Higher order SUSY, linearized non-linear Heisenberg algebras and coherent states*

David J. Fernández C.† Véronique Hussin‡

CRM-2599

February 1999

*Submitted to Journal of Physics A

†Departamento de Física, CINVESTAV-IPN, A.P. 14-740, 07000 México D.F., Mexico

‡Département de Mathématiques et Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-Ville, Montréal (Québec) H3C 3J7, Canada

Abstract

Using an iterative construction of the first order intertwining technique, we find k-parametric families of exactly solvable anharmonic oscillators whose spectra consist of a part isospectral to the oscillator plus k additional levels at arbitrary positions below $E_0 = \frac{1}{2}$. It is seen that the 'natural' ladder operators for these systems give place to polynomial non-linear algebras, and it is shown that these algebras can be linearized. The coherent states construction is performed in the non-linear and linearized cases.

Keywords: Intertwining technique; Heisenberg algebra; Coherent states

PACS: 03.65.-w, 11.30.Pb, 42.50.Dv

Résumé

Nous trouvons, grâce à une construction itérative utilisant la technique d'intertwining du premier ordre, des familles k-paramétriques d'oscillateurs anharmoniques exactement résolubles dont le spectre consiste en une partie isospectrale à l'oscillateur harmonique et k niveaux additionnels à des positions arbitraires en dessous de $E_0 = \frac{1}{2}$. On montre aussi que les opérateurs d'échelles naturels de ces systèmes donnent lieu à des algèbres polynômiales non linéaires qui peuvent être linéarisées. La construction des états cohérents est réalisée dans les cas non linéaires et linéarisés.

1 Introduction

The generation of exactly solvable potentials using the well known factorization method, supersymmetric quantum mechanics (SUSY QM) and related subjects is becoming a paradigm in Schrödinger quantum mechanics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Nowadays it is realized that the majority of these procedures arise from a general setting in which a first order differential operator intertwines two Hamiltonians [5, 6, 21, 22, 23]. This so called first order intertwining technique (FOIT) suggests further generalizations, the most obvious one involving a k-th order differential intertwining operator. By expressing this operator as a sum of $k+1$ terms $f_i(x)d^i/dx^i, i = 0, \ldots, k$, introducing it in the intertwining relationship and solving the resulting system of equations for the $f_i(x)$'s, assuming that one of the Hamiltonians is solvable, a new solvable Hamiltonian and its eigenstates are generated [24, 25, 26].

There is an alternative to deal with the above problem: instead of looking for directly the k-th order operator one can make the construction by iterating k first order transformations. This last procedure can be implemented either by means of the well known determinant formulas (see e.g. [25] and references therein) or by the simple iterative construction that we have recently introduced [23]. From the side of explicit examples, it has been shown that our procedure works very well to generate k-parametric families of potentials almost isospectral to the harmonic oscillator, the radial hydrogen-like potentials, and in the free particle case [20, 23, 27].

A parallel development concerning coherent states (CS) for potentials derived by means of the intertwining technique is on the way [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. Thus, Fukui and Aizawa were able to derive those states for the Infeld and Hull potentials [28], i.e., for particular cases of the general families of potentials which can be derived by means of the FOIT. The first work involving CS for the simplest non-trivial family of potentials strictly isospectral to the oscillator (Abraham-Moses-Mielnik (AMM) [2, 3]) was done by us in 1994 [29]. Later on various developments going deep inside the subject have appeared [30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. Thus, Bagrov and Samsonov constructed the CS for a class of anharmonic oscillators with quasi-equidistant spectra composed by a part isospectral to the oscillator plus one level below the first excited state at a multiple of the spacing between the oscillator levels [34]. Almost simultaneously Aizawa and Sato have found some CS for the most general family of potentials almost isospectral to the oscillator that one is able to derive using the FOIT [37]. Those potentials arise if a new level, at any place below the ground state energy of the oscillator, is used for the generation process [6, 18, 23].

Of particular interest for this work is the realization that the ladder operator used to derive our CS in [29] and its adjoint give place to the so called non-linear algebras [18, 37, 40, 41, 42]. Notice that the non-linear algebra generated by the 'natural' ladder operators for the AMM potentials can be partially linearized [30]. This means that through appropriate modifications on those ladder operators one can reconstruct the Heisenberg-Weyl algebra restricted to the subspace spanned by the eigenstates intertwined directly to the oscillator eigenstates (see also [38]).

The goals of this paper are as diverse as the subjects mentioned above. In the first place, we want to illustrate how the iteration of k FOIT's works in order to generate k-parametric families of anharmonic oscillators almost isospectral to the oscillator (see section 2). We will follow [23] with slight modifications in notation in order to guarantee the most general results. In section 3 we will show that the 'natural' ladder operators for the k-th Hamiltonian, introduced by Mielnik in 1984 for $k = 1$ [3], lead to polynomial non-linear algebras of order $2k$, as Dubov, Eleonsky and Kulagin realized for the first time for $k = 1$ [40]. In section 4 we will discuss the linearization process for arbitrary k and its relationship with the distorted Heisenberg algebra introduced in [30]. In section 5 we will construct two sets of CS as eigenstates of the non-linear and linearized 'annihilation' operators with a discussion about advantages and disadvantages of both sets. We will finish in

section 6 with our conclusions and some comments on the literature.

2 k-th Order Intertwining Technique (k-SUSY)

Let us consider two Hamiltonians

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x), \quad H_1 = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x), \tag{2.1}$$

and suppose that there exist a first order differential operator $A_1^\dagger$ intertwining them

$$H_1 A_1^\dagger = A_1^\dagger H_0, \tag{2.2}$$

where

$$A_1^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \alpha_1(x,\epsilon)\right). \tag{2.3}$$

Thus, interelations between $\alpha_1,\ V_0,\ V_1$ and a *factorization energy* ϵ arise:

$$\alpha_1'(x,\epsilon) + \alpha_1^2(x,\epsilon) = 2(V_0(x) - \epsilon), \tag{2.4}$$

$$V_1(x) = V_0(x) - \alpha_1'(x,\epsilon). \tag{2.5}$$

Let us notice that (2.4)–(2.5) guarantee that H_0 and H_1 become factorized:

$$H_0 = A_1 A_1^\dagger + \epsilon, \qquad H_1 = A_1^\dagger A_1 + \epsilon, \tag{2.6}$$

where

$$A_1 = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \alpha_1(x,\epsilon)\right) \tag{2.7}$$

is the operator adjoint to $A_1^\dagger$.

Suppose now that $V_0(x)$ is a known solvable potential with eigenfunctions $\psi_n^{(0)}(x)$ and eigenvalues $E_n,\ n = 0,1,2,\ldots$ Furthermore, let us assume that we have found a solution $\alpha_1(x,\epsilon_1)$ to the Riccati equation (2.4) for a given value of the factorization energy $\epsilon = \epsilon_1 < E_0$, where E_0 is the ground state energy of H_0. Thus, the $V_1(x)$ of (2.5) is a completely specified solvable potential with normalized eigenfunctions:

$$\psi_{\epsilon_1}^{(1)}(x) \propto \exp\left(-\int_0^x \alpha_1(y,\epsilon_1)dy\right), \qquad \psi_n^{(1)}(x) = \frac{A_1^\dagger \psi_n^{(0)}(x)}{\sqrt{E_n - \epsilon_1}}, \tag{2.8}$$

and eigenvalues $\{\epsilon_1, E_n,\ n = 0,1,2,\ldots\}$. Let us remark that the restriction above, $\epsilon_1 < E_0$, is imposed in order to avoid the non-normalizability of the $\psi_n^{(1)}(x)$ of Eq. (2.8). This is also related with the possibility of avoiding the arising of singularities in $\alpha_1(x,\epsilon_1)$ which would enter into the new potential $V_1(x)$ of (2.5) and the eigenfunctions (2.8). For a detailed discussion of this point the reader can seek, e.g., the work of Sukumar [6]. By simplicity, here and throughout the paper we shall assume that the ground state energy of any new Hamiltonian generated by means of the FOIT is below the ground state energy of the initial Hamiltonian. We shall suppose as well that the arbitrary parameter of a general solution of an equation of the kind (2.4) for a fixed ϵ has been successfully adjusted in order to avoid the singularities in the α's.

We would like to iterate the previous technique, taking now $V_1(x)$ as the known solvable potential and trying to generate a new one $V_2(x)$ using an intertwining operator $A_2^\dagger$ and a different factorization

energy ϵ_2, with $\epsilon_2 < \epsilon_1$. The corresponding intertwining relationship, $H_2 A_2^\dagger = A_2^\dagger H_1$, leads to equations similar to (2.4)–(2.5):

$$\alpha_2'(x,\epsilon_2) + \alpha_2^2(x,\epsilon_2) = 2(V_1(x) - \epsilon_2), \tag{2.9}$$

$$V_2(x) = V_1(x) - \alpha_2'(x,\epsilon_2). \tag{2.10}$$

It is the matter of a substitution to show that we have a solution to (2.9) in form of a finite difference formula if we know the solutions $\alpha_1(x,\epsilon_1)$, $\alpha_1(x,\epsilon_2)$ to the Riccati equation (2.4) for two factorization energies $\epsilon_1,\ \epsilon_2$ and $V_1(x) = V_0(x) - \alpha_1'(x,\epsilon_1)$ (see [23]):

$$\alpha_2(x,\epsilon_2) = -\alpha_1(x,\epsilon_1) - 2\frac{(\epsilon_1 - \epsilon_2)}{\alpha_1(x,\epsilon_1) - \alpha_1(x,\epsilon_2)}. \tag{2.11}$$

Notice that a similar formula has been used by Adler in order to discuss the Backlund transformations of the Painlevé equations [43]. The eigenfunctions associated to $V_2(x)$ are given by:

$$\psi_{\epsilon_2}^{(2)}(x) \propto \exp\left(-\int_0^x \alpha_2(y,\epsilon_2)dy\right), \qquad \psi_{\epsilon_1}^{(2)}(x) = \frac{A_2^\dagger \psi_{\epsilon_1}^{(1)}(x)}{\sqrt{\epsilon_1 - \epsilon_2}}, \tag{2.12}$$

$$\psi_n^{(2)}(x) = \frac{A_2^\dagger \psi_n^{(1)}(x)}{\sqrt{E_n - \epsilon_2}} = \frac{A_2^\dagger A_1^\dagger \psi_n^{(0)}(x)}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}. \tag{2.13}$$

The corresponding eigenvalues are $\{\epsilon_2,\ \epsilon_1,\ E_n,\ n = 0,1,2,\ldots\}$.

It is clear that we can continue the iteration of the FOIT as many times as solutions for different values ϵ_i to the initial Riccati equation (2.4) we can get. If we know k of these, $\{\alpha_1(x,\epsilon_i),\ i = 1,2,\ldots,k,\ \epsilon_{i+1} < \epsilon_i\}$, we can iterate the process k times, and a new solvable Hamiltonian H_k will be gotten whose potential reads:

$$V_k(x) = V_{k-1}(x) - \alpha_k'(x,\epsilon_k) = V_0(x) - \sum_{i=1}^k \alpha_i'(x,\epsilon_i), \tag{2.14}$$

where $\alpha_i(x,\epsilon_i)$ is given by a recursive finite difference formula generalizing (2.11):

$$\alpha_{i+1}(x,\epsilon_{i+1}) = -\alpha_i(x,\epsilon_i) - 2\frac{(\epsilon_i - \epsilon_{i+1})}{\alpha_i(x,\epsilon_i) - \alpha_i(x,\epsilon_{i+1})}, \quad i = 1,\ldots,k-1. \tag{2.15}$$

The eigenfunctions are given by:

$$\psi_{\epsilon_k}^{(k)}(x) \propto \exp\left(-\int_0^x \alpha_k(y,\epsilon_k)dy\right), \tag{2.16}$$

$$\psi_{\epsilon_{k-1}}^{(k)}(x) = \frac{A_k^\dagger \psi_{\epsilon_{k-1}}^{(k-1)}(x)}{\sqrt{\epsilon_{k-1} - \epsilon_k}}, \tag{2.17}$$

$$\vdots$$

$$\psi_{\epsilon_1}^{(k)}(x) = \frac{A_k^\dagger \ldots A_2^\dagger \psi_{\epsilon_1}^{(1)}(x)}{\sqrt{(\epsilon_1 - \epsilon_2)\ldots(\epsilon_1 - \epsilon_k)}}, \tag{2.18}$$

$$\psi_n^{(k)}(x) = \frac{A_k^\dagger \ldots A_1^\dagger \psi_n^{(0)}(x)}{\sqrt{(E_n - \epsilon_1)\ldots(E_n - \epsilon_k)}}. \tag{2.19}$$

The corresponding eigenvalues are $\{\epsilon_i,\ E_n,\ i=k,\ldots,1,\ n=0,1,2,\ldots\}$.

In order to have the scheme complete, let us remember how the H_i's are intertwined:

$$H_i A_i^\dagger = A_i^\dagger H_{i-1}, \quad i=1,\ldots,k. \tag{2.20}$$

Thus, departing from H_0 we have generated a chain of factorized Hamiltonians:

$$H_i = A_i^\dagger A_i + \epsilon_i = A_{i+1}A_{i+1}^\dagger + \epsilon_{i+1}, \quad i=1,\ldots,k-1, \tag{2.21}$$

$$H_k = A_k^\dagger A_k + \epsilon_k, \tag{2.22}$$

where the end potential $V_k(x)$ can be recursively determined by means of (2.14)–(2.15) if we will be able to find k solutions $\alpha_1(x,\epsilon_i),\ i=1,\ldots,k$ to the Riccati equation (2.4), which means to have k non-equivalent factorizations of the initial Hamiltonian H_0:

$$H_0 = \frac{1}{2}\left(\frac{d}{dx} + \alpha_1(x,\epsilon_i)\right)\left(-\frac{d}{dx} + \alpha_1(x,\epsilon_i)\right) + \epsilon_i, \qquad i=1,\ldots,k. \tag{2.23}$$

Let us notice that there is a k-th order differential operator, $B_k^\dagger = A_k^\dagger \ldots A_1^\dagger$, intertwining the initial H_0 and final Hamiltonians H_k:

$$H_k B_k^\dagger = B_k^\dagger H_0. \tag{2.24}$$

From equation (2.19) we get:

$$B_k^\dagger \psi_n^{(0)}(x) = \sqrt{(E_n-\epsilon_1)\ldots(E_n-\epsilon_k)}\,\psi_n^{(k)}(x), \tag{2.25}$$

while from the adjoint to (2.24) it turns out that:

$$B_k \psi_n^{(k)}(x) = \sqrt{(E_n-\epsilon_1)\ldots(E_n-\epsilon_k)}\,\psi_n^{(0)}(x). \tag{2.26}$$

These equations are the key towards the k-th order supersymmetric quantum mechanics, k-SUSY by short [24, 25, 26]. In this formalism, a representation of the standard SUSY algebra [7] with two generators

$$[Q_i, H_{\rm ss}] = 0, \qquad \{Q_i, Q_j\} = \delta_{ij} H_{\rm ss}, \qquad i,j=1,2, \tag{2.27}$$

is constructed with the aid of B_k and $B_k^\dagger$:

$$Q = \begin{pmatrix} 0 & 0 \\ B_k & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & B_k^\dagger \\ 0 & 0 \end{pmatrix}, \tag{2.28}$$

$$H_{\rm ss} = \{Q, Q^\dagger\} = \begin{pmatrix} B_k^\dagger B_k & 0 \\ 0 & B_k B_k^\dagger \end{pmatrix} = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix}, \tag{2.29}$$

where $Q_1 = (Q^\dagger + Q)/\sqrt{2},\ Q_2 = (Q^\dagger - Q)/i\sqrt{2}$. The SUSY quasi-Hamiltonian $H_{\rm ss}$ is a k-th order polynomial

$$H_{\rm ss} = (H_{\rm s}^p - \epsilon_1)\ldots(H_{\rm s}^p - \epsilon_k), \tag{2.30}$$

of the physical Hamiltonian $H_{\rm s}^p$ involving the k-intertwined Hamiltonians H_0 and H_k:

$$H_{\rm s}^p = \begin{pmatrix} H_k & 0 \\ 0 & H_0 \end{pmatrix}. \tag{2.31}$$

If $k=1$ we will get the standard representation of the SUSY algebra (2.27), closely related with the factorization method [5, 6, 7, 8, 9, 10, 11, 12, 13]. If $k=2$ we will get the quadratic superalgebra, or SUSUSY QM [26], which has proved useful to show that the Witten index criterion not always characterizes spontaneous SUSY breaking [24].

The previous technique can be applied to the harmonic oscillator potential $V_0(x)=x^2/2$ if we will find solutions to equation (2.4) for some values of $\epsilon<1/2$. The first work for which a general solution to (2.4) was successfully used in order to generate a 1-parametric family of potentials isospectral to the oscillator was done by Mielnik for $\epsilon_1=-1/2$ [3]. That family had been derived previously by Abraham and Moses using the techniques of inverse scattering [2]. That is the reason because we have been referring to those potentials as the Abraham-Moses-Mielnik (AMM) family [2, 3]. Later on, Sukumar was able to find the most general solution to (2.4) with $V_0(x)=x^2/2$ and an arbitrary $\epsilon<1/2$ [6], and he generated new 1-parametric families of potentials having spectra $\{\epsilon, E_n=n+1/2, n=0,1,2,\ldots\}$. After that work, rediscoveries of either some particular cases or the full Sukumar results have been elaborated [18, 23]. Of our special interest is the reformulation of Sukumar results made by Junker and Roy, who have expressed the most general solution to (2.4) with $V_0(x)=x^2/2$ and an arbitrary $\epsilon<1/2$ in terms of confluent hypergeometric functions [18]:

$$\begin{aligned}\alpha_1(x,\epsilon) &= -x+\frac{d}{dx}\left\{\ln\left[{}_1F_1\left(\frac{1-2\epsilon}{4},\frac{1}{2};x^2\right)+2\nu\frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\,x\,{}_1F_1\left(\frac{3-2\epsilon}{4},\frac{3}{2};x^2\right)\right]\right\}\\ &= x+\frac{d}{dx}\left\{\ln\left[{}_1F_1\left(\frac{1+2\epsilon}{4},\frac{1}{2};-x^2\right)+2\nu\frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\,x\,{}_1F_1\left(\frac{3+2\epsilon}{4},\frac{3}{2};-x^2\right)\right]\right\},\end{aligned} \tag{2.32}$$

where, in order to avoid singularities in $\alpha_1(x,\epsilon)$, the domain of $\nu\in\mathrm{R}$ has to be restricted to $|\nu|<1$.

Suppose now that we choose k of these general solutions (2.32), associated to k fixed values of the factorization energies $\{\epsilon_i,\ i=1,\ldots,k,\ \epsilon_{i+1}<\epsilon_i\}$ and characterized by the k arbitrary constants $\{\nu_i, i=1,\ldots,k\}$. After the iteration of k FOIT's we will have generated a k-parametric family of solvable anharmonic potentials, labeled by the k parameters $\{\nu_i,\ i=1,\ldots,k\}$:

$$V_k(x)=\frac{x^2}{2}-\sum_{i=1}^{k}\alpha_i'(x,\epsilon_i). \tag{2.33}$$

The spectrum of the end Hamiltonian H_k, intertwined to the harmonic oscillator Hamiltonian by means of the operator $B_k^\dagger$, will be $\{\epsilon_i, E_n=n+1/2,\ i=k,\ldots,1,\ n=0,1,\ldots\}$, i.e., it consists of a part isospectral to the oscillator plus k additional levels $\epsilon_i,\ i=1,\ldots,k$ below $E_0=1/2$.

3 Non Linear Algebra of H_k

We are going to analyze the algebraic structure inherent to the Hamiltonians H_k and their corresponding potentials (2.33). As the spectrum of H_k has a part formed by equally spaced energies, it emerges the idea of looking for some ladder operators that would connect the eigenstates associated to those levels. There is a natural construction for a pair of these operators [3, 29, 30], which is guessed from equation (2.24), its adjoint and the standard intertwining relationship involving the oscillator Hamiltonian H_0 and its creation $a^\dagger$ and annihilation operator a:

$$(H_0-1)a^\dagger=a^\dagger H_0,\qquad (H_0+1)a=aH_0. \tag{3.1}$$

The construction is composed of three stages (see figure 1): i) first we 'move' the eigenvectors $|\psi_n^k\rangle$ of H_k, represented in the previous section by the wavefunctions $\psi_n^{(k)}(x)$, to the eigenvectors $|\psi_n^0\rangle$

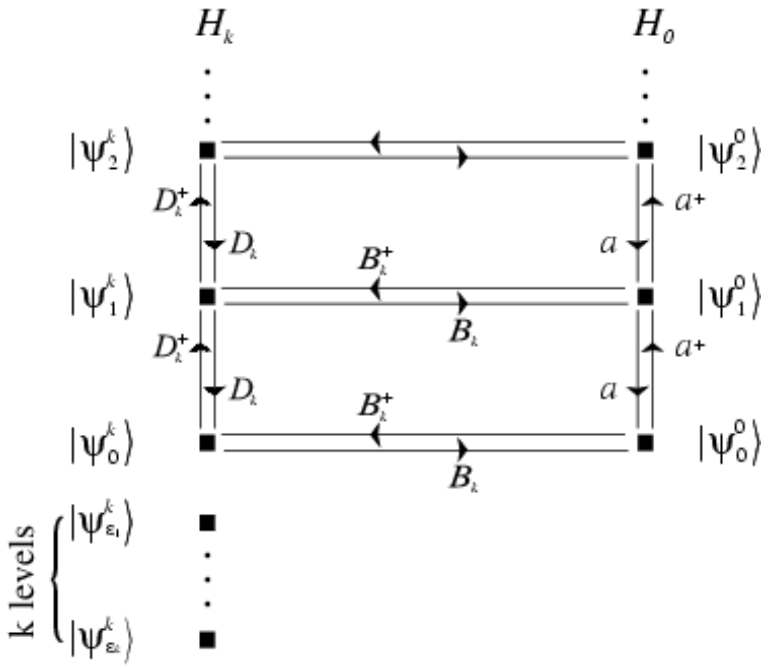

Figure 1: Schematic representation of the k-th order intertwining operators B_k, $B_k^\dagger$ and the ladder operators a, $a^\dagger$, D_k, $D_k^\dagger$ for the Hamiltonians H_0 and H_k.

of the oscillator Hamiltonian H_0 by means of the intertwining operator B_k. ii) Then, we move up ($|\psi_{n+1}^0\rangle$) or down ($|\psi_{n-1}^0\rangle$) on the ladder of H_0 by using $a^\dagger$ or a respectively, which will cause the effective 'motion' up or down on the ladder of H_k. iii) Finally, we come back to the ladder of H_k by acting $B_k^\dagger$ on $|\psi_{n+1}^0\rangle$ or $|\psi_{n-1}^0\rangle$. Thus, the 'natural' ladder operators for H_k can be chosen:

$$D_k = B_k^\dagger a B_k, \qquad D_k^\dagger = B_k^\dagger a^\dagger B_k, \qquad k = 0, 1, 2, \ldots \tag{3.2}$$

where, by completeness, we have extended the intertwining relationship (2.24) in order to include the case with $k = 0$ by assuming that $B_0^\dagger = B_0 = I$, I is the identity operator. The action of D_k and $D_k^\dagger$ is drawn just onto the points associated to $E_n = n + 1/2, n = 0, 1, \ldots$ because the k isolated eigenstates $\{|\psi_{\epsilon_i}^k\rangle,\ i = 1, \ldots, k\}$ are annihilated by both D_k and $D_k^\dagger$ due to the fact that they are annihilated by B_k.

The ladder operators D_k and $D_k^\dagger$ are differential operators of order $(2k+1)$-th satisfying:

$$[H_k, D_k] = -D_k, \qquad [H_k, D_k^\dagger] = D_k^\dagger. \tag{3.3}$$

Following the works on the non-linear generalization of the Fock method made by researchers at the Lukin Institute [40, 41, 42] (see also [44, 45, 46]), it is introduced the Hermitian operator $N(H_k) \equiv D_k^\dagger D_k$ generalizing the standard number operator N of the harmonic oscillator. It can be easily shown that $N(H_k)$ is a polynomial in H_k of $(2k+1)$-th order:

$$N(H_k) \equiv D_k^\dagger D_k = \left(H_k - \frac{1}{2}\right) \prod_{i=1}^{k} (H_k - \epsilon_i - 1)\,(H_k - \epsilon_i)\,, \tag{3.4}$$

and

$$D_k D_k^\dagger = N(H_k + 1) = \left(H_k + \frac{1}{2}\right) \prod_{i=1}^{k} (H_k - \epsilon_i)\,(H_k - \epsilon_i + 1)\,. \tag{3.5}$$

Thus, the operators $D_k,\ D_k^\dagger$ and H_k close a polynomial non-linear algebra or order $2k$:

$$[D_k, D_k^\dagger] = N(H_k+1) - N(H_k) = P_k(H_k)\prod_{i=1}^{k}(H_k - \epsilon_i), \tag{3.6}$$

where $P_k(x)$ is a polynomial in x of order k of the form:

$$\begin{aligned} P_k(x) =& (x+\frac{1}{2})\prod_{i=1}^{k}(x-\epsilon_i+1) - (x-\frac{1}{2})\prod_{i=1}^{k}(x-\epsilon_i-1) \\ =& \sum_{j=0}^{k}(-1)^j x^{k-j}\sum_{l=0}^{[\frac{j}{2}]}\frac{2(k-j+l)+1}{2l+1}\binom{k-j+2l}{2l}\sum \epsilon_{i_1 i_2 \ldots i_{j-2l}} \end{aligned} \tag{3.7}$$

and we have used the compact notation

$$\sum \epsilon_{i_1 i_2 \ldots i_j} = \begin{cases} 0 & \text{if } j<0 \\ 1 & \text{if } j=0 \\ \sum_{\substack{i_1<\ldots<i_j \\ i_1,\ldots,i_j=1}}^{k} \epsilon_{i_1}\ldots\epsilon_{i_j} & \text{if } j>0 \end{cases} \tag{3.8}$$

By completeness, the anticommutator is written below:

$$\{D_k, D_k^\dagger\} = N(H_k+1) + N(H_k) = Q_k(H_k)\prod_{i=1}^{k}(H_k-\epsilon_i)\,, \tag{3.9}$$

where

$$\begin{aligned} Q_k(x) =& (x+\frac{1}{2})\prod_{i=1}^{k}(x-\epsilon_i+1) + (x-\frac{1}{2})\prod_{i=1}^{k}(x-\epsilon_i-1) \\ =& 2x^{k+1} - 2\left(\sum_{i=1}^{k}\epsilon_i\right)x^k + \sum_{j=1}^{k-1}(-1)^{j+1}x^{k-j}\Bigg[2\sum\epsilon_{i_1\ldots i_{j+1}} + (k-j+1)^2\sum\epsilon_{i_1\ldots i_{j-1}} \\ &+ \sum_{l=1}^{[\frac{j+1}{2}]}\frac{k-j+l+1}{l+1}\binom{k-j+2l+1}{2l+1}\sum\epsilon_{i_1\ldots i_{j-2l-1}}\Bigg] \\ &+ (-1)^{k+1}\sum_{l=0}^{[\frac{k-1}{2}]}\sum\epsilon_{i_1\ldots i_{k-2l-1}}. \end{aligned} \tag{3.10}$$

By consistency, when $k=0$ we should get the standard Heisenberg-Weyl algebra because $D_0 = a$ and $D_0^\dagger = a^\dagger$. This linear case is indeed recovered from our formulae due to the fact that $P_0(H_0) = I$ and $Q_0(H_0) = 2H_0$, which implies that

$$[H_0, D_0] = -D_0, \qquad [H_0, D_0^\dagger] = D_0^\dagger, \qquad [D_0, D_0^\dagger] = P_0(H_0) = I. \tag{3.11}$$

The corresponding Fock operator becomes the standard linear expression in terms of the oscillator Hamiltonian H_0:

$$N(H_0) = H_0 - \frac{1}{2} = N. \tag{3.12}$$

On the other hand, when $k=1$ and ϵ_1 is arbitrary we recover the expression of Aizawa and Sato for $[D_1, D_1^\dagger]$, i.e., it arises a quadratic algebra [37] (see also [40, 41, 42]):

$$[D_1, D_1^\dagger] = (H_1 - \epsilon_1)(3H_1 - \epsilon_1). \tag{3.13}$$

The Fock operator becomes now cubic in H_1:

$$N(H_1) = \left(H_1 - \frac{1}{2}\right)(H_1 - \epsilon_1)\,(H_1 - \epsilon_1 - 1)\,. \tag{3.14}$$

If $k=2$ we will get a polynomial algebra of order 4:

$$[D_2, D_2^\dagger] = (H_2 - \epsilon_1)(H_2 - \epsilon_2)\left[5H_2^2 - 3(\epsilon_1+\epsilon_2)H_2 + \epsilon_1\epsilon_2 + 1\right], \tag{3.15}$$

and a polynomial of order 5-th for $N(H_2)$:

$$N(H_2) = \left(H_2 - \frac{1}{2}\right)(H_2 - \epsilon_1)\,(H_2 - \epsilon_2)\,(H_2 - \epsilon_1 - 1)\,(H_2 - \epsilon_2 - 1)\,. \tag{3.16}$$

For general k, it arises a polynomial non-linear algebra of order $2k$ whose properties are characterized by the $(2k+1)$-th order polynomial $N(H_k)$ of (3.4).

Let us notice that the polynomial algebras (3.3)–(3.6) are particular cases of the W_{2k+1} algebras [18, 47, 48, 49, 50], and they have been related to the $W_{1+\infty}$ algebras for $k=1$ [37]. As already mentioned, they represent also concrete realizations of the generalized Fock method introduced at the beginning of the 90's [40, 41, 42] (some of these ideas can be found in previous works [44, 45]). In order to clarify some points, let us mention some facts of that method which will be useful for our treatment.

Suppose that relations (3.3)–(3.4) involve more general ladder operators E^+, E^- and a Hamiltonian H. If it is assumed that E^+ and E^- are differential operators of order $(2k+1)$-th, we could obtain thus potentials whose spectra would consist of at most $(2k+1)$ superposed ladders because the generalized number operator would be a polynomial of order $(2k+1)$-th:

$$[H, E^+] = E^+, \quad [H, E^-] = -E^-, \quad N(H) = E^+E^- = \prod_{i=1}^{2k+1}(H - r_i)\,, \tag{3.17}$$

where it is assumed that all the roots $\{r_i, i = 1, \ldots, 2k+1\}$ of $N(H)$ are real. The number and length of the ladders depends on the properties of the Kernel of E^-, i.e., of the solutions to the $(2k+1)$-th order linear differential equation:

$$E^-\psi = 0. \tag{3.18}$$

Suppose that there exist m square integrable linearly independent solutions of (3.18). Due to the fact that:

$$E^+E^-\psi = \prod_{i=1}^{2k+1}(H - r_i)\,\psi = 0, \tag{3.19}$$

we can choose m square integrable linear combinations of such solutions, $\psi_i^g(x)$, orthogonal to each other and which are simultaneously eigenfunctions of H with eigenvalues $r_i, i = 1, \ldots, m$. If there is no any special values for the other $k-m$ roots $r_i, i = m+1, \ldots, k$, then the spectrum of H will consists of m infinite ladders with spacing $\Delta E = 1$, each one of them starting from one of the r_i's, $i = 1, \ldots, m$. It could happens, however, that after applying l times the operator E^+ onto some of

the ground states, let us say the j-th one, we would have that $(E^+)^l\psi_j^g \neq 0$ but $(E^+)^{l+1}\psi_j^g = 0$. As $H[(E^+)^l\psi_j^g] = (r_j + l)\psi_j^g$, we will have:

$$0 = E^-(E^+)^{l+1}\psi_j^g = \prod_{i=1}^{2k+1}(H - r_i + 1)\,[(E^+)^l\psi_j^g] = \prod_{i=1}^{2k+1}(r_j + l - r_i + 1)\,[(E^+)^l\psi_j^g]. \tag{3.20}$$

This means that one of the r_i's, $i = m+1, \ldots, k$ has to be of the form $r_i = r_j + l + 1, l \geq 0$. If this happens, instead of having m infinite ladders we will have just $m-1$ infinite and a finite one of lenght $l+1$, which will start from r_j and will end in $r_j + l, l \geq 0$.

By comparing this ideas with our k-SUSY treatment it is clear now why the roots of the polynomial (3.4) are precisely $\{1/2, \epsilon_i, \epsilon_i + 1, i = 1, \ldots, k\}$: our k-SUSY Hamiltonians H_k have precisely $k+1$ ground states associated to the $k+1$ roots (eigenvalues) $\{1/2, \epsilon_i, i = 1, \ldots, k\}$. As the ladder starting of $1/2$ is infinite, this does not impose any restriction to the other k roots of the $(2k+1)$-th order polynomial (3.4). However, as the ladders starting of ϵ_i are finite of lenght equal to 1 (they end at the initial energy ϵ_i), the other k roots have to be precisely of the form $\epsilon_i + 1, i = 1, \ldots, k$, as in our polynomial (3.4).

An interesting point concerning the non-linear nature of the polynomial algebras (3.3)–(3.6) in the standard SUSY case (with $k = 1$) is that they can be linearized [30]. We shall show next that the same procedure can be implemented for arbitrary k.

4 Linearization of the Non-linear Algebra of H_k

As pointed out at sections 2 and 3, the k isolated eigenstates $|\psi_{\epsilon_i}^k\rangle, i = 1, \ldots, k$ of H_k are disconnected between themselves and of the ones associated to the part of the spectrum isospectral to H_0. Hence, it is natural to perform the linearization on the subspace spanned by $\{|\psi_n^k\rangle, n = 0, 1, 2, \ldots\}$. The essence of this procedure, introduced in [30] for $k = 1$, is to modify the ladder operators D_k and $D_k^\dagger$ of (3.2) in order to construct an algebraic structure similar to the Heisenberg-Weyl algebra. As for the sub-basis $\{|\psi_n^k\rangle, n = 0, 1, \ldots\}$ the commutator $[D_k, D_k^\dagger]$ is already diagonal (see equation (3.6)), we should make a modification that would not change $[H_k, D_k] = -D_k$ and $[H_k, D_k^\dagger] = D_k^\dagger$ but would convert most of the diagonal elements of $[D_k, D_k^\dagger]$ to 1. With this aim, we propose two new ladder operators D_L and $D_L^\dagger$ in the form:

$$D_L = B_k^\dagger f(N) a B_k, \qquad D_L^\dagger = B_k^\dagger a^\dagger f(N) B_k, \tag{4.1}$$

where $N = a^\dagger a$ is the standard number operator of equation (3.12), $f(x)$ is a real function to be determined, and the subscript L indicates linearization. We ask that $[D_L, D_L^\dagger] = I$ on the subspace spanned by $\{|\psi_n^k\rangle, n = 1, 2, \ldots\}$, which will be denoted $\mathcal{H}_{\geq 1}$. Notice that we leave open the possibility that $[D_L, D_L^\dagger]|\psi_0^k\rangle = c|\psi_0^k\rangle$, $c \in \mathrm{R}$, $c \neq 1$. Recently, Seshadri et.al. have relaxed further this possibility for $k = 1$ by asking that $[D_L, D_L^\dagger]$ takes arbitrary independent values on $|\psi_n^k\rangle, n = 0, 1, 2, \ldots$ when H_1 is isospectral to the oscillator [38]. In this paper we will restrict to the simplest variant of the linearization, which coincides with the assumptions made initially.

Making use of equations (2.25)-(2.26) and (3.1) it is easy to show that

$$[D_L, D_L^\dagger]|\psi_n^k\rangle = [g(n+1) - g(n)]|\psi_n^k\rangle, \tag{4.2}$$

where

$$g(n) = \left[\prod_{i=1}^{k}\left(n - \epsilon_i - \frac{1}{2}\right)\left(n - \epsilon_i + \frac{1}{2}\right)\right][f(n-1)]^2\, n. \tag{4.3}$$

As we are asking that $[D_L, D_L^\dagger] = I$ on $\mathcal{H}_{\geq 1}$, we end up with the following finite difference equation:

$$g(n+1) - g(n) = 1, \quad n = 1, 2, \ldots \tag{4.4}$$

whose general solution is given by:

$$g(n) = n + w(n), \tag{4.5}$$

where $w(n)$ is periodic with period 1, $w(n+1) = w(n), n = 1, 2, \ldots$ Hence:

$$f(n-1) = \sqrt{\frac{n + w(n)}{n \prod_{i=1}^{k} (n - \epsilon_i - \frac{1}{2})(n - \epsilon_i + \frac{1}{2})}}. \tag{4.6}$$

As $w(n)$ takes the same value for all $n = 1, 2, \ldots$, it is important just $w \equiv w(1)$. Moreover, as $f(n-1)$ should be real $\Rightarrow w \geq -1$. Collecting all this information, we arrive finally to the ladder operators we were looking for:

$$D_L = B_k^\dagger \sqrt{\frac{N + 1 + w}{(N+1) \prod_{i=1}^{k} (N - \epsilon_i + \frac{1}{2})(N - \epsilon_i + \frac{3}{2})}}\, a B_k, \tag{4.7}$$

$$D_L^\dagger = B_k^\dagger a^\dagger \sqrt{\frac{N + 1 + w}{(N+1) \prod_{i=1}^{k} (N - \epsilon_i + \frac{1}{2})(N - \epsilon_i + \frac{3}{2})}}\, B_k. \tag{4.8}$$

Although apparently more complicated than the D_k and $D_k^\dagger$ of the non-linear algebra (see (3.2)), D_L and $D_L^\dagger$ act simpler than those operators on the energy eigenstates $|\psi_n^k\rangle, n = 0, 1, \ldots$ (excepting the case with $k = 0$ which is discussed at the end of this section):

$$D_L |\psi_n^k\rangle = (1 - \delta_{n0}) \sqrt{n + w}\, |\psi_{n-1}^k\rangle, \tag{4.9}$$

$$D_L^\dagger |\psi_n^k\rangle = \sqrt{n + w + 1}\, |\psi_{n+1}^k\rangle, \tag{4.10}$$

$$[D_L, D_L^\dagger] |\psi_n^k\rangle = (1 + w\delta_{n0}) |\psi_n^k\rangle. \tag{4.11}$$

Contrary to what happens with D_k and $D_k^\dagger$, this action is independent of k, i.e., of the number of iterations of the FOIT's needed to go from H_0 to H_k. Thus, this kind of linearization gives place to a universal representation of the algebra characteristic of any solvable Hamiltonian intertwined to the harmonic oscillator through the iteration of k FOIT's. As we can see, we have constructed once again the 'distorted' Heisenberg algebra introduced some time ago to linearize the non-linear algebra of order 2 characteristic of the AMM potentials, where $w \geq -1$ is the distortion parameter [30]. Here we have shown that this algebra is also the quasi-linearized version of the non-linear algebras of order $2k$ if we restrict ourselves to $\mathcal{H}_{\geq 1}$ and w is left arbitrary. If we want a 'complete' linearization on $\mathcal{H}_{\geq 0}$ (the subspace spanned by $\{|\psi_n^k\rangle, n = 0, 1, \ldots\}$), we should take $w = 0$ in order to get precisely the Heisenberg-Weyl algebra:

$$D_L |\psi_n^k\rangle = \sqrt{n} |\psi_{n-1}^k\rangle, \quad D_L^\dagger |\psi_n^k\rangle = \sqrt{n+1} |\psi_{n+1}^k\rangle, \quad [D_L, D_L^\dagger] |\psi_n^k\rangle = |\psi_n^k\rangle. \tag{4.12}$$

Let us notice that if $w = -1$ we will get once again the Heisenberg-Weyl algebra on $\mathcal{H}_{\geq 1}$, but now the state $|\psi_0^k\rangle$ will be annihilated by D_L and $D_L^\dagger$. In this way we can isolate *by hand* $|\psi_0^k\rangle$ of the rest of eigenstates of H_k; this isolation is *natural* for the other k eigenstates $|\psi_{\epsilon_i}^k\rangle, i = 1, \ldots, k$.

The curious case with $k = 0$ is worth of discussion. The intertwining in this case is trivial: each eigenstate of the oscillator is mapped into itself without creating any new level because $B_0 = B_0^\dagger = I$. The quasi-linearization introduced above for w arbitrary can be seen as a distortion of

the representation of the Heisenberg-Weyl algebra which changes the operators $a,\ a^\dagger$ into $D_L,\ D_L^\dagger$ by changing the values of the non-null matrix elements of a and $a^\dagger$ in the basis $|\psi_n^0\rangle$ but without changing the diagonal elements of $[a, a^\dagger]$ in the same basis excepting the one associated to $|\psi_0^0\rangle$, which becomes equal to $w+1$. This is clear from the explicit expressions of D_L and $D_L^\dagger$ for $k=0$:

$$D_L = \sqrt{\frac{N+1+w}{N+1}}\, a, \qquad D_L^\dagger = a^\dagger \sqrt{\frac{N+1+w}{N+1}}. \tag{4.13}$$

Notice once again that when $w=0$ we recover the original Heisenberg-Weyl algebra because in this case $D_L = a,\ D_L^\dagger = a^\dagger$. Moreover, when $w=-1$ we will get a reduced reducible representation becoming the Heisenberg-Weyl algebra representation on $\mathcal{H}_{\geq 1}$ and the null representation on the subspace generated by $|\psi_0^0\rangle$ because this state is annihilated by D_L and $D_L^\dagger$.

5 Coherent States of H_k

The beautiful properties of the coherent states for the harmonic oscillator motivated the interest in looking for them in other physical situations [51, 52, 53, 54, 55]. It is well known that there are various definitions, each one of them leading to sets of CS with, in general, different properties. Concerning the intertwining technique, CS which are eigenstates of certain annihilation operator for the potentials of the Infeld and Hull classification [1] were derived by Fukui and Aizawa [28]. As is well known, however, those potentials are particular cases of the general families which can be generated by means of the intertwining technique. The first set of CS associated to a full family of potentials generated in this way was derived by ourselves as eigenstates of the annihilation operator D_k of (3.2) in the case with $k=1$ for the AMM family of potentials isospectral to the oscillator [29]. Soon after, the linearization process for the same family of potentials was performed, as presented in section 4, and the corresponding CS derivation was also elaborated [30]. Since then, a lot of works have arisen looking for interrelations between CS and quantum groups, pseudodifferential operators, non-linear algebras, etc [31, 32, 33, 34, 35, 36, 37, 38, 39].

Here, we will look for the CS as eigenstates of the 'annihilation' operators D_k and D_L of the previous sections. First, let us determine the CS which are eigenstates of D_k (the non-linear case):

$$D_k|z\rangle = z|z\rangle, \quad z \in \mathrm{C}. \tag{5.1}$$

As usual, we express $|z\rangle$ as a linear combination of the subset of eigenstates $|\psi_n^k\rangle$ of H_k associated to the part of the spectrum isospectral to the oscillator:

$$|z\rangle = \sum_{n=0}^{\infty} c_n |\psi_n^k\rangle. \tag{5.2}$$

After inserting (5.2) in (5.1) we will get a recurrence relationship for the coefficients c_n

$$c_{n+1} = \frac{z}{\sqrt{(n+1)\prod_{i=1}^k (n-\epsilon_i+\frac{1}{2})(n-\epsilon_i+\frac{3}{2})}}\, c_n, \tag{5.3}$$

and all of them can be expressed in terms of c_0, which is fixed by the normalization condition $\langle z|z\rangle = 1$ and the requirement that $c_0 \in \mathrm{R}^+$. Hence, these CS become:

$$|z\rangle = \sum_{n=0}^{\infty} \frac{\sqrt{\prod_{i=1}^k \Gamma(-\epsilon_i+\frac{1}{2})\Gamma(-\epsilon_i+\frac{3}{2})}\ z^n |\psi_n^k\rangle}{\sqrt{n!\,{}_0F_{2k}(-\epsilon_1+\frac{1}{2},\ldots,-\epsilon_k+\frac{1}{2},-\epsilon_1+\frac{3}{2},\ldots,-\epsilon_k+\frac{3}{2};r^2)\prod_{i=1}^k \Gamma(n-\epsilon_i+\frac{1}{2})\Gamma(n-\epsilon_i+\frac{3}{2})}} \tag{5.4}$$

where $\Gamma(x)$ is the gamma function, $r=|z|$, and ${}_pF_q$ is a generalized hypergeometric function:

$$ {}_pF_q(a_1,\ldots,a_p,b_1,\ldots,b_q;x)=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)}{\Gamma(b_1+n)\ldots\Gamma(b_q+n)}\frac{x^n}{n!}. \tag{5.5}$$

Notice that $z=0$ is a $(k+1)$-th degenerate eigenvalue of D_k because of (5.4) we see that $|z=0\rangle = |\psi_0^k\rangle$ while $D_k|\psi_{\epsilon_i}^k\rangle = 0,\ i=1,\ldots,k$ because the $|\psi_{\epsilon_i}^k\rangle$ are isolated of the other eigenstates. Thus, the resolution of the identity should be looked for as:

$$ I=\sum_{i=1}^{k}|\psi_{\epsilon_i}^k\rangle\langle\psi_{\epsilon_i}^k| + \int |z\rangle\langle z| d\mu(z), \tag{5.6}$$

where $d\mu(z)$ is to be determined. Suppose now that

$$ d\mu(z)={}_0F_{2k}\left(-\epsilon_1+\frac{1}{2},\ldots,-\epsilon_k+\frac{1}{2},-\epsilon_1+\frac{3}{2},\ldots,-\epsilon_k+\frac{3}{2};r^2\right)h(r^2)rdrd\varphi. \tag{5.7}$$

Inserting this equation in (5.6) and using the fact that $\{|\psi_{\epsilon_i}^k\rangle, |\psi_n^k\rangle,\ i=1,\ldots,k,\ n=0,1,\ldots\}$ is complete, we arrive at the following requirement for $h(x)$:

$$ \int_0^{\infty}x^n h(x)dx=\frac{\Gamma(n+1)\prod_{i=1}^{k}\Gamma(n-\epsilon_i+\frac{1}{2})\Gamma(n-\epsilon_i+\frac{3}{2})}{\pi\prod_{i=1}^{k}\Gamma(-\epsilon_i+\frac{1}{2})\Gamma(-\epsilon_i+\frac{3}{2})}. \tag{5.8}$$

Hence, $h(x)$ is the inverse Mellin transform of the RHS of (5.8). It turns out that $h(x)$ is proportional to a Meijer G-function [56]:

$$ h(x)=\frac{G^{\,2k+1\ 0}_{\,0\ 2k+1}(x|0,-\epsilon_1-\frac{1}{2},\ldots,-\epsilon_k-\frac{1}{2},-\epsilon_1+\frac{1}{2},\ldots,-\epsilon_k+\frac{1}{2})}{\pi\prod_{i=1}^{k}\Gamma(-\epsilon_i+\frac{1}{2})\Gamma(-\epsilon_i+\frac{3}{2})}. \tag{5.9}$$

Let us notice that in the case $k=1$ and $\epsilon_1=-1/2$ it reduces to the result we have derived in [29], which was expressed in a more compact form recently by Cannata et.al. for an arbitrary $\epsilon_1<1/2$ [39].

Some other properties of the standard coherent states have their analogue for ours. For instance, any CS of the form (5.4) can be expressed in terms of the others:

$$ |z'\rangle=\int|z\rangle\langle z|z'\rangle d\mu(z), \tag{5.10}$$

where the reproducing Kernel $\langle z|z'\rangle$ can be easily evaluated:

$$\begin{aligned}\langle z|z'\rangle = {}_0F_{2k}(-\epsilon_1+\tfrac{1}{2},\ldots,-\epsilon_k+\tfrac{1}{2},-\epsilon_1+\tfrac{3}{2},\ldots,-\epsilon_k+\tfrac{3}{2};\bar{z}z')\\ \times {}_0F_{2k}(-\epsilon_1+\tfrac{1}{2},\ldots,-\epsilon_k+\tfrac{1}{2},-\epsilon_1+\tfrac{3}{2},\ldots,-\epsilon_k+\tfrac{3}{2};r^2)^{-1/2}\\ \times {}_0F_{2k}(-\epsilon_1+\tfrac{1}{2},\ldots,-\epsilon_k+t\frac{1}{2},-\epsilon_1+\tfrac{3}{2},\ldots,-\epsilon_k+\tfrac{3}{2};r'^2)^{-1/2}\end{aligned} \tag{5.11}$$

meaning that any two CS $|z\rangle$ and $|z'\rangle$ of (5.4) are non-orthogonal. From the resolution of the identity it is clear that any state vector can be expressed in terms of our CS if we include the atypical orthogonal CS $|\psi_{\epsilon_i}^k\rangle,\ i=1,\ldots,k$ naturally inherent to this treatment.

Let us evaluate now the coherent states associated to the linearized annihilation operator D_L of (4.7). Similarly as in the previous case, we look for states $|z,w\rangle$ such that:

$$D_L|z,w\rangle = z|z,w\rangle, \tag{5.12}$$

where we are showing explicitly the CS dependence on the distortion parameter w. Following the same procedure as before, we arrive at the final expression for $|z,w\rangle$:

$$|z,w\rangle = \sqrt{\frac{\Gamma(w+1)}{{}_1F_1(1,w+1;r^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\Gamma(n+w+1)}}|\psi_n^k\rangle. \tag{5.13}$$

Once again, the resolution of the identity becomes similar to (5.6):

$$I = \sum_{i=1}^{k}|\psi_{\epsilon_i}^k\rangle\langle\psi_{\epsilon_i}^k| + \int |z,w\rangle\langle z,w|d\mu_L(z), \tag{5.14}$$

where

$$d\mu_L(z) = \sigma(r,w)rdrd\varphi, \qquad \sigma(r,w) = \frac{{}_1F_1(1,w+1;r^2)}{\pi\Gamma(w+1)}e^{-r^2}r^{2w}. \tag{5.15}$$

The reproducing Kernel is now

$$\langle z,w|z',w\rangle = \frac{{}_1F_1(1,w+1;\bar{z}z')}{\sqrt{{}_1F_1(1,w+1;r^2){}_1F_1(1,w+1;r'^2)}}. \tag{5.16}$$

Let us notice that the CS (5.13) can be gotten from the ones of [30] by making $w \to w+1$. In the case $w=0$ (the full linearized case) the same formulae as for the standard coherent states are recovered by noticing that ${}_1F_1(1,1;r^2) = e^{r^2}$. By taking carefully the limit $w \to -1$ (the full linearized case once again) it can be shown that the standard expression for the CS is also recovered, but the eigenstate $|\psi_0^k\rangle$ associated to the eigenvalue $E_0 = 1/2$ will be isolated of the other ones, i.e., the series (5.13) will start from $|\psi_1^k\rangle$ [30].

A comparison of the 'annihilation' operators D_k and D_L and of both sets of coherent states derived in this section shows the following: from the side of their explicit expressions, the non-linear operator D_k is simpler than the linearized one D_L. As can be seen of equations (5.4) and (5.13), however, the CS associated to D_L are much simpler that the ones associated to D_k, which is due to the simplest algebra generated by D_L and $D_L^\dagger$. In order to give more support to this conclusion, let us compare the uncertainty product $(\Delta x)(\Delta p)$ for both sets of CS. As for $k=1$ such a comparison has been already performed for the AMM family of potentials isospectral to the oscillator in the oscillator limit [29, 30], taking $k=1$, $\epsilon_1 = -1/2$ and $\nu_1 = 0$ in the potentials (2.32)–(2.33), we shall stick just to an analogue situation in the case with $k=2$. Thus, by taking $\epsilon_1 = -1/2$, $\epsilon_2 = -3/2$ and labeling as ν_1 and ν_2 the parameters of the corresponding solutions (2.32), we will get once again, up to a displacement of the energy origin, the 2-parametric family of potentials (2.33) isospectral to the oscillator recently derived [26], where in order to avoid singularities we have to make the restrictions $|\nu_1| < 1$ and $|\nu_2| > 1$. In order to pick out the oscillator potential, we have to take $\nu_1 = 0$ and $\nu_2 \to \infty$. Assuming all this, we arrive finally to the two sets of coherent states which will be compared:

$$|z\rangle_{NL} = \sqrt{\frac{2}{{}_0F_4(1,2,2,3;r^2)}}\sum_{n=0}^{\infty}\frac{z^n}{n!(n+1)!\sqrt{(n+2)!}}|\psi_{n+2}^0\rangle, \tag{5.17}$$

$$|z,w\rangle_L = \sqrt{\frac{\Gamma(w+1)}{2\,{}_1F_1(1,w+1;r^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\Gamma(n+w+1)}}|\psi_{n+2}^0\rangle. \tag{5.18}$$

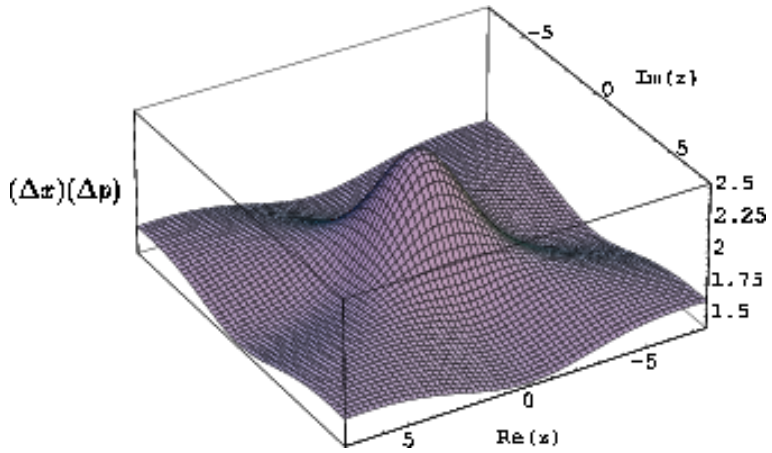

Figure 2: The uncertainty product $(\Delta x)(\Delta p)$ as function of z for the non-linear coherent states (5.17) associated to the member of the 2-parametric family of potentials isospectral to the oscillator arising for $k = 2$, $\epsilon_1 = -1/2$, $\epsilon_2 = -3/2$, $\nu_1 = 0$, $\nu_2 \to \infty$.

Above, the subscripts NL and L mean non-linear and linear respectively. A direct calculation leads to the uncertainties Δx and Δp in the non-linear case:

$$\Delta x = \sqrt{\frac{5}{2} - [Re(z)]^2 \rho(r)}, \tag{5.19}$$

$$\Delta p = \sqrt{\frac{5}{2} - [Im(z)]^2 \rho(r)}, \tag{5.20}$$

where $Re(z)$ and $Im(z)$ represent the real and imaginary parts of z respectively and

$$\rho(r) = \frac{1}{2}\left[\frac{{}_0F_4(2,2,3,3;r^2)}{{}_0F_4(1,2,2,3;r^2)}\right]^2 - \frac{1}{6}\left[\frac{{}_0F_4(2,3,3,4;r^2)}{{}_0F_4(1,2,2,3;r^2)}\right]. \tag{5.21}$$

A plot of the uncertainty product $(\Delta x)(\Delta p)$ is given in figure (2).

On the other hand, in the linear case with w arbitrary the uncertainties Δx and Δp of (5.18) will have terms involving square roots of rational functions of the summation index. In order to avoid that, we decided to make $w = 2$ (this is an interesting value additional to the ones previously mentioned $w = 0$ and $w = 1$ [30]), and in such a case we have:

$$(\Delta x)^2 = (\Delta p)^2 = (\Delta x)(\Delta p) = \frac{1}{2} + \frac{2}{{}_1F_1(1,3;r^2)}. \tag{5.22}$$

A plot of the product $(\Delta x)(\Delta p)$ is given in figure 3.

As we can see, $(\Delta x)(\Delta p)$ has more involved behaviour in the non-linear than in the linear case with $w = 2$. Notice also that $(\Delta x)(\Delta p)$ in the linear case differs of the standard result $(\Delta x)(\Delta p) = 1/2$ just in a vicinity of z around $z = 0$, and it quickly approaches the standard behaviour when $|z| \to \infty$ (see Fig.3). This does not happens for the non-linear CS for which the asymptotic value of $(\Delta x)(\Delta p)$ depends on the direction in which we are moving out of $z = 0$, and it is in general different from $1/2$. This reinforces the idea that the linear CS are closer to the standard CS than the non-linear ones.

This discussion lead us to conclude that, from an algebraic point of view, the most appropriate annihilation and creation operators for the k-parametric families of potentials almost isospectral to

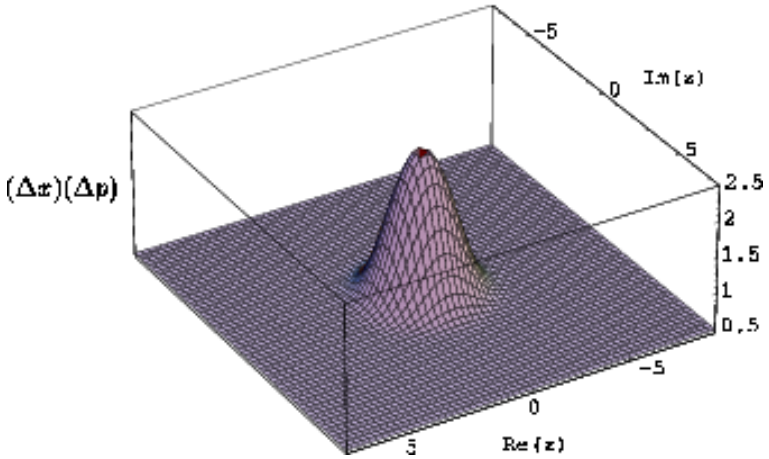

Figure 3: The uncertainty product $(\Delta x)(\Delta p)$ as function of z for the linear coherent states (5.18) with $w = 2$ associated to the member of the 2-parametric family of potentials isospectral to the oscillator arising for $k = 2$, $\epsilon_1 = -1/2$, $\epsilon_2 = -3/2$, $\nu_1 = 0$, $\nu_2 \to \infty$.

the oscillator derived by the k-th order intertwining technique are the linearized ones D_L and $D_L^\dagger$. They mimic the annihilation and creation operators a and $a^\dagger$ of the harmonic oscillator and lead to the standard expression for the CS in the case when the distortion parameter w takes the two values $w = 0$ and $w = -1$ when acting on the subspaces $\mathcal{H}_{\geq 0}$ and $\mathcal{H}_{\geq 1}$ respectively. Moreover, D_L and $D_L^\dagger$ become exactly equal to a and $a^\dagger$ when $k = w = 0$, and the corresponding CS, generated by using D_L or D_0, are precisely the standard CS for the harmonic oscillator.

6 Conclusions and Remarks

We have shown that, for k-SUSY potentials intertwined to the harmonic oscillator potential through k-th order differential operators, it can be constructed annihilation and creation operators D_L and $D_L^\dagger$ obeying the Heisenberg-Weyl algebra (see equation (4.12)) when restricted to the subspace $\mathcal{H}_{\geq 0}$ spanned by the eigenstates associated to the levels $E_n = n + 1/2$, $n = 0, 1, \ldots$ Both of those operators annihilate in a natural way the other k energy eigenstates $|\psi_{\epsilon_i}^k\rangle$, $i = 1, \ldots k$, and the coherent states associated to D_L have the form of the standard CS working on $\mathcal{H}_{\geq 0}$.

Now, some comments about the terminology used to designate the potentials (2.33) with $k = 1$ should be done. Some people name the potentials (2.33) conditionally exactly solvable because the parameters appearing inside $(\epsilon_1,\ \nu_1)$ have to be restricted in order to get a potential and eigenfunctions physically relevant (see e.g. [18]), where ν_1 denotes the ν-parameter arising in (2.32). For instance, taking $k = 1$ and $\epsilon_1 = -1/2$ one will get the AMM family of potentials, which are physically relevant (and thus conditionally exactly solvable) if $|\nu_1| < 1$ because then they are free of singularities and their eigenfuntions are continuous for all $x \in \mathrm{R}$, as for the initial harmonic oscillator potential. However, this interpretation is narrow because it excludes a physically interesting exactly solvable case arising when $|\nu_1| \to \infty$: in such a limit $V_1(x)$ has a singularity at $x = 0$, and thus it is possible to take instead of the oscillator in the full real line as the initial exactly solvable potential, the oscillator potential for $x > 0$ with an infinite barrier at $x = 0$ [19]. The corresponding SUSY partner potential will be also exactly solvable. Thus, care should be exercised when using that terminology.

An additional point concerns the coherent states for the k-SUSY potentials $V_k(x)$. After [29, 30] had arisen, Kumar and Khare considered as unnecessary our CS construction with $k = 1$ and

$\epsilon_1 = -1/2$ because in this case H_1 and H_0 are (up to a displacement of the energy origin) exactly isospectral. Thus, H_1 and H_0 are in principle unitarily equivalent, and the most appropriate CS for H_1 should be gotten from the action of such unitary transformation on the standard CS of the harmonic oscillator [32]. However, even for the simple case with $k = 1$ and an arbitrary $\epsilon_1 < 1/2$ the construction of Kumar and Khare can hardly be done, while our technique can be implemented without any problem (see also [37, 39]). In such a case it turns out more appropriate (although much more complicated than ours) the Bagrov and Samsonov CS construction [34]. From an algebraic point of view, the technique presented in this paper is (we hope) clearer, more general, and mainly more natural than the alternatives developed up to the present by other authors.

Acknowledgments

DJFC acknowledges support by CONACyT (México), project 26329-E and the kind hospitality at CRM. VH acknowledges support by research grants from NSERC of Canada and FCAR du gouvernement du Québec. The referee's suggestions are also acknowledged.

References

[1] L. Infeld and T.E. Hull, Rev. Mod. Phys. **23**, 21 (1951).

[2] P.B. Abraham and H.E. Moses, Phys. Rev. A **22**, 1333 (1980).

[3] B. Mielnik, J. Math. Phys. **25**, 3387 (1984).

[4] D.J. Fernández C., Lett. Math. Phys. **8**, 337 (1984); D.J. Fernández C., J. Negro and M.A. del Olmo, Ann. Phys. **252**, 386 (1996).

[5] A.A. Andrianov, N.V. Borisov and M.V. Ioffe, Phys. Lett. A **105**, 19 (1984).

[6] C.V. Sukumar, J. Phys. A **18**, L57 (1985); A **18**, 2917 (1985); A **18**, 2937 (1985).

[7] E. Witten, Nucl. Phys. B **188**, 513 (1981).

[8] M.M. Nieto, Phys. Lett. B **145**, 208 (1984).

[9] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 267 (1995).

[10] J. Beckers and V. Hussin, Phys. Lett. A **118**, 319 (1986); J. Beckers, D. Dehin and V. Hussin, J. Phys. A **20**, 1137 (1987); **21**, 651 (1988).

[11] N.A. Alves and E. Drigo-Filho, J. Phys. A **21**, 3215 (1988); E. Drigo-Filho, J. Phys. A **21**, L1025 (1988); Mod. Phys. Lett. A **8**, 63 (1993); **9**, 411 (1994); **10**, 1613 (1995).

[12] A. Stahlhofen and K. Bleuler, Nuov. Cim. **104B**, 447 (1989); A. Stahlhofen, *The Riccati-equation: the common basis for supersymmetric quantum mechanics and the factorization method*, Duke University preprint (1989).

[13] L.J. Boya, Eur. J. Phys. **9**, 139 (1988).

[14] J.F. Cariñena, G. Marmo, A.M. Perelomov and M.F. Rañada, Int. J. Mod. Phys. A **13**, 4913 (1998).

[15] O.L. de Lange and R.E. Raab, *Operator Methods in Quantum Mechanics*, Clarendon, Oxford (1991).

[16] B.N. Zakhariev and A.A. Suzko, *Direct and Inverse Problems*, Springer-Verlag, Berlin (1990); V.M. Chabanov and B.N. Zakhariev, Phys. Rev. A **49**, 3159 (1994); B.N. Zakhariev and V.M. Chabanov, Inv. Prob. **13**, R47 (1997).

[17] H.C. Rosu and M.A. Reyes, Phys. Rev. E **51**, 5112 (1995); H.C. Rosu, Phys. Rev. A **54**, 2571 (1996); E **56**, 2269 (1997); L.J. Boya, H.C. Rosu, A.J. Seguí-Santonja and F.J. Vila, Nuov. Cim. **113B**, 409 (1998).

[18] G. Junker and P. Roy, Phys. Lett. A **232**, 155 (1997); Phys. At. Nucl. **61**, 1736 (1998); Ann. Phys. **270**, 155 (1998).

[19] I.F. Márquez, J. Negro and L.M. Nieto, J. Phys. A **31**, 4115 (1998).

[20] J. Oscar Rosas-Ortiz, J. Phys. A **31**, L507 (1998); **31**, 10163 (1998).

[21] P.A. Deift, Duke Math. J. **45**, 267 (1978).

[22] R.W. Carrol, *Transmutation and Operator Differential Equations*, North-Holland Mathematics Studies **37**, The Netherlands (1979).

[23] D.J. Fernández C., V. Hussin and B. Mielnik, Phys. Lett. A **244**, 309 (1998).

[24] A.A. Andrianov, M.V. Ioffe and V. Spiridonov, Phys. Lett. A **174**, 273 (1993); A.A. Andrianov, M.V. Ioffe, F. Cannata and J.P. Dedonder, Int. J. Mod. Phys. A **10**, 2683 (1995).

[25] V.G. Bagrov and B.F. Samsonov, Theor. Math. Phys. **104**, 1051 (1995); Phys. Part. Nucl. **28**, 374 (1997).

[26] D.J. Fernández C., M.L. Glasser and L.M. Nieto, Phys. Lett. A **240**, 15 (1998); D.J. Fernández C., Int. J. Mod. Phys. A **12**, 171 (1997).

[27] B. Mielnik, L.M. Nieto and O. Rosas-Ortiz, *The finite difference algorithm for higher order supersymmetry*, preprint Universidad de Valladolid (1998).

[28] T. Fukui and N. Aizawa, Phys. Lett. A **180**, 308 (1993).

[29] D.J. Fernández C., V. Hussin and L.M. Nieto, J. Phys. A **27**, 3547 (1994).

[30] D.J. Fernández C., L.M. Nieto and O. Rosas-Ortiz, J. Phys. A **28**, 2693 (1995); J.O. Rosas-Ortiz, J. Phys. A **29**, 3281 (1996).

[31] V. Spiridonov, Phys. Rev. A **52**, 1909 (1995).

[32] M.S. Kumar and A. Khare, Phys. Lett. A **217**, 73 (1996).

[33] H.C. Fu and R. Sasaki, J. Phys. A **29**, 4049 (1996).

[34] V.G. Bagrov and B.F. Samsonov, JETP **82**, 593 (1996); J. Phys. A **29**, 1011 (1996).

[35] S. Seshadri, S. Lakshmibala and V. Balakrishnan, Phys. Rev. A **55**, 869 (1997).

[36] V.M. Eleonsky and V.G. Korolev, Phys. Rev. A **55**, 2580 (1997).

[37] N. Aizawa and H.T. Sato, in *Proceedings of the IV Wigner Symposium*, N.M. Atakishiyev et. al. Eds., World Scientific, Singapore (1996), p. 450; Prog. Theor. Phys. **98**, 707 (1997).

[38] S. Seshadri, V. Balakrishnan and S. Lakshmibala, J. Math. Phys. **39**, 838 (1998).

[39] F. Cannata, G. Junker and J. Trost, in *Particles, fields and gravitation*, J. Rembielinski Ed., AIP Conf. Proc. **453**, Woodbury (1998), p. 209; G. Junker and P. Roy, quant-ph/9807081.

[40] S.Y. Dubov, V.M. Eleonsky and N.E. Kulagin, Sov. Phys. JETP **75**, 446 (1992); Chaos **4**, 47 (1994).

[41] V.M. Eleonsky, V.G. Korolev and N.E. Kulagin, Chaos **4**, 583 (1994).

[42] V.M. Eleonsky and V.G. Korolev, J. Phys. A **28**, 4973 (1995); **29**, L241 (1996).

[43] V.E. Adler, Physica D **73**, 335 (1994).

[44] D.J. Fernández C., Master's Thesis, CINVESTAV (1984).

[45] J. Beckers, N. Debergh and A.G. Nikitin, J. Phys. A **24**, L1269 (1991).

[46] A.P. Veselov and A.B. Shabat, Funkz. Analiz Ego Pril. **27**, 1 (1993).

[47] V.A. Fateev and S.L. Lykyanov, Int. J. Mod. Phys. A **3**, 507 (1988).

[48] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. **124**, 87 (1989).

[49] M. Roček, Phys. Lett. B **255**, 554 (1991).

[50] C. Daskaloyannis, J. Phys. A **24**, L789 (1991).

[51] J.R. Klauder and B.S. Skagerstam, *Coherent States. Applications in Physics and Mathematical Physics*, World Scientific, Singapore (1985);

[52] A. Perelomov, *Generalized Coherent States and Their Applications*, Springer, Berlin (1986).

[53] W.M. Zhang, D.H. Feng and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).

[54] A.O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971);

[55] J. Beckers and N. Debergh, J. Math. Phys. **30**, 1732 (1989).

[56] H. Bateman, *Tables of Integral Transforms* Vol. 1, A. Erdélyi Ed., McGraw Hill, New York (1954).