Higher Order Local Accuracy by Averaging in the Finite Element Method

By J. H. Bramble and A. H. Schatz[*]

Abstract. Let u_h be a Ritz-Galerkin approximation, corresponding to the solution u of an elliptic boundary value problem, which is based on a uniform subdivision in the interior of the domain. In this paper we show that by "averaging" the values of u_h in the neighborhood of a point x we may (for a wide class of problems) construct an approximation to $u(x)$ which is often a better approximation than $u_h(x)$ itself. The "averaging" operator does not depend on the specific elliptic operator involved and is easily constructed.

1. Introduction. In this paper we shall discuss some local "superconvergence" results for a large class of Ritz-Galerkin methods, which are used to approximate solutions of elliptic boundary value problems. Briefly, let Ω be a bounded domain in $\mathbf{R}^N$ with boundary $\partial\Omega$ and for simplicity consider the boundary value problem

$$(1.1) \qquad Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f \quad \text{in } \Omega,$$

$$(1.2) \qquad bu = g \quad \text{on } \partial\Omega,$$

where L is uniformly elliptic on Ω and b is some boundary operator such that the problem (1.1), (1.2) has a unique solution.

Suppose that for each $0 < h < 1$ we are given a linear space of finite elements S_h (which we roughly think of as being a space of piecewise polynomials defined on some partition of Ω) and an approximate solution $u_h \in S_h$ of (1.1), (1.2) determined by one of the many Ritz-Galerkin methods which have been proposed. For a survey of some of these we refer the reader to [2]. Let $\Omega_1 \subset\subset \Omega$. For many of these methods, u_h satisfies the interior equations

$$B(u_h, \varphi) = \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}\frac{\partial u_h}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^{N} b_i\frac{\partial u_n}{\partial x_i}\varphi + cu_h\varphi\right) dx$$

$$= B(u, \varphi) = \int_\Omega f\varphi \, dx$$

for all $\varphi \in \overset{\circ}{S}_h(\Omega_1)$, where $\overset{\circ}{S}_h(\Omega_1) = \{\varphi \mid \varphi \in S_h(\Omega), \operatorname{supp}(\varphi) \subset \Omega_1\}$. Roughly speaking, suppose that S_h has the property (and some others described in Section 3) that for each u belonging to the Sobolev space H^r,

$$\inf_{\psi \in S_h} \|u - \psi\|_{1,\Omega_1} \leqslant Ch^{r-1}\|u\|_{r,\Omega_1}.$$

Here $r \geqslant 2$ is a given integer and for $s \geqslant 0$, $\| \cdot \|_{s, \Omega_1}$ denotes the norm on $H^s(\Omega_1)$. It was shown in [15] that for $\Omega_0 \subset\subset \Omega_1$ the interior estimate

$$(1.4) \qquad \|u - u_h\|_{0, \Omega_0} \leqslant C\{h^r \|u\|_{r, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}\}$$

holds. Here p is a nonnegative integer and $\| \cdot \|_{-p, \Omega_1}$ is the norm on the dual space of $\overset{\circ}{H}{}^p(\Omega_1)$ (the completion of $C_0^\infty(\Omega_1)$ under $\| \cdot \|_p$). Under the further assumption that the restriction of elements of S_h to Ω_1 are piecewise polynomials defined on a uniform mesh, it was shown in [4] that

$$(1.5) \qquad |u - u_h|_{0, \Omega_0} \leqslant C\{h^r \|u\|_{r+[N/2]+1, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}\}.$$

Here $|\cdot|_{0, \Omega_0}$ denotes the maximum norm on $\overline{\Omega}_0$ and $[N/2]$ is the integral part of $N/2$.

For many methods, estimates for $\|e\|_{-p, \Omega_1}$ already exist in the literature. Noting that $\|e\|_{-p, \Omega_1} \leqslant \|e\|_{-p, \Omega}$ and taking $p = r - 2$, one often finds that $\|u - u_h\|_{2-r, \Omega} \leqslant Ch^{2r-2} \|u\|_{r, \Omega}$. Hence in these cases, if $u \in H^r(\Omega)$ and $r \geqslant 3$, this latter term is of higher order in h than the first terms on the right in (1.4) and (1.5).

Instead of considering u_h, in this paper we shall consider certain "averages" of u_h as approximations to u. More precisely, the averages are formed by computing $K_h * u_h$, where K_h is a fixed function and $*$ denotes convolution. We shall see that the function K_h has the following properties:

(i) K_h has small support,

(ii) K_h is independent of the specific choice of S_h or the operator L,

(iii) $K_h * u_h$ is easily computable from u_h.

Furthermore we shall prove the following estimates (where again we assume S_h to be defined on a uniform mesh on Ω_1):

$$\|u - K_h * u_h\|_{0, \Omega_0} \leqslant C\{h^{2r-2} \|u\|_{2r-2, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}\}$$

and

$$|u - K_h * u_h|_{0, \Omega_0} \leqslant C\{h^{2r-2} \|u\|_{2r-2+[N/2]+1, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}\}.$$

Hence in the cases in which $r \geqslant 3$ and $\|u - u_h\|_{2-r, \Omega_1} \leqslant Ch^{2r-2} \|u\|_{r, \Omega}$, $K_h * u$ is of order h^{2r-2} locally and therefore, for h sufficiently small, better approximates u than does u_h.

It is important to note that $K_h * u_h$ may be computed at each point in Ω_0. However if we restrict our attention to specific points, say, for example, mesh points, then it very often takes on a very simple form; this will be discussed in Section 5.

An announcement of some of the results obtained in this paper was given in [7]. For some superconvergence type results for two point boundary value problems the reader is referred to [9], [10] and [11]. For some interior superconvergence type results for multi-dimensional elliptic problems see [6]. For parabolic problems we refer the reader to [17] and [18].

An outline of this paper is as follows: In Section 2 we collect some notation.

In Section 3 we discuss the necessary properties of the subspaces and in Section 4 we discuss the interior equations and collect some further preliminaries. In Section 5 the averaging operator is defined and its computation is discussed. Section 6 contains our main superconvergence results. In Section 7 we give some applications of the results and in one example show how to obtain superconvergence up to the boundary when Ω is the unit square, $Lu = -\Delta u$ in Ω and $bu \equiv u$ on $\partial\Omega$.

2. Notation and Some Preliminaries. Let Ω be a bounded open set in $\mathbf{R}^N$. For m a nonnegative integer, $C^m(\Omega)$ will denote the space of real-valued functions having continuous derivatives up to order m on $\overline{\Omega}$ with the norm

$$(2.1) \qquad |u|_m = \sum_{|\alpha| \leqslant m} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|.$$

$C_0^\infty(\Omega)$ will denote the infinitely differentiable functions on Ω whose support is contained in Ω. $H^m(\Omega)$ (resp. $\overset{\circ}{H}{}^m(\Omega)$) is the completion of $C^\infty(\Omega)$ (resp. $C_0^\infty(\Omega)$) with respect to the norm

$$(2.2) \qquad \|u\|_{m,\Omega} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |D^\alpha u|^2 \, dx \right)^{1/2}.$$

Note that $H^0(\Omega) = \overset{\circ}{H}{}^0(\Omega) = L_2(\Omega)$.

For m a nonnegative integer $H^{-m}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$(2.3) \qquad \|u\|_{-m,\Omega} = \sup_{v \in C_0^\infty(\Omega)} \frac{\int_\Omega uv \, dx}{\|v\|_{m,\Omega}} = \sup_{v \in C_0^\infty(\Omega)} \frac{(u,v)}{\|v\|_{m,\Omega}}.$$

For β a multi-integer define the translation operator $T_h^\beta u(x) = u(x + \beta h)$ and the difference quotients,

$$\partial_{h,j} u = h^{-1}(T_{h/2}^{e_j} - T_{h/2}^{-e_j}) u.$$

Here e_j is the multi-index whose jth component is 1 and all others 0. For any multi-index α we set

$$\partial_h^\alpha u = (\partial_{h,1}^{\alpha_1} \cdots \partial_{h,N}^{\alpha_N}) u.$$

We shall also need the following results:

LEMMA 2.1 (cf. e.g. [12]). *Let $\Omega_0 \subset\subset \Omega_1$. If $u \in H^{[N/2]+1}(\Omega_1)$, then (after possible modification on a set of measure zero) $u \in C^0(\Omega_0)$ and*

$$(2.4) \qquad |u|_{0,\Omega_0} \leqslant C \|u\|_{[N/2]+1,\Omega_0}$$

where $C = C(\Omega_0, \Omega_1)$ and $[N/2]$ is the integral part of $N/2$.

LEMMA 4.2. *Let $\Omega_0 \subset\subset \Omega_1$ and s be an arbitrary but fixed nonnegative integer. Then there is a constant C such that*

$$(2.5) \qquad \|u\|_{0,\Omega_0} \leqslant C \sum_{|\alpha| \leqslant s} \|D^\alpha u\|_{-s,\Omega_1}.$$

Proof. Without loss of generality we may assume that Ω_1 is smooth and $\Omega_0 \subset\subset \Omega_1 \subset\subset \mathbf{R}^N$. First suppose that supp $u \subset\subset \Omega_1$. In this case the proof proceeds by induction. For $s = 0$ (2.5) is trivial. Assume (2.5) for $s \geqslant 0$, then for any α with $|\alpha| \leqslant s$

$$(2.6) \qquad \|D^\alpha u\|_{-s,\Omega_1} = \sup_{v \in C_0^\infty(\Omega_1)} \frac{(D^\alpha u, v)}{\|v\|_{s,\Omega_1}}.$$

Let g be a solution of $-\Delta g = v$, in Ω_1, $v = 0$ on $\partial\Omega_1$. Since u has compact support, integration by parts and Schwarz's inequality yields

$$(2.7) \qquad \begin{aligned} (D^\alpha u, v) &= \sum_{i=1}^{N} \left(\frac{\partial}{\partial x_i} D^\alpha u, \frac{\partial g}{\partial x_i} \right) \\ &\leqslant C \left(\sum_{|\beta| = |\alpha|+1} \|D^\beta u\|_{-s-1,\Omega_1} \right) \|g\|_{s+2}. \end{aligned}$$

But by elliptic regularity (cf. [13])

$$\|g\|_{s+2,\Omega_1} \leqslant C\|v\|_{s,\Omega_1}.$$

In view of this, (2.7) and (2.6), it follows that

$$\sum_{|\alpha| \leqslant s} \|D^\alpha u\|_{-s,\Omega_1} \leqslant \sum_{|\beta| \leqslant s+1} \|D^\beta u\|_{-s-1,\Omega_1},$$

which completes the proof for u with compact support in Ω_1. More generally let $w \in C^\infty(\Omega_1)$ with $w = 1$ on Ω_0 and supp $w \subset\subset \Omega_1$. Applying (2.5) to the function wu we obtain

$$\|u\|_{0,\Omega_0} \leqslant C \sum_{|\alpha| \leqslant s} \|D^\alpha(wu)\|_{-s,\Omega_1}.$$

Using Leibniz's rule and the definition of the $H^{-s}(\Omega_1)$ norm it is easily seen that

$$\sum_{|\alpha| \leqslant s} \|D^\alpha(wu)\|_{-s,\Omega_1} \leqslant C \sum_{|\alpha| \leqslant s} \|D^\alpha u\|_{-s,\Omega_1},$$

which completes the proof.

3. Piecewise Polynomial Subspaces. In the following sections we shall be concerned with Ritz-Galerkin methods in which the functions used for approximation are piecewise polynomials on a uniform mesh. More precisely, let $\{Q_\beta\}$, $\beta \in Z^N$ (the multi-integers) denote a family of disjoint open sets which partition $\mathbf{R}^N$. We shall assume that the partition is invariant under translations by ν, for any $\nu \in Z^N$. Let $r \geqslant 2$ be an integer. S_1^r will denote a linear space of functions which are polynomials on each Q_β and belong to $H^1(\Omega)$ for any $\Omega \subset\subset \mathbf{R}^N$. Furthermore we assume that S_1^r is translation invariant in the sense that if $\varphi \in S_1^r$, then $T_1^\alpha \varphi \in S_1^r$, for all $\alpha \in Z^N$.

For each $0 < h \leqslant 1$, let $Q_\beta^h = hQ_\beta$. Then the family $\{Q_\beta^h\}$ forms a partition of $\mathbf{R}^N$ which is invariant under translations by $h\nu$, $\nu \in Z^N$. For $\Omega_1 \subset\subset \mathbf{R}^N$, $S_h^r(\Omega_1)$ will denote the finite-dimensional subspace of $H^1(\Omega_1)$ consisting of functions which are the restrictions to Ω_1 of functions of the form $\varphi(x/h)$, where $\varphi(x) \in S_1^r$. Let $\overset{\circ}{S}_h^r(\Omega) =$

$\{\varphi \mid \varphi \in S_h^r(\Omega)$, supp $\varphi \subseteq \Omega\}$. We note that if $\Omega_0 \subset\subset \Omega_1$ and $\alpha \in Z^N$, then for h sufficiently small (depending on α), $\partial_h^\alpha \varphi \in \mathring{S}_h^r(\Omega_1)$ for all $\varphi \in \mathring{S}_h^r(\Omega_0)$. We shall also make the following approximability assumptions:

A.1. Let $\Omega_0 \subset\subset \Omega_1$. There exists an $h_0 > 0$ such that for all $h \in (0, h_0]$ and $u \in \mathring{H}^j(\Omega_0)$

$$(3.1) \qquad \inf_{\eta \in \mathring{S}_h^r(\Omega_0)} \|u - \eta\|_{1,\Omega_0} \leqslant Ch^{j-1}\|u\|_{j,\Omega_1}, \qquad 1 \leqslant j \leqslant r,$$

where C is independent of u and h.

A.2. If $\varphi \in S_h^r(\Omega_1)$, $h \in (0, h_0]$, $\Omega_0 \subset\subset \Omega_1$, $\omega \in C_0^\infty(\Omega_0)$, then there exists an $\eta \in \mathring{S}_h^r(\Omega_1)$ such that

$$(3.2) \qquad \|\omega\varphi - \eta\|_{1,\Omega_0} \leqslant Ch\|\varphi\|_{1,\Omega_1},$$

where C is independent of h and φ.

Often in practice subspaces which satisfy the above conditions can be described in the following manner: There exist functions $\varphi_1, \ldots, \varphi_m$, in H^1, which are piecewise polynomials with compact support, such that $\varphi \in S_h^r(\Omega)$ is of the form

$$(3.3) \qquad \varphi(x) = \sum_{j=1}^m \sum_{\beta \in Z^N} a_\beta^j \varphi_j(h^{-1}x - \beta) \quad \text{for } x \in \Omega.$$

Here the coefficients a_β^j are real.

Example 1. An example of subspaces of this type are those generated by the B-splines of Schoenberg [16]. These will play a central role in what follows. Let us describe them more precisely.

For t real, define

$$\chi(t) = \begin{cases} 1, & |t| \leqslant \tfrac{1}{2}, \\ 0, & |t| > \tfrac{1}{2}, \end{cases}$$

and for l an integer, set

$$(3.4) \qquad \psi_1^{(l)}(t) = \chi * \chi * \cdots * \chi, \quad \text{convolution } l - 1 \text{ times.}$$

The function $\psi_1^{(l)}$ is the one-dimensional B-spline basis function of order l. Then

$$(3.5) \qquad \psi^{(l)}(x) = \prod_{j=1}^N \psi_1^{(l)}(x_j)$$

is the N-dimensional B-spline basis function of order l. In this case every element of S_h on Ω is of the form

$$(3.6) \qquad \varphi(x) = \sum_{\alpha \in Z^N} a_\alpha \psi^{(l)}(h^{-1}x - \alpha), \quad x \in \Omega.$$

Example 2. A more general class of splines than (3.6) which satisfy our assumptions are the tensor products of one-dimensional splines. They may be described as follows: Let us subdivide $\mathbf{R}^1$ into intervals I_j of length h. Let $M_{k,r}^h(\mathbf{R}^1) = \{\varphi \mid \varphi \in P_{r-1}(I_j), \varphi \in C^k(\Omega)$ for any $\Omega \subset\subset \mathbf{R}^1\}$, where $P_{r-1}(I_j)$ are the polynomials of degree

$\leqslant r - 1$ restricted to I_j. $M_{k,r}^h(\mathbf{R}^N)$ is then defined to be the N-fold tensor product of $M_{k,r}^h(\mathbf{R}^1)$ and for any $\Omega \subset\subset \mathbf{R}^N$, $M_{k,r}^h(\Omega)$ is the restriction of $M_{k,r}^h(\mathbf{R}^N)$ to Ω. We note that if $k = r - 2$ then these are the B-splines defined above. If m is a positive integer, $r = 2m$ and $k = m - 1$, then these are the so-called piecewise Hermite polynomials (cf. [4]).

Example 3. It was shown in [15] that the triangular elements of Bramble and Zlámal [8] satisfy our conditions. Here we take them on a uniform triangulation of $\mathbf{R}^N$.

4. Interior Equations for Ritz-Galerkin Methods and Some Preliminaries. Let Ω_1 be a bounded open set in $\mathbf{R}^N$ and $B(u, v)$ a bilinear form defined on $H^1(\Omega_1) \times H^1(\Omega_1)$ given by

$$(4.1) \qquad B(u, v) = \int_{\Omega_1} \left(\sum_{i,j=1}^N a_{ij}(x) D_i u D_j v + \sum_{i=1}^N b_i(x)(D_i u)v + c(x)uv \right) dx,$$

where for simplicity the coefficients $a_{ij}, b_i,$ are assumed to be of class $C^\infty(\Omega_1)$ and $a_{ij} = a_{ji}$. We note that in general $B(u, v)$ may not be symmetric. We assume throughout that $B(u, v)$ is uniformly elliptic on Ω_1; i.e., there exists a constant $C > 0$ such that for all $x \in \bar{\Omega}_1$ and real vectors $\xi = (\xi_1, \dots, \xi_N) \neq 0$,

$$C \sum_{i=1}^N \xi_i^2 \leqslant \sum_{i,j=N}^N a_{ij}\xi_i\xi_j.$$

Let $u \in H^1(\Omega_1)$ and $u_h \in S_h^r(\Omega_1)$ and suppose that $u - u_h$ satisfies the "interior" Ritz-Galerkin equations

$$(4.2) \qquad\qquad B(u - u_h, \varphi) = 0$$

for all $\varphi \in \overset{\circ}{S}_h^r(\Omega_1)$. Here u_h may be thought of as an approximation to u on Ω_1 obtained by using some Ritz-Galerkin method on a larger set Ω. Instead of investigating the properties of the error $u - u_h$ we shall be interested in investigating the properties of the difference between u and certain averages of u_h. In order to do so we shall need some results concerning $u - u_h$ derived in [15] and [4].

LEMMA 4.1. *Let* $\Omega_0 \subset\subset \Omega_1 \subset\subset \mathbf{R}^N$, $u \in H^{r+|\alpha|}(\Omega_1)$, α *a multi-index*, $u_h \in S_r^h(\Omega_1)$, $r \geqslant 2$ *and* p *be a nonnegative integer, arbitrary but fixed. Suppose that* A.1 *and* A.2 *are satisfied and that* $u - u_h$ *satisfies* (4.2). *Then for* h *sufficiently small*

$$(4.3) \qquad \| \partial_h^\alpha(u - u_h) \|_{2-r, \Omega_0} \leqslant C(h^{2r-2}\|u\|_{r+|\alpha|, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}).$$

Here C *is independent of* u, u_h *and* h *but in general depends on* p, α, Ω_0, Ω_1 *and the coefficients* a_{ij}, b_i *and* c.

Proof. Clearly it is sufficient to prove (4.3) for ∂_h^α replaced by the forward difference operator $\widetilde{\partial}_h^\alpha = T_{h/2}^\alpha \partial_h^\alpha$, and Ω_0 replaced by Ω_0' with $\Omega_0 \subset\subset \Omega_0'$. Now let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_1' \subset\subset \Omega_1$. It follows from (6.8) of [15], that for h sufficiently small and $e = u - u_h$

$$(4.4) \quad B(\widetilde{\partial}_h^{\alpha} e, \varphi) = \int_{\Omega_0} \sum_{i,j=1}^{N} \sum_{\beta < \alpha} \binom{\alpha}{\alpha - \beta} \widetilde{\partial}_h^{\alpha-\beta} a_{ij} T_h^{\alpha-\beta} \widetilde{\partial}_h^{\beta} D_i e D_j \varphi \, dx,$$

$$\forall \varphi \in \overset{\circ}{S}^h(\Omega_0').$$

Hence, referring to [15] in particular, from (5.6) of Theorem 5.2, (4.3) of Lemma 4.2, (6.5) following Theorem 6.1 and (4.4) above we have that

$$(4.5) \quad \|\widetilde{\partial}_h^{\alpha} e\|_{2-r, \Omega_0'} \leqslant C\left(h^{2r-2} \|u\|_{r-2+|\alpha|, \Omega_1'} + \|e\|_{-p, \Omega_1'} + \sum_{\beta < \alpha} \|\widetilde{\partial}_h^{\beta} e\|_{2-r, \Omega_1'} \right).$$

The proof now proceeds by induction on $|\alpha|$. If $\alpha = 0$ then (4.3) follows from (4.5). Assume now that (4.3) holds for $|\alpha| \leqslant m$. If $|\alpha| = m + 1$, then applying (4.3) to the third term on the right of (4.5) on the domains Ω_1' and Ω_1 the inequality (4.3) for $|\alpha| = m + 1$ is obtained, which completes the proof.

LEMMA 4.2 [4]. *Under the conditions of Lemma 4.1 there exists a constant $C > 0$ such that*

$$(4.6) \quad |\partial_h^{\alpha}(u - u_h)|_{0, \Omega_0} \leqslant C(h^r \|u\|_{r+|\alpha|+[N/2]+1, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}),$$

where C is independent of u, u_h and h but in general depends on p, α, Ω_0, Ω_1 and the coefficients a_{ij}, b_i and c.

5. A Class of Convolution Operators.

In this section we shall introduce a particular class of convolution operators and discuss some of their properties. In the next section we shall use these to show (under certain conditions) that a certain convolution of the Ritz-Galerkin solution u_h is closer to u than u_h is itself.

We shall begin by defining the kernels of the above-mentioned convolution operators. In each case it is an N-dimensional smooth spline which is the product of a single suitably chosen one-dimensional smooth spline. More precisely let $\psi_1^{(l)}$ be the one-dimensional smooth spline of order l defined by (3.4). For $l = r - 2, r \geqslant 2$ given, and $x \in \mathbf{R}^N$ set

$$(5.1) \quad K_h(x) = \prod_{m=1}^{N} \left(\sum_{j=-(r-2)}^{r-2} h^{-1} k_j' \psi_1^{(r-2)}(h^{-1}x_m - j) \right).$$

Here the constants K_j' are defined as follows:

(i) $k_{-j}' = k_j', j = 0, \ldots, r - 2$.

(ii) $k_0' = k_0$ and $k_j' = k_{j/2}, j = 1, \ldots, r - 2$,

where the $k_j, j = 0, \ldots, r - 2$, are determined as the unique solution (see Lemma 5.1 below) of the linear system of algebraic equations.

$$(5.2) \quad \sum_{j=0}^{r-2} k_j \int_{R_1} \psi_1^{(r-2)}(y)(y+j)^{2m} \, dy = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \ldots, r - 2. \end{cases}$$

We note that the constants k_j can be easily computed and depend only on the choice of r.

The convolution operators in which we are interested are of the form

$$(5.3) \quad (K_h * u)(x) = h^{-N} \int_{\mathbf{R}^N} \left[\prod_{m=1}^{N} \sum_{j=2-r}^{r-2} k'_j \, \psi_1^{(r-2)}(h^{-1}(x_m - y_m) - j) \right] u(y) \, dy.$$

If we set $k'_j = 0$ for $|j| \geq r - 1$ and for β a multi-integer define $k'_\beta = \prod_{j=1}^{N} k'_{\beta_j}$, then (5.2) and (5.3) may be rewritten as

$$(5.4) \qquad K_h(x) = h^{-N} \sum_\beta k'_\beta \, \psi^{(r-2)}(h^{-1}x - \beta)$$

and

$$(5.5) \qquad (K_h * u)(x) = h^{-N} \int_{\mathbf{R}^N} \sum_\beta k'_\beta \, \psi^{(r-2)}(h^{-1}(x - y) - \beta) u(y) \, dy,$$

respectively.

In what follows we shall be interested in $K_h * u_h$ as an approximation to u. Before investigating the properties of K_h, let us first consider the computation of $K_h * u_h$. If, in the region to be studied, u_h is of the form (3.3), that is

$$(5.6) \qquad u_h = \sum_{j=1}^{K} \sum_{\alpha \in Z^N} a_\alpha^j \, \varphi_j(h^{-1}x - \alpha),$$

then using (5.5) and making a change in variables we have

$$(5.7) \quad \begin{aligned} & (K_h * u_h)(x) \\ & = \sum_{j=1}^{m} \sum_{\alpha \in Z^N} a_\alpha^j \left(\sum_{\beta \in Z^N} k'_\beta \int_{\mathbf{R}^N} \psi^{r-2}(h^{-1}x - \eta - \beta - \alpha) \varphi_j(\eta) \, dx \right). \end{aligned}$$

Although the terms $k'_\beta \int_{\mathbf{R}^N} \psi^l(h^{-1}x - \eta - \beta - \alpha) \varphi_j(\eta) \, d\eta$ and hence $K_h * u_h$ may be calculated at any point, we shall, for simplicity restrict our attention to points of the form $x = h\gamma$, $\gamma \in Z^N$. In this case

$$(K_h * u_h)(h\gamma) = \sum_{j=1}^{m} \sum_{\alpha \in Z^N} a_\alpha^j \left(\sum_{\beta \in Z^N} k'_\beta \int_{\mathbf{R}^N} \psi^{r-2}(\gamma - \beta - \alpha - \eta) \varphi_j(\eta) \, d\eta \right)$$

or

$$(5.8) \qquad (K_h * u_h)(h\gamma) = \sum_{j=1}^{m} \sum_{\gamma, \delta \in Z^N} a_{\gamma-\delta}^j \, d_\delta^j,$$

where

$$d_\delta^j = \sum_{\beta \in Z^N} k'_\beta \int_{\mathbf{R}^N} \psi^{r-2}(\delta - \beta - \eta) \varphi_j(\eta) \, d\eta.$$

Thus the values of $K_h * u_h$ at mesh points are determined by taking a fixed finite linear combination of the values of the coefficients a_α^j. There are only a finite number of nonzero coefficients d_δ^j. They may be computed a priori and are independent of h and the particular mesh point.

As an example, let us take the case where u_h itself is a smooth spline. Then

$k = 1$, $\varphi_1 = \psi^{(r)}$ and u_h is of the form

$$(5.9) \qquad\qquad u_h = \sum_{\alpha \in Z^N} a_\alpha \psi^{(r)}(h^{-1}x - \alpha).$$

Setting $d_\delta^j = d_\delta$ in this case, we have

$$(5.10) \quad d_\delta = \sum_{\beta \in Z^N} k_\beta' \, \psi^{2r-2}(\delta - \beta) = \prod_{l=1}^{N} d_{\delta_l}' \equiv \prod_{l=1}^{N} \left(\sum_{\eta=-(t-1)}^{t-1} k_\eta' \, \psi_1^{2r-2}(\delta_l - \eta) \right).$$

Here the d_l' correspond to the weights d_δ in the one-dimensional case. A table of the one-dimensional coefficients d_l' from which the N-dimensional weights d_δ in (5.10) may be formed will be given in the appendix. We remark that the computation of the d_δ^j may also be reduced to the product of the one-dimensional case if we take S_h to be the more general class of tensor products of one-dimensional splines given in Example 2 of Section 3.

For technical reasons in what follows we shall make use of a slightly more general class of convolution operators than those defined by (5.3). We shall now introduce these and discuss some of their properties. The following result will be needed.

LEMMA 5.1 (cf. [6, Lemma 8.1]). *Let $t \geq 1$ and $l \geq 1$ be arbitrary but fixed integers. There exist uniquely determined real constants k_j, $j = 0, \ldots, t - 1$, which satisfy the linear system of algebraic equations.*

$$(5.11) \qquad \sum_{j=0}^{t-1} k_j \int_{R^N} \psi_1^{(l)}(y) \, (y + j)^{2m} \, dy = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \ldots, t - 1. \end{cases}$$

Here, with a slight abuse of notation, we have suppressed the dependence of the solutions of (5.11) on t and l. We note that (5.2) and (5.11) coincide in the case $l = r - 2$ and $t = r - 1$.

As before let $k_0' = k_0$, $k_j' = k_j/2$, $k_{-j}' = k_j'$ for $j = 1, \ldots, t - 1$ and $k_j = 0$ for $|j| \geq t$. For $\beta \in Z^N$ set $k_\beta' = \Pi_{j=1}^{N} k_{\beta_j}'$ and define

$$(5.12) \qquad
\begin{aligned}
K_{h,l}^{2t}(x) &= h^{-N} \sum_{\beta \in Z^N} k_\beta' \, \psi^l(h^{-1}x - \beta) \\
&= \prod_{m=1}^{N} \left[h^{-1} \sum_{j=-(t-1)}^{t-1} k_j' \, \psi_1^{(l)}(h^{-1}x_m - j) \right].
\end{aligned}$$

We remark that $\operatorname{supp}(K_{h,l}^{2t}(x))$ is the cube with side of length $(2t + l)h$ centered at the origin. Note also that $K_{h,r-2}^{2r-2}(x) = K_h(x)$.

The function $K_{h,l}(x)$ was constructed in such a way that $K_{h,l}^{2t} * u$, where $*$ denotes convolution, is an approximation of order h^{2t} to u. More precisely we have the following:

LEMMA 5.2. *Let $t \geq 1$ and $l \geq 1$ be fixed integers and $\Omega_0 \subset\subset \Omega_1$. Then for h sufficiently small*

$$(5.13) \qquad |u - K_{h,l}^{2t} * u|_{0,\Omega_0} \leq Ch^s |u|_{s,\Omega_1}, \qquad 0 \leq s \leq 2t,$$

and

$$(5.14) \qquad \|u - K_{h,l}^{2t} * u\|_{0,\Omega_0} \leqslant Ch^s \|u\|_{s,\Omega_1}, \qquad 0 \leqslant s \leqslant 2t.$$

The proofs will not be given. We only remark that (5.13) and (5.14) follow as a simple consequence of the Bramble-Hilbert Lemma (cf. [3]), using the fact that Lemma 5.1 implies that $K_{h,l}^{2t} * u$ reproduces polynomials of degree not exceeding $2t - 1$ in each variable (cf. [6, Lemma 8.1]). We also note that (5.13) may also be proved using this latter fact and Taylor's Theorem.

We shall need some other properties of $K_{h,l}^{2t} * u$.

LEMMA 5.3. *Let t, l, h, Ω_0 and Ω_1 be as above. Then*

(i) *For any multi-index α and $u \in L_2(\Omega_1)$*

$$(5.15) \qquad \partial_h^\alpha (K_{h,l}^{2t} * u)(x) = K_{h,l}^{2t} * \partial_h^\alpha u(x), \qquad x \in \Omega_0.$$

(ii) *If s is any fixed integer (positive or negative) and α is any multi-index with $\alpha_j \leqslant l$, $j = 1, \ldots, N$,*

$$(5.16) \qquad \|D^\alpha(K_{h,l}^{2t} * u)\|_{s,\Omega_0} \leqslant C \|\partial_h^\alpha u\|_{s,\Omega_1} \quad \textit{for all } u \in H^s(\Omega_1)$$

and

$$(5.17) \qquad |D^\alpha(K_{h,l}^{2t} * u)|_{0,\Omega_0} \leqslant C |\partial_h^\alpha u|_{0,\Omega_1} \quad \textit{for all } u \in C^0(\Omega_1).$$

Here C is a constant which is independent of h and u.

Proof. The identity (5.15) follows very simply using a change of variables.

For simplicity in notation we shall prove (5.16) and (5.17) in the case $N = 1$. The multi-dimensional case follows in essentially the same manner. For $N = 1$ we have

$$(K_{h,l}^{2t} * u)(x) = h^{-1} \int_{\mathbf{R}^1} \sum_{j=-(t-1)}^{t-1} k_j' \psi_1^{(l)}(h^{-1}(x - y) - j) u(y) \, dy.$$

If $0 \leqslant \alpha \leqslant l$ define

$$V_{h,l-\alpha}^{2t}(x) = h^{-1} \sum_{j=-(t-1)}^{t-1} k_j' \psi_1^{(l-\alpha)}(h^{-1}x - j).$$

Then it is easy to see from (5.15) and (3.4) that

$$D^\alpha(K_{h,l}^{2t} * u) = V_{h,l-\alpha}^{2t} * \partial_h^\alpha u.$$

Now since $\psi_1^{(l-\alpha)} \geqslant 0$, $h^{-1} \int_{\mathbf{R}^1} \psi_1^{(l-\alpha)}(h^{-1}x - j) \, dx = 1$ and $\operatorname{supp}(\psi_1^{(l-\alpha)}(h^{-1}x))$ is an interval of length $(l + 1 - \alpha)h$, it follows that for h sufficiently small

$$(5.18) \qquad |V_{h,l-\alpha}^{2t} * \partial_h^\alpha u|_{0,\Omega_0} \leqslant C |\partial_h^\alpha u|_{0,\Omega_1}$$

and

$$(5.19) \qquad \|V_{h,l-\alpha}^{2t} * \partial_h^\alpha u\|_{0,\Omega_0} \leqslant C \|\partial_h^\alpha u\|_{0,\Omega_1}.$$

The inequality (5.8) is just the inequality (5.17) and (5.19) is the inequality (5.16) in the case that $s = 0$. In the case that s is nonnegative, the inequality (5.16) follows from (5.9) on observing that for any nonnegative integer m,

$$D^m(D^\alpha(K_{h,l}^{2t} * u)) = V_{h,l-2}^{2t} * D^m \partial_h^\alpha u.$$

For s a negative integer we have

$$\|D^\alpha K_{h,l}^{2t} * u\|_{s,\Omega_0} = \sup_{v \in C_0^\infty(\Omega_0)} \frac{(D^\alpha(K_{h,l}^{2t} * u), v)}{\|v\|_{-s,\Omega_0}}$$

$$= \sup_{v \in C_0^\infty(\Omega_0)} \frac{(V_{h,l-\alpha}^{2t} * \partial_h^\alpha u, v)}{\|v\|_{-s,\Omega_0}} = \sup_{v \in C_0^\infty(\Omega_0)} \frac{(\partial_h^\alpha u, V_{h,l-\alpha}^{2t} * v)}{\|v\|_{-s,\Omega_0}}$$

$$\leqslant \sup_{v \in C_0^\infty(\Omega_0)} \frac{\|\partial_h^\alpha u\|_{s,\Omega_1} \|V_{h,l-\alpha}^{2t} * v\|_{-s,\Omega_1}}{\|v\|_{-s,\Omega_0}}.$$

It is easily seen that

$$\|v_{h,l-\alpha}^{2t} * v\|_{-s,\Omega_1} \leqslant C\|v\|_{-s,\Omega_0}$$

where C is independent of h and $v \in C_0^\infty(\Omega_0)$. This completes the proof.

6. Superconvergence Estimates. Let u and $u_h \in S_h^r$ satisfy the interior equations (4.2). The inequalities (1.4) and (1.5) provide estimates for the error in the L_2 and maximum norm respectively. It is often the case that for specific problems the term $\|e\|_{-p,\Omega_1}$ is $O(h^r)$ and hence the estimates (1.4) and (1.5) yield $O(h^r)$ convergence locally. For many important problems $\|e\|_{-p,\Omega_1}$ is in fact $O(h^{2r-2})$. We shall show that if one considers $K_h * u_h$ instead of u_h as an approximation to u, and if $\|e\|_{-p,\Omega_1}$ is $O(h^{2r-2})$, then $u - K_h * u_h$ is locally of order h^{2r-2} in both L_2 and maximum norms. Hence in these cases processing the solution u_h yields higher order accuracy for $r \geqslant 3$.

We emphasize that $u - K_h * u_h$ will be $O(h^{2r-2})$ at every point of the region in which the error is measured. In practice one usually computes the solution at specific points, for example at mesh points. In this case $K_h * u_h$ takes a particularly simple form. This will also be discussed. We shall start with finding estimates for $u - K_h * u$ in L_2 norm (Theorem 1) and then in the maximum norm (Theorem 2).

THEOREM 1. *Let $\Omega_0 \subset\subset \Omega_1$, $u \in H^{2r-2}(\Omega_1)$, $u_h \in S_h^r(\Omega_1)$, $r \geqslant 3$, where S_h^r satisfies A.1 and A.2 and $u - u_h$ satisfies (4.2). Let K_h be chosen as in (5.1) and let p be an arbitrary but fixed nonnegative integer. Then for all h sufficiently small*

$$(6.1) \qquad \|u - K_h * u_h\|_{0,\Omega_0} \leqslant C(h^{2r-2}\|u\|_{2r-2,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}),$$

where C is independent of u and h.

Proof. By the triangle inequality

$$(6.2) \qquad \|u - K_h * u_h\|_{0,\Omega_0} \leqslant \|u - K_h * u\|_{0,\Omega_0} + \|K_h * (u - u_h)\|_{0,\Omega_0}.$$

Let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_0'' \subset\subset \Omega_1$. Since $K_h(x) = K_{h,r-2}^{2r-2}(x)$ it follows from (5.14) that

$$(6.3) \qquad \|u - K_h * u\|_{0,\Omega_0} \leqslant Ch^{2r-2}\|u\|_{2r-2,\Omega_1}.$$

Now using (2.5), we have that

(6.4) $$\|K_h * (u - u_h)\|_{0,\Omega_0} \leqslant \sum_{|\beta| \leqslant r-2} \|D^\beta K_h * (u - u_h)\|_{2-r,\Omega_0'}.$$

The inequality (6.1) now follows from (5.16), (4.3) and (6.3).

In order to prove a maximum norm error estimate we shall make use of the following.

LEMMA 6.1. *Let* $\Omega_0 \subset\subset \Omega_1$, $v \in C^0(\Omega)$ *and* $N_0 = [N/2] + 1$. *Then for all* h *sufficiently small*

(6.5) $$|K_h * v|_{0,\Omega_0} \leqslant C \left[\sum_{|\alpha| \leqslant N_0 + r-2} \|\partial_h^\alpha v\|_{2-r,\Omega_1} + h^{r-2} \sum_{|\alpha| \leqslant r-2} |\partial_h^\alpha v|_{0,\Omega_1} \right]$$

where C *is independent of* v *and* h.

Proof. By the triangle inequality

(6.6) $$|K_h * v|_{0,\Omega_0} \leqslant |K_{h,N_0}^{2r-2} * K_h * v|_{0,\Omega_0} + |K_h * v - K_{h,N_0}^{2r-2} * K_h * v|_{0,\Omega_0}$$

where $K_{h,N_0}^{2r-2} * u$ is defined by (5.12). Let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_1$. Then using (2.4), (5.16), (5.15) and (2.5) we have for the first term on the right that

(6.7) $$|K_{h,N_0}^{2r-2} * K_h * v|_{0,\Omega_0} \leqslant C \sum_{|\alpha| \leqslant N_0} \|D^\alpha K_{h,N_0}^{2r-2} * K_h * v\|_{0,\Omega_0'}$$
$$\leqslant C \left(\sum_{|\alpha| \leqslant N_0} \|K_h * \partial_h^\alpha v\|_{0,\Omega_0'} \right) \leqslant C \left(\sum_{|\alpha| \leqslant N_0 + r-2} \|\partial_h^\alpha v\|_{2-r,\Omega_1} \right).$$

For the second term on the right of (6.6) we have

(6.8) $$|K_h * v - K_{h,N_0}^{2r-2} * K_h * v|_{0,\Omega_0} \leqslant Ch^{r-2} |K_h * v|_{r-2,\Omega_0'}$$
$$\leqslant Ch^{r-2} \left(\sum_{|\alpha| \leqslant r-2} |\partial_h^\alpha v|_{0,\Omega'} \right),$$

where we have used (5.13) and (5.17). The inequality (6.5) now follows from (6.6), (6.7) and (6.8).

THEOREM 2. *Suppose that the conditions of Theorem 1 are satisfied and that* $u \in H^{2r-2+N_0}$ $(N_0 = [N/2] + 1)$. *Then*

(6.9) $$|u - K_h * u_h|_{0,\Omega_0} \leqslant C(h^{2r-2} \|u\|_{2r-2+N_0,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}),$$

where C *is independent of* u *and* h.

Proof. Using the triangle inequality and setting $u - u_h = e$,

$$|u - K_h * u_h|_{0,\Omega_0} \leqslant |u - K_h * u|_{0,\Omega_0} + |K_h * e|_{0,\Omega_0}.$$

Let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_1$. Then using (5.13) and (2.4) we have

$$|u - K_h * u|_0 \leqslant Ch^{2r-2} |u|_{2r-2,\Omega_0'} \leqslant Ch^{2r-2} \|u\|_{2r-2+N_0,\Omega_1}.$$

Applying Lemma 6.1, (4.3) and (4.4)

$$|K_h * e|_{0,\Omega_0} \leqslant C\left(\sum_{|\alpha|\leqslant r-2+N_0}\|\partial_h^\alpha e\|_{2-r,\Omega_0'} + h^{r-2}\sum_{|\alpha|\leqslant r-2}|\partial_h^\alpha e|_{0,\Omega_0'}\right)$$

$$\leqslant C(h^{2r-2}\|u\|_{2r-2+N_0,\Omega_1} + \|u - u_h\|_{-p,\Omega_1})$$

which completes the proof.

Although $K_{h,r-2}^{2r-2} * u_h$ can be calculated at arbitrary points in Ω_0, Theorem 2 takes a particularly simple form if u_h is of the form (3.3) and if, for example, we are interested in points in Ω_0 of the form $h\gamma$, $\gamma \in \mathbf{Z}^N$.

COROLLARY. *Suppose the conditions of Theorem 2 are satisfied and u_h is of the form* (3.3). *Then at points $h\gamma \in \overline{\Omega}_0$, $\gamma \in \mathbf{Z}^N$*

$$(6.10) \qquad \sup_{h\gamma\in\overline{\Omega}_0;\gamma\in\mathbf{Z}^N}\left|u(h\gamma) - \sum_{j=1}^m\sum_\alpha a_{\gamma-\alpha}^j d_\alpha^j\right|$$

$$\leqslant C(h^{2r-2}\|u\|_{2r-2+N_0,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}),$$

where the d_α^j are given by (5.10).

7. Examples. Here the theory given in Section 6 will be exemplified, where for simplicity we shall restrict ourselves to discussing Dirichlet's problem. In Example 1 interior superconvergence estimates for two different methods will be discussed. In Example 2 a way of obtaining superconvergence up to the boundary, when the domain Ω is the unit square and the subspaces are taken to satisfy the boundary conditions, will be given.

Let Ω be a bounded domain in $\mathbf{R}^N$ with boundary $\partial\Omega$. For simplicity consider

$$(7.1) \qquad\qquad -\Delta u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

For the purposes of the applications given here an additional assumption on the subspace $S_h^r(\Omega)$ will be made. Namely we suppose that there exists a constant C independent of u and h such that for all $u \in H^t(\Omega)$; $1 \leqslant t \leqslant r$,

$$(7.2) \qquad\qquad \inf_{\chi\in S_h^r(\Omega)}\|u - \chi\|_{1,\Omega} \leqslant Ch^{t-1}\|u\|_{t,\Omega}.$$

In Example 2 the elements of $S_r^h(\Omega)$ are required to vanish on $\partial\Omega$. In this case (7.2) is assumed to hold only for $u \in \overset{\circ}{H}^1(\Omega) \cap H^r(\Omega)$.

In what follows we shall assume that the hypothesis of Theorems 1 and 2 are satisfied, where $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$.

Example 1. *Dirichlet's Problem on a Smooth Domain.* Here we shall assume for simplicity that $\partial\Omega \in C^\infty$. In Babuška [1] and Nitsche [14], methods were introduced for approximating the solution of (7.1) in which the approximating subspaces need not satisfy the boundary condition. These methods have the same interior equations; i.e. if u_h is the approximate solution determined by either of these methods, then

$$\int_\Omega \sum_{i=1}^N \frac{\partial u_h}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega f\varphi \, dx$$

(7.3)

$$= \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx, \quad \forall \varphi \in \overset{\circ}{S}{}_h^r (\Omega_1).$$

What is important here is that we may choose $S_h^r(\Omega)$ to satisfy all of our conditions (cf. [4, Example 2], for more details). Now it was shown in [5] that the estimate

(7.4)
$$\|u - u_h\|_{2-r, \Omega} \leqslant h^{r-1-t} \|u\|_{t, \Omega}$$

is valid for $1 \leqslant t \leqslant r$. Using this inequality and the fact that $\|u - u_h\|_{2-r, \Omega_1} \leqslant \|u - u_h\|_{2-r, \Omega}$, we obtain from Theorems 1 and 2 the error estimates

(7.5)
$$\|u - K_h * u_h\|_{0, \Omega_0} \leqslant Ch^{2r-2}(\|u\|_{2r-2, \Omega_1} + \|u\|_{r, \Omega}),$$

and in the maximum norm

$$|u - K_h * u_h|_{0, \Omega_0}$$

(7.6)
$$\leqslant Ch^{2r-2}(\|u\|_{2r-2+[N/2]+1, \Omega_1} + \|u\|_{r, \Omega}).$$

If we take $N = 2$ and for example S_h^r to be piecewise cubic Hermite polynomials or cubic smooth splines, then $r = 4$ and (7.5) and (7.6) become

$$\|u - K_h * u_h\|_{0, \Omega_0} \leqslant Ch^6(\|u\|_{6, \Omega_1} + \|u\|_{4, \Omega}),$$

$$|u - K_h * u_h|_{0, \Omega_0} \leqslant Ch^6(\|u\|_{8, \Omega_1} + \|u\|_{4, \Omega}).$$

Example 2. Superconvergence Up to the Boundary for Dirichlet's Problem on a Square. Let Ω be the unit square in $\mathbf{R}^2$ (i.e. $\Omega = \{x \mid 0 < x_i < 1, i = 1, 2\}$), with boundary $\partial\Omega$ and let u be the solution of (7.1).

Let $h = 1/n$, $n = 1, 2, 3, \ldots$, and let l be either 0 or 1. We shall approximate u taking our subspace $S_h^r(\Omega) = \bar{M}_{k,r}^h(\Omega) = \{\varphi \mid \varphi \in M_{l,r}^h(\Omega), \varphi|_{\partial\Omega} = 0, k = 0, 1\}$. That is, $\bar{M}_{k,r}^h(\Omega)$ is the subspace of the space of tensor products of one-dimensional C^0 or C^1 splines (discussed in Section 3) consisting of elements which vanish on $\partial\Omega$.

Let $u_h \in S_h^r(\Omega)$ be the approximate solution of (7.1) defined by

$$\sum_{i=1}^2 \int_\Omega \frac{\partial u_h}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega f\varphi \, dx$$

(7.7)

$$= \sum_{i=1}^2 \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx, \quad \forall S_h^r(\Omega).$$

We shall show that u_h can be extended, in a simple manner, from Ω to a larger domain so that $u - K_h * u_h = O(h^{2r-2})$ on $\bar\Omega$ in both the L_2 and maximum norms. The extension u_h can be described as follows:

Ω_2 Γ_2	Ω_3 Γ_4	Ω_4
Ω_1	Γ_3 Γ_1 Ω Γ_5 Γ_7	Ω_5 Γ_6
Ω_8	Γ_8 Ω_7	Ω_6
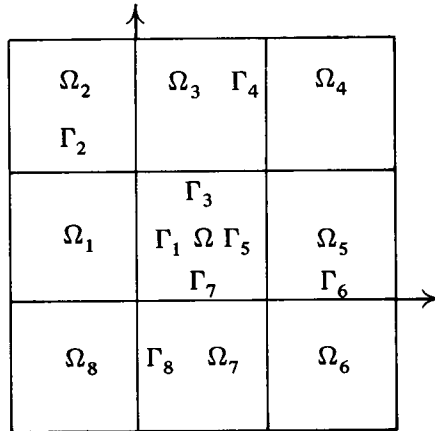
(i) Extend u_h from Ω to Ω_1, Ω_3, Ω_5 and Ω_7 as an odd function by reflection across the faces Γ_1, Γ_3, Γ_5 and Γ_7 respectively.

(ii) Then extend u_h to Ω_2, Ω_4, Ω_6 and Ω_8 as an odd function by reflection across the faces Γ_2, Γ_4, Γ_6 and Γ_8 respectively.

Notice that since k is either 0 or 1, $U_h \in \bar{M}_{k,r}^h(\Omega^E)$, where $\Omega^E = \{x \mid -1 < x_i < 2\}$.

THEOREM 3. *Let S_h^r and U_h be defined as above with $r \geqslant 3$. If h is sufficiently small and*

(i) $f \in \overset{\circ}{H}{}^{2r-4}(\Omega)$, *then*

(7.8) $$\|u - K_h * U_h\|_{0,\Omega} \leqslant C(h^{2r-2}\|f\|_{2r-4,\Omega}).$$

(ii) *If* $f \in \overset{\circ}{H}{}^{2r-2}(\Omega)$, *then*

(7.9) $$|u - K_h * U_h|_{0,\Omega} \leqslant C(h^{2r-2}\|f\|_{2r-2,\Omega}),$$

where C is independent of h and f.

Proof. Extend u to Ω^E in the same manner as U_h and call its extension U. Let Λ_i, $i = 1, 2, 3, 4$, be the squares

$$\Lambda_1 = \{x \mid -1 < x_1 < 1, 0 < x_2 < 2\}, \qquad \Lambda_2 = \{x \mid 0 < x_1 < 2, 0 < x_2 < 2\},$$

$$\Lambda_3 = \{x \mid -1 < x_1 < 1, -1 < x_2 < 1\}, \qquad \Lambda_3 = \{x \mid 0 < x_1 < 2, -1 < x_2 < 1\}.$$

We first claim that

(7.10) $$\sum_{j=1}^{2} \int_{\Lambda_i} \frac{\partial(U - U_h)}{\partial x_j} \frac{\partial \varphi}{\partial x_j}\, dx = 0, \quad \forall \varphi \in \bar{M}_k^h(\Lambda_i), i = 1, 2, 3, 4.$$

Note that $U_h \in \bar{M}_k^h(\Lambda_i)$ and if $f \in \overset{\circ}{H}{}^s(\Omega)$ ($s \geqslant 0$ an integer) then $U \in H^{s+2}(\Lambda_i) \cap \overset{\circ}{H}{}^1(\Lambda_i)$, $i = 1, 2, 3, 4$. Furthermore

(7.11) $$\|U\|_{s+2,\Lambda_i} \leqslant C\|f\|_{s,\Omega}, \quad i = 1, 2, 3, 4.$$

Before proving (7.10), let us show how (7.8) and (7.9) follow from it. Equation (7.10) says that U_h is the Ritz-Galerkin approximation in $\bar{M}_k^h(\Lambda_i)$ of U on each of the

domains Λ_i, $i = 1, 2, 3, 4$. Hence on each of these domains the interior estimates obtained in Theorems 1 and 2 are applicable. Let $\Omega_0^i \subset\subset \Lambda_i$, $i = 1, 2, 3, 4$, be chosen such that $\overline{\bigcup_{i=1}^4 \Omega_0^i}$ cover Ω. From (6.1) and (6.5) we have for h sufficiently small and $i = 1, 2, 3, 4$

(7.12) $\qquad \|U - K_h * U_h\|_{0, \Omega_0^i} \leqslant C(h^{2r-2}\|U\|_{2r-2, \Lambda_i} + \|U - U_h\|_{2-r, \Lambda_i})$

and

(7.13) $\qquad |U - K_h * U_h|_{0, \Omega_0^i} \leqslant C(h^{2r-2}\|U\|_{2r, \Lambda_i} + \|U - U_h\|_{2-r, \Omega^E})$.

It was shown in [15] that

(7.14) $\qquad\qquad \|U - U_h\|_{2-r, \Lambda_i} \leqslant Ch^{2r-2}\|U\|_{r, \Lambda_i}$.

The inequalities (7.8) and (7.9) now easily follow from (7.12), (7.13), (7.14), (7.11) and the fact that $\bigcup_{i=1}^4 \Omega_0^i$ covers Ω.

It remains to prove (7.10). We shall do so in the case $i = 1$, the other cases follow in the same manner. Let $\Lambda = \Omega \cup \Omega_1 \cup \Gamma_1$. We shall first show that

(7.15) $\qquad\qquad \sum_{i=1}^2 \int_\Lambda \frac{\partial(U - U_h)}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = 0, \quad \forall \varphi \in \bar{M}_{k,r}^h(\Lambda)$.

If $\varphi \in \bar{M}_{k,r}^h(\Lambda)$, then let $\varphi = \varphi^0 + \varphi^e$ where

$$\varphi^0(x_1, x_2) = \frac{\varphi(x_1, x_2) - \varphi(-x_1, x_2)}{2}$$

and

$$\varphi^e(x_1, x_2) = \frac{\varphi(x_1, x_2) + \varphi(-x_1, x_2)}{2}.$$

Now $\varphi^0 \in \bar{M}_{k,r}^h(\Omega_1)$ and $\bar{M}_{k,r}^h(\Omega)$, hence using (7.7)

$$\sum_{i=1}^2 \int_\Lambda \frac{\partial(U - U_h)}{\partial x_i} \frac{\partial \varphi^0}{\partial x_i} \, dx = 2 \sum_{i=1}^2 \int_\Omega \frac{\partial(U - U_h)}{\partial x_i} \frac{\partial \varphi^0}{\partial x} \, dx = 0.$$

Since φ^e is an even function of x_1 and $U - U_h$ is an odd function of x_1 it follows that

$$\sum_{i=1}^2 \int_\Lambda \frac{\partial(U - U_h)}{\partial x_i} \frac{\partial \varphi^e}{\partial x_i} \, dx = 0,$$

which proves (7.15).

Since U and U_h on Ω_2 and Ω_3 may be obtained from U and U_h on Λ by an odd reflection about the line $y = 1$, the argument given to prove (7.15) also shows that (7.10) holds which completes the proof.

Appendix. Here we present two tables. Table 1 gives values for the weights k_j' for various values of r. For a given choice of basis functions these may be used to calculate $K_h * u_h(x)$ at each point x or just at mesh points using (5.8). In Table 2 we give values of the weights d_j' for various values of r in the case where S_h^r is chosen to be the splines generated by the B-spline basis $\psi^{(r)}$.

TABLE 1. k_j', $l = r - 2$, $t = r - 1$

$j \backslash r$	3	4	5	6
0	13/12	37/30	346517/241920	76691/45360
1	−1/24	−23/180	−81329/322560	−48061/113400
2		1/90	6337/161280	20701/226800
3			− 3229/967680	−1573/113400
4				479/453600

TABLE 2. d_j'

$j \backslash r$	3	4	5	6
0	51/72	673/1080	$\dfrac{33055739}{58060800}$	$\dfrac{967,356,037}{1,828,915,200}$
1	11/72	4283/21600	$\dfrac{3589969}{16257024}$	$\dfrac{3,841,481,473}{16,460,236,800}$
2	−1/144	−61/5400	$\dfrac{-12162977}{1625702400}$	$\dfrac{31,253,191}{82,301,184,000}$
3		29/21600	$\dfrac{4795283}{2438553600}$	$\dfrac{48,179,483}{27,433,728,000}$
4		1/10800	$\dfrac{26273}{270950400}$	$\dfrac{89711}{514,382,400}$
5			$\dfrac{-58243}{812851200}$	$\dfrac{-2905789}{16,460,236,800}$
6			$\dfrac{-3229}{4877107200}$	$\dfrac{25867}{1,097,349,120}$
7				$\dfrac{117083}{82,301,184,000}$
8				479/164,602,368,000

Department of Mathematics
Cornell University
Ithaca, New York 14853

1. I. BABUŠKA, "The finite element method with Lagrangian multipliers," *Numer. Math.*, v. 20, 1973, pp. 179−192.   MR 50 #11806.

2. J. BRAMBLE, "A survey of some finite element methods proposed for treating the Dirichlet problem," *Advances in Math.*, v. 16, 1975, pp. 187−196.

3. J. H. BRAMBLE & S. HILBERT, "Bounds for a class of linear functionals with applications to Hermite interpolation," *Numer. Math.*, v. 16, 1971, pp. 362−369.

4. J. H. BRAMBLE, J. A. NITSCHE & A. H. SCHATZ, "Maximum-norm interior estimates for Ritz-Galerkin methods," *Math. Comp.*, v. 29, 1975, pp. 677−688.

5. J. H. BRAMBLE & J. E. OSBORNE, "Rate of convergence estimates for nonselfadjoint eigenvalue approximations," *Math. Comp.*, v. 27, 1973, pp. 525–549. MR 51 #2280.

6. J. H. BRAMBLE & A. H. SCHATZ, "Estimates for spline projection," *Rev. Française Automat. Informat. Recherche Opérationnelle Analyse Numérique*, v. 10, n° 8, août 1976, pp. 5–37.

7. J. H. BRAMBLE & A. H. SCHATZ, "Higher order local accuracy by averaging in the finite element method,"*Mathematical Aspects of Finite Elements in Partial Differential Equations* (Proc. Sympos., Math. Res. Center, Univ. of Wisconsin, Madison, 1974), edited by Carl de Boor, Academic Press, New York, 1974, pp. 1–14. MR 50 #1525.

8. J. H. BRAMBLE & M. ZLÁMAL, "Triangular elements in the finite element method," *Math. Comp.*, v. 24, 1970, pp. 809–820. MR 43 #8250.

9. C. de BOOR & B. SWARTZ, "Collocation at Gaussian points," *SIAM J. Numer. Anal.*, v. 10, 1973, pp. 582–606. MR 51 #9528.

10. J. DOUGLAS, JR. & T. DUPONT, "Some superconvergence results for Galerkin methods for the approximate solution of two point boundary problems." (Preprint.)

11. J. DOUGLAS, JR., T. DUPONT & M. F. WHEELER, "Some super-convergence results for an H^1-Galerkin procedure for the heat equation," *Computing Methods in Applied Sciences and Engineering*. Part I (Proc. Internat. Sympos., Versailles, 1973), Springer-Verlag, Berlin and New York, 1974, pp. 288–311. MR 49 #4197.

12. A. FRIEDMAN, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

13. J. L. LIONS & E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*. Vol. I, Springer-Verlag, Berlin and New York, 1972. MR 50 #2670.

14. J. A. NITSCHE, "Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind,"*Abh. Math. Sem. Univ. Hamburg*, v. 36, 1971, pp. 9–15. MR 49 #6649.

15. J. A. NITSCHE & A. H. SCHATZ, "Interior estimates for Ritz-Galerkin methods," *Math. Comp.*, v. 28, 1974, pp. 937–958. MR 51 #9525.

16. I. J. SCHOENBERG, "Contributions to the problem of approximation of equidistant data by analytic functions," Parts A, B, *Quart. Appl. Math.*,v. 4, 1946, pp. 45–99, 112–141. MR 7, 487; 8, 55.

17. V. THOMÉE, "Spline approximation and difference schemes for the heat equation," *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (Proc. Sympos., Univ. of Maryland, 1972), edited by A. K. Aziz, Academic Press, New York, 1972, pp. 711–746. MR 49 #11824.

18. V. THOMÉE & B. WENDROFF, "Convergence estimates for Galerkin methods for variable coefficient initial value problems," *SIAM J. Numer. Anal.*, v. 11, 1974, pp. 1059–1068. MR 51 #7309.