J. DIFFERENTIAL GEOMETRY
55 (2000) 385-440

HIGHER TYPE ADJUNCTION INEQUALITIES IN SEIBERG-WITTEN THEORY

PETER OZSVÁTH & ZOLTÁN SZABÓ

Abstract

In this paper, we derive new adjunction inequalities for embedded surfaces with non-negative self-intersection number in four-manifolds. These formulas are proved by using relations between Seiberg-Witten invariants which are induced from embedded surfaces. To prove these relations, we develop the relevant parts of a Floer theory for four-manifolds which bound circle-bundles over Riemann surfaces.

1. Introduction

In this paper, we prove certain adjunction inequalities, which give relations between the Seiberg-Witten invariants of a four-manifold X and the genus of embedded surfaces in X. These results are generalizations of results from [12], [21], [24], see also [13].

The investigations center on a construction of an appropriate Seiberg-Witten-Floer functor for manifolds which bound circle bundles Y over Riemann surfaces (with sufficiently large Euler number), which relies on the calculations of [22]. Special cases of this theory were studied in [24], where the authors used similar techniques to prove the symplectic Thom conjecture. That problem requires an analysis of those $\mathrm{Spin}_{\mathbb{C}}$ structures over Y for which the Seiberg-Witten moduli space contains only reducible solutions, which simplifies the corresponding Floer homology. In this paper, we work out the theory in the other, more complicated cases. We will give more applications of these techniques in [23].

Received July 13, 1998, and, in revised form, February 2, 1999. The first author was partially supported by NSF grant number DMS 9304580, and the second author was partially supported by NSF grant number DMS-9704359, a Sloan Foundation Fellowship, and a Packard Fellowship.

Before stating the results, we set up some notation. Let X be a closed, connected, smooth four-manifold equipped with an orientation for which $b_2^+(X) > 0$ (where $b_2^+(X)$ is the dimension of a maximal positive-definite linear subspace $H^+(X;\mathbb{R})$ of the intersection pairing on $H^2(X;\mathbb{R})$) and an orientation for $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$. Given such a four-manifold, together with a $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$, the Seiberg-Witten invariants (see [31], [19], [26]) form an integer-valued function

$$SW_{X,\mathfrak{s}}\colon \mathbb{A}(X) \longrightarrow \mathbb{Z},$$

where $\mathbb{A}(X)$ denotes the graded algebra obtained by tensoring the exterior algebra on $H_1(X)$ (graded so that $H_1(X)$ has grading one) with the polynomial algebra $\mathbb{Z}[U]$ on a single two-dimensional generator. The invariants are constructed via intersection theory on the moduli space $\mathcal{M}_X(\mathfrak{s})$ of solutions (A, Φ) modulo gauge to the Seiberg-Witten equations in $\mathfrak{s}$:

$$\rho(\mathrm{Tr} F_A^+) = i\{\Phi,\Phi\}_0 - \rho(i\eta) \tag{1}$$

$$\not{D}_A \Phi = 0, \tag{2}$$

where Φ is a section of W^+, A is a spin-connection in the spinor bundle W^+ of $\mathfrak{s}$, $\not{D}_A$ denotes the associated Dirac operator, ρ denotes Clifford multiplication, η is some fixed self-dual two-form, and $\{\Phi,\Phi\}_0$ is the usual quadratic map (see [31]). Note that the invariants are zero on homogeneous elements whose degree is not $d(\mathfrak{s})$, where

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - (2\chi(X) + 3\sigma(X))}{4}$$

denotes the formal dimension of the moduli space $\mathcal{M}_X(\mathfrak{s})$. When $b_2^+(X) > 1$, $SW_{X,\mathfrak{s}}$ is a diffeomorphism invariant of the four-manifold; when $b_2^+(X) = 1$, the invariants depend on a chamber structure (see [19], [24]). There are two distinguished chambers corresponding to the two components of $\mathcal{K}(X) = \{\omega \in H^2(X;\mathbb{R}) - 0 \mid \omega^2 \geq 0\}$. Given a component $\mathcal{K}_0$ of $\mathcal{K}(X)$, the corresponding invariant (still denoted $SW_{X,\mathfrak{s}}$) is calculated using the moduli space of solutions to the Seiberg-Witten equations perturbed by any generic self-dual two-form η, provided that the sign of $-2\pi c_1(\mathfrak{s})\cdot\omega_g + \int_X \eta\wedge\omega_g$ agrees with the sign of $\gamma\cdot\omega_g$, where γ is any class in $\mathcal{K}_0$, and $\omega_g \neq 0$ is a harmonic (with respect to the metric g), self-dual two-form over X. Note that $SW_{X,\mathfrak{s}}$ is a diffeomorphism invariant of X (and the component $\mathcal{K}_0$).

Those $\mathrm{Spin}_{\mathbb{C}}$ structures $\mathfrak{s}$ for which the invariant $SW_{X,\mathfrak{s}}$ is non-trivial are called *basic classes*.

Our results are easiest to state when $b_1(X) = 0$, where we have the following.

Theorem 1.1. *Let X be a smooth, closed, connected, oriented four-manifold with $b_2^+(X) > 0$ and $b_1(X) = 0$, and let $\Sigma \subset X$ be a smoothly-embedded surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with self-intersection number $[\Sigma] \cdot [\Sigma] \geq 0$. If $b_2^+(X) > 1$, then we have the following adjunction inequality*

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) \leq 2g(\Sigma) - 2,$$

for each basic class $\mathfrak{s} \in \mathrm{Spin}_{\mathbb{C}}(X)$. Furthermore, when $b_2^+(X) = 1$, for each basic class $\mathfrak{s}$ of X for the component of $\mathcal{K}(X)$ which contains $\mathrm{PD}[\Sigma]$ with

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma] \cdot [\Sigma] \geq 0,$$

we have an inequality

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma] \cdot [\Sigma] + 2d(\mathfrak{s}) \leq 2g(\Sigma) - 2.$$

Remark 1.2. The above theorem should be seen as a refinement of the adjunction inequality proved by Kronheimer-Mrowka and Morgan-Szabó-Taubes (see [12], [21], [3]). Analogous results for immersed spheres were obtained by Fintushel and Stern, see [7].

In fact, Theorem 1.1 follows from a more general version. To state this, note first that an inclusion $i\colon \Sigma \longrightarrow X$ induces a map

$$i_*\colon \mathbb{A}(\Sigma) \longrightarrow \mathbb{A}(X).$$

Theorem 1.3. *Let X be a smooth, closed, connected, oriented four-manifold with $b_2^+(X) > 0$. Let $\Sigma \subset X$ be a surface with genus $g(\Sigma) > 0$ representing a non-torsion homology class with self-intersection number $[\Sigma]\cdot[\Sigma] \geq 0$. Let ℓ be an integer so that there is a symplectic basis $\{A_j, B_j\}_{j=1}^g$ for $H_1(\Sigma)$ so that $i_*(A_j) = 0$ in $H_1(X;\mathbb{R})$ for $i = 1, \ldots, \ell$. Fix any element $a \in \mathbb{A}(X)$, and let and $b \in \mathbb{A}(\Sigma)$ be an element whose degree satisfies $d(b) \leq \ell$. If $b_2^+(X) > 1$ then for each $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$ so that $SW_{X,\mathfrak{s}}(a \cdot i_*(b))$ is non-zero, we have*

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma] \cdot [\Sigma] + 2d(b) \leq 2g(\Sigma) - 2.$$

Furthermore, when $b_2^+(X) = 1$ *then for each* $\mathrm{Spin}_{\mathbb{C}}$ *structure* $\mathfrak{s}$ *of* X *with*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0,$$

for which $SW_{X,\mathfrak{s}}(a\cdot i_*(b))$ *is non-zero, when calculated in the component of* $\mathcal{K}(X)$ *containing* $\mathrm{PD}[\Sigma]$*, we have an inequality*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] + 2d(b) \leq 2g(\Sigma) - 2. \tag{3}$$

The Adjunction Inequality (3) does not hold without homological restrictions on X, as we can see by looking at the ruled surface $X = S^2 \times \Sigma$. In general, one can obtain only a weaker inequality (losing the factor of 2 on the dimension $d(b)$), as follows.

Theorem 1.4. *Let* X *be a smooth, closed, connected, oriented four-manifold with* $b_2^+(X) > 1$*. Let* $\Sigma \subset X$ *be a surface with genus* $g(\Sigma) > 0$ *representing a non-torsion homology class with self-intersection number* $[\Sigma]\cdot[\Sigma] \geq 0$*. Let* $a \in \mathbb{A}(X)$ *and* $b \in \mathbb{A}(\Sigma)$*. If* $b_2^+(X) > 1$ *and if* $SW_{X,\mathfrak{s}}(a\cdot i_*(b))$ *is non-zero for some* $b \in \mathbb{A}(\Sigma)$ *of degree* $d(b)$*, then we have*

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma]\cdot[\Sigma] + d(b) \leq 2g(\Sigma) - 2.$$

If $b_2^+(X) = 1$ *and* $\mathfrak{s}$ *is a* $\mathrm{Spin}_{\mathbb{C}}$ *structure with*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0,$$

for which $SW_{X,\mathfrak{s}}(a\cdot i_*(b))$ *is non-zero, when calculated in the component of* $\mathcal{K}(X)$ *containing* $\mathrm{PD}[\Sigma]$*, then we have*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] + d(b) \leq 2g(\Sigma) - 2. \tag{4}$$

Remark 1.5. Adjunction inequalities for surfaces of positive square in Donaldson's theory were first obtained in the influential paper of Kronheimer and Mrowka (see [13]). These inequalities were strengthened under similar, but more restrictive, hypotheses in their preprint [14]; see also [6]. The conjectured relationship between the Donaldson and Seiberg-Witten invariants gives a correspondence between the adjunction inequalities arising in these two theories. For more on this correspondence, see [27], [31], [18], [5], [25], and [8].

We illustrate Theorem 1.4 with the following application:

Corollary 1.6. *Let* $T \subset S^1 \times S^3$ *be an embedded torus for which the restriction of the one-dimensional cohomology from* $H^1(S^1\times S^3)$ *is* *non-trivial. Let (X,ω) be a symplectic four-manifold and $\Sigma \subset X$ be a symplectic submanifold with non-negative self-intersection. Then, in the connected sum $X\#(S^1 \times S^3)$, the internal connected sum $\Sigma\#T \subset X\#(S^1 \times S^3)$ minimizes genus among all embedded surfaces $\Sigma' \subset X\#(S^1 \times S^3)$ which are homologous to $\Sigma\#T$ and for which the one-dimensional cohomology from $S^1 \times S^3$ restricts non-trivially to Σ'.*

Theorem 1.3 follows from a relation which holds for embedded surfaces with arbitrary self-intersection number. This relation can be viewed as a generalization of the relation appearing in [24]. Once again, we begin by stating the case when $b_1(X) = 0$, in the interest of exposition.

Theorem 1.7. *Let X be a smooth, closed, connected, oriented four-manifold with $b_1(X) = 0$, and let $\Sigma \subset X$ be a smoothly embedded surface with genus $g(\Sigma) > 0$. Then, for each $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$ with*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0$$

and

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] + 2d(\mathfrak{s}) > 2g(\Sigma) - 2,$$

we have

$$SW_{X,\mathfrak{s}}(U^d) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(U^{d'}),$$

where d and d' denote the dimensions of $\mathfrak{s}$ and $\mathfrak{s} - \mathrm{PD}[\Sigma]$ respectively. In the case where $b_2^+(X) = 1$, both invariants are to be calculated in the same component of $\mathcal{K}(X)$.

More generally, we have the following.

Theorem 1.8. *Let X be a smooth, closed, connected, oriented four-manifold with $b_2^+(X) > 0$. Let $\Sigma \subset X$ be a surface with genus $g(\Sigma) > 0$. Let ℓ be an integer so that there is a symplectic basis $\{A_j, B_j\}_{j=1}^g$ for $H_1(\Sigma)$ so that $i_*(A_j) = 0$ in $H_1(X;\mathbb{R})$ for $i = 1,\ldots,\ell$. For each $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$ with*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0$$

and each $b \in \mathbb{A}(\Sigma)$ of degree $d(b) \leq \ell$ with

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] + 2d(b) > 2g(\Sigma) - 2, \tag{5}$$

there is an element $b' \in \mathbb{A}(\Sigma)$ with $d(b') \geq d(b)$ so that for any $a \in \mathbb{A}(X)$, we have

$$SW_{X,\mathfrak{s}}(a \cdot i_*(b)) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a \cdot i_*(b')). \tag{6}$$

Furthermore, if $b = U^{d/2}$, then $b' - U^{d'/2}$ lies in the ideal generated by $H_1(\Sigma)$ in $\mathbb{A}(\Sigma)$. Once again, in the case where $b_2^+(X) = 1$, both invariants are to be calculated in the same component of $\mathcal{K}(X)$.

Theorem 1.3 is a simple consequence of Theorem 1.8, as the following proof shows.

Theorem 1.8 $\Rightarrow$ Theorem 1.3. Suppose Theorem 1.3 were false; i.e., suppose there were X, Σ, $\mathfrak{s}$, a, and b which satisfy the hypotheses of the theorem, but which violate Adjunction Inequality (3). We can assume without loss of generality that

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0,$$

by reversing the orientation of Σ if necessary (when $b_2^+(X) > 1$). Thus, Theorem 1.8 applies. Let b' be the element which satisfies Relation (6), so we have that $SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a\cdot i_*(b')) \neq 0$. Since b and b' are homogeneous elements with the same degree modulo two, and $d(b') \geq d(b)$, it follows that we can find elements $a' \in \mathbb{A}(X)$ and $b'' \in \mathbb{A}(\Sigma)$ with $d(b'') = d(b)$, and $SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a'\cdot i_*(b'')) \neq 0$. Now, since $[\Sigma]\cdot[\Sigma] \geq 0$ and $d(b'') = d(b)$, we see that Σ also violates the adjunction inequality for $\mathfrak{s} - \mathrm{PD}[\Sigma]$, $a' \in \mathbb{A}(X)$, and $b'' \in \mathbb{A}(\Sigma)$. Proceeding in this way, we see that $\mathfrak{s} - n\mathrm{PD}[\Sigma]$ is a Seiberg-Witten basic class for all $n \geq 0$. If $b_2^+(X) > 1$, then there are only finitely many basic classes of X, so since Σ is not a torsion class, we get a contradiction, proving Theorem 1.3 in this case.

The above argument works also when $b_2^+(X) = 1$, since there are still only finitely many basic classes of the form $\mathfrak{s}-n\mathrm{PD}[\Sigma]$ with $n \geq 0$ in the chamber corresponding to $\mathrm{PD}[\Sigma]$. We see this as follows. Fix a metric g on X and a generic self-dual two-form η. Clearly, if $\mathfrak{s}$ is fixed and n is sufficiently large, the sign of $\mathrm{PD}[\Sigma]\cdot\omega_g$ agrees with the sign of $-2\pi c_1(\mathfrak{s}-n\mathrm{PD}[\Sigma])\cdot\omega_g + \int \eta\wedge\omega_g$; i.e., for all large n, the η-perturbed moduli spaces for $\mathfrak{s} - n\mathrm{PD}[\Sigma]$ can be used calculate the invariant in the component which contains $\mathrm{PD}[\Sigma]$. But the usual compactness argument shows that all but finitely many of these moduli spaces are empty. Again, we have the contradiction completing the proof of Theorem 1.3. q.e.d.

By blowing up, Theorem 1.8 is reduced to the case where the self-intersection number of Σ is sufficiently negative. The theorem is then proved by expressing the Seiberg-Witten invariants of a four-manifold with such an embedded surface Σ in terms of relative invariants, which take values in a Seiberg-Witten-Floer homology associated to non-trivial circle bundles over Σ. In the presence of the topological hypotheses on the inclusion of $H_1(\Sigma)$ in $H_1(X)$, the above relation then follows from properties of this Floer homology.

The outline of this paper is as follows. In Section 2, we give examples of the adjunction formulae. Our examples include four-manifolds with $b_2^+(X) = 1$, and also examples where both $b_2^+(X) > 1$ and $b_1(X) > 0$, and a proof of Corollary 1.6. In Section 3, we show how Theorem 1.8 can be deduced from properties of a product formula, which relates the Seiberg-Witten invariants of a four-manifold containing an embedded surface with sufficiently negative self-intersection number with certain relative invariants associated to $X - \Sigma$. For completeness, we also show how a modified version of Theorem 1.8 implies Theorem 1.4. In Section 4 we review the gauge theory for circle bundles over Riemann surfaces as developed in [22]. There is one $\mathrm{Spin}_{\mathbb{C}}$ structure in which the moduli space of reducibles has singularities (to which we return in a later section). In Section 5, we prove the product formula introduced in Section 3, assuming technical facts about the moduli spaces over N, the tubular neighborhood of Σ. In Section 6, we define an invariant with irreducible boundary values and use properties of this relative invariant to analyze the terms appearing in the product formula, completing the proof of Theorem 1.8. In Section 7, we prove the technical facts about the moduli spaces over N which were used in earlier sections. In Section 8, we show how to extend the results of Sections 4 and 7 to deal with the remaining $\mathrm{Spin}_{\mathbb{C}}$ structure. Finally, in Section 9, which should be viewed as an appendix, we discuss representatives for the cohomology classes used throughout the paper.

Acknowledgements. The authors wish to thank Vicente Muñoz for his very helpful comments on an early version of this paper. They also wish to thank the referee for providing useful suggestions.

2. Examples

We give examples of embedded surfaces $\Sigma \subset X$ with $b_2^+(X) > 1$ and $b_1(X) > 0$, where the inequality in Theorem 1.3 is in fact an equality. (It is an open problem whether manifolds with $b_2^+(X) > 1$ and $b_1(X) = 0$ can admit basic classes of non-zero dimension.) But first, we need to find such four-manifolds X which admit basic classes of non-zero dimension.

To construct these examples, we use the following construction.

Definition 2.1. Let X be smooth four-manifold and let $S \subset X$

be an embedded two-sphere with zero self-intersection number. Let X' denote the manifold obtained as surgery on S; i.e.,

$$X' = (X - \mathrm{nd}(S)) \cup_\phi S^1 \times D^3,$$

where $\mathrm{nd}(S)$ is an open tubular neighborhood of S and

$$\phi\colon \partial(X - \mathrm{nd}(S)) \longrightarrow S^1 \times S^2$$

is a orientation-reversing diffeomorphism. Let $C \subset X'$ denote the closed curve which is the core of the added $S^1 \times D^3$. Note that there is a diffeomorphism $X - S \cong X' - C$.

Of course, $X' = X\#(S^1 \times S^3)$ is an example of this construction.

Proposition 2.2. *Let X be a closed, smooth, oriented four-manifold with $b_2^+(X) > 0$, and let $S \subset X$ be a homologically trivial embedded two-sphere. For each* $\mathrm{Spin}_\mathbb{C}$ *structure $\mathfrak{s}$ on X, there is a unique induced* $\mathrm{Spin}_\mathbb{C}$ *structure $\mathfrak{s}'$ on X' with the property that*

$$\mathfrak{s}|_{X-S} = \mathfrak{s}'|_{X'-C}.$$

Then, $d(\mathfrak{s}') = d(\mathfrak{s}) + 1$; and for all $a \in \mathbb{A}(X)$

$$SW_{X',\mathfrak{s}'}(a \cdot \mu(C)) = SW_{X,\mathfrak{s}}(a),$$

for some homology orientation on X'. (When $b_2^+(X) = 1$, both invariants are to be calculated in the same chamber.)

Proof. The dimension statement is straightforward.

To prove the relation, we pull X apart along $S^1 \times S^2 = \partial\, \mathrm{nd}(S)$, and study the corresponding moduli spaces (see Section 5 for more discussion on such matters). Let X_0 denote the complement $X - S$, given a cylindrical-end metric modeled on the product metric $[0, \infty) \times S^1 \times S^2$, where S^2 is given its standard, round metric. Note that this metric can be extended over both $S^1 \times D^3$ and $D^2 \times S^2$ to give metrics with non-negative scalar curvature. Consequently, the moduli spaces of solutions over $S^1 \times S^2$, $S^1 \times D^3$, and $D^2 \times S^2$ consist entirely of smooth reducibles (i.e., the moduli spaces are identified with S^1, S^1, and a point respectively).

Let $\mathcal{M}_{X_0}(\mathfrak{s}_0)$ denote the moduli space of finite energy solutions to the Seiberg-Witten equations over X_0 in the $\mathrm{Spin}_\mathbb{C}$ structure $\mathfrak{s}_0 = \mathfrak{s}|_{X_0}$. Thus, we can think of the boundary map as a map

$$\rho\colon \mathcal{M}_{X_0}(\mathfrak{s}_0) \longrightarrow S^1.$$

Gluing theory gives a diffeomorphism for all sufficiently large $T > 0$:

$$\mathcal{M}_{X(T)}(\mathfrak{s}) \cong \rho^{-1}(x_0),$$

where $X(T)$ denotes the metric on X with neck-length T and $x_0 \in S^1$ corresponds to the unique reducible on $S^1 \times S^2$ which extends to $D^2 \times S^2$. Consequently,

$$SW_{X,\mathfrak{s}}(a) = \langle \mathcal{M}_{X_0}(\mathfrak{s}_0), \mu(a) \cup \mu(C) \rangle, \tag{7}$$

since $\mu(C)$ is represented by the holonomy class around C (see Proposition 9.1).

Similarly, gluing gives a diffeomorphism of

$$\mathcal{M}_{X_0}(\mathfrak{s}_0) \cong \mathcal{M}_{X'(T)}(\mathfrak{s}'),$$

and consequently

$$SW_{X',\mathfrak{s}'}(a \cdot C) = \langle \mathcal{M}_{X_0}(\mathfrak{s}_0), \mu(a \cdot C) \rangle. \tag{8}$$

Together, Equations (7) and (8) prove the proposition. q.e.d.

With this proposition in hand, we can provide a proof of Corollary 1.6:

Proof of Corollary 1.6. According to Taubes' non-vanishing theorem (see [29]), the canonical $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}_0$ of the symplectic manifold has $SW_{(X,\mathfrak{s}_0)} = \pm 1$; thus, according to Proposition 2.2, if $\mathfrak{s}'$ is any $\mathrm{Spin}_{\mathbb{C}}$ structure over $X' = X \# (S^1 \times S^3)$ whose restriction to the X side agrees with $\mathfrak{s}_0$, and $b \in H_1(X'; \mathbb{Z})$ is a non-zero homology class coming from the $S^1 \times S^3$ factor, then $SW_{(X',\mathfrak{s}')}(b) \neq 0$. Thus, in view of the usual adjunction formula for the symplectic submanifold

$$-\langle c_1(\mathfrak{s}_0), [\Sigma] \rangle + \Sigma \cdot \Sigma = 2g(\Sigma) - 2,$$

Inequality (4) gives that $g(\Sigma') > g(\Sigma)$, as required.

Note that in the case where $b_2^+(X) = 1$, the additional hypothesis $-\langle c_1(\mathfrak{s}'), [\Sigma'] \rangle + [\Sigma'] \cdot [\Sigma'] > 0$ follows immediately from the adjunction formula except, of course, when $g(\Sigma) = 0$, in which case the result is vacuously true. q.e.d.

2.1 Examples of Theorem 1.3 with $b_2^+(X) > 1$

We can use Proposition 2.2 to construct examples where Theorem 1.3 is sharp, as well.

Fix natural numbers n, k, and m with $2k \geq n > 1$, and let X be the four-manifold $E(n)\#m(S^3 \times S^1)$, where $E(n)$ is a simply-connected elliptic surface with no multiple fibers and with geometric genus $n-1$. Let $\Sigma_0 \subset E(n)$ denote a symplectic submanifold representing the homology class $S + kF$, where S and F denote the homology classes of a section and a fiber respectively of the elliptic fibration. Let $T_i \subset X$ denote a fiber in the elliptic fibration of the i^{th} summand $S^3 \times S^1$. Let $\Sigma \subset X$ denote the internal connected sum of $\Sigma_0 \# F_1 \# \ldots \# F_m$. Note that $g(\Sigma) = k + m$ and $\Sigma \cdot \Sigma = 2k - n \geq 0$. Let $\mathfrak{s}$ be the $\mathrm{Spin}_{\mathbb{C}}$ structure over X induced from the canonical $\mathrm{Spin}_{\mathbb{C}}$ structure on $E(n)$, and let $b = \mathbb{A}(\Sigma)$ be the product $B_1 \cdot \ldots \cdot B_m$ where $B_i \in H_1(X)$ generates H_1 of the i^{th} copy of $S^1 \times S^3$. Note that $d(b) = m$ and Σ has a symplectic basis $\{A_i, B_i\}_{i=1}^{k+m}$ for which $A_1, \ldots, A_m$ are homologically trivial in X. By Proposition 2.2,

$$SW_{X,\mathfrak{s}}(B_1 \cdot \ldots \cdot B_m) = 1,$$

so the data X, b, Σ, $\mathfrak{s}$ satisfy the hypotheses of Theorem 1.3. In fact, we see that

$$\Sigma \cdot \Sigma + \langle c_1(\mathfrak{s}), [\Sigma] \rangle + 2d(b) = 2g(\Sigma) - 2,$$

which shows that the inequality of the theorem is sharp, for all choices of $g(\Sigma) > 0$, $\Sigma \cdot \Sigma \geq 0$, and $d(b)$.

2.2 Ruled surfaces: the homological hypotheses on $H_1(\Sigma)$

By looking at ruled surfaces, we give examples where Inequality (4) is sharp, and hence that some homological hypotheses are necessary for the stronger inequality (which appears in Theorem 1.3) to hold.

As mentioned before, one cannot hope for the adjunction inequality of Theorem 1.3 to be valid without additional topological hypotheses on the inclusion of Σ in X. Indeed, fix $n \geq 0$ and $g > 0$, and let X be the two-sphere bundle over a surface Σ of genus g, associated to the circle bundle with Euler number n. In particular, X contains an embedded copy of Σ with $\Sigma \cdot \Sigma = n$. In the chamber corresponding to $\mathrm{PD}[\Sigma]$, there is a zero-dimensional basic class $\mathfrak{s}_0$ with $c_1(\mathfrak{s}_0) = -K_X$, where K_X is the canonical class of X viewed as Kähler manifold. Moreover, letting F be the class of the two-sphere fiber in X, we see that the moduli space associated to $\mathfrak{s}_0 + d\mathrm{PD}[F]$ is identified with $\mathrm{Sym}^d(\Sigma)$, and U is the symmetric product of the volume form of Σ (see Proposition 6.10 for a related discussion). Thus, if $\mathfrak{s} = \mathfrak{s}_0 + d\mathrm{PD}[F]$, then $SW_{\mathfrak{s}}(U^d) \neq 0$, and

$$\langle c_1(\mathfrak{s}), [\Sigma]\rangle = 2d.$$

Clearly, Adjunction Inequality (4) is sharp for all values of k, d, n, and g provided that $-n \leq k$, where $k = \langle c_1(\mathfrak{s}), [\Sigma]\rangle$, $2d = d(b)$, $n = \Sigma \cdot \Sigma$, and $g = g(\Sigma)$. (This construction, strictly speaking, only gives us even values of $d(b)$. For odd values, one can attach an $S^1 \times S^3$.) In particular, we see that some homological criterion on the embedding of $\Sigma \subset X$ is necessary for the stronger Inequality (3) to hold.

3. From product formulas to relations

The aim of this section is to outline the proof of Theorem 1.8. By employing the blowup formula in a manner analogous to [24], we reduce to the case where the self-intersection number of Σ is very negative (Proposition 3.1). The invariants in this latter case are studied via a product formula, which we state (and prove in Section 5), whose terms are then related with other Seiberg-Witten invariants of X. In the end of the section, we discuss the modifications which are needed to prove Theorem 1.4.

We reduce Theorem 1.8 to the following special case.

Proposition 3.1. *Theorem* 1.8 *holds, under the additional hypothesis that*

$$0 \leq -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2 \quad \textit{and} \quad -[\Sigma]\cdot[\Sigma] > 2g - 2.$$

The reduction involves the following basic result of Fintushel and Stern.

Theorem 3.2. (Blowup Formula: [7] and [26]). *Let X be a smooth, closed four-manifold, and let $\widehat{X} = X\#\overline{\mathbb{CP}}^2$ denote its blow-up, with exceptional class $E \in H^2(\widehat{X};\mathbb{Z})$. If $b_2^+(X) > 1$, then for each* $\mathrm{Spin}_{\mathbb{C}}$ *structure $\widehat{\mathfrak{s}}$ on $\widehat{X}$ with $d(\widehat{\mathfrak{s}}) \geq 0$, and each $a \in \mathbb{A}(X) \cong \mathbb{A}(\widehat{X})$, we have*

$$SW_{\widehat{X},\widehat{\mathfrak{s}}}(a) = SW_{X,\mathfrak{s}}(U^m a),$$

where $\mathfrak{s}$ is the $\mathrm{Spin}_{\mathbb{C}}$ *structure induced on X obtained by restricting $\widehat{\mathfrak{s}}$, and $2m = d(\mathfrak{s}) - d(\widehat{\mathfrak{s}})$. If $b_2^+(X) = 1$, there is a one-to-one correspondence between components of $\Omega^+(X)$ and $\Omega^+(\widehat{X})$, and the above relation holds provided both invariants are calculated in chambers associated to corresponding components.*

Before showing how to reduce Theorem 1.8 to the special case, we point out that another special case of Theorem 1.8 was already proved in Theorem 1.3 [24]. More specifically, the following was shown:

Theorem 3.3. [24] *Let X be a smooth, closed, connected, oriented four-manifold with $b_2^+(X) > 0$. Let $\Sigma \subset X$ be a surface with genus $g(\Sigma) > 0$ and negative self-intersection. For each* $\mathrm{Spin}_{\mathbb{C}}$ *structure $\mathfrak{s}$ with*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] > 2g(\Sigma) - 2,$$

there is an element $b' \in \mathbb{A}(\Sigma)$ so that that for any $a \in \mathbb{A}(X)$, we have

$$SW_{X,\mathfrak{s}}(a) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a \cdot i_*(b')).$$

Furthermore, $b' - U^{d'/2}$ lies in the ideal generated by $H_1(\Sigma)$ in $\mathbb{A}(\Sigma)$.

Remark 3.4. In the language of Theorem 1.8, this case corresponds to $\ell = 0$ and $b = 1$.

Proposition 3.1 $\Rightarrow$ Theorem 1.8. Let $g = g(\Sigma)$, fix an integer m with

$$m > [\Sigma]\cdot[\Sigma] + 2g - 2,$$

let $\widehat{X} = X \# m\overline{\mathbb{CP}}^2$, and let $\widehat{\Sigma}$ be the "proper transform" of Σ, the embedded surface obtained by internal connected sum of Σ with the m exceptional spheres in the $\overline{\mathbb{CP}}^2$ summands; i.e.,

$$\mathrm{PD}[\widehat{\Sigma}] = \mathrm{PD}[\Sigma] - E_1 - \cdots - E_m.$$

Finally, let $\widehat{\mathfrak{s}}$ denote the $\mathrm{Spin}_{\mathbb{C}}$ structure on $\widehat{X}$ which agrees with $\mathfrak{s}$ in the complement of the exceptional spheres, whose Chern class satisfies

$$c_1(\widehat{\mathfrak{s}}) = c_1(\mathfrak{s}) - E_1 - \cdots - E_m.$$

It is easy to check that:

$$\begin{aligned}
-[\widehat{\Sigma}]\cdot[\widehat{\Sigma}] = m - [\Sigma]\cdot[\Sigma] &> 2g - 2, \\
-\langle c_1(\widehat{\mathfrak{s}}), [\widehat{\Sigma}]\rangle + [\widehat{\Sigma}]\cdot[\widehat{\Sigma}] &= -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma]; \\
d(\mathfrak{s}) &= d(\widehat{\mathfrak{s}}).
\end{aligned}$$

Now, if

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 2g,$$

the hypotheses of Theorem 3.3 are satisfied; and otherwise, the hypotheses of Proposition 3.1 are. In either case, for each $b \in \mathbb{A}(\Sigma)$ of degree $d(b) \leq \ell$, we can find $b' \in \mathbb{A}(\Sigma)$ with

$$SW_{\widehat{X},\widehat{\mathfrak{s}}}(a\cdot i_*(b)) = SW_{\widehat{X},\widehat{\mathfrak{s}}-\mathrm{PD}[\widehat{\Sigma}]}(a\cdot i_*(b')). \tag{9}$$

According to the blow-up formula,

$$SW_{\widehat{X},\widehat{\mathfrak{s}}}(a\cdot i_*(b)) = SW_{X,\mathfrak{s}}(a\cdot i_*(b)); \tag{10}$$

and, since $\widehat{\mathfrak{s}} - \mathrm{PD}[\widehat{\Sigma}]$ agrees with $\mathfrak{s} - \mathrm{PD}[\Sigma]$ away from the exceptional spheres and

$$c_1(\widehat{\mathfrak{s}} - \mathrm{PD}[\widehat{\Sigma}]) = c_1(\mathfrak{s} - \mathrm{PD}[\Sigma]) - E_1 - \cdots - E_m,$$

we see from another application of the blowup formula that

$$SW_{\widehat{X},\widehat{\mathfrak{s}}-\mathrm{PD}[\widehat{\Sigma}]}(a\cdot i_*(b')) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a\cdot i_*(b')). \tag{11}$$

Theorem 1.8 then follows by combining Equations (9), (10) and (11).

q.e.d.

We now turn to the special case considered in Proposition 3.1. We will study the Seiberg-Witten invariant of X by decomposing it into two pieces

$$X = N \cup_Y (X - N),$$

where Y a circle bundle over Σ (as in the proposition), and N is the associated disk bundle. Following [22], the moduli space of Seiberg-Witten monopoles over Y decomposes into an irreducible and a reducible component. (Actually, there is one $\mathrm{Spin}_{\mathbb{C}}$ structure over Y, where it is necessary to perturb the equations for this decomposition to occur; this perturbation is studied Section 8.) Correspondingly, we construct relative invariants of $X - \Sigma$, denoted $SW^{irr}_{\mathfrak{s}}$ and $SW^{red}_{\mathfrak{s}}$, arising from the L^2 moduli spaces on $X - \Sigma$ with irreducible and reducible boundary values. In Section 5 (see Lemma 5.6, and the discussion following it), we prove the following:

Proposition 3.5. *Suppose*

$$0 \leq -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2 \quad \textit{and} \quad -[\Sigma]\cdot[\Sigma] > 2g - 2.$$

Then,

$$SW_{X,\mathfrak{s}} = SW^{irr}_{\mathfrak{s}} + SW^{red}_{\mathfrak{s}}.$$

We can interpret the latter invariant in terms of the closed manifold as follows.

Definition 3.6. Let Σ be a surface of genus g, and let $\{A_i, B_i\}_{i=1}^{g}$ be a standard symplectic basis for $H_1(\Sigma;\mathbb{Z})$. For $j = 0,\ldots,g$, let $\xi_j(\Sigma) \in \mathbb{A}(\Sigma)$ be the degree $2j$ component of

$$\prod_{i=1}^{g}\Big(1 + U + A_i \cdot B_i\Big) \in \mathbb{A}(\Sigma);$$

i.e., $\xi_0 = 1$, $\xi_1(\Sigma) = gU + \sum A_i \cdot B_i$, $\ldots$, $\xi_g(\Sigma) = \prod_{i=1}^{g}\Big(U + A_i \cdot B_i\Big)$.

Proposition 3.7. *Suppose*

$$0 \leq -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2 \quad \textit{and} \quad -[\Sigma]\cdot[\Sigma] > 2g - 2.$$

Then, letting

$$e = g - 1 + \frac{\langle c_1(\mathfrak{s}), [\Sigma]\rangle - [\Sigma]\cdot[\Sigma]}{2},$$

we have that

$$SW^{red}_{\mathfrak{s}}(a) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a \cdot \xi_{g-1-e}(\Sigma))$$

for all $a \in \mathbb{A}(X)$.

Furthermore, under the homological condition of Theorem 1.8, we will express $SW^{irr}_{\mathfrak{s}}$ in terms of $SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}$, as follows.

Proposition 3.8. *Suppose*

$$0 \leq -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2 \quad \textit{and} \quad -[\Sigma]\cdot[\Sigma] > 2g - 2,$$

and let ℓ be an integer so that there is a symplectic basis $\{A_i, B_i\}_{i=1}^{g}$ for $H_1(\Sigma)$ so that $i_*(A_i) = 0$ in $H_1(X;\mathbb{R})$ for $i = 1,\ldots,\ell$. Then, for each $b \in \mathbb{A}(\Sigma)$ of degree $e < d(b) \leq \ell$, there is an element $b'' \in \mathbb{A}(\Sigma)$ so that

$$SW^{irr}_{\mathfrak{s}}(a \cdot i_*(b)) = SW_{\mathfrak{s}-\mathrm{PD}[\Sigma]}(a \cdot i_*(b'')).$$

Furthermore, b_2 lies in the ideal generated by $H_1(\Sigma)$ in $\mathbb{A}(\Sigma)$.

The proof of Proposition 3.8 is given in the end of Section 6.

Proposition 3.1 follows immediately from Propositions 3.5–3.8. In the proof of these latter propositions, we will construct a natural Seiberg-Witten-Floer functor for four-manifolds which bound Y.

Before proceeding, we pause to tie up one more loose end: Theorem 1.4. That result can be reduced to a relation which replaces Theorem 1.8, using the same argument given in the proof of Theorem 1.3. The relevant relation in this case is:

Theorem 3.9. *Let X be a smooth, closed, connected, oriented four-manifold with $b_2^+(X) > 0$. Let $\Sigma \subset X$ be a surface with genus $g(\Sigma) > 0$. For each* $\mathrm{Spin}_{\mathbb{C}}$ *structure $\mathfrak{s}$ with*

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \geq 0$$

and each $b \in \mathbb{A}(\Sigma)$ of degree $d(b)$ with

$$-\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] + d(b) > 2g(\Sigma) - 2, \tag{12}$$

there is an element $b' \in \mathbb{A}(\Sigma)$ with $d(b') \geq d(b)$ so that for any $a \in \mathbb{A}(X)$, we have

$$SW_{X,\mathfrak{s}}(a \cdot i_*(b)) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a \cdot i_*(b')). \tag{13}$$

Furthermore, if $b = U^d$, then $b' - U^{d'}$ lies in the ideal generated by $H_1(\Sigma)$ in $\mathbb{A}(\Sigma)$.

Once again, via the blowup formula, this relation can be reduced to the case where the self-intersection number Σ is very negative; i.e.,

$$0 \leq -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2 \quad \text{and} \quad -[\Sigma]\cdot[\Sigma] > 2g - 2.$$

(compare Proposition 3.1). Like Proposition 3.1, this special case also follows from the product formula in Proposition 3.5, the relation in Proposition 3.7, together with the following analogue of Proposition 3.8 (whose proof is also given in the end of Section 6):

Proposition 3.10. *Suppose*

$$0 \leq -\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2 \quad \textit{and} \quad -[\Sigma]\cdot[\Sigma] > 2g - 2.$$

Then, for each $b \in \mathbb{A}(\Sigma)$ of degree $2e < d(b)$, there is an $b'' \in \mathbb{A}(\Sigma)$ so that

$$SW^{irr}_{\mathfrak{s}}(a \cdot i_*(b)) = SW_{\mathfrak{s}-\mathrm{PD}[\Sigma]}(a \cdot i_*(b'')).$$

Furthermore, b'' lies in the ideal generated by $H_1(\Sigma)$.

4. Gauge theory on $\mathbb{R} \times Y$

The Seiberg-Witten moduli spaces over Y and $\mathbb{R} \times Y$ were studied for Seifert fibered three-manifolds Y in [22]. We summarize these results here, for Y a circle-bundle over a Riemann surface Σ with $g(\Sigma) > 0$ and Euler number $-n$, where $n > 2g - 2$.

Y admits a canonical $\mathrm{Spin}_{\mathbb{C}}$ structure whose bundle of spinors is $\mathbb{C} \oplus \pi^*({K_\Sigma}^{-1})$, which we use to identify the $\mathrm{Spin}_{\mathbb{C}}$ structures on Y with $H^2(Y;\mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n\mathbb{Z}$.

Let $\mathcal{N}_Y(\mathfrak{t})$ denote the moduli space of solutions to the Seiberg-Witten equations over Y in the $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{t}$. Here, we use the metric g_Y and $SO(3)$-connection over TY of [22]. Given a pair of components C_1, C_2 in $\mathcal{N}_Y(\mathfrak{t})$, let $\mathcal{M}(C_1, C_2)$ denote the moduli space of solutions $[A, \Phi]$ to the Seiberg-Witten equations on $\mathbb{R} \times Y$ for which

$$\lim_{t \mapsto -\infty} [A, \Phi]|_{\{t\} \times Y} \in C_1, \quad \text{and} \quad \lim_{t \mapsto \infty} [A, \Phi]|_{\{t\} \times Y} \in C_2.$$

This moduli space admits a translation action by $\mathbb{R}$. Let $\widehat{\mathcal{M}}(C_1, C_2)$ denote the quotient of $\mathcal{M}(C_1, C_2)$ by the translation action.

In general, these spaces admit a Morse-theoretic interpretation. If $c_1(\mathfrak{t})$ is a torsion class, there is a real-valued functional

$$\mathrm{CSD}\colon \mathcal{B}(Y, \mathfrak{t}) \longrightarrow \mathbb{R}$$

defined over the configuration space $\mathcal{B}(Y, \mathfrak{t})$ of pairs (B, Ψ) of spin-connections B in $\mathfrak{t}$ and spinors Ψ modulo gauge (cf. Equation (28) below; when $e = g - 1$, we will use the perturbation given in Equation (32)). The critical manifolds are the moduli spaces $\mathcal{N}(Y; \mathfrak{t})$. When $c_1(\mathfrak{t})$ is not torsion, the functional is circle-valued. The Seiberg-Witten equations on $\mathbb{R} \times Y$ are the upward gradient-flow equations for this functional. In keeping with this interpretation, we call $\widehat{\mathcal{M}}(C_1, C_2)$ the space of unparameterized flows from C_1 to C_2.

Theorem 4.1. [22] *Let Y be a circle-bundle over a Riemann surface with genus $g > 0$ and Euler number $-n < 2 - 2g$. The moduli space $\mathcal{N}_Y(\mathfrak{t})$ is empty unless $\mathfrak{t}$ corresponds to a torsion class in $H^2(Y;\mathbb{Z})$. So, suppose $\mathfrak{t}$ corresponds to $e \in \mathbb{Z}/n\mathbb{Z} \subset H^2(Y;\mathbb{Z})$.*

(1) *If $0 \leq e < g - 1$ then $\mathcal{N}_Y(\mathfrak{t})$ contains two components, a reducible one $\mathcal{J}$, identified with the Jacobian torus $H^1(\Sigma; \mathbb{R}/\mathbb{Z})$, and a smooth irreducible component C diffeomorphic to* $\mathrm{Sym}^e(\Sigma)$.

Both of these components are non-degenerate in the sense of Morse-Bott. There is an inequality $\mathrm{CSD}(\mathcal{J}) > \mathrm{CSD}(C)$*, so the space* $\widehat{\mathcal{M}}(\mathcal{J}, C)$ *is empty. The space* $\widehat{\mathcal{M}}(C, \mathcal{J})$ *is smooth of expected dimension* $2e$*; indeed it is diffeomorphic to* $\mathrm{Sym}^e(\Sigma)$*.*

(2) *If* $g - 1 < e \leq 2g - 2$*, the Seiberg-Witten moduli spaces over both* Y *and* $\mathbb{R} \times Y$ *in this* $\mathrm{Spin}_{\mathbb{C}}$ *structure are naturally identified with the corresponding moduli spaces in the* $\mathrm{Spin}_{\mathbb{C}}$ *structure* $2g - 2 - e$*, which we just described.*

(3) *For all other* $e \neq g - 1$*,* $\mathcal{N}_Y(\mathfrak{t})$ *contains only reducibles. Furthermore, it is smoothly identified with the Jacobian torus.*

In the $\mathrm{Spin}_{\mathbb{C}}$ structure corresponding to $g - 1 \in \mathbb{Z}/n\mathbb{Z}$, the unperturbed Seiberg-Witten equations used in Theorem 4.1 are inconvenient, since the corresponding reducible manifold is not smooth in the sense of Morse-Bott. To overcome this difficulty, when working in this $\mathrm{Spin}_{\mathbb{C}}$ structure, we use a perturbation of the equations where the theory resembles the case where $0 \leq e < g - 1$ (and, in particular, the reducibles are smooth). A thorough discussion of the perturbation is given in Section 8.

5. The product formula

In this section, we define two quantities, SW^{irr} and SW^{red}, and prove that the Seiberg-Witten invariant decomposes into a sum of these (Propostion 3.5). Furthermore, we express SW^{red} in terms of another Seiberg-Witten invariant of X (Proposition 3.7).

Decompose X as
$$X = N \cup_Y X_0,$$
where Y is unit circle bundle over Σ with Euler number $-n$, with $n > 2g - 2$. N is a tubular neighborhood of the surface Σ (which is diffeomorphic to the disk bundle associated to Y), and X_0 is the complement in X of the interior of N. Fix metrics g_{X_0}, g_N, and g_Y for which g_{X_0} and g_N are isometric to
$$dt^2 + g_Y^2$$
in a collar neighborhood of their boundaries (where t is a normal coordinate to the boundary). Let $X(T)$ denote the Riemannian manifold

which is diffeomorphic to X and whose metric g_T is obtained from the description

$$X(T) = N \cup_{\partial N = \{-T\}\times Y} [-T,T] \times Y \cup_{\{T\}\times Y = -\partial X_0} X_0;$$

i.e., $g_T|_N = g_N$, $g_T|_{[-T,T]\times Y} = dt^2 + g_Y^2$, and $g_T|_{X_0} = g_{X_0}$. Our goal here is to provide, for all sufficiently large T, a description of the moduli space $\mathcal{M}_{X(T)}(\mathfrak{s})$ on $X(T)$ in terms of the moduli spaces for Y, $\mathcal{N}_Y(\mathfrak{s}|_Y)$, and the finite-energy, cylindrical-end moduli spaces associated to X_0 and N, denoted $\mathcal{M}_{X_0}(\mathfrak{s}|_{X_0})$, and $\mathcal{M}_N(\mathfrak{s}|_N)$ respectively. In this context, finite energy means that the total variation of the Chern-Simons-Dirac functional over the infinite cylinder is bounded. Henceforth, X_0 and N will denote the cylindrical-end manifolds obtained by attaching $[0,\infty)\times Y$ (with appropriate orientations) to the corresponding subsets of X.

In the case where $b_2^+(X) = 1$, we choose the perturbing form η to be compactly supported in X_0 in such a way that

$$-2\pi c_1(\mathfrak{s})\cdot\omega_\infty + \int_{X_0} \eta \wedge \omega_\infty$$

has the same sign as $\gamma\cdot\omega_\infty$, where γ is a compactly supported representative for a class in the chosen component $\mathcal{K}_0 \subset \mathcal{K}(X)$, and ω_∞ is a self-dual harmonic two-form over X_0 with $\int_{X_0}\omega_\infty\wedge\omega_\infty = 1$. Note that such a γ and ω_∞ can be found since $\Sigma\cdot\Sigma < 0$, forcing $b_2^+(X_0) = 1$ (see [1]). Now, the moduli spaces of the η-perturbed Seiberg-Witten equations over $X(T)$ calculate the invariant in the chosen chamber for all sufficiently large T.

We collect useful facts about the moduli spaces $\mathcal{M}_N(\mathfrak{s}|_N)$, most of which we defer to Section 7 (see also [24]), but first we introduce some notation. The map

$$\mathrm{Spin}_{\mathbb{C}}(N) \to \mathbb{Z}$$

given by

$$\mathfrak{s} \mapsto \langle c_1(\mathfrak{s}), [\Sigma]\rangle$$

induces a one-to-one correspondence between $\mathrm{Spin}_{\mathbb{C}}$ structures and integers which are congruent to n modulo 2. Note that the $\mathrm{Spin}_{\mathbb{C}}$ structure over Y $\mathfrak{s}|_Y$ corresponds to the mod n reduction of

$$e = g - 1 + \frac{\langle c_1(\mathfrak{s}), [\Sigma]\rangle + n}{2}$$

appearing in Theorem 4.1.

By taking limits at the end of the tube, one can define maps

$$\rho\colon \mathcal{M}_N(\mathfrak{s}) \longrightarrow \mathcal{N}_Y(\mathfrak{s}|_Y) \quad \text{and} \quad \rho\colon \mathcal{M}_{X_0}(\mathfrak{s}) \longrightarrow \mathcal{N}_Y(\mathfrak{s}|_Y)$$

(see [20]). If C is a connected manifold of $\mathcal{N}_Y(\mathfrak{s}|_Y)$, then $\mathcal{M}_N(\mathfrak{s}, C)$ and $\mathcal{M}_{X_0}(\mathfrak{s}, C)$ denotes the pre-image of C under ρ. Throughout the following discussion, we will use the perturbation discussed in Section 8 over X_0, N, and Y, when $\mathfrak{s}|_Y$ corresponds to $e = g-1$ (in the notation of Section 4); i.e., in this case, $\mathcal{N}_Y(\mathfrak{s}|_Y)$, $\mathcal{M}_N(\mathfrak{s})$ and $\mathcal{M}_{X_0}(\mathfrak{s}|_{X_0})$ will denote the perturbed versions of these moduli spaces, with perturbation parameter u in the range $0 < u < 2$, in the notation of Section 8. (We will show in Section 8 that this is an allowable perturbation to use when $b_2^+(X) = 1$; i.e., we are computing the Seiberg-Witten invariants in the correct chamber.) When they are clear from the context, we leave the $\mathrm{Spin}_{\mathbb{C}}$ structures out of the notation. Note that on the cylinders, the analogous boundary value maps factor through the unparameterized spaces, defining

$$\rho_{\mathcal{J}}\colon \widehat{\mathcal{M}}(C, \mathcal{J}) \longrightarrow \mathcal{J} \quad \text{and} \quad \rho_C\colon \widehat{\mathcal{M}}(C, \mathcal{J}) \longrightarrow C,$$

where $\mathcal{J}$ and C are the critical manifolds of Theorem 4.1.

Proposition 5.1. *Suppose that $-n-2g+2 \leq \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq -n$, and let*

$$e = g - 1 + \frac{\langle c_1(\mathfrak{s}), [\Sigma]\rangle + n}{2},$$

Then according to Theorem 4.1, and Theorem 8.1 when $e = g-1$, $\mathcal{N}_Y(\mathfrak{s}|_Y)$ has two components, $\mathcal{J}$ and C, where C is diffeomorphic to $\mathrm{Sym}^e(\Sigma)$. Furthermore, the expected dimensions of the moduli spaces over N and X_0 are given by:

$$\text{e-dim}\mathcal{M}_N(\mathcal{J}) = 2e+1 \tag{14}$$

$$\text{e-dim}\mathcal{M}_N(C) = 2e \tag{15}$$

$$\text{e-dim}\mathcal{M}_{X_0}(\mathcal{J}) = 2d+2g-2e-2 \tag{16}$$

$$\text{e-dim}\mathcal{M}_{X_0}(C) = 2d, \tag{17}$$

where $d = d(\mathfrak{s})$ and $g = g(\Sigma)$. Moreover, $\mathcal{M}^*_N(\mathcal{J})$, $\mathcal{M}_N(C)$, $\mathcal{M}_{X_0}(\mathcal{J})$, and $\mathcal{M}_{X_0}(C)$ are transversally cut out by the Seiberg-Witten equations (in particular, they are manifolds of the expected dimension).

Proof. This is a combination of Proposition 7.9 and 7.10 when $\langle c_1(\mathfrak{s}), [\Sigma]\rangle \neq n$, and Proposition 8.3 in the remaining case. q.e.d.

When studying the deformation theory of reducibles inside $\mathcal{M}_N(\mathcal{J})$, the L^2 kernel and the cokernel of the Dirac operator (on the cylindrical-end version of N) play a central role. These spaces can be concretely understood, thanks to the holomorphic interpretation of the Dirac operator (see also [24]).

Proposition 5.2. *Suppose that* $-n-2g+2 \leq \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq -n$, *then there is a natural correspondence between reducibles* $[(A,0)] \in \mathcal{M}_N(\mathcal{J})$ *with holomorphic line bundles* $\mathcal{E}$ *of degree* e *over* Σ *which identifies*

$$\mathrm{Ker} \not{D}_A = H^0(\Sigma, \mathcal{E}) \quad \textit{and} \quad \mathrm{Coker} \not{D}_A = H^1(\Sigma, \mathcal{E}).$$

Proof. This follows from Theorem 7.4 and Proposition 7.5 (see also the proof of Theorem 8.1 in the perturbed case). q.e.d.

The above proposition allows us to understand an important class of reducibles.

Definition 5.3. The *jumping locus* $\Theta \subset \mathcal{M}_N(\mathcal{J})$ is the locus of reducible solutions $[(A,0)] \in \mathcal{M}_N(\mathcal{J})$ for which $\mathrm{Ker} \not{D}_A$ is non-trivial.

Corollary 5.4. *Suppose that* $-n-2g+2 \leq \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq -n$, *then the jumping locus* $\Theta \subset \mathcal{J} = \mathcal{M}_N^{\mathrm{red}}(\mathcal{J})$ *is the image of a smooth map* $\mathrm{Sym}^e(\Sigma) \longrightarrow \mathcal{J}$.

Proof. According to Proposition 5.2, the space $\Theta \subset \mathcal{J}$ is identified with the space of degree e line bundles over Σ with non-trivial H^0. The forgetful map $\mathrm{Sym}^e(\Sigma) \longrightarrow \mathcal{J}$ which takes a degree e divisor, thought of as a complex line bundle with section, to the underlying complex line bundle gives the surjection to this locus. q.e.d.

We will also need to understand those $\mathrm{Spin}_{\mathbb{C}}$ structures $\mathfrak{s} \in \mathrm{Spin}_{\mathbb{C}}(N)$ for which $-n < \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq n$.

Proposition 5.5. *If*

$$-n < \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq n,$$

then the moduli space $\mathcal{M}_N(\mathcal{J})$ *contains only reducibles. Moreover, the space of reducibles is smoothly identified with the Jacobian torus* $\mathcal{J}$ *(i.e., the kernel and the cokernel of the Dirac operator coupled to any reducible vanishes). Furthermore,* $\mathcal{M}_N(C)$ *is empty.*

Proof. When $|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| < n$, this is proved in Section 7, where it appears as Proposition 7.6. The remaining case is covered by Proposition 8.2. q.e.d.

With these preliminaries in place, we turn to the Seiberg-Witten invariants of X, by investigating the moduli spaces over $X(T)$. Specifically, choose some $a \in \mathbb{A}(X)$ of degree $d(\mathfrak{s})$, and indeed choose representatives for the corresponding homology classes which are compactly supported in X_0. Let $V(a)$ denote the corresponding representatives for $\mu(a)$ in the configuration spaces for X_0 and $X(T)$ as appropriate (see Section 9 for a discussion of such representatives). Recall that $SW_{X,\mathfrak{s}}(a)$ is the number of points in $\mathcal{M}_{X(T)}(\mathfrak{s}) \cap V(a)$, counted with appropriate sign.

Lemma 5.6. *Suppose that $-n - 2g + 2 \leq \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq -n$, then for each $\epsilon > 0$, there is a $T_0 > 0$ so that for all $T \geq 2T_0$ the restriction of $[(A, \Phi)] \in \mathcal{M}_{X(T)}(\mathfrak{s}) \cap V(a)$ to any slice $\{t\} \times Y$ with $t \in [-T_0, T_0]$ lies within ϵ (in the C^∞ topology) from either $\mathcal{J}$ or C. Accordingly, if ϵ is sufficiently small, then $[(A, \Phi)]$ satisfies exactly one of the following two conditions:*

(H-1) *$[(A, \Phi)]|_N$ is C^∞ close to smooth reducible and $[(A, \Phi)]|_{X_0}$ is C^∞ close to (the restriction to X_0) of a configuration in $\mathcal{M}_{X_0}(\mathcal{J}) \cap V(a)$.*

(H-2) *$[(A, \Phi)]|_N$ is C^∞ close to a configuration in $\mathcal{M}_N(C)$, and $[(A, \Phi)]|_{X_0}$ is C^∞ close to a configuration in the cut-down moduli space $\mathcal{M}_{X_0}(C) \cap V(a)$.*

Proof. This is a dimension-counting argument. Suppose we have a sequence $[A_i, \Phi_i] \in \mathcal{M}_{X(T_i)}(\mathfrak{s}) \cap V(a)$, for some increasing, unbounded sequence $\{T_i\}_{i=1}^\infty$ of real numbers. By local compactness, there is a subsequence which converges in C^∞_{loc} to a pair of configurations (A_N, Φ_N) and (A_{X_0}, Φ_{X_0}) over N and X_0 respectively. By the usual compactness arguments (see [12]), the total variation of the Chern-Simons-Dirac functional of (A_i, Φ_i) over the cylinder $[-T_i, T_i] \times Y$ remains globally bounded (independent of i), so (A_N, Φ_N) and (A_{X_0}, Φ_{X_0}) both have finite energy.

First, we prove that either Hypothesis (H-1) or (H-2) is satisfied. There are *a priori* four cases, according to which critical manifolds $\rho[A_{X_0}, \Phi_{X_0}]$ and $\rho[A_N, \Phi_N]$ lie in.

(P-1) The case where $\rho[A_N, \Phi_N] \in \mathcal{J}$ while $\rho(A_{X_0}, \Phi_{X_0}) \in C$ is excluded because $\mathrm{CSD}(C) > \mathrm{CSD}(\mathcal{J})$.

(P-2) The case where $\rho[A_N,\Phi_N]\in C$ while $\rho(A_{X_0},\Phi_{X_0})\in\mathcal{J}$ is excluded by a dimension count, as follows. In this case, we see that $\rho[A_{X_0},\Phi_{X_0}]\in\rho_{\mathcal{J}}(\mathcal{M}(C,\mathcal{J}))\cap\rho(\mathcal{M}_{X_0}(\mathcal{J})\cap V(a))$. But

$$\rho_{\mathcal{J}}(\mathcal{M}(C,\mathcal{J}))=\rho_{\mathcal{J}}(\widehat{\mathcal{M}}(C,\mathcal{J})),$$

so

$$\text{e-dim}(\rho_{\mathcal{J}}(\mathcal{M}(C,\mathcal{J}))\cap\rho(\mathcal{M}_{X_0}(\mathcal{J})\cap V(a)) \quad = \quad -2.$$

It follows from Theorems 4.1 and 8.1 that $\mathcal{M}(C,\mathcal{J})$ is smooth of the expected dimension, so from the usual transversality results, the above intersection is generically empty.

(P-3) Suppose that $\rho(A_N,\Phi_N)\in\mathcal{J}$ and $\rho(A_{X_0},\Phi_{X_0})\in\mathcal{J}$. Then we see that

$$\rho[A_N,\Phi_N]=\rho[A_{X_0},\Phi_{X_0}]\in\rho(\mathcal{M}_N(\mathcal{J}))\cap\rho(\mathcal{M}_{X_0}(\mathcal{J})\cap V(a));$$

but, according to Proposition 5.1

$$\begin{aligned}&\text{e-dim}\rho(\mathcal{M}^*_N(\mathcal{J}))\cap\rho(\mathcal{M}_{X_0}(\mathcal{J})\cap V(a))\\ &\quad=\quad \text{e-dim}\mathcal{M}_N(\mathcal{J})+\mathcal{M}_{X_0}(\mathcal{J})-2d-2g\\ &\quad=\quad -1,\end{aligned}$$

which is generically empty. Thus, it follows that $[A_N,\Phi_N]$ must be reducible. Moreover, according to Corollary 5.4,

$$\begin{aligned}\text{e-dim}\rho(\Theta)\cap\rho(\mathcal{M}_{X_0}(\mathcal{J})\cap V(a)) \quad &= \quad 2e+\text{e-dim}\mathcal{M}_{X_0}(\mathcal{J})-2d-2g\\ &= \quad -2,\end{aligned}$$

which is also generically empty. Hence, $[A_N,\Phi_N]$ and $[A_{X_0},\Phi_{X_0}]$ satisfy Hypotheses (H-1).

(P-4) If $\rho[A_N,\Phi_N]$ and $\rho[A_{X_0},\Phi_{X_0}]$ both lie in C, then the Hypotheses (H-2) are satisfied.

The assertion at the beginning of the proposition follows easily.

q.e.d.

The above proposition says that we can partition the points in the cut-down moduli space (which is an oriented, zero-dimensional manifold) for sufficiently large T into two disjoint sets, the subsets of configurations which satisfy (H-1) and (H-2) respectively. Thus, if we let

$SW^{red}_{\mathfrak{s}}(a)$ and $SW^{irr}_{\mathfrak{s}}(a)$ be the signed number of points satisfying (H-1) and (H-2) respectively, then

$$SW_{X,\mathfrak{s}}(a) = SW^{red}_{\mathfrak{s}}(a) + SW^{irr}_{\mathfrak{s}}(a). \tag{18}$$

As we shall see, gluing theory allows us to compute both of these quantities in terms of cylindrical-end moduli spaces. So, in the next step, we study these cylindrical-end moduli spaces.

Lemma 5.7. *For all* $\mathrm{Spin}_{\mathbb{C}}$ *structures* $\mathfrak{s}$ *on* X *the corresponding moduli spaces* $\mathcal{M}_N(C)$*,* $\mathcal{M}_{X_0}(\mathcal{J})$*, and* $\mathcal{M}_{X_0}(C) \cap V(a)$ *are all compact manifolds.*

Proof. The compactness of $\mathcal{M}_{X_0}(\mathcal{J})$ and $\mathcal{M}_N(C)$ follows from the usual compactness arguments, together with the facts that the Chern-Simons-Dirac functional is real-valued, $\mathrm{CSD}(\mathcal{J}) > \mathrm{CSD}(C)$, and there are no other critical manifolds. Compactness of $\mathcal{M}_{X_0}(C) \cap V(a)$ follows from this, together with a straightforward dimension count (see the discussion above in the proof of Lemma 5.6, part (P-2)). q.e.d.

Compactness of $\mathcal{M}_{X_0}(\mathcal{J})$ allows us to define a relative invariant with reducible boundary values. We pause to discuss some relevant properties of this invariant.

Definition 5.8. Let $\mathfrak{s}_0$ be a $\mathrm{Spin}_{\mathbb{C}}$ structure on X_0 which extends over X. Since the moduli space $\mathcal{M}_{X_0,\mathfrak{s}_0}(\mathcal{J})$ is compact, there is a *relative Seiberg-Witten invariant*

$$SW_{(X_0,\mathfrak{s}_0,\mathcal{J})} \colon \mathbb{A}(X_0) \longrightarrow \mathbb{Z},$$

defined by the pairing $SW_{(X_0,\mathfrak{s}_0,\mathcal{J})}(a) = \langle [\mathcal{M}_{X_0,\mathfrak{s}_0}(\mathcal{J})], \mu(a) \rangle$.

This relative invariant is related to an absolute invariant, according to the following.

Proposition 5.9. *If* $\mathfrak{s}$ *satisfies* $-n < \langle c_1(\mathfrak{s}), [\Sigma] \rangle \leq n$*, then for all* $a \in \mathbb{A}(X)$*,*

$$SW_{X,\mathfrak{s}}(a) = SW_{(X_0,\mathfrak{s}_0,\mathcal{J})}(a),$$

where $\mathfrak{s}_0 = \mathfrak{s}|_{X_0}$.

Proof. Recall that $\mathcal{M}(\mathfrak{s}|_N)$ consists entirely of reducibles all of which are smooth, according to Proposition 5.5; thus, gluing theory identifies the moduli spaces $\mathcal{M}_{X(T)}(\mathfrak{s})$ for large T with $\mathcal{M}_{X_0,\mathfrak{s}_0}(\mathcal{J})$. (See also [24], where this result appears as Proposition 2.7.) q.e.d.

We now return to the discussion of SW^{red} and SW^{irr}. Although the definitions of both terms implicitly use T, we show now that if T is sufficiently large, then the terms can be computed from absolute invariants (and hence are independent of the parameter).

Proposition 5.10. *Suppose that $\mathfrak{s}$ satisfies*

$$-n-2g+2 \leq \langle c_1(\mathfrak{s}), [\Sigma]\rangle \leq -n,$$

where Σ has self-intersection number $-n$, and let

$$e = g-1+\frac{\langle c_1(\mathfrak{s}), [\Sigma]\rangle + n}{2}.$$

Then, for all sufficiently large T,

$$SW^{red}_{\mathfrak{s}}(a) = SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}(a\cdot \xi_{g-1-e}(\Sigma)),$$

where $\xi_{g-1-e}(\Sigma)\in \mathbb{A}(\Sigma)$ is the element defined in Definition 3.6*.*

Proof. The moduli space $\mathcal{M}^{\mathrm{red}}_N(\mathcal{J})-\Theta$ comes equipped with an obstruction bundle $\Xi \longrightarrow \mathcal{M}^{\mathrm{red}}_N(\mathcal{J})-\Theta$, defined by $\Xi_{[(A,0)]} = \mathrm{Coker}\not{D}_A$. (whose K-theory class canonically extends over all of $\mathcal{M}^{\mathrm{red}}_N(\mathcal{J})$). The dimension count in Lemma 5.6 guarantees that each solution in $\mathcal{M}_{X_0}(\mathcal{J})\cap V(a)$ extends uniquely to a smooth reducible over N. Thus, gluing theory gives that

$$\begin{aligned} SW^{red}_{\mathfrak{s}}(a) &= \langle [\mathcal{M}_{X_0}(\mathcal{J})\cap V(a)], \mathbf{e}(\mathcal{L}\otimes\rho^*(\Xi))\rangle \\ &= \langle [\mathcal{M}_{X_0}(\mathcal{J})\cap V(a)], c_{g-1-e}(\mathcal{L}\otimes\rho^*(\Xi))\rangle \\ &= \langle [\mathcal{M}_{X_0}(\mathcal{J})], \mu(a)\cup c_{g-1-e}(\mathcal{L}\otimes\rho^*(\Xi))\rangle, \end{aligned}$$

where $\mathbf{e}(\mathcal{L}\otimes\rho^*(\Xi))$ denotes the Euler class of the bundle over $\mathcal{M}^{\mathrm{red}}_N(\mathcal{J})-\Theta$. According to the Riemann-Roch formula, $\dim(\Xi) = 2g-2-2e$ so the Euler class agrees with the top Chern class c_{g-1-e}, which in turn extends over all of $\mathcal{J}$. Using the index theorem for families, together with the holomorphic interpretation of the obstruction bundle Ξ given in Proposition 5.2, it is a straightforward computation that the total Chern class of Ξ is

$$\prod_{i=1}^{g}(1+\mu(A_i)\mu(B_i))$$

(see also [24] Proposition 2.6); thus,

$$c_{g-1-e}(\mathcal{L}\otimes\rho^*(\Xi)) = \xi_{g-1-e}(\Sigma).$$

Putting all this together, we have that

$$SW^{red}_{\mathfrak{s}}(a) = SW_{(X_0,\mathfrak{s}_0,\mathcal{J})}(a\cdot\xi_{g-1-e}(\Sigma)), \tag{19}$$

where $\mathfrak{s}_0 = \mathfrak{s}|_{X_0}$. Since $n-2g+2 \leq \langle c_1(\mathfrak{s}-\mathrm{PD}[\Sigma]),[\Sigma]\rangle \leq n$ and $\mathfrak{s}-\mathrm{PD}[\Sigma]|_{X_0} = \mathfrak{s}_0$, the proposition then follows from Proposition 5.9.

q.e.d.

Proposition 5.11. *For sufficiently large T,*

$$SW^{irr}_{\mathfrak{s}}(a) = \#\mathcal{M}_{X_0}(C)\cap V(a).$$

Proof. Gluing shows that

$$SW^{irr}_{\mathfrak{s}}(a) = \Big(\#\mathcal{M}_{X_0}(C)\cap V(a)\Big)\Big(\deg(\rho\colon \mathcal{M}_N(C)\to C)\Big).$$

According to Propositions 7.9 and 8.3, $\rho\colon \mathcal{M}_N(C)\to C$ either has degree $+1$, or $\mathcal{M}_N(C)$ is empty. The latter case would force $SW^{irr}_{\mathfrak{s}}(a)\equiv 0$ (for the given genus and self-intersection number).

To rule out this latter case, we need only look at an example where the irreducible term is non-zero. Let X be a ruled surface X over Σ associated to the line bundle with Euler number $-n$. Let $\Sigma\subset X$ denote the section with self-intersection number $-n$, and fix any $0\leq e\leq g-1$. Let $\mathfrak{s}$ denote the $\mathrm{Spin}_{\mathbb{C}}$ structure over X given by $\mathfrak{s} = \mathfrak{s}_0 + e\mathrm{PD}[F]$, where $\mathfrak{s}_0$ is the canonical $\mathrm{Spin}_{\mathbb{C}}$ structure on X associated to the Kähler structure, and F denotes a fiber in the ruling. It is easy to see that $SW_{X,\mathfrak{s}-\mathrm{PD}[\Sigma]}\equiv 0$, as the corresponding space of divisors is empty (see Proposition 7.5). Moreover, we know that $SW_{X,\mathfrak{s}}\not\equiv 0$ (compare the example in Section 2.2). Thus, in light of Equation (18) and Proposition 5.10, we have examples where $SW^{irr}\not\equiv 0$, forcing the degree to be non-zero. q.e.d.

We will give the seemingly *ad hoc* quantity $\#\mathcal{M}_{X_0}(C)\cap V(a)$ a more intrinsic formulation in Section 6. With the help of this formulation, we can then prove a vanishing result for this term under suitable algebro-topological hypotheses on the embedding of $\Sigma\subset X$ (Proposition 3.8).

6. Relative invariants

Let X_0 be a smooth, oriented manifold-with-boundary with $b_2^+(X_0)>0$, whose boundary is identified with $\partial X_0 = -Y$, a circle bundle over a Riemann surface Σ of genus $g > 0$ with Euler number $-n$, where $n > 2g-2$.

In Section 5, we studied the moduli space $\mathcal{M}_{X_0}(C)$, and used it to define a relative invariant

$$SW^{irr}_{\mathfrak{s}} \colon \mathbb{A}(X_0) \longrightarrow \mathbb{Z},$$

by cutting down the moduli space $\mathcal{M}_{X_0}(C)$ by submanifolds representing $\mu(a)$ which are induced from compactly supported representatives for homology in X_0 (see Proposition 5.11). When $a \in \mathbb{A}(Y)$, there are alternate representatives which are supported "at infinity." The advantage of these representatives is that the corresponding relative invariant inherits relations arising from the cohomology ring of C. In view of the non-compactness of $\mathcal{M}_{X_0}(C)$, the two types of representatives do not necessarily give rise to the same invariant. However, the difference can be explicitly computed in terms of other Seiberg-Witten invariants. In this section, we recast this discussion in a more algebraic setting, defining an invariant

$$SW_{(X_0,C)} \colon \mathbb{A}(X_0) \otimes H^*(C) \longrightarrow \mathbb{Z}.$$

which simultaneously captures both types of representatives; in particular,

$$SW^{irr}_{\mathfrak{s}}(a) = SW_{(X_0,C)}(a \otimes 1).$$

Proposition 3.8 then follows from properties of this invariant.

A subtlety arises in the definition of $SW_{(X_0,C)}$, since the moduli space $\mathcal{M}_{X_0}(C)$ is not compact. However, we have the following weak compactness theorem.

Definition 6.1. A sequence of configurations $\{[A_i,\Phi_i]\}_{i=1}^{\infty}$ is said to *converge weakly* to a configuration

$$[B,\Psi] \times [A,\Phi] \in \widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J})$$

if $[A_i,\Phi_i]$ converges to $[A,\Phi]$ in $C^{\infty}_{\mathrm{loc}}$, and there is an increasing, unbounded sequence of real numbers $\{T_i\}_{i=1}^{\infty}$ with $T_i > i$, so that the translates of $\{[A_i,\Phi_i]|_{[0,2T_i]\times Y}\}_{i=1}^{\infty}$, viewed as a sequence of configurations on $[-T_i,T_i]\times Y$, converge in $C^{\infty}_{\mathrm{loc}}$ to a configuration which is equivalent (under translations) to $[B,\Psi]$.

Proposition 6.2. *Weak convergence gives the space*

$$\overline{\mathcal{M}}_{X_0}(C) = \mathcal{M}_{X_0}(C) \coprod \widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J})$$

the structure of a compact Hausdorff space.

Proof. This a standard argument from Morse-Floer theory. A general discussion of compactness results for the anti-self-duality equation can be found in [20] (see especially Theorem 6.3.3 of [20]); so we sketch the argument here only briefly.

A sequence $[A_i, \Phi_i] \in \mathcal{M}_{X_0}(C)$ converges in C^∞_{loc}, after passing to a subsequence, to some solution $[A, \Phi]$ to the Seiberg-Witten equations on X_0. Since each of the $[A_i, \Phi_i]$ have finite energy, so does $[A, \Phi]$; thus, it has a boundary value. If $\rho[A, \Phi] \in C$, then the length-energy estimates of L. Simon [28] can be used to show that the convergence is C^∞ as in [20].

If, on the other hand, $\rho[A, \Phi] \notin C$, it must be the case that $\rho[A, \Phi] \in \mathcal{J}$. Now, let $T_i \in \mathbb{R}$ be the number so that

$$\mathrm{CSD}[A_i, \Phi_i]_{\{T_i\}\times Y} = \frac{\mathrm{CSD}(\mathcal{J}) + \mathrm{CSD}(C)}{2}.$$

Clearly, $T_i \mapsto \infty$. After passing to a subsequence, we can find a configuration $[B, \Psi]$ so that the sequence $[A_i, \Phi_i]|_{[0,2T_i]\times Y}$, viewed as a sequence of configurations over $[-T_i, T_i]$, converges in C^∞_{loc} to $[B, \Psi]$. In fact, $[B, \Psi]$ must solve the Seiberg-Witten equations and it must have finite energy, so $[B, \Psi] \in \mathcal{M}(C, \mathcal{J})$. The usual length-energy estimates then guarantee that the boundary values match up. q.e.d.

The topological space $\overline{\mathcal{M}}_{X_0}(C)$ defined in Proposition 6.2 is called the compactified moduli space. The following result follows immediately from its definition.

Proposition 6.3. *The inclusion maps*

$$i\colon \mathcal{M}_{X_0}(C) \longrightarrow \mathcal{B}^*(X_0 - (0,\infty)\times Y)$$

and

$$i \circ \Pi_2\colon \widehat{\mathcal{M}}(C, \mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J}) \longrightarrow \mathcal{B}^*(X_0 - (0,\infty)\times Y)$$

fit together to give a continuous map

$$\bar{i}\colon \overline{\mathcal{M}}_{X_0}(C) \longrightarrow \mathcal{B}^*(X_0 - (0,\infty)\times Y),$$

where $\mathcal{B}^*$ denotes the irreducible configurations.

Similarly, we can extend the restriction map over the compactified moduli space, as follows.

Proposition 6.4. *The restriction maps*

$$\rho_C\colon \mathcal{M}_{X_0}(C) \longrightarrow C$$

and

$$\rho_C \circ \Pi_1\colon \widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J}) \longrightarrow C$$

fit together to give a continuous map

$$\overline{\rho}_C\colon \overline{\mathcal{M}}_{X_0}(C) \longrightarrow C.$$

Proof. If a sequence $[A_n, \Phi_n] \in \mathcal{M}_{X_0}(C)$ converges to an ideal point

$$[B,\Psi] \times [A,\Phi] \in \widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J}),$$

then there is a divergent sequence $\{T_n\}_{n=1}^{\infty}$ of real numbers so that

$$\lim_{n\mapsto\infty} \tau_n^*[A_n,\Phi_n]|_{[T_n,\infty)\times Y} = [B,\Psi]|_{[0,\infty)\times Y},$$

where

$$\tau_n\colon [0,\infty)\times Y \longrightarrow [T_n,\infty)\times Y$$

is the map induced by translation by T_n on the first coordinate. Since each path has finite energy, continuity of the restriction maps (see [20]) guarantees that

$$\lim_{n\mapsto\infty} \rho[A_n,\Phi_n]|_{\{t\}\times Y} = \lim_{n\mapsto\infty} \rho\tau_n^*[A_n,\Phi_n] = \rho[B,\Psi].$$

q.e.d.

Gluing gives this space more structure.

Proposition 6.5. *Gluing endows $\overline{\mathcal{M}}_{X_0}(C)$ with the structure of a manifold. The space of ideal solutions*

$$\widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J})$$

has the structure of a smooth submanifold of codimension two. In particular, a fundamental class for $\mathcal{M}_{X_0}(C)$ gives rise to a unique fundamental class for $\overline{\mathcal{M}}_{X_0}(C)$.

Proof. Gluing describes the end of $\mathcal{M}(X,C)$ as a fibered product

$$\left(\widehat{\mathcal{M}}^0(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}^0_{X_0}(\mathcal{J}) \times (0,\infty)\right)/S^1,$$

where the superscript denotes based versions of the moduli spaces. This gives the space of ideal solutions a disk-bundle neighborhood in $\overline{\mathcal{M}}_{X_0}(C)$. q.e.d.

In light of the above result, we can define the relative Seiberg-Witten invariant $SW_{(X_0,C)}$, as follows.

Definition 6.6. The *relative Seiberg-Witten invariant*

$$SW_{(X_0,C)}\colon \mathbb{A}(X_0) \otimes H^*(C) \longrightarrow \mathbb{Z}$$

is defined by

$$SW_{(X_0,C)}(a \otimes \omega) = \langle [\overline{\mathcal{M}}_{X_0}(C)], \overline{i}^*(\mu(a)) \cup \overline{\rho}_C^*(\omega)\rangle.$$

We now spell out the strategy for proving Proposition 3.8. First, it is shown that for $b \in \mathbb{A}(Y)$, $SW_{(X_0,C)}(a \cdot b \otimes \omega)$ can be expressed in terms of $SW_{(X_0,C)}(a \otimes b \cdot \omega)$ and $SW_{(X_0,\mathcal{J})}$ (Lemma 6.7 and Proposition 6.9). Here, $b \cdot \omega$ denotes the action of $\mathbb{A}(Y)$ on $H^*(C)$ induced from the inclusion of C in $\mathcal{B}^*(Y)$. (Note the cohomology classes over $\overline{\mathcal{M}}_{X_0}(C)$ induced from $\mathbb{A}(Y)$ through the action on $H^*(C)$, and pulled back via $\overline{\rho}$, correspond to divisor representatives over X_0 which are supported "at infinity.") Then, it is shown that $SW_{(X_0,C)}(a \otimes b \cdot \omega)$ vanishes, when b has sufficiently high degree. This follows from algebraic considerations, according to which $b \cdot \omega = b' \cdot \omega$, where $b' \in \mathbb{A}(Y)$ lies in the ideal generated by the cycles in Y which bound in X_0 (Proposition 6.12). It is then easy to see that $SW(a \otimes b' \cdot \omega)$ vanishes (Corollary 6.13).

Now, we express the "commutator" $SW(a \otimes b \cdot \omega) - SW(a \cdot b \otimes \omega)$. First note that if b is induced from $H_1(Y)$, the commutator vanishes, as follows.

Lemma 6.7. *Let $[\gamma] \in H_1(Y)$, then for all $a \in \mathbb{A}(X)$ and $\omega \in H^*(C)$,*

$$SW_{(X_0,C)}(a \otimes \mu[\gamma] \cdot \omega) = SW_{(X_0,C)}(a \cdot \mu[\gamma] \otimes \omega).$$

Proof. We must show that $\overline{\rho}_C^*(\mu[\gamma])$ is homologous to $\overline{i}^*(\mu[\gamma])$. It suffices to verify this over the subset $\mathcal{M}(X_0, C) \subset \overline{\mathcal{M}}_{X_0}(C)$, since the complement has codimension two, and the classes in question are one-dimensional. Over the subset, now, the claim is easy to verify. On $\mathcal{M}_{X_0}(C)$, $\rho^*(\mu[\gamma])$ is represented by $(\mathrm{Hol}_\gamma \circ \rho_C)^*(d\theta)$, the holonomy around a representative of γ "at infinity" (see Proposition 9.1); while $i^*(\mu[\gamma])$ is represented by $\mathrm{Hol}^*_{\gamma_0}(d\theta)$, where $\gamma_0 = 0 \times \gamma \subset 0 \times Y \subset$

$[0,\infty) \times Y \subset X$. Now, the cylinder $[0,\infty) \times \gamma$ provides a homotopy between $\mathrm{Hol}_\gamma \circ \rho_C$ and Hol_{γ_0}. q.e.d.

It remains to see how the point class commutes. For this class, we can express the commutator in terms of $SW_{(X,\mathcal{J})}$ and another Seiberg-Witten invariant, defined below.

Definition 6.8. There is a *Seiberg-Witten invariant of the tube*

$$\widehat{SW}_{(C,\mathcal{J})}\colon H^*(C) \longrightarrow H^*(\mathcal{J}) \subset \mathbb{A}(X_0)$$

which raises degree by dimension $2g - \dim \widehat{\mathcal{M}}(C,\mathcal{J}) = 2g - \dim C$, defined by

$$\widehat{SW}_{(C,\mathcal{J})}(\omega) = (P_2)_*\Big((\rho_{\mathcal{J}} \times \mathrm{Id})^*\mathrm{PD}[\Delta] \cup (\rho_C \circ P_1)^*\omega\Big),$$

where P_1 and P_2 are the projection maps

$$P_1\colon \widehat{\mathcal{M}}(C,\mathcal{J}) \times \mathcal{J} \longrightarrow \widehat{\mathcal{M}}(C,\mathcal{J}) \quad \text{and} \quad P_2\colon \widehat{\mathcal{M}}(C,\mathcal{J}) \times \mathcal{J} \longrightarrow \mathcal{J},$$

and $\mathrm{PD}[\Delta]$ denotes the Poincaré dual of the diagonal $\Delta \subset \mathcal{J} \times \mathcal{J}$. Thus, $\widehat{SW}_{(C,\mathcal{J})}$ satisfies:

$$\langle \widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J}), \overline{i}^*(\mu(a)) \cup \overline{\rho}_C^*(\omega)\rangle = SW_{(X_0,\mathcal{J})}(a \cdot \widehat{SW}_{(C,\mathcal{J})}(\omega)).$$

We can now calculate the commutator, which involves comparing the cohomology classes $\overline{\rho}_C^*\mu(y)$ and $\overline{i}^*\mu(x)$ over $\overline{\mathcal{M}}_{X_0}(C)$, where y is a point in Y and x is a point in X_0.

Proposition 6.9. *Choose points $x \in X_0$ and $y \in Y$. In $\overline{\mathcal{M}}_{X_0}(C)$, we have*

$$\overline{\rho}_C^*(\mu(y)) - \overline{i}^*(\mu(x)) = \mathrm{PD}[\widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J})].$$

Consequently, there is a relation between Seiberg-Witten invariants:

$$\begin{aligned} SW_{(X_0,C)}(a \otimes \mu(y) \cdot \omega) - SW_{(X_0,C)}(a \cdot \mu(x) \otimes \omega) \\ = SW_{(X_0,\mathcal{J})}(a \cdot \widehat{SW}_{(C,\mathcal{J})}(\omega)). \end{aligned}$$

Proof. Clearly, the difference $\overline{\rho}_C^*(\mu(y)) - \overline{i}^*(\mu(x))$ is the first Chern class of the circle bundle $\mathrm{Hom}_{S^1}(\mathcal{L}_x, \mathcal{L}_y)$. Here, $\mathcal{L}_z$ denotes the moduli space based at z; see Section 9. To prove the proposition, we must verify that this bundle admits a section σ in the complement of

$$\widehat{\mathcal{M}}(C,\mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J}) \subset \overline{\mathcal{M}}_{X_0}(C)$$

(i.e., over $\mathcal{M}_{X_0}(C) \subset \overline{\mathcal{M}}_{X_0}(C)$) and that, with respect to a trivialization of the circle bundle over a disk transverse to the submanifold, the restriction of the section to the boundary induces a map from the circle to the circle which has degree one.

The section σ is induced by parallel transport, as follows. Let γ be a half-infinite arc formed by joining $[0, \infty) \times y$ to any arc which connects x to $0 \times y$. Over the point $[A, \Phi] \in \mathcal{M}_{X_0}(C)$, parallel transport via A along γ induces a homomorphism in $\mathrm{Hom}_{S^1}(\mathcal{L}_x, \mathcal{L}_y)$.

We now verify that the trivialization induces a degree one map around circles transverse to the submanifold. For any point in the submanifold

$$[A_1, \Phi_1] \times [A_2, \Phi_2] \in \widehat{\mathcal{M}}(C, \mathcal{J}) \times_{\mathcal{J}} \mathcal{M}_{X_0}(\mathcal{J}),$$

fix fibers

$$[A_1, \Phi_1, \lambda_1] \in \mathcal{L}_x|_{[A_1,\Phi_1]} \quad \text{and} \quad [A_2, \Phi_2, \lambda_2] \in \mathcal{L}_y|_{[A_2,\Phi_2]}.$$

These choices induce a trivialization of $\mathrm{Hom}_{S^1}(\mathcal{L}_x, \mathcal{L}_y)$ over a disk in $\overline{\mathcal{M}}_{X_0}(C)$ transverse to $[A_1, \Phi_1] \times [A_2, \Phi_2]$ (obtained by varying the gluing and translation parameters). Calculating the desired degree amounts to seeing how the holonomy along γ varies as the gluing parameter is rotated. But holonomy along any path which crosses the gluing region once varies as a degree one function of the gluing parameter. q.e.d.

We can understand the action of $\mathbb{A}(Y)$ on $H^*(C)$ explicitly, under the identification $C \cong \mathrm{Sym}^k(\Sigma)$.

Before describing this, we begin with a few preliminaries about the homology of symmetric products of Σ (for an extensive discussion of this topic, see [17]). Recall that $\mathrm{Sym}^k(\Sigma)$ can be viewed as the quotient of the k-fold Cartesian product $\Sigma^{\times k}$ by the action of the symmetric group on k letters. We denote the quotient map by

$$q \colon \Sigma^{\times k} \longrightarrow \mathrm{Sym}^k(\Sigma).$$

According to elementary properties of the transfer homomorphism,

$$q_* \colon H_*(\Sigma^{\times k}) \longrightarrow H_*(\mathrm{Sym}^k(\Sigma))$$

is surjective. Dually, we have a map

$$q^* \colon H^*(\mathrm{Sym}^k(\Sigma)) \longrightarrow H^*(\Sigma^{\times k})$$

which identifies $H^*(\mathrm{Sym}^k(\Sigma))$ with the elements of $H^*(\Sigma^{\times k}) \cong H^*(\Sigma)^{\otimes k}$ which are invariant under the symmetric group action. In particular, by summing over the action, we obtain a map

$$\mathrm{Sym}^k\colon H^*(\Sigma) \longrightarrow H^*(\mathrm{Sym}^k(\Sigma)).$$

Thus, if we fix any collection of points $\{p_2, \ldots, p_k\} \subset \Sigma$, given $\omega \in H^*(\Sigma)$, $\mathrm{Sym}^k(\Sigma)$ is the class characterized by the property that

$$\langle \mathrm{Sym}^k(\omega), q_*(Z \times p_2 \times \cdots \times p_k)\rangle = \langle \omega, Z\rangle,$$

for any cycle $Z \subset \Sigma$. Equivalently, given a cycle $Z \in H_*(\Sigma)$, $\mathrm{Sym}^k(\mathrm{PD}[Z])$ is Poincaré dual to the cycle $q(Z \times \Sigma \times \cdots \times \Sigma)$. The above discussion works over rational coefficients (which suffices for our purposes), but in fact it works over $\mathbb{Z}$ as well, since $H_*(\mathrm{Sym}^k(\Sigma))$ has no torsion (see [17]).

Proposition 6.10. *Under the identification $C \cong \mathrm{Sym}^k(\Sigma)$, the canonical map*

$$\mathbb{A}(Y) \longrightarrow H^*(\mathrm{Sym}^k(\Sigma)),$$

induced from the inclusion of $\mathrm{Sym}^k(\Sigma) = C \longrightarrow \mathcal{B}^*(Y)$, takes $\mu(y)$ for $y \in H_*(Y)$ to the cohomology class $\mathrm{Sym}^k(\mathrm{PD}[\pi_*(y)]) \in H^{2-*}(\mathrm{Sym}^k(\Sigma))$, where $\pi\colon Y \to \Sigma$ is the projection map.

Proof. We can reduce to a corresponding statement for configurations over Σ, as follows. Let E be a line bundle over Σ, so that $W \cong \pi^*(E \otimes (\mathbb{C} \oplus K_\Sigma))$. Then, pull-back induces a map

$$\pi^*\colon \mathcal{B}(\Sigma, E) = \mathcal{A}(E) \times \Gamma(E)/\mathrm{Map}(\Sigma, S^1) \longrightarrow \mathcal{B}(Y, W),$$

to the configurations where the fiber-wise holonomy of the connection is constant, and the section is covariantly constant around each fiber. The identification between the critical manifolds and the symmetric powers $C \cong \mathrm{Sym}^k(\Sigma)$ described in [22] is obtained by proving that C lies in the image of this pull-back map, and indeed that it lies in the pull-back of the vortex moduli space, which, according to [11] (see also [2]), is in turn identified with the space of divisors, by looking at the zero-set of the section. The key points we need presently are that C lies in $\pi^*(\mathcal{B}(\Sigma, E))$, and that configurations are the pull-backs of configurations $[A, \Phi] \in \mathcal{B}(\Sigma, E)$, where Φ is $\overline{\partial}_A$-holomorphic section.

Over $\mathcal{B}(\Sigma, E)$, there is a universal line bundle $\mathcal{L}(\Sigma)$, defined in the usual manner. Note that

$$\mathcal{L}(Y)|_{\pi^*(\mathcal{B}(\Sigma))\times Y} \cong \pi^*(\mathcal{L}(\Sigma)),$$

so $\mu(y)|_{\pi^*(\mathcal{B}(\Sigma))}$ for $y \in H_*(Y)$ agrees with $\mu([\pi_*(y)])$, where the former μ-map is induced from $\mathcal{L}(Y)$, and the latter from $\mathcal{L}(\Sigma)$. We have thus reduced the proof of the proposition to a statement purely over Σ; so for the duration of the proof, $\mathcal{L}$ will refer to $\mathcal{L}(\Sigma)$, $\mathcal{B}$ will refer to $\mathcal{B}(\Sigma, E)$, and all μ-maps will be calculated over Σ.

To facilitate the proof over Σ, we pause for a discussion about the canonical section σ of the universal line bundle $\mathcal{L}$, which takes the configuration $[A, \Phi] \times \{x\} \in \mathcal{B} \times \Sigma$ to the based configuration $[A, \Phi, \Phi(x)]$. This section has the property that, under the canonical identification of $\mathcal{L}|_{[A,\Phi]\times\Sigma} \cong E$ (where E is the bundle over Σ with Chern number k), the restriction $\sigma|_{[A,\Phi]\times\Sigma}$ is identified with the section Φ of E. In particular, if $\Phi \not\equiv 0$ is a holomorphic section, then $\sigma|_{[A,\Phi]\times\Sigma}$ has at most k zeros; moreover, if it has k zeros, then each is transverse.

Now, if $y \in \Sigma$ is a point (i.e., a generator of $H_0(\Sigma; \mathbb{Z})$), then by definition, $\mu(y)$ is the element of $H^2(C)$ whose pairing against any homology class $[S] \in H_2(C)$ is given by

$$\langle \mu(y), [S] \rangle = \langle c_1(\mathcal{L}_y), [S] \rangle.$$

where, as usual, $\mathcal{L}_y$ denotes the restriction of $\mathcal{L}$ to $\mathcal{B}(\Sigma, E) \times \{y\}$. Choose points $\{p_2, \dots, p_k\} \subset \Sigma$ which are distinct from y. Recall that $H_2(\mathrm{Sym}^k(\Sigma))$ is generated by the surface $q(\Sigma \times p_2 \times \cdots \times p_k)$ (where $p_2, \dots, p_k$ are points on Σ), and the tori of the form $q(C_1 \times C_2 \times p_3 \times \cdots \times p_k)$, where $C_1, C_2 \subset \Sigma$ are closed curves in Σ, which we can choose to miss y. The canonical section σ restricted to a torus of the form $q(C_1 \times C_2 \times p_3 \times \cdots \times p_k) \times \{y\}$ clearly vanishes nowhere (as all k of the zeros have been constrained to lie in the set $C_1 \cup C_2 \cup \{p_3, \dots, p_k\}$ which does not include the point y); thus,

$$\langle c_1(\mathcal{L}_y), [q(C_1 \times C_2 \times \cdots \times p_k)] \rangle = 0.$$

Over $q(\Sigma \times p_2, \cdots \times p_k)$ the canonical section vanishes at the single point $q(y \times p_2 \times \cdots \times p_k)$. We verify transversality of this zero, as follows. View σ as a section over $\Sigma \times \Sigma = q(\Sigma \times p_2 \times \cdots \times p_k) \times \Sigma$; we know that $\sigma(y, y) \equiv 0$, and that $D\sigma_{(y,y)}$ induces an isomorphism from $0 \oplus T_y\Sigma$ to E_y (i.e., that the zero of $\sigma|_{\{y\}\times\Sigma}$ at y is transverse). Differentiating the equation that $\sigma(y, y) \equiv 0$, we see that

$$D\sigma_{(y,y)}(0, v) = -D\sigma_{(y,y)}(v, 0).$$

Thus, the section $\sigma|_{\Sigma\times\{y\}}$ of $\mathcal{L}_y|_{[\Sigma]}$ has a single, transverse zero, which shows that

$$\langle c_1(\mathcal{L}), [q(\Sigma \times p_2 \times \cdots \times p_k)] \rangle = \pm 1.$$

Moreover, the sign is positive since the section is holomorphic.

Hence, we have proved the result when $y \in H_0(\Sigma)$. Proving the result for classes coming from $H_1(\Sigma)$ amounts to proving that, if C_1 and C_2 closed curves in Σ which meet transversally, then

$$\langle c_1(\mathcal{L}), q(C_1 \times p_2 \times \cdots \times p_n) \times C_2 \rangle = -\#C_1 \cap C_2 = \#C_2 \cap C_1.$$

Note first that the zeros of the canonical section σ, restricted to $q(C_1 \times p_2 \times \cdots \times p_k) \times C_2$ are the points $C_1 \cap C_2$ (a zero of σ corresponds to a point where the section Φ vanishes at some point of C_2, but the zeros of Φ lie in $C_1 \cup \{p_2, \ldots, p_k\}$, and $\{p_2, \ldots, p_k\} \cap C_2$ is empty). We must now consider the local contribution of each zero (and check transversality).

Consider the map $C_1 \times C_2 \colon S^1 \times S^1 \longrightarrow \mathrm{Sym}^k(\Sigma) \times \Sigma$ defined by $C_1 \times C_2(s,t) = q(C_1(s) \times p_2 \times \cdots \times p_k) \times C_2(t)$. Suppose for notational simplicity that $C_1(0) = C_2(0) = y$. We can view σ as a section of $\mathcal{L}$ pulled back to this torus. Now, evaluated on a typical tangent vector to the torus $a\frac{\partial}{\partial s} + b\frac{\partial}{\partial t}$, the derivative of σ at the intersection point is given by

$$\begin{aligned} D_{(y,y)}\sigma \circ (C_1 \times C_2)(a\frac{\partial}{\partial s} + b\frac{\partial}{\partial t}) &= aD_{(y,y)}\sigma(\frac{dC_1}{ds}(0), 0) \\ &\quad + bD_{(y,y)}\sigma(0, \frac{dC_2}{dt}(0)) \\ &= -aD_{(y,y)}\sigma(0, \frac{dC_1}{ds}(0)) \\ &\quad + bD_{(y,y)}\sigma(0, \frac{dC_2}{dt}(0)). \end{aligned} \tag{20}$$

(We have used the chain rule and the derivative of the relation that $\sigma(C_1(s), C_1(s)) \equiv 0$.) Transversality of the intersection of C_1 and C_2 at 0 ensures that the image of this differential is

$$D_{(y,y)}\sigma(0 \oplus T\Sigma_y);$$

so transversality of the section corresponding to $q(y \times p_2 \times \cdots \times p_k)$ at its zero y ensures that the image of the differential surjective onto the fiber of E over y; i.e., the canonical section is transverse. The sign is correct, as one can see by inspecting Equation (20). q.e.d.

Remark 6.11. With the help of the above results, we can describe explicitly the invariant of the tube:

$$\widehat{SW}_{(C,\mathcal{J})} \colon H^*(C) \longrightarrow H^*(\mathcal{J}) \subset \mathbb{A}(X_0),$$

which we do now for completeness. Let $\Lambda = \Lambda^* H_1(\Sigma) \subset \mathbb{A}(Y)$. According to the proof of Lemma 6.7, $\widehat{SW}_{(C,\mathcal{J})}$ is a homomorphism of Λ-modules; so, since $\mathbb{A}(Y) = \Lambda[U]$ surjects onto $H^*(C)$, the invariant is determined by $\widehat{SW}_{(C,\mathcal{J})}(U^i)$, as i ranges over the non-negative integers. Since the Poincaré dual of $\mathrm{Sym}^k(\Sigma) \subset T^{2g}$ (which is the image of $\widehat{\mathcal{M}}(C,\mathcal{J})$ under $\rho_{\mathcal{J}}$, according to Theorem 4.1), is

$$\frac{(\sum_{i=1}^g \mu(A_i)\mu(B_i))^k}{k!},$$

it follows that

$$\widehat{SW}_{(C,\mathcal{J})}(U^\ell) = \frac{(\sum_{i=1}^g A_i \cdot B_i)^{k+\ell}}{(k+\ell)!}.$$

We will not use this formula, however. The results we prove in this paper require only the general properties of $\widehat{SW}_{(C,\mathcal{J})}$ which follow from its definition, together with Proposition 6.9.

According to Lemma 6.7, if $\gamma \subset Y$ is a curve which is null-homologous in X_0, then it annihilates the relative invariants, in the sense that

$$SW_{(X_0,C)}(a \otimes \mu(\gamma)\cdot\omega) = 0.$$

If sufficiently many curves in Y become null-homologous in X_0, then any class of sufficiently high degree in $\mathbb{A}(Y)$ annihilates the invariant, as follows.

Proposition 6.12. *Fix natural numbers k, ℓ with $\ell \geq k$. Let I denote the ideal generated by $\mu(A_1), \mu(A_2), \dots, \mu(A_\ell)$ in $H^*(\mathrm{Sym}^k(\Sigma))$. Then every element of $H^*(\mathrm{Sym}^k(\Sigma))$ of degree greater than k lies in I.*

Proof. The vector space $H^*(\mathrm{Sym}^k(\Sigma))$ is generated by homogeneous elements of the form

$$U^a \cdot \prod_{q=1}^{b}(A_{i_q}\cdot B_{i_q}) \cdot \prod_{r=1}^{c} A_{i_{b+r}} \cdot \prod_{s=1}^{d} B_{i_{b+c+s}},$$

where $\{i_1, \dots, i_{b+c+d}\}$ is a subset of $\{1, \dots, g\}$, and a, b, c, d are integers with $a+b+c+d \leq k$.

Clearly, it suffices to prove the proposition for homogeneous generators of degree $k+1$. Modulo I, such an element is equivalent to the element

$$\prod_{p=1}^{a}(U - A_p \cdot B_p) \cdot \prod_{q=1}^{b}(A_{i_q}\cdot B_{i_q} - A_{a+q}\cdot B_{a+q}) \cdot \prod_{r=1}^{c} A_{i_{b+r}} \cdot \prod_{s=1}^{d} B_{i_{b+c+s}}.$$

Indeed, in light of the fact that $a+b \leq k-d \leq \ell - d$, we can arrange (after possibly simultaneously permuting the indices of the $\{A_i\}_{i=1}^g$ and $\{B_i\}_{i=1}^g$) that for each $s = 1, \ldots d$, $a+b < i_{b+c+s}$. Moreover, the original homogeneous element would automatically lie in I unless we had that $a+b < k \leq \ell < i_{b+r}$ for all $r = 1, \ldots, c$. Put together, must consider elements of the above form which satisfy the constraint that $a+b < i_j$ for all $j > b$. If the degree of such an element is $k+1$, it must vanish in $H^*(\mathrm{Sym}^k(\Sigma))$.

This vanishing can be seen geometrically: U is Poincaré dual to the subset (identified with $\mathrm{Sym}^{k-1}(\Sigma)$) of $\mathrm{Sym}^k(\Sigma)$ where one point is constrained to lie in a specified point on Σ: A_i (resp. B_i) is Poincaré dual to the cycle where one point is constrained to lie on A_i (resp. B_i). Thus, (if one chooses the point representing U to be $A_i \cap B_i$), then $U - A_i \cdot B_i$ is Poincaré dual to the locus where two distinct points are constrained; one is to lie on A_i, the other on B_i. Similarly, the manifold Poincaré dual to $A_i \cdot B_i - A_j \cdot B_j$ gives a constraint on two distinct points in the symmetric power. Finally, the remaining $A_{i_{b+r}}$ and $B_{i_{b+c+s}}$ give additional, disjoint constraints (these are disjoint, if one chooses that representing curves to be disjoint from the A_i and B_i for $i = 1, \ldots, a+b$, which can be arranged since $a+b < i_{b+r}$ for all $r \geq 1$). Thus, since the total degree of the expression considered is $k+1$, we have put constraints on $k+1$ distinct points, forcing the intersection to be empty. q.e.d.

Corollary 6.13. *Suppose that $C = \mathrm{Sym}^k(\Sigma)$, and let $\ell \geq k$ be an integer so that there is a symplectic basis $\{A_i, B_i\}_{i=1}^g$ for $H_1(\Sigma)$ so that $i_*(A_i) = 0$ in $H_1(X_0; \mathbb{R})$ for $i = 1, \ldots, \ell$. Then, for each $b \in \mathbb{A}(\Sigma)$ of degree $d(b) > k$, and each $a \in \mathbb{A}(X_0)$, $\omega \in H^*(C)$, we have*

$$SW_{(X_0,C)}(a \otimes b \cdot \omega) \equiv 0.$$

Proof. By Proposition 6.12, b lies in the ideal generated by $\mu(A_1), \ldots, \mu(A_\ell)$. Now the proposition follows from Lemma 6.7. q.e.d.

We now have the promised proof of Proposition 3.8.

Proof of Proposition 3.8. Recall that we have constructed $SW_{(X_0,C)}$ so that

$$SW^{irr}_{\mathfrak{s}}(a \cdot i_*(b)) = SW_{(X_0,C)}(a \cdot i_*(b) \otimes 1).$$

By Lemma 6.7 and Proposition 6.9, we can write

$$SW_{(X_0,C)}(a \cdot i_*(b) \otimes 1) = SW_{(X_0,C)}(a \otimes b) + SW_{(X_0,\mathcal{J})}(a \cdot c).$$

for some $c \in \mathbb{A}(\Sigma)$. Note that c lies in the ideal generated by $H_1(\Sigma)$, as it can be expressed in terms of Seiberg-Witten invariants of the tube, which take values in $H^*(\mathcal{J}) \cong \Lambda^*(H_1(\Sigma)) \subset H_1(\Sigma) \cdot \mathbb{A}(\Sigma)$. By Corollary 6.13, the first term vanishes (using the homological hypothesis of the inclusion of Σ in X). The remaining term is identified with an absolute invariant, according to Proposition 5.9. q.e.d.

The proof of Proposition 3.10, follows from the same argument as Proposition 3.8; only in that case, one must use the following (much simpler) analogue of Corollary 6.13.

Lemma 6.14. *Suppose that $C = \mathrm{Sym}^k(\Sigma)$. Then, for each $b \in \mathbb{A}(\Sigma)$ of degree $d(b) > 2k$, and each $a \in \mathbb{A}(X_0)$, $\omega \in H^*(C)$, we have*

$$SW_{(X_0,C)}(a \otimes b \cdot \omega) \equiv 0.$$

Proof. This follows immediately from the fact that $\dim C = 2k$. q.e.d.

7. The moduli spaces over N

The purpose of this section is to give the results about the neighborhood of Σ which were used in Section 5. Most of these results are applications of [22] and [24]. We assume for the duration of this section that the $\mathrm{Spin}_{\mathbb{C}}$ structure over N satisfies $\langle c_1(\mathfrak{s}), [\Sigma]\rangle \not\equiv n \pmod{2n}$. We return to the excluded cases in Section 8.

Over N, endowed with a cylindrical-end metric and a certain torsion connection on TN , the Seiberg-Witten equations admit a complex interpretation analogous to the complex interpretation of the equations over a Kähler manifold (see Section 5 of [24] for an explicit description of this connection, and especially Proposition 5.6 where the complex interpretation is proved). The Seiberg-Witten equations over N can be written as equations for a connection A over E, $\alpha \oplus \beta \in (\Omega^{0,0} \oplus \Omega^{0,1})(N, E)$:

$$2\Lambda F_A - \Lambda F_{K_N} = \frac{i}{2}(|\alpha|^2 - |\beta|^2) \tag{21}$$

$$\mathrm{Tr} F_A^{0,2} = \overline{\alpha} \otimes \beta \tag{22}$$

$$\overline{\partial}_A \alpha + \overline{\partial}_A^* \beta = 0, \tag{23}$$

where Λ denotes projection onto the $(1,1)$ form of the metric. As noted in [24], for finite energy solutions, decay estimates justify the usual integration-by-parts which shows that one of α or β must vanish identically; i.e., the solutions over N correspond to vortices over N.

When $\beta \equiv 0$, then A induces an integrable $\overline{\partial}$-operator on E, $\overline{\partial}_A$, with respect to which α is holomorphic. Moreover, by the usual exponential decay results, together with the understanding of the solutions over Y (Theorem 4.1), (A, α) exponentially approaches the pull-back of a vortex solution over Σ. According to [22], the underlying holomorphic data extends to the ruled surface R obtained by attaching a copy of Σ (denoted Σ_+) to N "at infinity." We state the results here for convenience.

Definition 7.1. Let $\Phi \in \Gamma(N, W^+)$, $\Psi \in \Gamma(Y, W)$ be a pair of spinors, and $\delta > 0$ be some real number. Then, Ψ is said to *δ-decay to* Ψ if for each $k \geq 0$,

$$\lim_{t \mapsto \infty} \sup_{\{t\} \times Y} e^{\delta t} |\nabla^{(k)} \Psi - \nabla^{(k)} \pi^*(\Psi)| = 0,$$

where $\nabla^{(k)}$ denotes the k-fold covariant derivative. More generally, Φ is said to *decay to* Ψ if there is some $\delta > 0$ so that Ψ δ-decays to Ψ. A similar notion can be defined for objects other than spinors, such as connections, differential forms, etc.

Definition 7.2. Given a line bundle E over Z, a *holomorphic pair* (A, α) *in* E is a pair consisting of a $\overline{\partial}$-operator $\overline{\partial}_A$ over E, and a section α of E, so that $F_A^{0,2} = 0$, and $\overline{\partial}_A \alpha = 0$.

Theorem 7.3. *Let (A, α) be a holomorphic pair on N which decays to a the pull-back of a holomorphic pair (A_0, α_0) over Σ. Then, there is a naturally associated line bundle $\widehat{E}$ over R and holomorphic pair $(\widehat{A}, \widehat{\alpha})$ in $\widehat{E}$, so that $(\overline{\partial}_{\widehat{A}}, \widehat{\alpha})|_{R - \Sigma_+} \cong (\overline{\partial}_A, \widehat{\alpha})$ and $(\overline{\partial}_{\widehat{A}}, \widehat{\alpha})|_{\Sigma_+} \cong (\overline{\partial}_{A_0}, \alpha_0)$.*

The above theorem is essentially a restatement of Theorem 7.7 of [22], where it is stated for the cylinder, thought of as R minus two copies of Σ, rather than the neighborhood of Σ, thought of as R minus one copy of Σ (though the proof is no different). Analogous results for the anti-self-dual equations were obtained by Guo [9].

In a similar vein we have the following result, which allows us to deal with solutions with reducible boundary values. We state the result slightly differently from the above, since we will apply it in other contexts later.

Theorem 7.4. *Let A be a connection on a line bundle E over N, with $F_A^{0,2} = 0$ and $E|_{(0,\infty) \times Y} \cong \pi^*(E_0)$, so curvature form F_A decays to* *the pull-back of a closed two-form* F_0 *over* Σ *with*

$$\left(\frac{i}{2\pi}\int_\Sigma F_0\right) - \langle c_1(E_0), [\Sigma]\rangle \notin n\mathbb{Z}.$$

Then, there is an associated line bundle $\widehat{E}$ *over* R *and complex structure* $\overline{\partial}_{\widehat{A}}$ *with*

$$(\widehat{E}, \overline{\partial}_{\widehat{A}})|_{R-\Sigma_+} \cong (E, \overline{\partial}_A),$$

and $\langle c_1(\widehat{E}), [\Sigma_+]\rangle$ *is the greatest integer congruent to* $\langle c_1(E), [\Sigma]\rangle$ *moduli* n *smaller than* $\frac{i}{2\pi}\int F_0$*. Furthermore, there is a natural identification*

$$\mathrm{Ker}\not{D}_A \cap L^2 \cong H^0(R, \widehat{E}) \oplus H^2(R, \widehat{E}),$$

and

$$\mathrm{Coker}\not{D}_A \cap L^2 \cong H^1(R, \widehat{E}).$$

The above is proved in Proposition 9.2 (see Corollary 9.11 and Theorem 10.6) of [22].

These results allow us to rule out the existence of certain solutions. Recall first the following standard fact about the cohomology of R (see for example [10]):

Proposition 7.5. *Let* R *denote the ruled surface over* Σ*, which is given as the projectivization of* $\mathbb{C} \oplus L$*,* $\mathbb{P}(\mathbb{C} \oplus L)$ *(here,* L *is some line bundle over* Σ*). Let* $\widehat{E}$ *be a line bundle over the ruled surface* R *and let* E_0 *denote the restriction of of* $\widehat{E}$ *to* $\mathbb{P}(\mathbb{C} \oplus 0) \cong \Sigma$ *and let* ℓ *be the evaluation of* $c_1(\widehat{E})$ *on a fiber in the ruling. Then, if* $\ell \geq 0$*,*

$$H^0(R, \widehat{E}) \cong \sum_{j=0}^{\ell} H^0(\Sigma, E_0 \otimes L^{\otimes j});\ H^1(R, \widehat{E}) \cong \sum_{j=0}^{\ell} H^1(\Sigma, E_0 \otimes L^{\otimes j});$$
$$H^2(R, \widehat{E}) = 0;$$

and if $\ell < 0$*,*

$$H^0(R, \widehat{E}) = 0;\ H^1(R, \widehat{E}) \cong \sum_{j=1}^{-\ell-1} H^0(\Sigma, E_0 \otimes L^{\otimes -j});$$
$$H^2(R, \widehat{E}) \cong \sum_{j=1}^{-\ell-1} H^1(\Sigma, E_0 \otimes L^{\otimes -j}).$$

In particular, if $\ell = -1$*, then* $H^*(R, \widehat{E}) = 0$*.*

We can apply these results to the case where N is a neighborhood of a surface of self-intersection number with $-\Sigma\cdot\Sigma > 2g-2$. Recall that according to Theorem 4.1, for each $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$ on N, there are at most two components to the moduli space of the boundary, the reducible component $\mathcal{J}$ and the irreducible component C.

Proposition 7.6. *If*

$$\Big|\langle c_1(\mathfrak{s}),[\Sigma]\rangle\Big| < n,$$

the moduli space $\mathcal{M}_N(\mathcal{J})$ contains only reducibles. Moreover, the space of reducibles is smoothly identified with the Jacobian torus $\mathcal{J}$ (i.e., the kernel and the cokernel of the Dirac operator coupled to any reducible vanishes). Furthermore, $\mathcal{M}_N(C)$ is empty.

Proof. We prove that both moduli spaces $\mathcal{M}^*_N(\mathcal{J})$ and $\mathcal{M}_N(C)$ are empty. Suppose there were some finite energy solution to the Seiberg-Witten equations in a $\mathrm{Spin}_{\mathbb{C}}$ structure with $|\langle c_1(\mathfrak{s}),[\Sigma]\rangle| < n$. We know that the spinor lies entirely in one of the two summands in the splitting of the spinor bundle $W^+\cong E\oplus({K_N}^{-1}\otimes E)$ (i.e., it is an α- or a β-spinor, in the notation of Equations (21)-(23)). By conjugating if necessary (which switches the two summands and sends the $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$ to another one $J\mathfrak{s}$ with $c_1(J\mathfrak{s}) = -c_1(\mathfrak{s})$), we can assume without loss of generality that the solution is an α-solution.

According to Theorem 7.3 (and Theorem 7.4, when the boundary value is reducible), we can extend the data $(E,\overline{\partial}_A,\alpha)$ over the associated ruled surface R, obtained by attaching the curve Σ_+ at infinity. The fact that $\widehat{E}$ is an extension of E says that

$$\begin{aligned}\langle c_1(\widehat{E}),[\Sigma_-]\rangle &= \langle c_1(E),[\Sigma]\rangle\\ &= \frac{1}{2}\langle c_1(\mathfrak{s})+c_1(K_N),[\Sigma]\rangle\\ &= g-1+\frac{n+\langle c_1(\mathfrak{s}),[\Sigma]\rangle}{2}\end{aligned}$$

where Σ_- is the curve in $\widehat{E}$ with self-intersection number $-n$ (which is identified with $\Sigma\subset N$). By our hypothesis, then,

$$g-1<\langle c_1(\widehat{E}),[\Sigma_-]\rangle < n+g-1.$$

On the other hand, Equation (21) says that $\frac{i}{2\pi}F_A$ converges to the pull-back of a form over Σ whose integral is $g-1$, so Theorem 7.4 guarantees that the Chern number of restriction to the other section of the ruling satisfies the bound

$$-n + g - 1 < \langle c_1(\widehat{E}), [\Sigma_+]\rangle < g - 1.$$

Now, since the Poincaré dual of a fiber is $(\mathrm{PD}[\Sigma_+] - \mathrm{PD}[\Sigma_-])/n$, we see that the evaluation of $c_1(\widehat{E})$ on a fiber is given by

$$\ell = \frac{\langle c_1(\widehat{E}), [\Sigma_+]\rangle - \langle c_1(\widehat{E}), [\Sigma_-]\rangle}{n} = -1.$$

According to Proposition 7.5, it follows that $\widehat{\alpha}$ (and hence also α) must vanish identically, contradicting the irreducibility hypothesis on (A, α).

The fact that the reducibles are smoothly cut out in this range follows in an analogous manner, using Theorem 7.4 and Proposition 7.5.

q.e.d.

Remark 7.7. Most of this result can be found in Proposition 2.5 of [24].

The above vanishing result is special to the particular $\mathrm{Spin}_{\mathbb{C}}$ structures considered, as it used the fact that the Dolbeault cohomology of certain line bundles over the ruled surface vanish. In general, the moduli spaces over N typically do contain irreducibles. To study the deformation theory around these irreducibles, we use an infinitesimal version of Theorem 7.3; but first, we pause for a brief discussion of deformation theory for the Seiberg-Witten equations in general.

In general, on a four-manifold X_0 with a cylindrical end, the deformation complex around a solution (A, Φ) whose boundary value is smooth and irreducible, is given by

$$\Omega^0(X_0, i\mathbb{R}) \longrightarrow \Omega^1(X_0, i\mathbb{R}) \oplus \Gamma(X_0, W^+) \longrightarrow \Omega^+(X_0, i\mathbb{R}) \oplus \Gamma(X_0, W^-).$$

Here, terms in $\Omega^0(X_0, i\mathbb{R})$ are required to lie in $L^2_{\delta,k}$, the δ-decaying Sobolev space with k derivatives (here we can choose any $k \geq 3$); i.e., functions for which

$$(\|f\|_{\delta,k})^2 = \int_{X_0} (|f|^2 + |\nabla f|^2 + \cdots + |\nabla^{(k)} f|^2) e^{\delta\tau} < \infty,$$

where τ is a smooth function on X_0 which agrees with the t coordinate over the cylindrical end. Terms in $\Omega^1(X_0, i\mathbb{R}) \oplus \Gamma(X_0, W^+)$ are required to lie in $L^2_{\delta,k-1}$ extended by the tangent space to the moduli space at infinity at $\rho[A,\Phi]$. Finally, terms in $\Omega^+(X_0, i\mathbb{R}) \oplus \Gamma(X_0, W^-)$ are required to lie in L^2_{k-2}. The first map in the deformation complex is the linearization of the gauge group action on $[A,\Phi]$ around the identity, while the second is the linearization of the Seiberg Witten equations around $[A,\Phi]$. When the boundary value of $[A,\Phi]$ is a smooth reducible, then the above specifies the deformation theory for the moduli space based at infinity. In either case, the moduli space of solutions about $[A,\Phi]$ is transversally cut out by the Seiberg-Witten equations on X_0 if the H^2 of the above complex vanishes. (This discussion is modeled on the theory developed in [20].)

The space of divisors in a compact, complex surface X admits a deformation theory, defined as follows. Consider the pair $(\overline{\partial}_{\widehat{A}}, \widehat{\alpha})$ where $\overline{\partial}_{\widehat{A}}$ is an integrable $\overline{\partial}$-operator, and $\widehat{\alpha}$ is $\overline{\partial}_{\widehat{A}}$-holomorphic; i.e.,

$$\begin{aligned} F^{0,2}_{\widehat{A}} &= 0 \\ \overline{\partial}_{\widehat{A}}\alpha &= 0. \end{aligned}$$

This has a deformation complex

$$\Omega^{0,0} \longrightarrow \Omega^{0,1} \oplus \Omega^{0,0}(E) \longrightarrow \Omega^{0,2} \oplus \Omega^{0,1}(E) \longrightarrow \Omega^{0,2},$$

whose cohomology groups are identified with the cohomology groups of the quotient sheaf $\mathcal{E}/\widehat{\alpha}$, obtained from the short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\widehat{\alpha}} \mathcal{E} \longrightarrow \mathcal{E}/\widehat{\alpha} \longrightarrow 0.$$

Theorem 7.8. *Let (A,α) correspond to a solution to the Seiberg-Witten equations over N, with irreducible boundary values. Then, the cohomology groups of the deformation complex of the Seiberg-Witten deformation complex are naturally isomorphic to the cohomology groups deformation complex of the divisor $[\overline{\partial}_{\widehat{A}}, \widehat{\alpha}]$ in the line bundle $\widehat{E}$ over R, provided by Theorem 7.3. When (A,α) has a reducible boundary value, then H^2 of the Seiberg-Witten deformation complex is identified with $H^1(R, \mathcal{E}/\widehat{\alpha})$, while the tangent space of the based moduli space is identified with $\mathbb{C} \oplus H^0(R, \mathcal{E}/\widehat{\alpha})$.*

Proof. This follows exactly as in Theorem 9.14 (for irreducible boundary values) and Theorem 10.12 (for reducible boundary values) of [22]. The key observation at this point is to note that

$$\Lambda\partial\overline{\partial} + |\alpha|^2 \colon L^2_{\delta,k} \longrightarrow L^2_{\delta,k-2}$$

is an isomorphism, which allows one to "unroll" parts of the Seiberg-Witten deformation complex to identify it with the deformation theory of divisors in N. As in [22] (see also [13]), we can identify $\Lambda\partial\overline{\partial}$ with the operator over the cylindrical end with

$$-e^{-2\lambda t}\frac{\partial}{\partial t}e^{2\lambda t}\frac{\partial}{\partial t}+\Delta_Y,$$

where $\lambda = \frac{\pi n}{\mathrm{Vol}(\Sigma)}$. According to the theory of [16], the operator

$$\Lambda\partial\overline{\partial}\colon L^2_{k,\delta}\longrightarrow L^2_{k-2,\delta}$$

is Fredholm for all weights $0<\delta<4\lambda$. In particular, it has the same index for all small $0<\delta$ as it has on the weight $\delta=2\lambda$, where it can be connected via Fredholm operators to the manifestly self-adjoint operator

$$d^{*_\lambda}d\colon L^2_{k,\lambda}\longrightarrow L^2_{k-2,\lambda},$$

where d^{*_λ} denotes the formal λ-weighted adjoint of d. It follows from the homotopy invariance of the index that $\Lambda\partial\overline{\partial}+|\alpha|^2$ has index zero on $L^2_{k,\delta}$. From the maximum principle, it has no kernel, so it induces an isomorphism as claimed, identifying the deformation theory of the Seiberg-Witten equations with the deformation theory of divisors in N. Passing to the ruled surface then follows from Corollary 9.4 of [22].

q.e.d.

Proposition 7.9. *Let N be a disk bundle over a surface Σ with*

$$\Sigma\cdot\Sigma=-n<2-2g,$$

endowed with a $\mathrm{Spin}_{\mathbb{C}}$ *structure $\mathfrak{s}$ with*

$$n<|\langle c_1(\mathfrak{s}),[\Sigma]\rangle|\le n+2g-2.$$

Let

$$e=\frac{n+2g-2-|\langle c_1(\mathfrak{s}),[\Sigma]\rangle|}{2}.$$

Then, the expected dimensions of the moduli spaces over N and X_0 are given by:

$$\text{e-dim}\mathcal{M}_N(\mathcal{J}) = 2e+1 \tag{24}$$

$$\text{e-dim}\mathcal{M}_N(C) = 2e. \tag{25}$$

Moreover, $\mathcal{M}^*_N(\mathcal{J})$, $\mathcal{M}_N(C)$, are transversally cut out by the Seiberg-Witten equations (in particular, they are manifolds of the expected dimension). Furthermore, the boundary map

$$\rho\colon \mathcal{M}_N(C) \longrightarrow C$$

is an orientation-preserving diffeomorphism onto its image.

Proof. The deformation theory around a solution $[A, \alpha] \in \mathcal{M}_N(C)$ is identified the deformation theory around a corresponding divisor in the line bundle $\widehat{E}$ with $\langle c_1(\widehat{E}), [\Sigma_\pm]\rangle = e$; i.e., with divisors in a line bundle which (topologically) pulls back from Σ. According to Proposition 7.5, all such divisors actually pull back from the base Σ; and indeed, the deformation theory corresponds to deformation theory of degree e divisors in the base Σ, which is unobstructed. Thus, $\mathcal{M}_N(C)$ is a manifold of real dimension $2e$, transversally cut out by the Seiberg-Witten equations.

The above transversality applies to $\mathcal{M}_N(\mathcal{J})$ as well, except that the expected dimension is greater by one, as we saw in Theorem 7.8.

This identification of deformation theories of $\mathcal{M}_N(C)$ proves that ρ is an orientation-preserving local diffeomorphism onto its image in $\mathrm{Sym}^e(\Sigma) \cong C$. In fact, it is injective, as follows. As we saw, any two solutions with the same boundary values actually vanish over the same disks (with the same multiplicities). By the usual analysis of the vortex equations, any two such solutions must differ by a complex gauge transformation; i.e., a function u which satisfies

$$\Lambda\overline{\partial}\partial u + |\alpha|^2(e^{2u} - 1) = 0,$$

where u is a function which decays on the cylinder. By the maximum principle, such a function must vanish identically. q.e.d.

Having analyzed the moduli spaces over neighborhoods of Σ, we close with a some general results concerning the rest of the moduli spaces of the complement of Σ.

Proposition 7.10. *Let X_0 be as in Proposition 5.1. Then, letting $\text{e-dim}\mathcal{M}(X) = d$, we have*

$$\text{e-dim}\mathcal{M}_{X_0}(\mathcal{J}) = d + 2g - 2e - 2 \tag{26}$$

$$\text{e-dim}\mathcal{M}_{X_0}(C) = d. \tag{27}$$

Moreover, $\mathcal{M}_{X_0}(\mathcal{J})$, and $\mathcal{M}_{X_0}(C)$ are transversally cut out by the Seiberg-Witten equations (in particular, they are manifolds of the expected dimension).

Proof. By a standard excision argument, we have

$$\text{e-dim}\mathcal{M}_{X_0}(\mathcal{J}) + \text{e-dim}\mathcal{M}_N(\mathcal{J}) - 2g + 1 = \text{e-dim}\mathcal{M}_X(\mathfrak{s}) = d,$$

which calculates e-dim$\mathcal{M}_{X_0}(\mathcal{J})$, given Proposition 7.9. Similarly, we have

$$\text{e-dim}\mathcal{M}_{X_0}(C) + \text{e-dim}\mathcal{M}_N(C) - 2e = d,$$

which gives us e-dim$\mathcal{M}_{X_0}(C)$.

Smoothness of $\mathcal{M}_{X_0}(\mathcal{J})$ and $\mathcal{M}_{X_0}(C)$ follows from adapting methods of [20]. q.e.d.

8. Perturbations when $e = g - 1$

In our earlier discussion, we had to exclude one $\text{Spin}_{\mathbb{C}}$ structure over Y. In this section, we introduce a perturbation of the equations which allows us to handle this case. We begin by adapting results of Section 4 to this perturbed equation, and then, we give a discussion which is parallel to that of Section 7. The perturbations used here are analogues of Taubes' perturbations in the symplectic category [29], [30]; see also [15] for a related discussion.

Recall that the Seiberg-Witten equations over Y are obtained as the critical points of the Chern-Simons-Dirac functional CSD defined over the configuration space

$$\mathcal{B}(Y, W) = \mathcal{A}(W) \oplus \Gamma(Y, W)/\text{Map}(Y, S^1),$$

where $\mathcal{A}(Y, W)$ denotes the space of connection in the spinor bundle W which are compatible with some fixed connection ∇ on TY. The functional is defined by

$$\text{CSD}(B, \Psi) = \int_Y (B - B_0) \wedge \text{Tr}(F_B + F_{B_0}) - \int_Y \langle \Psi, \not{D}_B \Psi \rangle, \tag{28}$$

where $B_0 \in \mathcal{A}(Y, W)$ is some reference connection, $B - B_0 \in \Omega^1(Y; i\mathbb{R})$ denotes the difference 1-form, and Tr denotes the trace of the corresponding connection on W. Its Euler-Lagrange equations (the three-dimensional Seiberg-Witten equations) are

$$*\text{Tr}(F_B) - i\tau(\Psi) = 0 \tag{29}$$

$$\not{D}_B \Psi = 0, \tag{30}$$

where

$$\tau\colon \Gamma(Y,W) \to \Omega^1(Y;\mathbb{R})$$

is adjoint to Clifford multiplication, in the sense that for all $\gamma \in \Omega^1(Y;\mathbb{R})$, $\Psi \in \Gamma(Y,W)$, we have

$$\frac{1}{2}\langle i\gamma \cdot \Psi, \Psi\rangle_W = -\langle \gamma, \tau(\Psi)\rangle_{\Lambda^1}. \tag{31}$$

Moreover, its upward gradient flow equations are the usual Seiberg-Witten equations on the four-manifold $\mathbb{R} \times Y$.

When Y is a circle-bundle over a Riemann surface with Euler number $-n$ satisfying

$$n > 2g-2,$$

recall that these equations are inconvenient in the $\mathrm{Spin}_{\mathbb{C}}$ structure when $e = g-1$ (in the notation of Section 4). We will find it useful to consider a perturbed functional

$$\mathrm{CSD}_u\colon \mathcal{B}(Y,\mathfrak{t}) \longrightarrow \mathbb{R},$$

where $u \in \mathbb{R}$, given by

$$\mathrm{CSD}_u(B,\Psi) = \mathrm{CSD}(B,\Psi) + u\int_Y i\eta \wedge (\mathrm{Tr}F_B - \mathrm{Tr}F_{B_0}), \tag{32}$$

where η is the connection form for Y over Σ, and the reference connection B_0 satisfies $\mathrm{Tr}(F_{B_0}) \equiv 0$ (i.e., $B_0 \in \mathcal{J}$). These give rise to perturbed Seiberg-Witten equations of the form

$$*\mathrm{Tr}(F_B) - i\tau(\Psi) + iu(*d\eta) = 0 \tag{33}$$

$$\not{D}_B\Psi = 0, \tag{34}$$

whose moduli space of solutions is denoted $\mathcal{N}_u(Y)$. The gradient flow equations of the perturbed functional are solutions to the Seiberg-Witten equations on $\mathbb{R} \times Y$, perturbed by the self-dual component of $iu(d\eta)$, which can be collected into moduli spaces, denoted $\mathcal{M}_u(C_1,C_2)$, or their unparameterized versions

$$\widehat{\mathcal{M}}_u(C_1,C_2) = \mathcal{M}_u(C_1,C_2)/\mathbb{R}.$$

We have the following analogue of Theorem 4.1.

Theorem 8.1. *Let Y be a circle-bundle over a Riemann surface with genus $g > 0$ and Euler number $-n < 2 - 2g$. Let $\mathfrak{t}$ be the* $\mathrm{Spin}_{\mathbb{C}}$ *structure corresponding to $g - 1 \in \mathbb{Z}/n\mathbb{Z} \subset H^2(Y;\mathbb{Z})$. For all u with $0 < u < 2$, the moduli space contains two components, a reducible one $\mathcal{J}$, identified with the Jacobian torus $H^1(\Sigma;\mathbb{R}/\mathbb{Z})$, and a smooth irreducible component C diffeomorphic to $\mathrm{Sym}^{g-1}(\Sigma)$. Both of these components are non-degenerate in the sense of Morse-Bott. There is an inequality $\mathrm{CSD}_u(\mathcal{J}) > \mathrm{CSD}_u(C)$, so the space $\mathcal{M}_u(\mathcal{J}, C)$ is empty. The space $\widehat{\mathcal{M}}_u(C, \mathcal{J})$ is smooth of expected dimension $2g - 2$; indeed it is diffeomorphic to $\mathrm{Sym}^{g-1}(\Sigma)$.*

Proof. Most of this is a straightforward adaptation of [22].

We begin with the identification of the moduli spaces over Y. As in [22], the equations over Y reduce to vortex equations over Σ. More specifically, the components of the moduli spaces $\mathcal{N}_Y(\mathfrak{t})$ correspond to line bundles E_0 over Σ with the property that

$$\pi^*(E_0 \oplus K^{-1} \otimes E_0) \cong W,$$

the spinor bundle of $\mathfrak{t}$ (here K denotes the canonical line bundle over Σ). The vortex equations are are equations for $B \in \mathcal{A}(E_0)$, $\alpha \oplus \beta \in \Gamma(\Sigma, E_0 \oplus K^{-1} \otimes E_0)$, which, in the case at hand, take the form

$$2F_B - F_K + iu(d\eta) = i(|\alpha|^2 - |\beta|^2)(*1) \tag{35}$$

$$\overline{\partial}_B \alpha + \overline{\partial}_B^* \beta = 0 \tag{36}$$

$$\alpha \otimes \beta = 0. \tag{37}$$

Thus, one of α or β must vanish. In fact, in our case,

$$\deg E_0 \equiv g - 1 \pmod{n}.$$

In fact, if

$$\deg E_0 \neq g - 1,$$

then the solution space to these equations ($0 < u < 2$) is empty. More specifically, letting $\deg E_0 = g - 1 + n\ell$, we see that when $\beta \not\equiv 0$, then by integrating Equation (35) over Σ against $i/2\pi$, we get

$$2(g - 1 + n\ell) - (2g - 2) + u \deg Y = 2n\ell - un \geq 0,$$

which forces $\ell \geq 1$ (since $u > 0$). Since in this case $\deg(E_0) > g - 1$, $H^1(\Sigma, E_0) = 0$, so β must vanish. If, on the other other hand, it is $\alpha \not\equiv 0$, then we obtain in the same manner that

$$2n\ell - un \leq 0,$$

which forces $\ell \leq 0$ (since $u < 2$). Since α represents a class in $H^0(\Sigma, E_0)$, it follows that $\ell = 0$.

So, all irreducibles correspond to α-vortices in the line bundle E_0 with $\deg E_0 = g-1$. The identification of this space of vortices with the symmetric product follows from [2] (see also [11]).

Non-degeneracy of the irreducible manifold C follows exactly as in [22]. To see non-degeneracy of $\mathcal{J}$, we appeal to results of Section 5.8 of [22]. Consider the Dirac operator on the $\mathrm{Spin}_{\mathbb{C}}$ structure with spinors $W = E \otimes (\mathbb{C} \oplus K^{-1})$ with connection induced from a connection $B \in \mathcal{A}(E)$ whose curvature pulls up from Σ. It is shown in Proposition 5.8.4 of [22] that this Dirac operator admits no harmonic spinors unless the holonomy around a fiber circle in Y is trivial. In fact this holonomy is trivial when the following integral is congruent to $g-1$ modulo $n\mathbb{Z}$:

$$\frac{i}{4\pi^2}\int_Y F_B \wedge \eta = g - 1 - \frac{u \deg(Y)}{2},$$

(we have used here Equation (33)). Since $0 < u < 2$, this holonomy is non-trivial, so the reducibles admit no harmonic spinors, i.e., $\mathcal{J}$ is smoothly cut out by the equations.

We now perform the Chern-Simons calculations (see the proof of Proposition 5.23 of [22]). Suppose $[(B_1, \Psi_1)] \in C$, and $[(B_0, 0)] \in \mathcal{J}$. Then, we have

$$\begin{aligned} 2\deg B_0 - \deg K + u \deg(Y) &= 0; \\ 2\deg B_1 - \deg K &= 0, \end{aligned}$$

where by $\deg B$, we mean the integral $\frac{i}{4\pi^2}\int_Y F_B \wedge \eta$, which when B is induced from a line bundle over Σ, agrees with the degree of that line bundle. So,

$$\begin{aligned} \mathrm{CSD}_u(B_1) &= \int_Y (B_1 - B_0) \wedge (2F_{B_1} + 2F_{B_0} - 2F_K) \\ &\quad + u \int i\eta \wedge (2F_{B_1} - 2F_{B_0}) \\ &= \frac{8\pi^2}{\deg Y}(\deg B_1 - \deg B_0)(\deg B_1 + \deg B_0 - \deg K) \\ &\quad + u \int i\eta \wedge (2F_{B_1} - 2F_{B_0}) \\ &= 2\pi^2 u^2 \deg Y, \end{aligned}$$

which is negative; while $\mathrm{CSD}_u(B_0) = 0$.

The smoothness of the space of flows, and its identification with the symmetric product, follows exactly as in the unperturbed case (see Section 4). q.e.d.

We now turn to the neighborhood of Σ. We use a perturbation over N which is compatible with the above perturbation over Y. Specifically, let

$$f\colon N \longrightarrow \mathbb{R}$$

be a smooth function which is identically zero on the complement of the cylinder $[0,\infty)\times Y \subset N$, and identically one on the subcylinder $[1,\infty)\times Y$. We consider the Seiberg-Witten equations perturbed by the self-dual part of $iuf(d\eta)$. Note that this perturbing two-form is $iu\lambda f$ times the $(1,1)$ form of the standard cylindrical-end metric on N (see [24]), where

$$\lambda = -\frac{2\pi \deg Y}{\mathrm{Vol}(\Sigma)}$$

(here, $\mathrm{Vol}(\Sigma)$ denotes the volume of Σ). Similarly, we can extend the perturbation over Y to a self-dual two-form perturbation of the equations over X_0 (and, consequently, $X(T)$ to all $T>2$). Denote the corresponding moduli spaces by $\mathcal{M}_{N,u}(\mathcal{J})$, $\mathcal{M}_{N,u}(C)$, $\mathcal{M}_{X_0,u}(\mathcal{J})$, $\mathcal{M}_{X_0,u}(C)$, and $\mathcal{M}_{X(T),u}$. Strictly speaking, we still have to show that these perturbed moduli spaces $\mathcal{M}_{X(T),u}(\mathfrak{s})$ can be used to calculate the Seiberg-Witten invariant in either chamber. This is clear because we can always choose a compactly-supported perturbing two-form η_0 whose integral against ω_g dominates the integral of ω_g against $iuf(d\eta)^+$. The key point is that the latter integral is finite, since ω_g decays exponentially (see [1]).

We now have the following analogue of Proposition 5.5:

Proposition 8.2. *Suppose $\langle c_1(\mathfrak{s}),[\Sigma]\rangle = n$, and let u be a real number with $0<u<2$. Then the perturbed moduli space $\mathcal{M}_{N,u}(\mathcal{J})$ contains only reducibles. Moreover, the space of reducibles is smoothly identified with the Jacobian torus $\mathcal{J}$ (i.e., the kernel and the cokernel of the Dirac operator coupled to any reducible vanishes). Furthermore, $\mathcal{M}_{N,u}(C)$ is empty.*

Proof. We begin by proving $\mathcal{M}_{N,u}(C)$ is empty. Note that C consists entirely of α-solutions, hence so must any section in $\mathcal{M}_{N,u}(C)$. Thus, a solution $(A,\alpha)\in\mathcal{M}_{N,u}(C)$ induces a non-zero element in $H^0(\widehat{E})$ with

$$\langle c_1(\widehat{E}),[\Sigma_-]\rangle = n+g-1 \quad\text{and}\quad \langle c_1(\widehat{E}),[\Sigma_+]\rangle = g-1.$$

But $H^*(R, \widehat{E}) \equiv 0$, according to Proposition 7.5. The same argument, now appealing to Theorem 7.4, shows that $\mathcal{M}^*_{N,u}(\mathcal{J})$ is empty, and that $\mathcal{J}$ is smooth. q.e.d.

Proposition 8.3. *Suppose that*

$$\langle c_1(\mathfrak{s}), [\Sigma] \rangle = -n,$$

and let u be any real number with $0 < u < 2$. Then according to Theorem 8.1, $\mathcal{N}_u(\mathfrak{s}|_Y)$ has two components, $\mathcal{J}$ and C, where C is diffeomorphic to $\mathrm{Sym}^{g-1}(\Sigma)$. Furthermore, the expected dimensions of the moduli spaces over N and X_0 are given by:

$$\text{e-dim}\mathcal{M}_{N,u}(\mathcal{J}) = 2g - 1 \tag{38}$$

$$\text{e-dim}\mathcal{M}_{N,u}(C) = 2g - 2 \tag{39}$$

$$\text{e-dim}\mathcal{M}_{X_0,u}(\mathcal{J}) = 2d \tag{40}$$

$$\text{e-dim}\mathcal{M}_{X_0,u}(C) = 2d. \tag{41}$$

Moreover, $\mathcal{M}^*_{N,u}(\mathcal{J})$, $\mathcal{M}_{N,u}(C)$, $\mathcal{M}_{X_0,u}(\mathcal{J})$, and $\mathcal{M}_{X_0,u}(C)$ are transversally cut out by the Seiberg-Witten equations (in particular, they are manifolds of the expected dimension). Furthermore, the boundary map

$$\rho\colon \mathcal{M}_{N,u}(C) \longrightarrow C$$

is an orientation-preserving diffeomorphism onto its image.

Proof. The proofs of Propositions 7.9 and 7.10 apply directly in this perturbed context. q.e.d.

9. Cohomology

The Seiberg-Witten invariant is obtained from pairings of certain canonical cohomology classes on the Seiberg-Witten moduli space. These cohomology classes are inherited from the configuration spaces in which the moduli spaces live. In this section, we recall the definitions of these classes and discuss natural geometric representatives for them. (See Chapter 5 of [4] for the corresponding discussion of cohomology relevant to Donaldson invariants.)

Let X be a Riemannian four-manifold with a $\mathrm{Spin}_{\mathbb{C}}$ structure $\mathfrak{s}$ specified by the pair of Hermitian $\mathbb{C}^2$ bundles W^+ and W^-, and the Clifford action

$$\rho\colon TX \otimes W^+ \longrightarrow W^-.$$

The Seiberg-Witten pre-configuration space is the space

$$\mathcal{C}(W^+) = \mathcal{A}(W^+) \times \Gamma(X; W^+) \cong \Omega^1(W; \mathbb{R}) \times \Gamma(X; W^+),$$

where $\mathcal{A}(W^+)$ denotes the space of connections compatible with some fixed connection ∇ on TX, and the isomorphism above is induced by comparing any connection A against some fixed connection A_0. The irreducible pre-configuration space $\mathcal{C}^*(W^+)$ is the subset of $\mathcal{C}(W^+)$ consisting of pairs (A, Φ), where $\Phi \not\equiv 0$. Now, $\mathcal{C}^*(W^+)$ is weakly contractible, and the space $\mathrm{Map}(X; S^1)$ acts freely on it, so the *irreducible configuration space*, which is

$$\mathcal{B}^*(W^+) = \mathcal{C}^*(W^+)/\mathrm{Map}(X; S^1)$$

is weakly homotopy equivalent to the classifying space of $\mathrm{Map}(X; S^1)$. Now,

$$\mathrm{Map}(X; S^1) \sim \mathrm{Map}(X; S^1)_e \times \pi_0(\mathrm{Map}(X; S^1)) \sim S^1 \times H^1(X; \mathbb{Z});$$

so

$$\mathcal{B}^*(W^+) \sim \mathbb{CP}^\infty \times \frac{H^1(X; \mathbb{R})}{H^1(X; \mathbb{Z})},$$

and

$$H^*(\mathcal{B}^*(W^+); \mathbb{Z}) \cong \mathbb{Z}[U] \otimes \Lambda^* H^1(X; \mathbb{Z}),$$

where U is a generator with grading two. More invariantly, we define

$$\mathbb{A}(X) = \mathbb{Z}[H_0(X; \mathbb{Z})] \otimes \Lambda^* H_1(X; \mathbb{Z}),$$

graded by declaring $H_0(X; \mathbb{Z})$ to have grading two and $H_1(X; \mathbb{Z})$ to have grading one. Then, we have seen that

$$H^*(\mathcal{B}^*(W^+); \mathbb{Z}) \cong \mathbb{A}(X).$$

We describe two functorial mechanisms for constructing generators in $H^*(\mathcal{B}^*(W^+); \mathbb{Z})$. Over the space $X \times \mathcal{B}^*(W^+)$, there is a universal line bundle $\mathcal{L} = X \times S^1 \times \mathcal{C}^*(W^+)/\mathrm{Map}(X, S^1)$, where the action is defined by

$$u(x, \zeta, A, \Phi) = (x, u(x)\zeta, A + u^{-1}du, u\Phi).$$

Using this class we can define a "μ-map"

$$\mu\colon (H_0 \oplus H_1)(X; \mathbb{Z}) \longrightarrow H^*(\mathcal{B}^*(W^+)),$$

which sends a homology class of degree i to a cohomology class of degree $2-i$, by

$$\mu(x) = c_1(\mathcal{L})/x;$$

i.e., $\mu(x)$ is the cohomology class on $\mathcal{B}^*(W^+)$ with the property that for any homology class $c \in H_*(\mathcal{B}^*(W^+))$,

$$\langle \mu(x), c\rangle = \langle c_1(\mathcal{L}), x \times c\rangle.$$

We describe another convenient mechanism for constructing one-dimensional cohomology in $\mathcal{C}(W^+)$ as follows. A closed curve $x\colon S^1 \longrightarrow X$ induces a map

$$\mathrm{Hol}_x\colon \mathcal{B}^*(W^+) \longrightarrow S^1$$

which is defined to be the holonomy of the connection A around the curve x. The pull-back of the volume form $d\theta$ of S^1 by this map gives rise to a one-dimensional cohomology class $\mathrm{Hol}_x^*(d\theta)$ associated to x, which we call the *holonomy class around* x.

Proposition 9.1. *The cohomology groups of the configuration space $\mathcal{B}^*(W^+)$ are generated by the image of the μ-map. Moreover, given $x \in H_1(X;\mathbb{Z})$, $\mu(x)$ is the holonomy class around x, $\mathrm{Hol}_x^*(d\theta)|_{\mathcal{B}^*(W^+)}$.*

Remark 9.2. Note that $\mathrm{Hol}_x^*(d\theta)$ is naturally defined over the entire configuration space

$$\mathcal{B}(W^+) = \mathcal{C}(W^+)/\mathrm{Map}(X, S^1) \sim \frac{H^1(X;\mathbb{R})}{H^1(X;\mathbb{Z})}.$$

Proof. We begin by proving the second claim. Note that $\mathcal{L}$ comes with a tautological connection along the X factor, with the property that for any path $\beta\colon S^1 \longrightarrow X$ and connection $A \in \mathcal{C}(W^+)$,

$$\mathrm{Hol}_{\beta\times A}(\mathcal{L}) = \mathrm{Hol}_\beta(A). \tag{42}$$

$\mathcal{C}(W^+)$ Now, fix a path in X,

$$\beta\colon S^1 \longrightarrow X.$$

We need to show that for all paths in the configuration space

$$\alpha\colon S^1 \longrightarrow \mathcal{B}(W^+)(\mathfrak{s}),$$

we have that

$$\langle c_1(\mathcal{L}), \alpha \times \beta\rangle = \deg(\mathrm{Hol}_\beta \circ \alpha\colon S^1 \longrightarrow S^1).$$

This follows from the fact that for a line bundle L over the the torus $S^1 \times S^1$, the first Chern number is the degree of the map from $S^1 \times S^1$ defined by

$$x \mapsto \mathrm{Hol}_{x \times S^1} L$$

(a map which makes sense only after one puts a connection on L, but the degree is independent of this connection, so we left it out of the notation), together with the universal property of Equation (42). Thus, we have identified μ on any one-dimensional homology class.

The rest of the proposition is established, once we see that for a point $x \in X$, $\mu[x]$ generates H^2 of the configuration space. But this follows easily from the fact that $\mathrm{Map}(X, S^1)_e$ acts freely on the space of irreducible configurations. q.e.d.

With this concrete understanding of the μ-classes, we turn to a discussion of submanifold representatives for them.

Given a point $x \in X$, let $\mathcal{L}_x$ denote the line bundle associated to the base fibration of X; i.e., it is the restriction of the universal line bundle $\mathcal{L}$ to the slice $\mathcal{B}^*(W^+) \cong \{x\} \times \mathcal{B}^*(W^+) \subset X \times \mathcal{B}^*(W^+)$. Given a point in the fiber $\Psi(x) \in W^+_x$, we can construct a canonical section over $\mathcal{B}(X, W^+)$ by

$$[A, \Phi] \mapsto [A, \Phi, \langle \Phi(x), \Psi(x) \rangle].$$

The zero set of this section in $\mathcal{B}(X, W^+)$ is a codimension-two submanifold representing $\mu[x]$. The restriction of this section to a moduli space $\mathcal{M}_X(\mathfrak{s}) \subset \mathcal{B}(X, \mathfrak{s})$ is not, in general, transverse. However, by mollifying the construction appropriately, we can find a section which is generic over the moduli space, and hence obtain a divisor $V(x)$ representing $\mu[x]$, as follows.

Definition 9.3. Fix a ball $B \subset X$ around x and a non-vanishing section Ψ of $W^+|_B$. Given a self-dual two-form λ which is compactly supported over B, the λ-*mollified section* is the section of $\mathcal{L}_x$ defined by

$$[A, \Phi] \mapsto [A, \Phi, \int_B \langle \lambda \cdot \Phi, \Psi \rangle].$$

Lemma 9.4. *There are L^2 sections λ compactly supported in B so that the λ-mollified section of $\mathcal{L}_x$, restricted to the moduli space $\mathcal{M}_X$, vanishes transversally.*

Proof. Fix a compactly-supported cut-off function β in B, and consider the section

$$[A,\Phi]\times\lambda\mapsto[A,\Phi,\int_B\langle\lambda\cdot\Phi,\Psi\rangle\beta]$$

of $\pi_2^*(\mathcal{L}_x)$, thought of as a line bundle over $\Omega^+(B)\times\mathcal{M}(X,\mathfrak{s})$, giving $\Omega^+(B)$ the L^2 topology. This transversality follows from the fact that, for any $[A,\Phi]\in\mathcal{M}_X$, as we vary λ, the integral $\int_B\langle\lambda\cdot\Phi,\Psi\rangle\beta$ can take on any complex value. This, in turn, follows from the unique continuation theorem for elliptic differential operators, which guarantees that the section Φ cannot vanish identically over B. q.e.d.

Remark 9.5. In effect, the above lemma tells us how to construct a divisor representative $V(x)$ for $\mu[x]$ when $[x]\in H_0(X)$; this divisor is represented by the zero-set of the λ-mollified section of $\mathcal{L}_x$. Finding codimension-one representatives for $\mu[\gamma]$, where $\gamma\in H_1(X)$ is even easier: one need only find a regular value θ for the map

$$\mathrm{Hol}_\gamma\colon\mathcal{M}_X(\mathfrak{s})\to S^1.$$

Then, $\mathrm{Hol}_\gamma^{-1}(\theta)$ is the submanifold $V(\gamma)$ representing $\mu[\gamma]$ over $\mathcal{M}_X(\mathfrak{s})$.

References

[1] M. F. Atiyah, V. K. Patodi & I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Camb. Phil. Soc. **77** (1975) 43–69.

[2] S. B. Bradlow, *Special metrics and stability for holomorphic bundles with global sections*, J. Differential Geom. **33** (1991) 169–213.

[3] S. K. Donaldson, *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc. **33** (1996) 45–70.

[4] S. K. Donaldson & P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford University Press, 1990.

[5] P. M. N. Feehan & T. G. Leness, *PU(2) monopoles and relations between four-manifold invariants*, to appear in Proc. Georgia Topology Conference, Atlanta, GA, June 1996.

[6] R. Fintushel & R. J. Stern, *Donaldson invariants of 4-manifolds with simple type*, J. Differential Geom. **42** (1995) 577–633.

[7] ———, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995) 145–157.

[8] L. Göttsche & D. Zagier, *Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+=1$*, Selecta Math. **4** (1998) 69–115.

[9] G.-Y. Guo, *Yang-Mills fields on cylindrical manifolds and holomorphic bundles. I*, Comm. in Math. Phys. **179** (1996) 737–776.

[10] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., N. 52, Springer, 1977.

[11] A. Jaffe & C. H. Taubes, *Vortices and monopoles*, Progr. in Phys., N. 2, Birkhäuser, 1980.

[12] P. B. Kronheimer & T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994) 797–808.

[13] ———, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995) 573–734.

[14] ———, *The structure of Donaldson's invariants for four-manifolds not of simple type*, Preprint, 1995.

[15] P. B. Kronheimer, T. S. Mrowka & P. S. Ozsváth. *Dirac operators, holomorphic Euler characteristics, and lattice points*, Preprint, 1997.

[16] R. Lockhart & R. McOwen. *Elliptic differential operators on non-compact manifolds*, Ann. Scuola Norm. Sup. de Pisa **IV-12** (1985) 409–448.

[17] I. G. MacDonald, *Symmetric products of an algebraic curve*, Topology **1** (1962) 319–343.

[18] G. Moore & E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. **1** (1997) 298–387.

[19] J. W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifold*, Math. Notes, N. 44, Princeton Univ. Press, Princeton, 1996.

[20] J. W. Morgan, T. S. Mrowka & D. Ruberman, *The L^2-moduli space and a vanishing theorem for Donaldson polynomial invariants*, Monographs in Geometry and Topology, N. II, Internat. Press, 1994.

[21] J. W. Morgan, Z. Szabó & C. H. Taubes, *A product formula for Seiberg-Witten invariants and the generalized Thom conjecture*, J. Differential Geom. **44** (1996) 706–788.

[22] T. S. Mrowka, P. S. Ozsváth & B. Yu, *Seiberg-Witten monopoles on Seifert fibered spaces*, Comm. Anal. Geom. **5** (1997) 685–793.

[23] P. S. Ozsváth & Z. Szabó, *On embedding circle-bundles in four-manifolds*, Math. Res. Lett. **7** (2000) 657–669.

[24] ———, *The symplectic Thom conjecture*, Ann. of Math. **151** (2000) 93–124.

[25] V. Pidstrigach & A. Tyurin, *Localisation of the Donaldson's invariants along Seiberg-Witten classes*, `dg-ga/9507004`.

[26] D. Salamon, *Spin geometry and Seiberg-Witten invariants*, Preprint, 1997.

[27] N. Seiberg & E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nuclear Phys. B **426** (1994) 19–52.

[28] L. Simon, *Asymptotics for a class of non-linear evolution equations with applications to geometric problems*, Ann. of Math. **118** (1983) 525–571.

[29] C. H. Taubes. *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994) 809–822.

[30] C. H. Taubes, *More constraints on symplectic forms from Seiberg-Witten invariants*, Math. Res. Lett. **2** (1995) 9–13.

[31] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994) 769–796.

Institute for Advanced Study, New Jersey
Princeton University