,1,\ldots,k-1$:

$$\frac{\partial^l}{\partial\varepsilon^l}F_i(t,x(t,z,\varepsilon))\Bigg|_{\varepsilon=0} = \sum_{S_l} \frac{l!}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\left(\frac{\partial^L F_i}{\partial x^L}(t,x(t,z,\varepsilon))\right)\Bigg|_{\varepsilon=0} \bigodot_{j=1}^{l} y_j(t,z)^{b_j}. \tag{8}$$

Here $S_l$ is the set of all $l$-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ which are solutions of the equation $b_1 + 2b_2 + \cdots + lb_l = l$, $L = b_1 + b_2 + \cdots + b_l$, and

$$y_j(t,z) = \left(\frac{\partial^j}{\partial\varepsilon^j}x(t,z,\varepsilon)\right)\Bigg|_{\varepsilon=0}. \tag{9}$$

Substituting (8) in (7) the Taylor expansion at $\varepsilon = 0$ of $F_i(s,x(t,z,\varepsilon))$ becomes

$$\begin{aligned} F_i(s,x(s,z,\varepsilon)) = \;& F_i(s,z) \\ & + \sum_{l=1}^{k-i}\sum_{S_l} \frac{\varepsilon^l}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial x^L}F_i(s,z)\bigodot_{j=1}^{l} y_j(s,z)^{b_j} \\ & + \varepsilon^{k-i+1}\mathcal{O}(1), \end{aligned} \tag{10}$$

for $i = 0,1,\ldots,k-1$. Moreover, for $i = k$ we have that

$$F_k(s,x(s,z,\varepsilon)) = F_k(s,z) + \varepsilon\mathcal{O}(1). \tag{11}$$

Indeed, by compactness of the set $[0,T]\times\overline{V}\times[-\varepsilon_0,\varepsilon_0]$ it follows

$$|F_k(s,x(s,z,\varepsilon)) - F_k(s,z)| < L|x(s,z,\varepsilon) - z| = \varepsilon\mathcal{O}(1),$$

because $F_k(s,x)$ is locally Lipschitz in the second variable.

Now, by expressions (6), (10) and (11), we have that

$$\begin{aligned} x(t,z,\varepsilon) = \;& z + \int_0^t Q(s,z,\varepsilon)ds + \sum_{i=0}^{k}\varepsilon^i\int_0^t F_i(s,z)ds \\ & + \varepsilon^{k+1}\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds + \varepsilon^{k+1}\mathcal{O}(1), \end{aligned} \tag{12}$$

where

$$Q(s,z,\varepsilon) = \sum_{i=0}^{k-1}\sum_{l=1}^{k-i}\varepsilon^{l+i}\sum_{S_l}\int_0^t \frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial x^L}F_i(s,z)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}ds.$$

We may write

$$Q(s,z,\varepsilon) = \sum_{l=1}^{k}\sum_{i=l}^{k}\varepsilon^i\sum_{S_l}\int_0^t \frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial x^L}F_{i-l}(s,z)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}\,ds$$
$$= \sum_{i=1}^{k}\varepsilon^i\sum_{l=1}^{i}\sum_{S_l}\int_0^t \frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial x^L}F_{i-l}(s,z)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}\,ds. \tag{13}$$

Finally, from (12) and (13), we get

$$\begin{aligned} x(t,z,\varepsilon) = {} & z + \int_0^t F_0(t,z)ds \\ & + \sum_{i=1}^{k}\varepsilon^i\left(\int_0^t F_i(s,z) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial x^L}F_{i-l}(s,z)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}\,ds\right) \\ & + \varepsilon^{k+1}\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds + \varepsilon^{k+1}\mathcal{O}(1). \end{aligned}$$

Now, using this last expression of $x(t,z,\varepsilon)$ we conclude that functions $y_i(t,z)$ defined in (9), for $i = 1,2,\ldots,k-1$, can be computed recurrently from the following integral equation:

$$\begin{aligned} y_i(t,z) & = \left(\frac{\partial^i x}{\partial\varepsilon^i}(t,z,\varepsilon)\right)\Bigg|_{\varepsilon=0} \\ & = i!\int_0^t\left(F_i(s,z) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial z^L}F_{i-l}(s,z)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}\right)ds. \end{aligned}$$

This completes the proof of lemma. □

*Proof of Theorem A.* Let $x(\cdot,z,\varepsilon)$ be a solution of system (1) such that $x(0,z,\varepsilon)=z$. For each $z\in\overline{V}$, there exists $\varepsilon_1>0$ such that if $\varepsilon\in[-\varepsilon_1,\varepsilon_1]$ then $x(\cdot,z,\varepsilon)$ is defined in $[0,T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [21]), $x(\cdot,z,\varepsilon)$ is defined for all $0\le t\le \inf(T,d/M(\varepsilon))$, where

$$M(\varepsilon)\ge\left|\sum_{i=1}^{k}\varepsilon^i F_i(t,x)+\varepsilon^{k+1}R(t,x,\varepsilon)\right|$$

for all $t\in[0,T]$, for each $x$ with $|x-z|\le b$ and for every $z\in\overline{V}$. When $\varepsilon$ is sufficiently small we can take $d/M(\varepsilon)$ sufficiently large in order that $\inf(T,d/M(\varepsilon))=T$ for all $z\in\overline{V}$.

By continuity of the solution $x(t,z,\varepsilon)$ and by compactness of the set $[0,T]\times\overline{V}\times[-\varepsilon_1,\varepsilon_1]$, there exits $K$ a compact subset of $D$ such that $x(t,z,\varepsilon)\in K$ for all $t\in[0,T]$, $z\in\overline{V}$ and $\varepsilon\in[-\varepsilon_1,\varepsilon_1]$. Now, by the continuity of the function $R$, $|R(s,x(s,z,\varepsilon))|\le\max\{|R(t,x,\varepsilon)|,(t,x,\varepsilon)\in[0,T]\times K\times[-\varepsilon_1,\varepsilon_1]\}=N$. Then

$$\left|\int_0^T R(s,x(s,z,\varepsilon),\varepsilon)ds\right|\le\int_0^T|R(s,x(s,z,\varepsilon),\varepsilon)|\,ds=TN,$$

which implies that

$$\int_0^T R(s,x(s,z,\varepsilon),\varepsilon)ds=\mathcal{O}(1). \tag{14}$$

We denote

$$\varepsilon f(z,\varepsilon)=x(T,z,\varepsilon)-z.$$

From Lemma 1 and equation (14), we have that

$$f(z,\varepsilon)=f_1(z)+\varepsilon f_2(z)+\varepsilon^2 f_3(z)+\cdots+\varepsilon^{k-1}f_k(z)+\varepsilon^k\mathcal{O}(1),$$

where the function $f_i$ is the one defined in (2) for $i=1,2,\cdots,k$. From the assumption (ii) of the theorem we have that

$$f(z,\varepsilon)=\varepsilon^{r-1}f_r(z)+\cdots+\varepsilon^{k-1}f_k(z)+\varepsilon^k\mathcal{O}(1),$$

Clearly $x(\cdot,z,\varepsilon)$ is a $T$–periodic solution if and only if $f(z,\varepsilon)=0$, because $x(t,z,\varepsilon)$ is defined for all $t\in[0,T]$.

From Lemma 6 of the appendix and hypothesis (ii) we have for $|\varepsilon|>0$ sufficiently small that

$$d_B\left(f_r(z),V,a\right)=d_B\left(f(z,\varepsilon),V,a\right)\neq 0.$$

Hence, by item (i) of Theorem 4 (see Appendix), $0\in f(V,\varepsilon)$ for $|\varepsilon|>0$ sufficiently small, i.e, there exists $a_\varepsilon\in V$ such that $f(a_\varepsilon,\varepsilon)=0$.

Therefore, for $|\varepsilon|>0$ sufficiently small, $x(t,a_\varepsilon,\varepsilon)$ is a periodic solution of system (1). Clearly we can choose $a_\varepsilon$ such that $a_\varepsilon\to a$ when $\varepsilon\to 0$, because $f(z,\varepsilon)\neq 0$ in $V\setminus\{a\}$. This completes the proof of the theorem. □

*Proof of Theorem B.* Let $x(\cdot,z,\varepsilon)$ be a solution of system (1) such that $x(0,z,\varepsilon)=z$. For each $z\in\overline{V}$, there exists $\varepsilon_1>0$ such that if $\varepsilon\in[-\varepsilon_1,\varepsilon_1]$ then $x(\cdot,z,\varepsilon)$ is defined in $[0,T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [21]), $x(\cdot,z,\varepsilon)$ is defined for all $0\leq t\leq \inf\left(T,d_z/M(\varepsilon)\right)$, where

$$M(\varepsilon)=\sup\left|\sum_{i=0}^{k}\varepsilon^iF_i(t,x)+\varepsilon^{k+1}R(t,x,\varepsilon)\right|$$

for all $t\in[0,T]$, for each $x$ with $|x-z|\leq d_z$ and for every $z\in\overline{V}$.

Denote

$$E(\varepsilon)=\sup\left|\sum_{i=1}^{k}\varepsilon^iF_i(t,x)+\varepsilon^{k+1}R(t,x,\varepsilon)\right|$$

for all $t\in[0,T]$, for each $x$ with $|x-z|\leq d_z$ and for every $z\in\overline{V}\subset\overline{W}$. Observe that, for $|\varepsilon|>0$ sufficiently small, $E(\varepsilon)$ can be taken arbitrarily small. Moreover, $M(\varepsilon)\leq||F_0||_{G_z}+E(\varepsilon)$, which implies that

$$\frac{d_z}{M(\varepsilon)}\geq\frac{d_z}{||F_0||_{G_z}+E(\varepsilon)}.$$

By the other hand, hypothesis (i) says that $||F_0||_{G_z}<d_z/T$. So, for $|\varepsilon|>0$ sufficiently small

$$||F_0||_{G_z}+E(\varepsilon)\leq\frac{d_z}{T},$$

thus

$$T\leq\frac{d}{||F_0||_{G_z}+E(\varepsilon)}\leq\frac{d_z}{M(\varepsilon)}.$$

Hence, when $\varepsilon$ is sufficiently small we can take $d_z/M(\varepsilon)\geq T$ in order that $\inf\left(T,d/M(\varepsilon)\right)=T$ for all $z\in\overline{V}$.

Observe that we have proved that the hypothesis (j) implies the assumption of Remark 2.

Now, denoting

$$f(z,\varepsilon)=x(T,z,\varepsilon)-z,$$

the proof follows similarly of Theorem A. □

## 3. Computing formulae

In this section we illustrate how to compute the formulae of Theorems A and B for some $k \in \mathbb{N}$.

### 3.1. Fifth order averaging theorem, assuming $F_0 \equiv 0$.

For instance consider $k = 1, 2, 3, 4, 5$.

First we should determine the sets $S_l$ for $l = 1, 2, 3, 4$.

$$
\begin{aligned}
S_1 &= \{1\},\\
S_2 &= \{(0,1),(2,0)\},\\
S_3 &= \{(0,0,1),(1,1,0),(3,0,0)\},\\
S_4 &= \{(0,0,0,1),(1,0,1,0),(2,1,0,0),(0,2,0,0),(4,0,0,0)\}.
\end{aligned}
$$

To compute $S_l$ is conveniently to exhibit a table of possibilities with the value $b_i$ in the column $i$. We starts it from the last column.

### 3.2. Construction of $S_5$.

Clearly the last column can be only filled by 0 and 1, because $5b_5 > 5$ for $b_5 > 1$. The same happens with the fourth and the third column, because $3b_3$, $4b_4 > 5$, for $b_3$, $b_4 > 1$. Taking $b_5 = 1$, the unique possibility is $b_1 = b_2 = b_3 = b_4 = 0$, thus any other solution satisfies $b_5 = 0$. Taking $b_5 = 0$ and $b_4 = 1$, the unique possibility is $b_1 = 1$ and $b_2 = b_3 = 0$, thus any other solution must have $b_4 = b_5 = 0$. Finally, taking $b_5 = b_4 = 0$ and $b_3 = 1$, we have two possibilities either $b_1 = 2$ and $b_2 = 0$, or $b_1 = 0$ and $b_2 = 1$. Thus any other solution satisfies $b_3 = b_4 = b_5 = 0$.

Now we observe that the second column can be only filled by 0, 1 or 2, since $2b_2 > 5$ for $b_2 > 2$; and taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 1$ the unique possibility is $b_1 = 3$. Taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 2$ the unique possibility is $b_1 = 1$, thus any other solution satisfies $b_2 = b_3 = b_4 = b_5 = 0$. Finally, taking $b_2 = b_3 = b_4 = b_5 = 0$ the unique possibility is $b_1 = 5$. Therefore the complete table of solutions is

$$
S_5 = \begin{array}{|c|c|c|c|c|}
b_1 & b_2 & b_3 & b_4 & b_5 \\
\hline
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0
\end{array}
$$

Hence, from (3) we obtain that

$$
\begin{aligned}
y_1(t,z) = {} & \int_0^s F_1(s,z)ds,\\
y_2(t,z) = {} & 2\int_0^s F_2(s,z) + \frac{\partial F_1}{\partial x}(s,z)y_1(s,z)ds,\\
y_3(t,z) = {} & \int_0^s \left(6F_3(s,z) + 6\frac{\partial F_2}{\partial x}(s,z)y_1(t,z) + 3\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z)^2 + 3\frac{\partial F_1}{\partial x}(s,z)\, y_2(s,z)\right) ds,\\
y_4(t,z) = {} & 24\int_0^s \left(F_4(s,z) + \frac{\partial F_3}{\partial x}(s,z)y_1(s,z)\right) ds\\
& +12\int_0^s \left(\frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z)^2 + \frac{\partial F_2}{\partial x}(s,z)y_2(s,z)\right) ds\\
& +12\int_0^s \frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z)\odot y_2(s,z)ds\\
& +4\int_0^s \left(\frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^3 + \frac{\partial F_1}{\partial x}(s,z)y_3(s,z)\right) ds,\\
y_5(t,z) = {} & 120\int_0^t \left(F_5(s,z) + \frac{\partial F_4}{\partial x}(s,z)y_1(s,z)\right) ds\\
& +60\int_0^t \left(\frac{\partial^2 F_3}{\partial x^2}(s,z)y_1(s,z)^2 + \frac{\partial F_3}{\partial x}(s,z)y_2(s,z) + \frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z)\odot y_2(s,z)\right) ds\\
& +20\int_0^t \left(\frac{\partial^3 F_2}{\partial x^3}(s,z)y_1(s,z)^3 + \frac{\partial F_2}{\partial x}(s,z)y_3(s,z) + \frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z)\odot y_3(s,z)\right) ds\\
& +15\int_0^t \frac{\partial^2 F_1}{\partial x^2}(s,z)y_2(s,z)^2 ds + 30\int_0^t \frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^2\odot y_2(s,z)ds\\
& +5\int_0^t \left(\frac{\partial^4 F_1}{\partial x^4}(s,z)y_1(s,z)^4 + \frac{\partial F_1}{\partial x}(s,z)y_4(s,z)\right) ds.
\end{aligned}
$$

So from (2) we have that

$$
\begin{aligned}
f_0(z) = {} & 0,\\
f_1(z) = {} & \int_0^T F_1(t,z)dt,\\
f_2(z) = {} & \int_0^T F_2(t,z)ds + \frac{\partial F_1}{\partial x}(t,z)y_1(t,z)dt,\\
f_3(z) = {} & \int_0^T \left(F_3(t,z) + \frac{\partial F_2}{\partial x}(t,z)y_1(t,z)\right) dt\\
& +\frac{1}{2}\int_0^T \left(\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)^2 + \frac{\partial F_1}{\partial x}(t,z)y_2(t,z)\right) dt,
\end{aligned}
$$

$$
\begin{aligned}
f_4(z) = & \int_0^T \left(F_4(t,z) + \frac{\partial F_3}{\partial x}(t,z)y_1(t,z)\right)dt \\
& + \frac{1}{2}\int_0^T \left(\frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)^2 + \frac{\partial F_2}{\partial x}(t,z)y_2(t,z)\right)dt \\
& + \frac{1}{2}\int_0^T \frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)dt \\
& + \frac{1}{6}\int_0^T \left(\frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^3 + \frac{\partial F_1}{\partial x}(t,z)y_3(t,z)\right)dt, \\
f_5(z) = & \int_0^T \left(F_5(t,z) + \frac{\partial F_4}{\partial x}(t,z)y_1(t,z)\right)dt \\
& + \frac{1}{2}\int_0^T \left(\frac{\partial^2 F_3}{\partial x^2}(t,z)y_1(t,z)^2 + \frac{\partial F_3}{\partial x}(t,z)y_2(t,z) + \frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)\right)dt \\
& + \frac{1}{6}\int_0^T \left(\frac{\partial^3 F_2}{\partial x^3}(t,z)y_1(t,z)^3 + \frac{\partial F_2}{\partial x}(t,z)y_3(t,z) + \frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y_3(t,z)\right)dt \\
& + \frac{1}{8}\int_0^T \frac{\partial^2 F_1}{\partial x^2}(t,z)y_2(t,z)^2 dt + \frac{1}{4}\int_0^T \frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^2\odot y_2(t,z)dt \\
& + \frac{1}{24}\int_0^T \left(\frac{\partial^4 F_1}{\partial x^4}(t,z)y_1(t,z)^4 + \frac{\partial F_1}{\partial x}(t,z)y_4(t,z)\right)dt.
\end{aligned}
$$

3.3. **Fifth order averaging theorem, assuming $F_0 \not\equiv 0$.** First of all, initial value problem, or equivalently an integral equation (see Remark ), must be solved to compute the expressions $y_i(t,z)$ for $i = 1, 2, \ldots, k$. We give the required equations for $k = 1, 2, 3, 4$.

Hence, from (3) we obtain that

$$
\begin{aligned}
y_1(t,z) = & \int_0^s F_1(s,z) + \frac{\partial F_0}{\partial x}(s,z)y_1(s,z)dt, \\
y_2(t,z) = & \int_0^s \left(2F_2(s,z) + 2\frac{\partial F_1}{\partial x}(s,z)y_1(s,z) + \frac{\partial^2 F_0}{\partial x^2}(s,z)y_1(s,z)^2 + \frac{\partial F_0}{\partial x}(s,z)\,y_2(s,z)\right)dt, \\
y_3(t,z) = & \int_0^s \left(6F_3(s,z) + 6\frac{\partial F_2}{\partial x}(s,z)y_1(s,z) + 3\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z)^2 + 3\frac{\partial F_1}{\partial x}(s,z)\,y_2(s,z)\right)dt \\
& + \int_0^s \left(3\frac{\partial^2 F_0}{\partial x^2}(s,z)y_1(s,z)\odot y_2(s,z) + \frac{\partial^3 F_0}{\partial x^3}(s,z)y_1(s,z)^3 + \frac{\partial F_0}{\partial x}(s,z)y_3(s,z)\right)dt,
\end{aligned}
$$

$$
\begin{aligned}
y_4(t,z)= \quad & 24\int_0^s \left(F_4(s,z)+\frac{\partial F_3}{\partial x}(s,z)y_1(s,z)\right)dt\\
&+12\int_0^s \left(\frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z)^2+\frac{\partial F_2}{\partial x}(s,z)y_2(s,z)\right)dt\\
&+12\int_0^s \frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z)\odot y_2(s,z)dt\\
&+4\int_0^s \left(\frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^3+\frac{\partial F_1}{\partial x}(s,z)y_3(s,z)+\frac{\partial^2 F_0}{\partial x^2}(s,z)y_1(s,z)\odot y_3(s,z)\right)dt\\
&+3\int_0^t \frac{\partial^2 F_0}{\partial x^2}(s,z)y_2(s,z)^2dt+6\int_0^t \frac{\partial^3 F_0}{\partial x^3}(s,z)y_1(s,z)^2\odot y_2(s,z)dt\\
&+\int_0^t \left(\frac{\partial^4 F_0}{\partial x^4}(s,z)y_1(s,z)^4+\frac{\partial F_0}{\partial x}(s,z)y_4(s,z)\right)dt.
\end{aligned}
$$

So from (2) we have that

$$
\begin{aligned}
f_0(z)= \quad & \int_0^T F_0(t,z)dt,\\
f_1(z)= \quad & \int_0^T F_1(t,z)+\frac{\partial F_0}{\partial x}(t,z)y_1(t,z)dt,\\
f_2(z)= \quad & \int_0^T \left(F_2(t,z)+\frac{\partial F_1}{\partial x}(t,z)y_1(t,z)+\frac{1}{2}\frac{\partial^2 F_0}{\partial x^2}(t,z)y_1(t,z)^2+\frac{1}{2}\frac{\partial F_0}{\partial x}(t,z)\,y_2(t,z)\right)dt,\\
f_3(z)= \quad & \int_0^T \left(F_3(t,z)+\frac{\partial F_2}{\partial x}(t,z)y_1(t,z)+\frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)^2+\frac{1}{2}\frac{\partial F_1}{\partial x}(t,z)\,y_2(t,z)\right)dt\\
&+\int_0^T \left(\frac{1}{2}\frac{\partial^2 F_0}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)+\frac{1}{6}\frac{\partial^3 F_0}{\partial x^3}(t,z)y_1(t,z)^3+\frac{1}{6}\frac{\partial F_0}{\partial x}(t,z)y_3(t,z)\right)dt,\\
f_4(z)= \quad & \int_0^T \left(F_4(t,z)+\frac{\partial F_3}{\partial x}(t,z)y_1(t,z)\right)dt\\
&+\frac{1}{2}\int_0^T \left(\frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)^2+\frac{\partial F_2}{\partial x}(t,z)y_2(t,z)\right)dt\\
&+\frac{1}{2}\int_0^T \frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)dt\\
&+\frac{1}{6}\int_0^T \left(\frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^3+\frac{\partial F_1}{\partial x}(t,z)y_3(t,z)+\frac{\partial^2 F_0}{\partial x^2}(t,z)y_1(t,z)\odot y_3(t,z)\right)dt\\
&+\frac{1}{8}\int_0^T \frac{\partial^2 F_0}{\partial x^2}(t,z)y_2(t,z)^2dt+\frac{1}{4}\int_0^T \frac{\partial^3 F_0}{\partial x^3}(t,z)y_1(t,z)^2\odot y_2(t,z)dt\\
&+\frac{1}{24}\int_0^T \left(\frac{\partial^4 F_0}{\partial x^4}(t,z)y_1(t,z)^4+\frac{\partial F_0}{\partial x}(t,z)y_4(t,z)\right)dt.
\end{aligned}
$$

## 4. Application 1: Generalized Liénard polynomial equation

In 1900 Hilbert [13] in the second part of his 16–th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree, and also to study their distribution or configuration in the plane. It has been one of the main problems in the qualitative theory of planar differential equations in the XX century. This problem remains open even for the quadratic polynomial differential systems. In Llibre and Rodriguez [20] it is proved that any finite configuration of limit cycles is realizable for some polynomial differential system.

Following Liénard [16] we consider a special class of polynomial differential equation, called the *generalized Liénard polynomial differential equation*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{15}$$

where $f(x)$ is a polynomial of degree $n$, and $g(x)$ is a polynomial of degree $m$. For this subclass of polynomial vector field we have a simplified version of Hilbert's problem, see [15] and [22].

We call the lower upper bound for the maximum number of limit cycles of the equation (15) by *Hilbert's Number*, which is denoted by $H(m,n)$. As far as we know the Hilbert's numbers $H(m,n)$ are determined only for five cases: $H(1,1)=0$ and $H(1,2)=1$ proved in 1977 by Lins, de Melo and Pugh [15]; $H(2,1)=1$ proved in 1998 by Copell [7]; $H(3,1)=1$ proved in 1990–1996 by Dumortier, Li and Rousseau in [10] and [8]; $H(2,2)=1$ proved in 1997 by Dumortier and Li [9]; and $H(1,3)=1$ proved by Li and Llibre [17] in 2012.

In [18] the number $\tilde{H}_k(m,n)$ is defined as the maximum number of limit cycles of the Liénard differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \sum_{i>0} \varepsilon^i \left( f_n^k(x) y + g_m^k(x) \right), \end{aligned} \tag{16}$$

found using the averaging method of order $k$. For $i = 1, 2, \ldots$, $f_n^i(x)$ is a polynomial of degree $n$, and $g_m^i(x)$ is a polynomial of degree $m$. Of course, from the definitions we have that $H(m,n) \geq \tilde{H}_k(m,n)$ for every $k \in \mathbb{N}$. In [18] it was obtained that

$$\begin{aligned} \tilde{H}_1(m,n) &= \left[\frac{n}{2}\right], \\ \tilde{H}_2(m,n) &= \max\left\{ \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right], \left[\frac{n}{2}\right] \right\}, \\ \tilde{H}_3(m,n) &= \left[\frac{n+m-1}{2}\right]. \end{aligned}$$

Now, using Theorem A, we have the suitable formulae for computing $\tilde{H}_k(m,n)$ for $k > 3$, but these computations are not easy.

## 5. Application 2: Perturbation of a non linear center

We consider the following system

$$\begin{aligned} \dot{x}(t) &= y(1 - \lambda\, x) + \varepsilon F(x,y), \\ \dot{y}(t) &= -x - \lambda\, y^2 + \varepsilon G(x,y), \end{aligned} \tag{17}$$

where $\lambda > 0$, $F, G : D \to \mathbb{R}$ are continuous functions and $D$ an open subset of $\mathbb{R}^2$. The unperturbed system, i.e. $\varepsilon = 0$, has a center at the origin for every $\lambda \in \mathbb{R}$ (see for instance Theorem 8.1 of [6]).

We define the differential equation

$$u'(\theta) = -2\lambda r \sin(\theta) u + \left(\lambda r \sin^2(\theta) - \cos(\theta)\right) \tilde{F}(\theta, r) - \sin(\theta)\left(1 + \lambda r \cos(\theta)\right) \tilde{G}(\theta, r). \tag{18}$$

Let $\theta \mapsto \mathcal{U}_\theta(r)$ be the solution of (18) such that $\mathcal{U}_0(r) = 0$.

A zero $r_0^* > 0$ of the equation

$$\mathcal{U}_{2\pi}(r_0) = 0, \tag{19}$$

such that $\mathcal{U}'_{2\pi}(r_0^*) \neq 0$ is called a *simple zero* of (19).

**Proposition 2.** *For $\varepsilon \neq 0$ sufficiently small and for every simple zero $r_0^* \in (0, 1/(8\pi|\lambda|))$ of the equation (19), the system (17) has a periodic solution $(x(\cdot,\varepsilon), y(\cdot,\varepsilon))$ such that $y(0,\varepsilon) = 0$ and $x(0,\varepsilon) \to r_0^*$ when $\varepsilon \to 0$.*

As an application of Proposition 2 we have the following corollary.

**Corollary 3.** *Consider $F(x,y) = ay^2 + by^4$ and $G(x,y) = 0$. Then for $\lambda = 1/200$, $a = -4$, $b = 1/10$ and for $\varepsilon \neq 0$ sufficiently small, the system (17) has a periodic solution $(x(\cdot,\varepsilon), y(\cdot,\varepsilon))$ such that $y(0,\varepsilon) =$ and $x(0,\varepsilon) \to r_0^* \approx 7.56$ when $\varepsilon \to 0$.*

*Proof of Proposition 2.* By applying the polar change of variables, system (17) becomes

$$\begin{aligned}
\dot{\theta}(t) &= -1 - \varepsilon \frac{\sin(\theta)\tilde{F}(\theta,r) - \cos(\theta)\tilde{G}(\theta,r)}{r}, \\
\dot{r}(t) &= \lambda \sin(\theta) r^2 + \varepsilon \left(\cos(\theta)\tilde{F}(\theta,r) + \sin(\theta)\tilde{G}(\theta,r)\right),
\end{aligned} \tag{20}$$

where $\tilde{F}(\theta,r) = F(r\cos(\theta), r\sin(\theta))$ and $\tilde{G}(\theta,r) = G(r\cos(\theta), r\sin(\theta))$.

Now taking $\theta$ as the new independent variable we have

$$\begin{aligned}
r'(\theta) = \quad & -\lambda \sin(\theta) r^2 \\
& +\varepsilon \left( \left(\lambda r \sin^2(\theta) - \cos(\theta)\right) \tilde{F}(\theta,r) - \sin(\theta)\left(1 + \lambda r \cos(\theta)\right) \tilde{G}(\theta,r) \right) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{21}$$

The system (21) is $2\pi$–periodic. Now, we consider

$$F_0(\theta,r) = -\lambda \sin(\theta) r^2, \quad W^\kappa = \left(\kappa, \frac{1}{8\pi|\lambda|} - \kappa\right), \text{and} \quad d_{r_0} = \frac{1 - 4\pi|\lambda|r_0 + \sqrt{1 - 8\pi|\lambda|r_0}}{4\pi|\lambda|} > 0,$$

where $\kappa$ is a small positive parameter. Thus, for every $r_0 \in \overline{W^\kappa}$ we have that

$$||F_0||_{G_{r_0}} < \frac{d_{r_0}}{2\pi}.$$

Moreover

$$f_0(r) = \int_0^{2\pi} F_0(t,r)dt = 0.$$

and

$$y_1(\theta,r) = \int_0^\theta F_1(s,r) + \frac{\partial}{\partial r}F_0(s,r)y_1(s,r)ds. \tag{22}$$

From (22) we conclude that $\theta \mapsto y_1(\theta,r)$ is the solution of system (18) such that $y_1(0,r) = 0$. Therefore $y_1(\theta,r) = \mathcal{U}_\theta(r)$. Which implies that $f_1(r) = \mathcal{U}_{2\pi}(r)$.

Hence, applying Theorem B, we have assured the existence of a periodic solution $r(\cdot,\varepsilon)$ of system (21) such that $r(0,\varepsilon) \to r_0^*$ when $\varepsilon \to 0$. Which implies the existence of a periodic solution $(x(\cdot,\varepsilon), y(\cdot,\varepsilon))$ of system (17) such that $(x(0,\varepsilon), y(0,\varepsilon)) = (r(0,\varepsilon), 0) \to (r_0^*, 0)$ when $\varepsilon \to 0$. $\square$

*Proof of Corollary 3.* Using the software Mathematica we compute $y_1(\theta, r)$ as the solution of differential equation (22). So

$$f_1(r) = y_1(2\pi, r) = \frac{2\pi\lambda(5a\lambda^2 - 21b)}{4\lambda^4} r I_0(2\lambda r) + \pi \frac{(21b(2+\lambda^2 r^2) - 10a\lambda^2)}{4la^4} I_1(2\lambda r), \tag{23}$$

where $I_n(z)$ is the modified Bessel function of the first kind, for more details see [3].

Now, for $\lambda = 1/200$, $a = -4$ and $b = 1/10$ we obtain numerically the existence of a simple solution $r_0^* \approx 7.56 \in (0, 200/(8\pi))$ of the system (19). Hence, applying Proposition 2 we conclude the proof of the corollary. □

## Appendix: Basic results on the Brouwer degree

In this appendix we present the existence and uniqueness result from the degree theory in finite dimensional spaces. We follow the Browder's paper [4], where are formalized the properties of the classical Brouwer degree. We also present some results that we shall need for proving the main results of this paper.

**Theorem 4.** *Let $X = \mathbb{R}^n = Y$ for a given positive integer $n$. For bounded open subsets $V$ of $X$, consider continuous mappings $f : \overline{V} \to Y$, and points $y_0$ in $Y$ such that $y_0$ does not lie in $f(\partial V)$ (as usual $\partial V$ denotes the boundary of $V$). Then to each such triple $(f, V, y_0)$, there corresponds an integer $d(f, V, y_0)$ having the following three properties.*

(i) *If $d(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If $f_0$ is the identity map of $X$ onto $Y$, then for every bounded open set $V$ and $y_0 \in V$, we have*

$$d\left(f_0\big|_V, V, y_0\right) = \pm 1.$$

(ii) *(Additivity) If $f : \overline{V} \to Y$ is a continuous map with $V$ a bounded open set in $X$, and $V_1$ and $V_2$ are a pair of disjoint open subsets of $V$ such that*

$$y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2)),$$

*then,*

$$d(f_0, V, y_0) = d(f_0, V_1, y_0) + d(f_0, V_1, y_0).$$

(iii) *(Invariance under homotopy) Let $V$ be a bounded open set in $X$, and consider a continuous homotopy $\{f_t : 0 \le t \le 1\}$ of maps of $\overline{V}$ in to $Y$. Let $\{y_t : 0 \le t \le 1\}$ be a continuous curve in $Y$ such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$. Then $d(f_t, V, y_t)$ is constant in $t$ on $[0, 1]$.*

**Theorem 5.** *The degree function $d(f, V, y_0)$ is uniquely determined by the conditions of Theorem 4.*

For the proofs of Theorems 4 and 5 see [4].

**Lemma 6.** *We consider the continuous functions $f_i : \overline{V} \to \mathbb{R}^n$, for $i = 0, 1, \cdots, k$, and $f, g, r : \overline{V} \times [\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$, given by*

$$g(\cdot, \varepsilon) = f_1(\cdot) + \varepsilon f_2(\cdot) + \varepsilon^2 f_3(\cdot) + \cdots + \varepsilon^{k-1} f_k(\cdot),$$

$$f(\cdot, \varepsilon) = g(\cdot, \varepsilon) + \varepsilon^k r(\cdot, \varepsilon).$$

*Assume that $g(z, \varepsilon) \neq 0$ for all $z \in \partial V$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. If for $|\varepsilon| > 0$ sufficiently small $d_B(f(\cdot, \varepsilon), V, y_0)$ is well defined, then*

$$d_B(f(\cdot, \varepsilon), V, y_0) = d_B(g(\cdot, \varepsilon), V, y_0).$$

For a proof of Proposition 6 see Lemma 2.1 in [5].

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