HIGHEST RANK OF A POLYTOPE FOR A_n

PETER J. CAMERON, MARIA ELISA FERNANDES, DIMITRI LEEMANS, AND MARK MIXER

Abstract. We prove that the highest rank of a string C-group constructed from an alternating group A_n is 3 if $n = 5$; 4 if $n = 9$; 5 if $n = 10$; 6 if $n = 11$; and $\lfloor \frac{n-1}{2} \rfloor$ if $n \geq 12$. Moreover, if $n = 3, 4, 6, 7$ or 8, the group A_n is not a string C-group. This solves a conjecture made by the last three authors in 2012.

Keywords: Abstract Regular Polytopes, String C-Groups, Alternating Groups, Permutation Groups.
2000 Math Subj. Class: 52B11, 20D06.

1. Introduction

Given a group G and a set of involutions $S := \{\rho_0, \dots, \rho_{r-1}\}$ which generate G, such that

$$\forall i, j \text{ with } |i - j| > 1, \rho_i \text{ and } \rho_j \text{ commute (the } \textit{string property}\text{)},$$

we call the pair (G, S) a *string group generated by involutions* (or *sggi* for short). We denote by Γ_I the group generated by $\{\rho_i : i \in I\}$ for $I \subseteq \{0, \dots, r-1\}$. The pair (G, S) satisfies the *intersection property* if for every $I, J \subseteq \{0, \dots, r-1\}$, $\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$. A sggi $\Gamma := (G, S)$ that satisfies the intersection property is called a *string C-group* of *rank* $|S|$. If $\Gamma := (G, S)$ is a string C-group, we sometimes will abuse language and talk about the group Γ and denote the *rank* of G as the largest size of a set S of involutions such that $\Gamma := (G, S)$ is a string C-group. For $i \in \{0, \dots, r-1\}$, we denote by Γ_i the group generated by all the elements of S except ρ_i. Similarly, for $i, j, k \in \{0, \dots, r-1\}$, we denote by Γ_{ij} the group generated by all the elements of S except ρ_i and ρ_j and by Γ_{ijk} the group generated by all the elements of S except ρ_i, ρ_j and ρ_k.

It is known that string C-groups are automorphism groups of abstract regular polytopes and that, given an abstract regular polytope and a base flag of the polytope, one can construct a string C-group whose group G is the automorphism group of the polytope [26, Section 2E]. Hence the study of string C-groups has interest not only for group theory, but also for geometry.

Classifications of string C-groups from almost simple groups started with experimental work of Leemans and Vauthier [25] (see also [18, 19, 21, 10] for more experimental results) and quickly led to the determination of the rank of a string C-group of Suzuki type [20]. A series of results then followed for the almost simple groups with socle $\mathrm{PSL}(2, q)$ [22, 23, 9], groups $\mathrm{PSL}(3, q)$ and $\mathrm{PGL}(3, q)$ [4], groups $\mathrm{PSL}(4, q)$ [3], small Ree groups [24] and finally, symmetric groups [11, 12] and alternating groups [13, 14]. In particular, only the last two families gave rise to string C-groups of arbitrary large rank. It was proved in [7] that the maximal rank of a string C-group for transitive subgroups of S_n that are neither A_n nor S_n is $\frac{n}{2} + 1$.

A symmetric group S_n is known to have rank $n-1$ [11] and an alternating group A_n is known to have rank at least $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$ [14]. It is conjectured in [14] that this is the highest possible rank for a string C-group of alternating type. In this paper, we prove this conjecture. Our main result is as follows.

Theorem 1.1. *The rank of* A_n *is* 3 *if* $n = 5$*;* 4 *if* $n = 9$*;* 5 *if* $n = 10$*;* 6 *if* $n = 11$ *and* $\lfloor \frac{n-1}{2} \rfloor$ *if* $n \geq 12$*. Moreover, if* $n = 3, 4, 6, 7$ *or* 8*, the group* A_n *is not a string C-group.*

The cases where $n \leq 11$ had already been dealt with the use of Magma [13]. In this paper, we show that if $\Gamma := (\mathrm{A}_n, S)$ is a string C-group and $n \geq 12$, then $|S| \leq \lfloor \frac{n-1}{2} \rfloor$. The proof is divided into three parts. In Sections 2 and 3, we deal with the case where some subgroup Γ_i is primitive or transitive imprimitive, respectively, and our main tool here is permutation group theory. In the remainder of the paper we have to deal with the case where all Γ_i's are intransitive. Our main tool for this case is the use of *fracture graphs*; but these are also used elsewhere, so we give a brief introduction here.

Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ be an sggi acting as a permutation group on a set $\{1, \ldots, n\}$. We define the *permutation representation graph* $\mathcal{G}$ (also called CPR-graph by Pellicer in [27] when the group is a C-group) as the r-edge-labeled multigraph with n vertices and with a single i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$. We will use special subgraphs of the permutation representation graph that we call fracture graphs and 2-fracture graphs and that we define now. Suppose we have a sggi Γ which is a transitive subgroup of the symmetric group S_n, such that every subgroup Γ_i is intransitive. Then, for each i, the involution ρ_i has a cycle whose points lie in different Γ_i-orbits. Choosing one such cycle for each i, and regarding them as the edges of a graph on the vertex set $\{1, \ldots, n\}$, we obtain a graph with r edges that we call a *fracture graph* for Γ. The fracture graph is of course not unique, and indeed much of our proof involves showing how to replace a fracture graph by a more convenient one.

If Γ is contained in the group of even permutations (as in our main theorem), then each involution ρ_i has at least two cycles. If it happens that for each i we can find two cycles of ρ_i such that, for each cycle, its points are in different Γ_i-orbits, then taking an i-edge between each of these pairs of points we obtain a graph on n vertices with $2r$ edges that we call a 2-*fracture graph*. Section 4 handles the case where a 2-fracture graph exists. Section 5 handles the case where it does not, and we then use fracture graphs instead.

We take as convention throughout the rest of the paper that $\Gamma := (\mathrm{A}_n, S)$ denotes a string C-group of rank r (i.e. with $|S| = r$) whose group is A_n and we use $\Phi := (G, T)$ to denote a string C-group with group G not necessarily A_n and with rank $d := |T|$.

In some parts of the proof of Theorem 1.1, we use induction over n. This is the case for instance in Proposition 3.2.

2. Γ_i is primitive for some i

Now we embark on the proof of the main theorem. In this section we prove the theorem in the case where some Γ_i is primitive.

Given a string C-group $\Phi := (G, T)$ with $T := \{\rho_0, \ldots, \rho_{d-1}\}$, the *diagram* of Φ is a graph with d vertices and an edge between vertices i and j whenever $\rho_i\rho_j$ is not an involution. Moreover, the edge $\{i, j\}$ is then labelled with the order of $\rho_i\rho_j$. Observe that, by the string property, the diagram of Φ is a union of paths. We say that a set $U \subseteq T$ is *connected* provided the labels of the generators of U form an interval.

Let us first state a theorem due to Maróti that will be useful in the proof of the next proposition and also later on.

Theorem 2.1 (Maróti [16]). *Let G be a primitive group of degree n which is not S_n or A_n. Then one of the following possibilities occurs:*

(a) *For some integers m, k, l, we have $n = \binom{m}{k}^l$, and G is a subgroup of $S_m \wr S_l$, where S_m is acting on k-subsets of $\{1, \dots, m\}$;*
(b) *G is M_{11}, M_{12}, M_{23} or M_{24} in its natural 4-transitive action;*
(c) $|G| \leq n \cdot \prod_{i=0}^{\lfloor \log_2 n \rfloor - 1} (n - 2^i)$.

Proposition 2.2. *Let $n \geq 12$. If $\Phi := (G, T)$ is a string C-group of rank d with $G < A_n$ and G primitive, then $d \leq (n-3)/2$.*

Indeed, in this case d is asymptotically much smaller than $n/2$.

Proof. We use the methods of [7].

Suppose first that the diagram of Φ is not connected. Then the primitive group G is the direct product of two proper subgroups, each of which is necessarily simple and acts regularly; so $|G| = n^2$, and $n \geq 60$. But clearly $|G| \geq 2^d$; so $d \leq 2\log_2 n < (n-3)/2$ for $n \geq 60$.

So we may suppose that the diagram of Φ is connected. Now we combine Conder's lower bound 2^{2d-1} for the order of a string C-group of rank d [8] with well-known upper bounds for the order of primitive groups, such as Maróti's (see Theorem 2.1). We deal with the three cases of Maróti's Theorem. Case (b) is handled by computer. In case (a), since we are only interested in an upper bound for $|G|$, we can assume that G is maximal in A_n, so that either G is S_m acting on k-sets, or $G = S_m \wr S_l$ with $l > 1$. In the first subcase, $d \leq m-1$, while $n = \binom{m}{k} \geq m(m-1)/2$, hence $d \leq \frac{n-3}{2}$ for $n \geq 12$. In the second subcase, we can use the main result of [7] to conclude that, $d \leq \frac{ml}{2} + 1$ while $n = \binom{m}{k}^l$ and $n \geq 12$; again this gives $d \leq \frac{n-3}{2}$. In these cases, G is embeddable in a smaller symmetric group, of degree m in the first case, or ml in the second. Finally, in case (c), we have

$$2^{2d-1} \leq |G| \leq n^{1+\log_2 n},$$

If we assume $d \geq \frac{n-2}{2}$, we get

$$2^{n-3} \leq 2^{2d-1} \leq |G| \leq n^{1+\log_2 n},$$

thus

$$n \leq (\log_2 n)(\log_2 n + 1) + 3$$

which gives a contradiction for $n \geq 34$.

For $n \leq 33$, we give in Table 1 the list of primitive groups of degree n such that their order is $\geq 2^{2\lfloor \frac{n}{2} \rfloor - 3}$, following numbering of Sims's list [5]. When MAGMA is mentioned in the references column, it means we computed all string C-groups representations of the corresponding group using MAGMA and the bound is sharp.

Degree	Number	G	Max rank	Reference
12	1	$PSL(2,11)$	4	[25, 22]
	2	$PGL(2,11)$	3	[25, 23]
	3	M_{11}	0	[25]
	4	M_{12}	4	[25]
13	7	$PSL(3,3)$	0	[25, 4]
14	2	$PGL(2,13)$	3	[25, 23]
15	3	A_7	0	[25]
	4	$PSL(4,2)$	0	Magma
16	18	$2^4 : S_6$	5	Magma
	19	$2^4 : A_7$	0	Magma
	20	$2^4 : PSL(4,2)$	0	Magma
17	8	$P\Gamma L(2,16)$	0	[25]
22	2	$M_{22} : 2$	4	[25]
23	5	M_{23}	0	[19]
24	3	M_{24}	5	[19]

Table 1. Primitive groups G of degree ≤ 33 with $|G| \geq 2^{2\lfloor \frac{n}{2} \rfloor - 3}$.

So $d \leq (n-3)/2$ in all cases. □

We remark that Maróti's bound uses the Classification of Finite Simple Groups. The use of such heavy machinery could in principle be avoided by using the slightly weaker bounds proved by 'elementary' means by Babai and Pyber [1, 17]; however, this would require examining of many more 'small' cases, some of which are too large for practical computation.

The previous proposition gives the following corollary that finishes a case for our main theorem, when some Γ_i is primitive.

Corollary 2.3. *Let $n \geq 12$ and let $\Gamma := (A_n, S)$ be a string C-group of rank r. If Γ_i is primitive for some i then $r \leq \frac{n-1}{2}$.*

Proof. If Γ_i is primitive for some i, then $\Gamma_i < A_n$ and satisfies the hypotheses of Proposition 2.2. Hence the rank $r-1$ of Γ_i is bounded by $\frac{n-3}{2}$. □

3. Γ_i is transitive imprimitive for some i

In this section, we prove the main theorem in the case where Γ_i is transitive but imprimitive for some i.

We recall that a set T of elements of a group G is an *independent set* if $t \notin \langle T \backslash \{t\} \rangle$ for each $t \in T$. We say that T is an *independent generating set* of G is T is an independent set and $\langle T \rangle = G$.

Let $\Gamma \cong A_n$ with $n \geq 12$ and Γ_i be transitive imprimitive for some $i \in \{0, \dots, r-1\}$. Let k and m be such that Γ_i is embedded into $S_k \wr S_m$. We assume that the blocks of imprimitivity are maximal (so Γ_i acts primitively on the set of blocks), but do not require that k is as big as possible.

Consider the following subsets of the generating set $\{\rho_0, \dots, \rho_{r-1}\} \setminus \{\rho_i\}$ of Γ_i:

- L an independent generating set for the block action;
- C the set of generators that commute with all elements of L;
- R the remaining generators.

Let us first recall some important results found in [7]. We have that $|L| \leq m-1$ and $|C| \leq k-1$. The group $\langle L \rangle$ is primitive on the set of blocks, and L has at most two connected components. When L has two components, $r-1 \leq 2\log_2 m + (k-1)+4$ and $m \geq 60$, thus in that case we have $r \leq \frac{n}{2} - 1$. So, in what follows, we assume that L is connected and generates a primitive group on the set of blocks.

Proposition 3.1. *If $m \neq 2$ then $\{\rho_i\} \cup L$ must be connected and $|R| \leq 1$.*

Proof. As Γ is primitive, ρ_i must break the imprimitivity of Γ_i, thus it must swap at least one pair of points in different blocks. On the other hand $\langle L \rangle$ is primitive, thus ρ_i cannot commute with every element of L. Hence $\{\rho_i\} \cup L$ must be connected and $|R| \leq 1$. □

3.1. **The case $k, m > 2$.**

Proposition 3.2. *If $k > 2$ and $m > 2$, then $r \leq \lfloor (n-1)/2 \rfloor$.*

Proof. As observed at the beginning of the section, $|L| \leq m-1$, $|C| \leq k-1$. By Proposition 3.1, $|R| \leq 1$, hence $r-1 \leq (m-1)+(k-1)+1$. When $n = 12$ the bound that we get for the rank is 7, but using Magma [2] we found out that there are no polytopes of ranks 6 or 7 for A_{12}. So we may assume that $n > 12$.

If $r > \lfloor (n-1)/2 \rfloor$, then

$$\frac{n-1}{2} = \frac{km-1}{2} \leq r \leq k+m,$$

so $(k-2)(m-2) \leq 5$. The solutions with $km > 12$ are $(k,m) = (3,5)$, $(3,6)$, $(3,7)$, $(4,4)$, $(5,3)$, $(6,3)$, $(7,3)$.

Now we consider these cases. If $(k,m) = (3,7)$ or $(7,3)$, then $r \leq 10 = (21-1)/2$, as required. If $(k,m) = (3,5)$, $(3,6)$, $(4,4)$, $(5,3)$ or $(6,3)$, then we have $r \leq \lfloor (n-1)/2 \rfloor$ unless $|L| = m-1$, $|C| = k-1$, and $|R| = 1$. So $\langle C \rangle \cong S_k$, and since $\langle C \rangle$ commutes with a group acting primitively on the blocks, it acts in the same way on each block.

We also see that $\langle L \rangle$ acts as S_m on the set of blocks, and since it commutes with S_k fixing the blocks, we have $\langle L, C \rangle \cong S_k \times S_m$. Transpositions in S_k (resp. S_m) act as products of m (resp. k) transpositions on the point set. So if either m or k is odd, then Γ contains an odd permutation, a contradiction.

Now suppose that $(k,m) = (4,4)$ and $r = 8$. We know that ρ_i commutes with a subgroup $S_3 \times S_4$ with orbits of sizes 4 and 12. We know from the previous paragraph that $S_4 \times S_4$ is acting on the product of two sets of size 4. So when we descend to $S_3 \times S_4$, the orbit of size 4 has S_4 acting in the usual action (and its centraliser is trivial), while the orbit of size 12 is the product of sets of sizes 3 and 4. A permutation which commutes with it must fix the two systems of imprimitivity, so its projection onto each factor commutes with the corresponding symmetric group, and so is trivial. □

3.2. **The case $k = 2$.** The estimate above gives $r \leq m+2 = \frac{n}{2}+2$. We have to knock three off this bound. The group induced on the blocks is primitive. It follows that the centraliser of $\langle L \rangle$ in the symmetric group is generated by the involution z which interchanges the points of each block. Now if m is odd, then z is an odd permutation, and so $C = \emptyset$. If m is even, then $|C| \leq 1$, and if $z \in \langle L \rangle$ then the intersection property forces $C = \emptyset$.

We separate the argument into three cases, according as the group H induced on blocks is S_m, A_m, or neither of these. The cases $H = S_m$ and $H = A_m$ use similar arguments, but differ in detail, so we have kept them separate.

Case $H = S_m$. We assume that $m \geq 7$ for this proof.

Note first that $|R \cup C| \leq 1$. Indeed, if both R and C are nonempty, m is even, and z is contained in $\langle L \cup R\rangle \cap \langle C\rangle$, a contradiction. [This is because the kernel of the action of Γ_i on blocks is an S_m-submodule of $(C_2)^m$; the only such submodules are the trivial ones, the module M_1 generated by z, and the module M_2 consisting of elements interchanging an even number of blocks; if m is even then $M_1 \leq M_2$, and there cannot exist two independent submodules.]

So $r \leq |L| + 2$, and if either $|L| \leq m - 3$ or $R \cup C = \emptyset$ and $|L| \leq m - 2$ then we have the required result. Up to duality, there are the three possibilities, either

(A) $\rho_i = \rho_0$, $L = \{\rho_1, \ldots, \rho_{r-2}\}$, $R = \{\rho_{r-1}\}$ and $C = \emptyset$,
(B) $\rho_i = \rho_{r-2}$, $L = \{\rho_0, \ldots, \rho_{r-3}\}$, $C = \{\rho_{r-1}\}$ and $R = \emptyset$, or
(C) $\rho_i = \rho_0$, $L = \{\rho_0, \ldots, \rho_{r-1}\}$, and $R = C = \emptyset$.

Let $G = \langle L\rangle$. Now G induces the symmetric group S_m on the set of blocks, and L is an independent set of generators for G as a string group (not necessarily a string C-group!). We have $|L| \leq m - 1$.

Assume that $|L| = m - 1$ and $m \geq 7$. The elements of L induce the Coxeter generators on the set of blocks: for a certain numbering of the blocks, ρ_j swaps blocks j and $j + 1$, for $j \in \{1, \ldots, m - 1\}$. (This is an easy deduction from the result of [6].) In (A) and (C), ρ_i commutes with $\rho_2, \ldots, \rho_{m-1}$, and these elements generate a group acting as S_{m-1} on blocks, fixing the first block. Since $m \geq 7$, we see that ρ_i must fix all the blocks numbered from 2 to m, and clearly also block 1; so it preserves the block system. In (B) ρ_i commutes with $\rho_0, \ldots, \rho_{m-3}$, which also acts as S_{m-1} on blocks, and the same applies. So Γ preserves the block system, and is imprimitive, a contradiction to the assumption that Γ is the alternating group.

So we can assume that $|L| = m - 2$ and $R \cup C \neq \emptyset$.

Let K be the kernel of the action of Γ_i on the blocks, and let $K_1 = K \cap G$. Then K and K_1 are S_m-submodules of the permutation module F^m, where F is the field with two elements. The only submodules have dimensions 0, 1 (spanned by the all-1 vector), $m - 1$ (the vectors of even weight), and m. Now since Γ_i consists of even permutations, we cannot have $K = F^m$.

We show that $K = K_1$ is impossible. If $K = K_1$, then $G = \Gamma_i$, and so $R \cup C = \emptyset$, contrary to our assumption.

Next we show that $K_1 = 1$ is impossible. In this case, L generates S_m as string C-group. By the main result of [11, 12], there is a unique possibility, up to duality. In (A) the group generated by $\rho_2, \ldots, \rho_{r-2}$ is S_{m-1}, and ρ_0 commutes with this group, so ρ_0 preserves the block system, a contradiction. In (B) the group generated by $\rho_0, \ldots, \rho_{r-4}$ is S_{m-1}, and ρ_i commutes with this group. We then get the same contradiction as before.

So we are left with the case $K = [(C_2)^m]^+$ (that is the submodule of vectors of even weight) and $K_1 = \langle z\rangle$. In this case G is an extension of C_2 by S_m and $C = \emptyset$ (as in case (A)).

The involutions z and ρ_0 both commute with $\Gamma_{0,1}$, since z is in the centre of Γ_0. So the dihedral group D they generate also commutes with $\Gamma_{0,1}$. Moreover, ρ_0 and z do not commute with each other; if they did, then $\Gamma = \langle \Gamma_0, \rho_0\rangle$ would be contained in the centralizer of z, contradicting the fact that Γ is the alternating group. So D has order $2d$ with $d \geq 3$. We now separate in two cases.

In the case where $\Gamma_{0,1}$ is transitive, the group D is semiregular; thus $\Gamma_{0,1}$ has m/d blocks of imprimitivity each of size $2d$, and is contained in $D \wr S_{m/d}$. Now since $d \geq 3$, we can replace the action of $\Gamma_{0,1}$ by one where each orbit of D has size d, rather than $2d$; this action is still faithful (since D acts faithfully on d points if $d \geq 3$). So $\Gamma_{0,1}$ is isomorphic to a transitive imprimitive group of degree m. By the main result of [7], we have $r - 2 \leq \frac{m}{2} + 1$ (whence $m \leq 6$, which is not so).

So suppose that $\Gamma_{0,1}$ is intransitive on blocks; then also $\Gamma_{0,1,r-1}$ is intransitive. Let H_1 be the group it induces on the blocks. Now the images of $\rho_2, \ldots, \rho_{r-2}$ form a set of $r-3$ generators for H_1 as an sggi (not necessarily a string C-group). If $r \geq m$, then we conclude that

- H_1 has at most three orbits;
- if it has three orbits, then it acts on each as the symmetric group;
- if it has two orbits, then it acts on one of them as the symmetric group.

Suppose there are three orbits. Then H_1 commutes with the group induced by D (which has at least one orbit of size $d \geq 3$), the three orbits must be isomorphic and a D-orbit meets each in one point. But then $H_1 \leq S_{m/3}$ and the number of independent generators for H_1 is at most $m/3 - 1$. So we have $m/3 - 1 \geq m - 3$, which is impossible.

Suppose that H_1 has two orbits O_1 and O_2 with the action of H_1 on O_1 being that of the symmetric group $S_{|O_1|}$. We have a dihedral group D commuting with the symmetric group such that each D-orbit meets O_1 in one point, as these intersections form a system of imprimitivity for D on O_1. Suppose that the action on the other orbit is not faithful. Then there is a non-trivial subgroup fixing all points in this orbit (and hence fixing all D-orbits) but non-trivially on O_1, and so moving the intersections of D-orbits with O_1 (since these have size 1, only the trivial group fixes them all). As to the size, each D-orbit has one point in O_1 and $d-1$ in O_2, so $|O_2|$ is $(d-1)/2$ times the degree, that is, $(d-1)m/d$. Thus H_1 has at most $(d-1)m/d - 1$ independent generators. If our inequality holds, then $(d-1)m/d - 1 \geq m - 3$, from which we get $m \leq 2d$. But then the dihedral group has at most two orbits, and $\Gamma_{0,1} \leq D \wr C_2$. A group of order $2m^2$ has largest independent set of size at most $2\log_2 m + 1$. This number cannot be $m-2$ or greater for $m > 8$; the remaining cases are resolved by a computer check.

Case $H = A_m$. As before, let L be an independent set of generators for the action of Γ_i on blocks, and let $G = \langle L \rangle$. If G is intransitive, then its orbits form a transversal for the blocks, and so $G \cong A_m$. By the induction hypothesis, if $m \geq 12$, then $|L| \leq \lfloor \frac{m-1}{2} \rfloor$, and so $r \leq \lfloor \frac{m+3}{2} \rfloor \leq m - 1$, since $m \geq 7$. For $m < 12$, if the bound fails, we have either $m = 10$ and $|L| = 5$, or $m = 11$ and $|L| = 6$; the required bound is satisfied in either case.

So we may assume that G is transitive.

Let K be the kernel of the action of Γ_i on blocks, and $K' = K \cap G$. Since Γ_i/K and G/K' are both isomorphic to A_m, we see that $K \neq K'$. Moreover, A_m cannot act transitively on $2m$ points, and so $K' \neq 1$. Since K and K' are submodules of the A_m-module $(C_2)^m$, and neither is the whole of $(C_2)^m$ (which contains odd permutations), we must have $|K| = 2^{m-1}$, $|K'| = 2$. The generator of K' is the involution z which interchanges the points in each block. Since this is an even permutation, m must be even. Moreover $C = \emptyset$. Thus up to duality we may assume that $i = 0$. If $R = \emptyset$ then $r \leq m-1$, hence we now assume that $R = \{\rho_{r-1}\}$.

Now z is in the centre of Γ_0, and so commutes with $\Gamma_{0,1}$. The involution ρ_0 also commutes with $\Gamma_{0,1}$, and by the intersection property $\rho_0 \neq z$. Let $D = \langle z, \rho_0 \rangle$, a dihedral group of order $2d$, say. Now ρ_0 and z do not commute: for, if they did, then $\langle \rho_0, \Gamma_0 \rangle$ would be contained in the centraliser of z, whereas in fact this group is A_{2m}. In particular, $d \geq 3$.

Suppose first that $\Gamma_{0,1}$ is transitive. Then D, which commutes with a transitive group, is semiregular; and $\Gamma_{0,1}$, which commutes with D, is isomorphic to a subgroup of $D \wr S_{m/d}$. Since D acts faithfully on d points (as $d \geq 3$), $\Gamma_{0,1}$ is isomorphic to a transitive imprimitive group on m points. By the main theorem of [7], we get $r - 2 \leq \frac{m}{2} + 1$, so $m \leq 6$.

Now suppose that $\Gamma_{0,1}$ is intransitive.

We know that $\langle \rho_1, \ldots, \rho_{r-2} \rangle \cong C_2 \times A_m$, and $\{\rho_1, \ldots, \rho_{r-2}\}$ are string C-group generators. We claim that their images, $\bar{\rho}_1, \ldots, \bar{\rho}_{r-2}$, in $\Gamma_{0,r-1}/\langle z \rangle \cong A_m$ are independent.

Suppose not. Note that they generate A_m as an sggi. If they fail to be independent, one of them can be expressed in terms of the others. Suppose that it is $\bar{\rho}_h$. We cannot have $1 < h < r-2$, since then these elements would generate a commuting product of two subgroups. We cannot have $h = 1$, since we are assuming that $\Gamma_{0,1}$ is intransitive. And finally, we cannot have $h = r-2$. For if so, then $C_{A_{2m}}(A_m) = \langle z \rangle$; but ρ_{r-1} centralises $\langle \rho_1, \ldots, \rho_{r-3} \rangle = A_m$, so $\rho_{r-1} = z$, contradicting the intersection property, since $z \in \langle \rho_1, \ldots, \rho_{r-2} \rangle$.

Now the images mod z of $\rho_2, \ldots, \rho_{r-2}$ are independent, and generate an intransitive subgroup of A_m. So $r - 3 \leq m - 3$. If equality holds, then this group has just two orbits; it acts on each orbit as the symmetric group. But this contradicts the fact that these elements belong to $\Gamma_{0,1}$, which centralises the dihedral group $D = \langle \rho_0, z \rangle$ having at least one orbit of size greater than 2 on the set of blocks.

Case $H \neq S_m, A_m$. In this case we prove the following result on independent sets.

Proposition 3.3. *Let G be a primitive group of degree $n \geq 8$, not isomorphic to A_n or S_n. Then the maximum size of an independent generating set of G is at most $n - 4$.*

Proof. Let $M(G)$ be the maximum size of an independent generating set of G.

We consider separately the three possibilities given by Theorem 2.1.

In case (a) when $l \geq 2$ we have $n - 4 = \binom{m}{k}^l - 4 \geq m^l - 4 \geq ml - 2 \geq M(G)$. When $l = 1$ we have $k \geq 2$, $m \leq n/2$ and the group is a subgroup of S_m or A_m, so $M(G) \leq m - 1$, much smaller than $n - 4$.

In case (b) we have to consider the groups M_{11}, M_{12}, M_{23} or M_{24}. The maximal length of a chain of subgroups of M_{11}, M_{23} or M_{24} is 7, 11 and 14 resp. (see [28]). If G is isomorphic to M_{12} then $M(G) \leq 9$ by [28]. Suppose that $M(G) = 9$. Then one of the following subgroups H of M_{12}, namely M_{11} or $P\Gamma L(2,9)$, has to have $M(H) = 8$. As $M(M_{11}) \leq 7$ (see [28]), we must have $M(P\Gamma L(2,9)) = 8$. A quick look at the subgroup lattice of M_{12} shows that this is impossible as two subgroups of order 1440 never intersect in a subgroup of order 720. Hence $M(M_{12}) \leq 8$.

In case (c) the chain length is bounded by $\log_2 \left[n.\prod_{i=0}^{\lfloor \log_2 n-1 \rfloor} (n - 2^i) \right]$ that is at most $n - 4$ for $n \geq 26$. We also know that if $|G| = p_1^{e_1} \ldots p_k^{e_k}$ then the chain length (and hence $M(G)$) is bounded by $e_1 + \ldots + e_k$. Combining these two bounds, and using MAGMA, we conclude the result holds for $n > 9$ and for $n = 8$ we are left with $P\Gamma L_1(8)$, $PSL_2(7)$, $PGL_2(7)$ and $ASL_3(2)$. But for those, looking at the subgroup lattice we get $M(G) \leq 4$. □

3.3. **The case** $m = 2$**.** Suppose that $m = 2$, so that $\Gamma_i \leq S_k \wr S_2$. An involution interchanging the blocks is fixed-point-free so k is even and $n = 0 \mod 4$.

We separate the argument into two cases, according as there is or is not a value of $j \neq i$ such that $\Gamma_{i,j}$ is transitive. First, suppose that such a j exists.

Proposition 3.4. *If* $\Gamma_{i,j}$ *is transitive for some* $j \neq i$ *and* Γ_i *is transitive imprimitive embedded into* $S_{n/2} \wr S_2$*, then* $r \leq \frac{n-1}{2}$*.*

Proof. Suppose $r > \frac{n-1}{2}$ and $\Gamma_{i,j}$ is transitive for $j \neq i$. Then the two groups Γ_i and Γ_j are transitive, so each has two blocks of size $n/2$. The stabilisers of the blocks for the two subgroups each have index 2, so their intersection is a normal subgroup of index 4 in $\Gamma_{i,j}$. The two block systems cannot be the same, since then they would be preserved by $\langle \Gamma_i, \Gamma_j \rangle = \Gamma$. These intersections are blocks for $\Gamma_{i,j}$, of size $n/4 = l$, say, and the action on the blocks (which is not primitive in this case) is isomorphic to the Klein group $C_2 \times C_2$ generated, modulo the normal subgroup, by a set L of size 2.

Now we play the usual game: let C be the set of generators commuting with L (so C acts in the same way on each block, and has rank at most $l - 1$), and R the remaining generators of $\Gamma_{i,j}$, so $|R| \leq 4$.

Thus, $r - 2 = |R| + |C| + |L| \leq l + 5$. As $r > \frac{n-1}{2}$, $l + 7 \geq 2l$, so $l \leq 7$, and $n \leq 28$. Since n is a multiple of 4, we only need to consider the cases $n = 16$, 20, 24 and 28.

To deal with the exceptions, we first subdivide into two cases, according as the permutations of L commute. Let $L = \{\rho_s, \rho_r\}$.

Suppose first that ρ_s and ρ_r do not commute. Then they are adjacent in the diagram; so we can improve our estimate to $|R| \leq 2$. Also, there is a vertex v such that, if we follow a path with labels r, s, r, s, we arrive at a different point w in the same block. Then the stabilizer of v in $\langle C \rangle$ also fixes w. So $\langle C \rangle$ is not the symmetric group, and we have $|C| \leq l - 2$. Then we have $r - 2 \leq l + 2$, so $n \leq 16$.

Now suppose that ρ_s and ρ_r commute. If $|R| = 4$, or if $|R| = 3$ and the diagram of C is connected, then at least one element of R, say ρ_h, also commutes with C. If $\langle C \rangle$ acts primitively on a block, then the centralizer of $\langle C \rangle$ is generated by ρ_r and ρ_s, and so $\rho_h \in \langle \rho_r, \rho_s \rangle$, a contradiction. So we conclude that either C is disconnected (giving $|C| \leq l - 2$), or $\langle C \rangle$ is imprimitive (giving $|C| \leq l/2 + 1$) or $|R| \leq 2$. Putting these into our estimates shows that $n \leq 24$.

If $n = 24$, in the worst case scenario, we have $|R| = 4$, $|C| = 4$ and $|L| = 2$ possibly giving $r = 12$. If there are two permutations ρ_a and ρ_b such that $\Gamma_{i,j,a,b}$ is transitive, then it may be assumed that Γ_a and Γ_b are both embedded into $S_{n/2} \wr S_2$. This forces n to be divisible by 16, a contradiction. If $|R| \geq 2$ then $\langle C \rangle$ is intransitive within the blocks, otherwise another two generators can be removed from $\Gamma_{i,j}$ and the group remains transitive. Let $\langle C \rangle$ be intransitive within the blocks. If $|C| = 4$ then one element of R fuses the orbits of $\langle C \rangle$ inside a block, hence $|R| \leq 2$, by the same reason. Thus the case $|C| = |R| = 4$ leads to a contradiction.

We eliminate the case $n = 20$ as follows. For $n = 20$, we must have either $|C| = |R| = 3$, or $|C| = 4$, $|R| = 2$. In the first case, the diagram of C is disconnected, so we must have $C \cong S_3 \times S_2$, with orbit lengths 3 and 2. Take an element of R which commutes with one component of the diagram of C, and modify it using ρ_i and ρ_j so that it fixes all the blocks; this element commutes with $S_2 \times S_3$, but this group contains its centraliser in S_5, a contradiction. In the second case, $\langle C \rangle = S_5$. Then $\langle C, \rho_j \rangle \cong S_5 \times S_2$ fixes the blocks of imprimitivity for Γ_i. But this group is maximal in the stabiliser of this block system. There can be at most one more generator in Γ_i, giving $|R| \leq 1$, a contradiction.

For $n = 16$, we used the computer to show the nonexistence of such a group. □

In what follows we consider that $\Gamma_{i,j}$ is intransitive for every $j \neq i$. We prove that $r \leq \frac{n-1}{2}$ using a certain fracture graph and we use the following results that are immediate consequences of the definition of a fracture graph of a given sggi G with permutation representation graph $\mathcal{G}$. By an alternating square, in $\mathcal{G}$, we mean a square having opposed edges with the same label.

- ([15], Proposition 3.2) if two edges of $\mathcal{G}$, e and e', have the same label and e is an edge of a fracture graph, then at least one vertex of e' is in a different component of that fracture graph;
- ([15], Proposition 3.6) when two edges of an alternating square of $\mathcal{G}$, belong to a fracture graph of G, the vertex of the square not in these edges is in another component of that fracture graph.

When representing $\mathcal{G}$, dashed edges are used to represent edges that are in $\mathcal{G}$ but not in the chosen fracture graph.

Proposition 3.5. *If $\Gamma_{i,j}$ is intransitive for every $j \neq i$ and Γ_i is transitive imprimitive embedded into $S_{n/2} \wr S_2$, then $r \leq \frac{n-1}{2}$.*

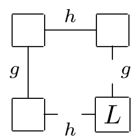
Proof. Let $L = \{\rho_l\}$. Suppose that $R = \emptyset$. In this case $\Gamma_i \leq S_2 \times S_{n/2}$ and therefore we could see Γ_i as an imprimitive group with blocks of size two which we dealt with before. Hence $r \leq \frac{n-1}{2}$.

We may now assume that R is nonempty.

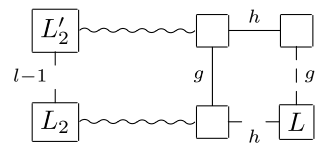
First observe that if $\rho_j \in R$ then ρ_j connects at least two pairs of $\langle \rho_l \rangle$-orbits. Indeed as ρ_j is an even permutation and ρ_j does not commute with ρ_l, there must be an even number (different from zero) of pairs of $\langle \rho_l \rangle$-orbits joined by a single ρ_j-edge.

Suppose $\rho_{l-1} \in R$. Consider the graph $\mathcal{L}$ whose vertices are the $\langle \rho_l \rangle$-orbits and with a j-edge for each element of $C \cup R$ connecting $\langle \rho_l \rangle$-orbits in different $\Gamma_{i,j}$-orbits. Note that $\mathcal{L}$ is a fracture graph for the group action on the $\langle \rho_l \rangle$-orbits of Γ_i. Since $\Gamma_{i,j}$ is intransitive for every $j \neq i$, we have that $\mathcal{L}$ has no cycles. Then ρ_{l-1} must connect at least two pairs of $\langle \rho_l \rangle$-orbits, as observed before. Let us denote two of them by (L_1, L_1'), and (L_2, L_2'). We have that $|\{L_1, L_1', L_2, L_2'\}| \geq 3$. Suppose that $\{L_1, L_1'\}$ is the $(l-1)$-edge of $\mathcal{L}$. Then either L_2 or L_2' is another connected component of $\mathcal{L}$, different from the component of L_1. Let L_1 and L_2 be in different connected components of $\mathcal{L}$. Therefore $|C \cup R| \leq n/2 - 2$. Suppose we have the equality. Then $\mathcal{L}$ has exactly two connected components, and either $|C| = n/2 - 3$ and $|R| = 1$, or $|C| = n/2 - 4$ and $R = 2$.

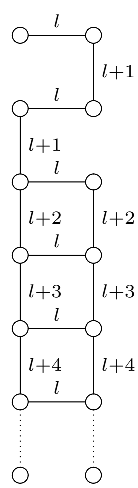
Let $|C| = n/2 - 3$ and $R = \{\rho_{l-1}\}$. We claim that the incident edges of $\mathcal{L}$ have consecutive labels. Suppose the contrary. Let g and h be non-consecutive labels of incident edges of $\mathcal{L}$. Consider first $g, h \neq l-1$. Then there exists an alternating square, with labels g and h, whose vertices are $\langle \rho_l \rangle$-orbits. Three vertices of the square belong to one connected component of $\mathcal{L}$ and the fourth belongs to another component of $\mathcal{L}$. Let us denote this fourth vertex by L.



Hence it may be assumed that L and L_2 are in the same connected component of $\mathcal{L}$.

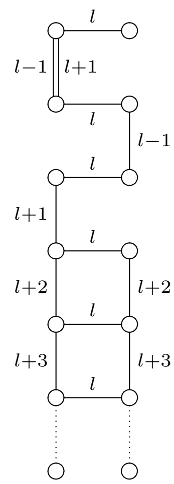


There are paths in $\mathcal{L}$ from the two ends of the g-edge to L_2 and L_2'. But then these two vertices are in the same Γ_g-orbit, a contradiction. Now suppose there is an h-edge with $h \neq l-2$ incident to an $(l-1)$-edge. These two edges are on an alternating square. Let h be minimal with this property. As $k > 4$, there is a vertex incident to that square. Now the label of the edge connecting that vertex to the square must be $h+1$. Hence there is also an alternating square adjacent to the first one, with labels $h+1$ and $l-1$. These two squares give three different connected components of $\mathcal{L}$, a contradiction. Thus two incident edges in $\mathcal{L}$ must be consecutive. This gives a unique possibility for the permutation representation graph of Γ_i.



Now there are two possibilities for i, namely either $i = 0$ or $i = r-1$. But as ρ_i must break the block system, the only possibility is $i = r-1$ and ρ_i a single transposition connecting the two bottom vertices of the picture below, a contradiction.

Let $|C| = n/2 - 4$ and $R = \{\rho_{l-1}, \rho_{l+1}\}$. As observed at the beginning of the proof of this proposition, ρ_{l+1} connects at least two pairs of $\langle\rho_l\rangle$-orbits. If it does not connect L_2 to L_2', then $\mathcal{L}$ has three connected components, a contradiction. Hence assume both ρ_{l-1} and ρ_{l+1} connect L_2 to L_2'. As $k \geq 3$ there exists another $\langle\rho_l\rangle$-orbit L that is adjacent to either L_2 or L_2'. Let $\rho_h \in R \cup C$ be a permutation connecting L and L_2. If $\rho_h \in C$ then, as it commutes with at least one of ρ_{l-1} and ρ_{l+1}, there is an alternating square in the permutation representation graph of Γ with labels h and one of $l-1$ and $l+1$. This implies that ρ_h also connects two pairs of $\langle\rho_l\rangle$-orbits different from (L_2, L_2'). Hence $|C \cup R| \le n/2 - 3$, a contradiction. Thus $\rho_h \in R$. We may assume that $h = l - 1$. If a $(l+1)$-edge is incident to a $(l-1)$-edge then there exists an alternating square and we get a contradiction as before. Hence there exists a double edge with labels $l+1$ and $l-1$ and no other incidence between edges with these labels. Also if incident edges have labels in C, then the labels must be consecutive. Moreover Γ_i has the following permutation representation graph.



Now there are two possibilities for i, namely either $i = 0$ or $i = r - 1$. In each case either ρ_i is an odd permutation or ρ_i fixes the blocks, a contradiction. □

4. All Γ_i's intransitive: 2-fracture graphs exist

In this section and the next, we handle the case where all subgroups Γ_i are intransitive. As explained in the introduction, we use the techniques of fracture graphs. In this section we deal with the case where 2-fracture graphs exist. In fact, for later use, our results are more general: we do not assume that $\Gamma \cong A_n$ until the end of the section.

Let $\mathcal{G}$ be a permutation representation graph on n vertices for Γ, which is connected (meaning that this permutation representation of Γ is transitive). From now on, we assume that Γ is indeed the permutation group defined by $\mathcal{G}$.

Suppose that Γ has no transitive maximal parabolic subgroup Γ_i. We make the further assumption that each ρ_i interchanges at least two pairs of points which lie in different Γ_i-orbits for all i. Then $\mathcal{G}$ has a subgraph with n vertices and $2r$ edges corresponding to two pairs of vertices in different Γ_i-orbits that are both swapped by ρ_i for every $i = 0, \ldots, r-1$. We call this graph a *2-fracture graph*.

For ease of notation, we denote by $q_{i,j}$ an alternating square with labels i and j, and call a cycle with more than four vertices a *big cycle*.

We give some basic results that follow immediately from the definition of a 2-fracture graph.

Proposition 4.1. *If $e = \{v, w\}$ is an i-edge of a 2-fracture graph Q of Γ, then any path from v to w in $\mathcal{G}$ which does not contain e must contain another i-edge.*

Proposition 4.2. *A cycle in a 2-fracture graph has either zero or two i-edges. In particular, a 2-fracture graph has no multiple edges.*

Proposition 4.3. *If one i-edge is in the intersection of a pair of cycles of a 2-fracture graph, then both i-edges are in the intersection of those cycles. In particular, an edge of a square of a 2-fracture graph cannot be common to any other cycle.*

Proof. If only one i-edge is common to two cycles of a 2-fracture graph, then there is a cycle with only one i-edge, which is not possible by Proposition 4.2. □

Proposition 4.4. *A big cycle of a 2-fracture graph has adjacent edges with nonconsecutive labels.*

Proof. Let i be the smallest edge label in a big cycle. By Proposition 4.2 there are two i-edges in that cycle. These i-edges must be adjacent to at least three further edges. The only edges in the cycle with labels consecutive with i have label $(i+1)$ (by minimality of i), and there are at most two of these; so the cycle has adjacent edges with nonconsecutive labels. □

In the following two propositions we show two useful ways of getting a 2-fracture graph from another. In the first one an i-edge of a 2-fracture graph that is in a cycle is replaced by another i-edge in the same cycle. In the second proposition, one i-edge of a cycle is in a 2-fracture graph and the other i-edge in that cycle is replaced by the i-edge not in the cycle in the 2-fracture graph.

Proposition 4.5. *If there is a cycle C in $\mathcal{G}$ containing exactly two i-edges e_1, e_2, such that e_1 is in a 2-fracture graph $\mathcal{Q}$ and e_2 is not, then there is another 2-fracture graph $\mathcal{Q}'$ obtained by removing e_1 and adding e_2.*

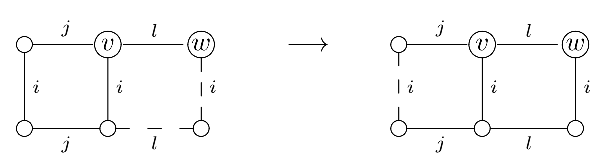
Proof. The graph $\mathcal{Q}'$ is a 2-fracture graph, because the edge e_2 is between vertices in different Γ_i-orbits. □

Proposition 4.6. *If there is a cycle C in $\mathcal{G}$ containing exactly two i-edges e_1, e_2, such that e_1 is in a 2-fracture graph $\mathcal{Q}$ and e_2 is not, then there is another 2-fracture graph $\mathcal{Q}'$ obtained by removing the i-edge of $\mathcal{Q}$ which is not e_1 and adding e_2.*

Proof. The proof is the same as that of the previous Proposition. □

Both of the above Propositions will be of particular use when the cycle C is an alternating square.

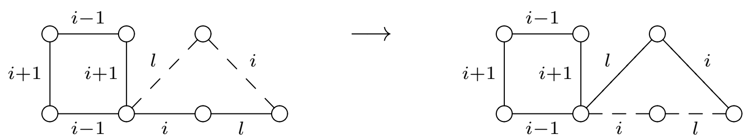
Proposition 4.7. *Let $q_{i,j}$ be an alternating square with a vertex v l-adjacent to a vertex w in a 2-fracture graph $\mathcal{Q}$. If l is not consecutive with i, then the square can be moved to include the edge $\{v, w\}$. That is, there is another 2-fracture graph $\mathcal{Q}'$ obtained from $\mathcal{Q}$ by changing exactly two edges, as pictured below. Furthermore $\mathcal{Q}'$ does not have more alternating squares than $\mathcal{Q}$.*



Proof. Let l be not consecutive with i. There is an alternating square in $\mathcal{G}$, sharing an i-edge with $q_{i,j}$. We apply Proposition 4.5 to the other i-edge of $q_{i,j}$, and Proposition 4.6 for the l-edges.

□

Proposition 4.8. *If a 2-fracture graph $\mathcal{Q}$ contains the subgraph on the left of the following figure, then there is a 2-fracture graph $\mathcal{Q}'$ containing the subgraph on the right, such that $\mathcal{Q}$ and $\mathcal{Q}'$ differ only in four edges.*



Proof. This is a consequence of applying Proposition 4.5 twice. □

Proposition 4.9. *If Γ has a 2-fracture graph, then it has one with no big cycle.*

Proof. Consider a 2-fracture graph $\mathcal{Q}$ with q big cycles. Suppose that C is a big cycle of $\mathcal{Q}$. By Proposition 4.4 there is at least one pair of adjacent edges with nonconsecutive labels i and j. Hence these edges belong to an alternating square $q_{i,j}$ of $\mathcal{G}$. The other two edges of this square are not edges of $\mathcal{Q}$. Indeed the remaining i and j edges of $\mathcal{Q}$ must belong to C by Proposition 4.2. Consider the 2-fracture graph $\mathcal{Q}'$ that is obtained from $\mathcal{Q}$ by applying Proposition 4.5 twice, replacing the i-edge and the j-edge of C which are not in $q_{i,j}$ by those that are not in C but are in $q_{i,j}$. By Proposition 4.3 the edges of the square cannot belong to another cycle, thus $\mathcal{Q}'$ has $q-1$ big cycles. Continuing this process we obtain a 2-fracture graph without big cycles. □

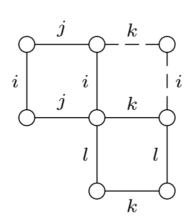
Proposition 4.10. *If $\mathcal{G}$ has a 2-fracture graph, then it has a 2-fracture graph such that each connected component has at most one cycle, and any cycle is an alternating square.*

Proof. By Proposition 4.9, $\mathcal{G}$ contains a 2-fracture graph $\mathcal{Q}_0$ having no big cycle. If $\mathcal{Q}_0$ has a cycle, then it is an alternating square by Proposition 4.2. Suppose that a connected component of $\mathcal{Q}_0$ has p alternating squares with $p > 1$. We will prove that there exists a 2-fracture graph with $p-1$ squares and without any big cycle.

The proof is a double induction. Suppose that $q_{i,j}$ and $q_{l,k}$ are squares of $\mathcal{Q}_0$ that have distance s which is the smallest distance between two squares in $\mathcal{Q}_0$. We produce another 2-fracture graph with either one fewer square, or the minimal distance s reduced by 1.

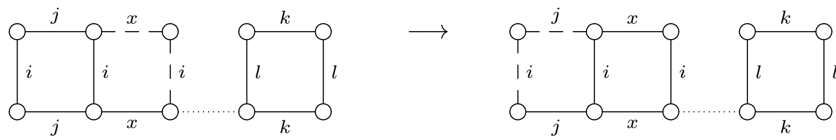
Either the squares $q_{i,j}$ and $q_{l,k}$ share a vertex, or there is a path in $\mathcal{Q}_0$ connecting them.

Suppose that they share a vertex. Then there exists another alternating square in $\mathcal{G}$. We may assume that it is $q_{i,k}$, which shares an edge with both $q_{i,j}$ and $q_{l,k}$.



By Proposition 4.5, we can replace one of the i-edges of $\mathcal{Q}_0$ with the i-edge of $q_{i,k}$ not in $\mathcal{Q}_0$ to obtain a 2-fracture graph $\mathcal{Q}_1$. Note that this new i-edge of $\mathcal{Q}_1$ does not belong to any cycle by Proposition 4.3. Thus $\mathcal{Q}_1$ has $p-1$ squares.

Now assume that the shortest path between $q_{i,j}$ and $q_{k,l}$ has $s \geq 1$ edges. If the first or the last edge of this path has a label not consecutive with the square that it meets, then we can use Proposition 4.7, to create a new 2-fracture graph which does not have more squares, and with a smaller minimal distance between squares.

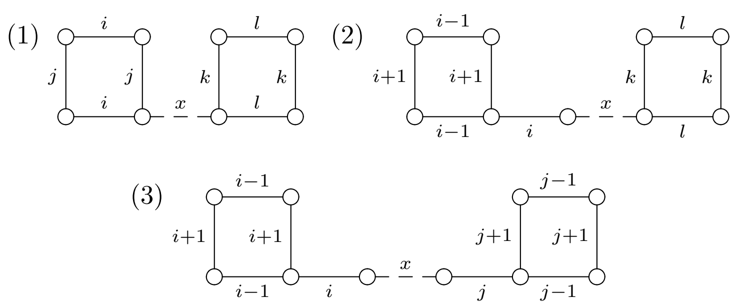


Otherwise, we can use Proposition 4.8 in order to create a new 2-fracture graph with the first or the last edge of a new path having a label not consecutive with the square that it meets, and then use Proposition 4.7 as before.

Continuing this process, we get a 2-fracture graph where the minimal distance between squares is zero, and then applying the argument above, we get a 2-fracture graph with fewer squares, and without big cycles. This construction terminates with a 2-fracture graph with no big cycles and with at most one square in each connected component.

□

Proposition 4.11. *Suppose that Γ has a disconnected 2-fracture graph $\mathcal{Q}$ with no big cycles such that every connected component has exactly one square. Then Γ has a 2-fracture graph $\mathcal{Q}'$ with the same characteristics as $\mathcal{Q}$ and such that the minimal distance between two squares in $\mathcal{G}$ is either one, two or three accordantly to one of the following three cases.*



Proof. As $\mathcal{Q}$ contains all vertices of $\mathcal{G}$, any component of $\mathcal{Q}$ is at distance one from another component of $\mathcal{Q}$. Consider two components in $\mathcal{Q}$ at distance one. Repeatedly using Propositions 4.7 and 4.8, we can obtain a 2-fracture graph that contains one of the above subgraphs. □

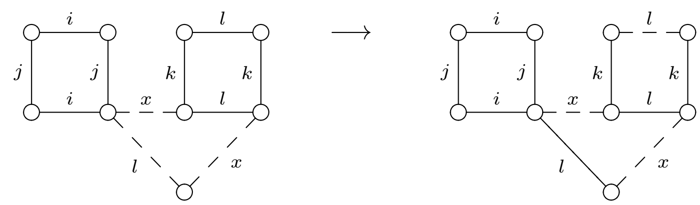
4.1. **Disconnected 2-fracture graphs.** We now separate the argument into two cases, dealing first with the case of a disconnected 2-fracture graph.

Proposition 4.12. *If $\mathcal{G}$ has no connected 2-fracture graph, then it has a 2-fracture graph that has at least one component which is a tree, all the others having only one cycle (which is an alternating square).*

Proof. Suppose that $\mathcal{Q}$ is a disconnected 2-fracture graph of Γ. By Proposition 4.10 there exists a 2-fracture graph $\mathcal{Q}$ of Γ having at most one cycle, that is an alternating square, in each connected component. We proceed by induction on the number of connected components of $\mathcal{Q}$.

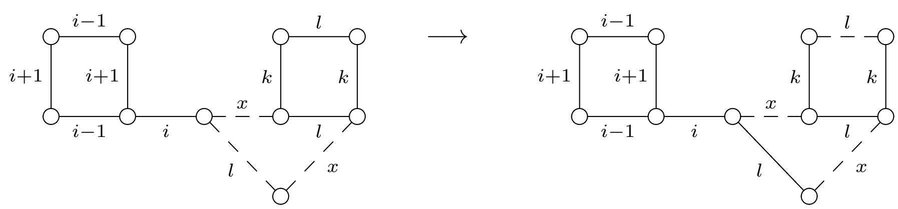
Suppose that $\mathcal{Q}$ has p connected components, each with an alternating square. By Proposition 4.11, we can choose $\mathcal{Q}$ so that it has a subgraph as shown in one of the three cases of Proposition 4.11. In what follows we deal with each of these cases separately.

Let us consider that $\mathcal{G}$ has the subgraph (1) of Proposition 4.11. Observe here that x is not consecutive with at least one of the labels of the squares. In addition since the squares of $\mathcal{Q}$ are in different connected components of $\mathcal{Q}$, the x-edge in not in $\mathcal{Q}$. Suppose x and l are nonconsecutive. Then we have an alternating square as in the following picture on the left. Using Proposition 4.5 we obtain $\mathcal{Q}'$ on the right.

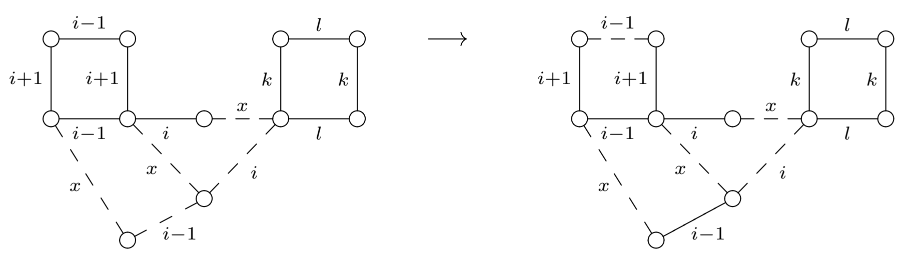


If the other x-edge is not in $\mathcal{Q}$, then we created a connected component which is a tree. On the other hand, if the other x-edge is in $\mathcal{Q}$, then the new 2-fracture graph has $p-1$ connected components.

Let us consider the second case of Proposition 4.11. First suppose that x is not consecutive either with l or k. Without loss of generality we assume that it is not consecutive with l; then we use Proposition 4.5 as shown in the following diagram.

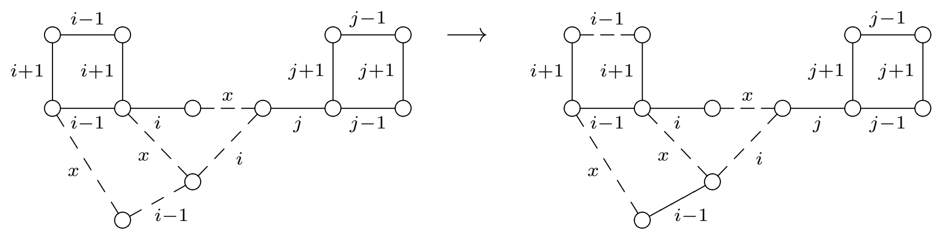


The argument for why this exchange does not create any new cycles is the same as the previous case. Similar to above, if neither of the x-edges is in $\mathcal{Q}$, then we created a connected component which is a tree. On the other hand, if one of the x-edges is in $\mathcal{Q}$, then the new 2-fracture graph has $p-1$ connected components. Next, consider that x is consecutive with both l and k, then we have the following diagram.



Note that x is not consecutive with i, and not consecutive with either $i-1$ or $i+1$. Without loss we assume it is not consecutive with $i-1$. Suppose that both x-edges of the square $q_{x,i-1}$ are in $\mathcal{Q}$. Then using Proposition 4.5, we create a square $q_{x,i-1}$ in $\mathcal{Q}'$. Now if both i-edges are in $\mathcal{Q}$ we have reduced the number of connected components; otherwise, we go back to case (1). Similarly, if an x-edge of $q_{x,i-1}$ is not in $\mathcal{Q}$, then we either created a 2-fracture graph with $p-1$ components or with a component that is a tree.

Finally, we consider the diagram below for case (3). Here, x is not consecutive with either i or j. We may assume it is not consecutive with i. Furthermore, x is not consecutive either with $i-1$ or $i+1$. Let us assume it is not consecutive with $i-1$. A similar argument shows that the new graph either has $p-1$ components or a component that is a tree.



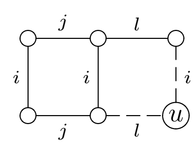
After any of the exchanges seen above, we have a 2-fracture graph with no big cycle, and either $p-1$ connected components or a component which is a tree. Therefore, by induction, there exists a 2-fracture graph that is either connected or disconnected with at least one component being a tree.

□

4.2. **Connected 2-fracture graphs.** In this subsection we will assume that Γ has a connected 2-fracture graph.

Proposition 4.13. *Let $q_{i,j}$ be an alternating square of a connected 2-fracture graph $\mathcal{Q}$ of Γ. Let $q_{i,l}$ be an alternating square in $\mathcal{G}$ sharing an i-edge with $q_{i,j}$. Then neither or both the l-edges of $q_{i,l}$ are in $\mathcal{Q}$.*

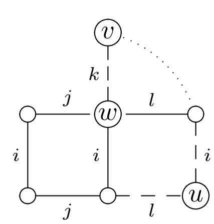
Proof. Suppose that exactly one l-edge of $q_{i,l}$ is in $\mathcal{Q}$. Let u be the vertex of $q_{i,l}$ which is adjacent to $q_{i,j}$ in $\mathcal{G}$ by the l edge of $q_{i,l}$ which is not in $\mathcal{Q}$.



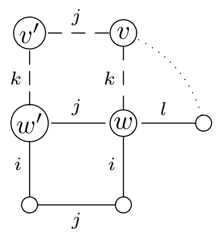
Any possibility to connect u to $q_{i,j}$ gives a contradiction to Proposition 4.1. □

Proposition 4.14. *Let $q_{i,j}$ be an alternating square of a connected 2-fracture graph $\mathcal{Q}$. All edges of $\mathcal{G}$ meeting the vertices of $q_{i,j}$ belong to $\mathcal{Q}$ and have labels consecutive either with i or j.*

Proof. Let w be a vertex of $q_{i,j}$ k-adjacent (in $\mathcal{G}$) to a vertex v. Suppose that $\{v, w\}$ is not in $\mathcal{Q}$.



There exists a path in $\mathcal{Q}$ from v to w. Let l be the label of the edge of this path meeting $q_{i,j}$. Suppose that there is a square $q_{i,l}$ or $q_{j,l}$ sharing an edge with $q_{i,j}$. Suppose first that $q_{i,l}$ contains w. Let u be the vertex of $q_{l,i}$ diagonally opposed to w. One of the l-edges of $q_{l,i}$ is not in $\mathcal{Q}$, because both are in the path from w to v. By Proposition 4.13, we have a contradiction. Similar arguments rule out the case where $q_{i,l}$ shares the other i-edge with $q_{i,j}$. Thus $|i-j| = 2$ and l is consecutive with both i and j. But then k is not consecutive either with i or with j. Suppose without loss of generality it is not consecutive with j, thus we have a square $q_{j,k}$ sharing an edge with $q_{i,j}$. Let $\{v', w'\}$ be the other k-edge of $q_{i,k}$, with v' not in $q_{i,j}$.

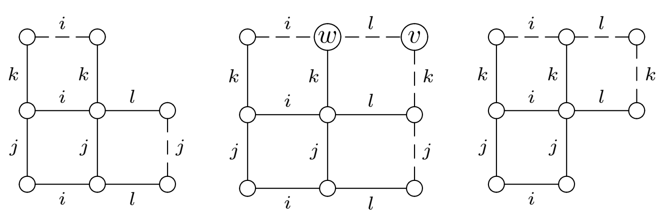


By Proposition 4.13, $\{v', w'\}$ is not in $\mathcal{Q}$. Thus there is a path in $\mathcal{Q}$, connecting w' to v', and this path does not have any l-edges. Let l' be the label of the edge of this path meeting w'. As l' is not consecutive with both i and j we have a contradiction as before. Thus $\{v, w\}$ is in $\mathcal{Q}$.

Furthermore, as all edges of $\mathcal{G}$ adjacent to $q_{i,j}$ are in $\mathcal{Q}$, k must be consecutive to either i or j. Otherwise, we would have at least four edges with the same label in $\mathcal{Q}$. □

Proposition 4.15. *If $q_{i,j}$ is an alternating square of a connected 2-fracture graph $\mathcal{Q}$ with $n \geq 9$ vertices, then each vertex of $q_{i,j}$ has degree at most three in $\mathcal{Q}$.*

Proof. Suppose that $q_{i,j}$ has a vertex adjacent, in $\mathcal{Q}$, to two other edges, and let k and l be the labels of those edges. By Proposition 4.14 k and l must be consecutive with one of the labels of $q_{i,j}$. We may assume that $i < j$ and $l < k$. Then there are three possibilities either $i < l < k < j$, $l < i < j < k$ or $i < l < j < k$ corresponding to the following graphs.



Consider the second case. As $\mathcal{Q}$ is connected, there is a path in $\mathcal{Q}$ between v and w. Then by Proposition 4.5, we can create a cycle in a 2-fracture graph having only one edge with some label, contradicting Proposition 4.2.

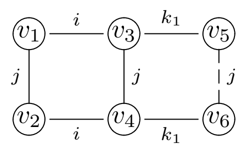
The third case becomes the same as the first case by using Proposition 4.5 on the i-edges.

Now we consider the first case. If the labels are not all consecutive, then another pair of labels give an alternating square and we get the same contradiction as in the second case. Suppose the labels are consecutive, then we use the fact that there is another vertex of $\mathcal{Q}$ connected to one of the vertices of one of the graphs above. Again using Propositions 4.5 and 4.14, independent of how this other vertex is connected, we can obtain the same contradiction as in the second case. □

Proposition 4.16. *Let $n \geq 9$. If Γ has a 2-fracture graph with a square, then it has a 2-fracture graph having a square $q_{i,j}$ such that $|i-j| = 2$.*

Proof. Choose $\mathcal{Q}$, i, and j so that the alternating square $q_{i,j}$ has the property that $|i-j|$ is minimal.

First suppose that $|i-j| \geq 3$. By Proposition 4.14 there is a vertex of the square $q_{i,j}$ which is k_1-adjacent to another vertex with $k_1 \in \{i-1, i+1, j-1, j+1\}$. Without loss of generality, we can assume $k_1 = i \pm 1$, and thus there is an alternating square $q_{k_1,j}$ sharing an edge with $q_{i,j}$. By Proposition 4.5, there is another connected 2-fracture graph $\mathcal{Q}'$ with $q_{k_1,j}$ as its alternating square. Since we assumed that $|i-j|$ is minimal we know that $|k_1 - j| > |i-j|$. Consider the vertices v_i, $i \in \{0, \ldots, 6\}$ as in the following figure.

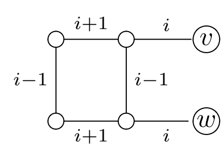


By Proposition 4.15 the degree of the vertices v_3 and v_4 is three. Suppose that v_1 has degree three, then the label k of the edge incident to v_1, not in $q_{i,j}$, would satisfy $|k-j| < |i-j|$. As above, using Proposition 4.5, we would obtain a contradiction. Thus the degree of v_1 is two, and similarly the degree of v_2 is two.

As $n \geq 9$ the degree of either v_5 or v_6 is three. Assume without loss of generality that v_5 is k_2-adjacent to a vertex v_7. We have that k_2 must be consecutive with k_1; furthermore, there is an alternating square $q_{k_2,j}$ with vertices v_5, v_6, v_7, v_8, and thus $|k_2 - j| > |k_1 - j| > |i-j|$. Continuing this process we get a sequence of labels $k_3, \ldots, k_s$ with $|k_s - j| > \ldots > |k_1 - j| > |i-j|$. Thus $\mathcal{Q}$ does not have edges with labels between i and j. We can then conclude that Γ is not connected, a contradiction. Hence $|i-j| \leq 2$.

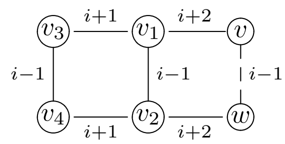
Suppose that $|i-j| = 1$, say $j = i+1$. Then by Proposition 4.14 there is a vertex either $i-1$ or $i+2$ adjacent to this square. This situation guarantees another alternating square either $q_{i-1,i+1}$ or $q_{i+2,i}$. By Proposition 4.5, there is another connected 2-fracture graph having a square $q_{i,j}$ with $j = i+2$, as wanted. □

Proposition 4.17. *Let $n \geq 9$. If $q_{i-1,i+1}$ is an alternating square of a connected 2-fracture graph $\mathcal{Q}$, then both i-edges are incident to the square as shown in the following figure.*

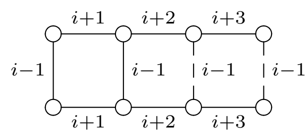


Moreover the degree, in $\mathcal{Q}$, of vertices v and w is one.

Proof. Suppose that there is no i-edge incident to a vertex of the square $q_{i+1,i-1}$. Then, by Proposition 4.14, an edge incident to $q_{i+1,i-1}$ has labels $i+2$ or $i-2$. By duality we may assume there is an edge incident to the square having label $i+2$. Then there is a square sharing an edge with $q_{i+1,i-1}$, as shown in the following figure.

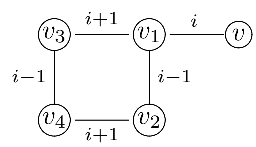


By Proposition 4.15 there is no i-edge incident to the vertices of the figure above. Furthermore, the vertices v_1 and v_2 have degree three and the vertices v_3 and v_4 have degree two. Thus one of the vertices on the right must have degree three. Suppose v has degree three. Then it is $(i+3)$-incident to another vertex, and we get a square $q_{i+3,i-1}$ as pictured below.



Again, by Proposition 4.15, there is no i-edges incident to the vertices of this figure. Continuing this process we get a graph that doesn't have i-edges, hence Γ is disconnected, a contradiction.

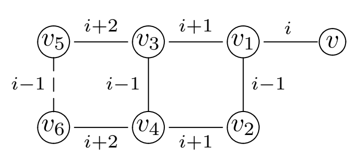
Hence there is a vertex of the square i-adjacent to another vertex v. Consider v_1, v_2, v_3 and v_4 as in the following figure.



If the degree of v, in $\mathcal{Q}$, is greater than one, then there exist an edge with label j not consecutive with i and incident to v. Then there is an alternating square $q_{i,j}$ containing v and v_1. Hence v_1 has degree at least four in $\mathcal{G}$, which is not possible by Proposition 4.15. Thus the degree of v is one in $\mathcal{Q}$.

Suppose that v_4 is i-adjacent, in $\mathcal{Q}$, to a vertex w. Then the degree of w, analogously to v, is one. Thus either v_2 or v_3 is k-adjacent to another vertex. As k is not consecutive either with $i+1$ or $i-1$ there is a square $q_{k,i\pm1}$ containing v_1 or v_4. Thus v_1 or v_4 is of degree four and we get a contradiction.

Now suppose that neither v_2 nor v_3 is i-adjacent to any vertex. Either v_2, v_3, or v_4 has degree three. Using duality we may assume that either v_3 or v_4 has degree three. Suppose at first that v_3 has degree three. By the same arguments as above v_3 is $(i+2)$-adjacent to another vertex v_5. Then there exists an alternating square $q_{i+2,i-1}$ containing v_3, v_4, v_5, and v_6 as seen in the following figure. Thus, both v_3 and v_4 have degree three.



On the other hand, if we assume that v_4 has degree three, then using duality, we may assume it is incident to an edge of label $i+2$. Thus there is an alternating square $q_{i-1,i+2}$ containing v_3; thus both situations give the same result.

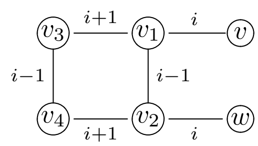
Now if the other i-edge of $\mathcal{Q}$ is incident to either v_5 or v_6, this creates an alternating square $q_{i,i+2}$ containing either v_3 or v_4, which is not possible because they have degree three. Continuing this process we never get a way to connect the other i-edge, giving a contradiction.

Hence either v_2 or v_3 is i-adjacent, in $\mathcal{Q}$, to some vertex w. Moreover the same argument used to prove that v has degree one holds for w. □

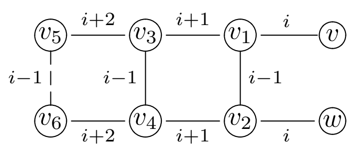
Proposition 4.18. *Let $n \geq 9$. If $\mathcal{G}$ has a connected 2-fracture graph, then either the 2-fracture graph is a tree or Γ is embedded in $C_2 \wr S_{n/2}$.*

Proof. Suppose $\mathcal{G}$ has a connected 2-fracture graph that is not a tree and let $\mathcal{Q}$ be a 2-fracture graph with a square $q_{i+1,i-1}$ accordantly with Proposition 4.16.

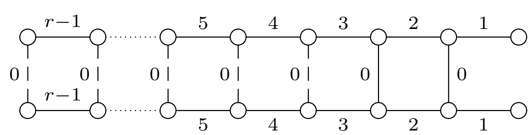
By Proposition 4.17 there are two i-edges incident to the square. Let v_1, v_2, v_3 and v_4 be the vertices of $q_{i+1,i-1}$, as in the following figure.



The degree of v and w are both one. Hence, by Proposition 4.17, either v_3 or v_4 has degree three. Suppose without loss of generality that v_3 has degree three. Now the label of the edge incident to v_3, not in the square $q_{i-1,i+1}$, must have label $i+2$. This creates an alternating square $q_{i-1,i+2}$ in $\mathcal{G}$. In particular the degree of v_4 is also three.



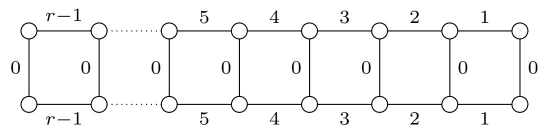
As $n \geq 9$ either v_5 or v_6 has degree three. Assume that v_5 is incident to another vertex v_7. Then the label of the edge $\{v_5, v_7\}$ must be $(i+3)$ and the degree of v_5 must be three. Then there exists an alternating square $q_{i-1,i+3}$ in $\mathcal{G}$, and therefore the degree of v_6 must be three. If $n = 10$ we are done. Otherwise we continue this process. The result is the graph below where $i = 1$.



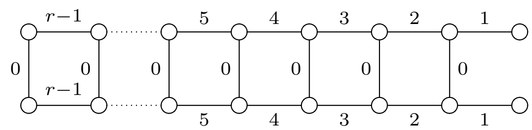
Now the permutation graph of Γ is the graph above or has another 0-edge connecting the vertices on the right. Indeed these are all the edges of $\mathcal{G}$ for otherwise there is a cycle containing an edge of $\mathcal{Q}$ and no other edge with the same label, a contradiction. In any case we get Γ embedded into $C_2 \wr S_{\frac{n}{2}}$ (since all the permutations ρ_i preserve the partition whose parts are the 0-edges and the pair of vertices on the right). □

Corollary 4.19. *With the assumptions of Proposition 4.18, if the 2-fracture graph is not a tree, there are two possibilities for Γ, namely*

(a) *The permutation representation graph is the following and* $\Gamma \cong C_2 \times S_{n/2}$.



(b) *The permutation representation graph is the following and* Γ *depends on parity of* n. *If* $n/2$ *is even, then* $\Gamma \cong C_2 \wr S_{n/2}$ *and if* $n/2$ *is odd then* $\Gamma \cong C_2^{n/2-1} : S_{n/2}$ *which is a subgroup of index* 2 *in* $C_2 \wr S_{n/2}$.



We now prove the main theorem in the case where all Γ_i are intransitive and there exists at least one 2-fracture graph. Hence our assumptions now are:

- $\Gamma \cong A_n$;
- all Γ_i's are intransitive, and each ρ_i interchanges at least two pairs of points in different Γ_i-orbits.

The conclusions we reached from the second assumption are given in Propositions 4.12 and 4.18: either

- there is a 2-fracture graph of which one component is a tree and the others are unicyclic; or
- the permutations are as given in Corollary 4.19.

In the first case, such a graph has $n-1$ edges. So $2r \leq n-1$, as required. In the second, the group Γ is not A_n.

5. All Γ_i's intransitive: no 2-fracture graphs

In this section we continue to handle the case where all subgroups Γ_i are intransitive. In particular, we deal with the case where Γ does not have any possible 2-fracture graph. Although some of our results are more general, throughout this section we will make the assumption that Γ is isomorphic to A_n.

Suppose that all maximal parabolic subgroups of Γ are intransitive but there exists $i \in \{0, \ldots, r-1\}$ such that ρ_i permutes only one pair $\{a, b\}$ of vertices in different Γ_i-orbits. Consequently, the only generators that can act non-trivially on a and b are ρ_{i-1} and ρ_{i+1}.

We will say that the orbit of a is the first Γ_i-orbit and the orbit of b is the second Γ_i-orbit. Let n_1 and n_2 be the sizes of the first and the second Γ_i-orbit, respectively; and let A and B be the correspondent groups, determined by the action of Γ_i on each orbit. Both A and B are string groups generating by involutions (or sggi's). Indeed let $\rho_j = \alpha_j \beta_j$ with α_j and β_j being the permutations in each Γ_i-orbit. Then $A := \langle \alpha_i \,|\, i \in \{0, \ldots, r-1\} \rangle$ and $B := \langle \beta_i \,|\, i \in \{0, \ldots, r-1\} \rangle$.

Proposition 5.1. *If* A *is primitive, then the set* $J_A := \{i \,|\, i \in \{0, \ldots, r-1\}$ *and* $\alpha_i \neq 1_A\}$ *is an interval. The same result holds for* B.

Proof. Suppose that J_A is not an interval. Then $A = H \times K$ for $H = \langle \alpha_j \mid j \in J_1 \rangle$ and $K = \langle \alpha_j \mid j \in J_2 \rangle$ for some disjoint index sets J_1 and J_2 such that $J_A = J_1 \cup J_2$.

As both H and K are transitive on the n_1 points, the cardinality of J_1 and J_2 is at least two. Moreover each generator α_j commutes with all generators of a transitive group on n_1 points, either H or K, which implies that it has full support on the first Γ_i-orbit. Therefore, it has a nontrivial action on a. However, we have seen that the only generators that can act nontrivially on a are ρ_{i-1} and ρ_{i+1}. This gives a contradiction, so J_A is an interval. The proof also works for B. □

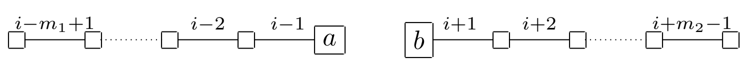
Thanks to Proposition 5.1 we consider (up to duality) separately the following cases : Case (1) A and B are both imprimitive; Case (2): J_A and J_B are intervals and $i \notin \{0, r-1\}$; Case (3): we deal with the remaining cases, particularly we assume that $J_B = \emptyset$ or an interval.

5.1. **Case (1):** ***A* and *B* are both imprimitive.** Let A be embedded into $S_{k_1} \wr S_{m_1}$ and B be embedded into $S_{k_2} \wr S_{m_2}$ with $n_1 = m_1k_1$ and $n_2 = m_2k_2$. Consider a minimal subset M of the set of generators of Γ_i generating the group induced on the two block systems. Let R be the set containing the remaining generators of Γ_i.

Consider the permutation representation graph $\mathcal{X}$ for the block action, that is, a graph having $m_1 + m_2$ vertices, corresponding to the blocks, and with a j-edge between two blocks whenever ρ_j swaps them. As Γ_i has exactly two orbits, the graph $\mathcal{X}$ has two connected components. Also, consider the subgraph $\bar{\mathcal{X}}$ of $\mathcal{X}$ with the same vertices and with a j-edge for each element of M, between blocks in different Γ_j-orbits. This is a fracture subgraph of $\mathcal{X}$, particularly $\bar{\mathcal{X}}$ has no cycles. Hence $|M| \leq m_1 + m_2 - 2$.

Similarly, consider the graph $\mathcal{Y}$ with k vertices corresponding to the $\langle M\rangle$-orbits, with a j-edge between a pair of $\langle M\rangle$-orbits L and L' whenever there is $x \in L$ such that $x\rho_j \in L'$ with $\rho_j \in R$. Let $\bar{\mathcal{Y}}$ be a fracture subgraph of $\mathcal{Y}$ having only one j-edge for each element $\rho_j \in R$ between $\langle M\rangle$-orbits in different Γ_j-orbits. As before, $\bar{\mathcal{Y}}$ has no cycles and has at least two components. Hence $|R| \leq k - 2$. Note that $k \leq k_1 + k_2$, hence $|R| \leq k_1 + k_2 - 2$.

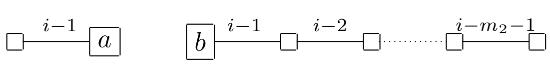
Proposition 5.2. *If $|M| = m_1 + m_2 - 2$ then $\mathcal{X}$ has two connected components and consecutive labels. Up to a duality, $\mathcal{X}$ is the following graph.*

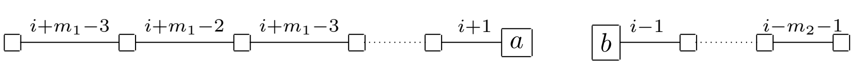


Proof. Since $|M| = m_1 + m_2 - 2$ and $\bar{\mathcal{X}}$ has two components, $\bar{\mathcal{X}} = \mathcal{X}$. As ρ_i only swaps the points a and b in different Γ_i-orbits, a j-edge of $\mathcal{X}$ incident to the blocks containing a and b, must be consecutive with i.

The graph $\mathcal{X}$ does not have any alternating squares, and thus only edges with consecutive labels are incident. Up to duality, we have determined $\mathcal{X}$. □

Proposition 5.3. *If $|M| = m_1 + m_2 - 3$ then, up to duality, either $m_1 = 2$ or $m_1 \geq 4$, accordantly to one of the following graphs.*

(1) 

(2) 

Proof. As $\bar{\mathcal{X}}$ has $m_1 + m_2 - 3$ edges, it has 3 connected components. Since $\mathcal{X}$ has two components, there exists a j-edge e_j of $\mathcal{X}$ such that $\bar{\mathcal{X}} \cup e_j$ has two components and no cycles. Hence, as in Proposition 5.2 incident edges of $\bar{\mathcal{X}} \cup e_j$ must have consecutive labels.

First suppose that the j-edges of $\bar{\mathcal{X}} \cup e_j$ are different Γ_i-orbits. We claim that $j = i \pm 1$. Assume the contrary. Then there is a path in the first Γ_i-orbit connecting the block containing a to the block moved by ρ_j; the same happens in the second Γ_i-orbit: there is a path connecting the block containing b to the block moved by ρ_j.

Assume that $j > i + 1$. Then both of these paths have to contain the label $i + 1$, a contradiction. Thus $j = i \pm 1$. Up to duality we may consider $j = i - 1$ corresponding to the first possibility for $\mathcal{X}$.

Now consider the j-edges of $\bar{\mathcal{X}} \cup e_j$ in the same Γ_i-orbit; assume it is the first orbit. There is a path in $\bar{\mathcal{X}} \cup e_j$ joining the four blocks swapped by ρ_j which has consecutive labels and no repeating labels other than j. Thus this path is a single edge with label $j \pm 1$. Assume that this edge has label $j + 1$. At least one of the four vertices of $\bar{\mathcal{X}} \cup e_j$ that are incident to the j-edges has degree one. Otherwise, there would be another repeated label. This gives the second possibility for the graph $\mathcal{X}$ where $j = i - m_1 - 3$ with $m_1 \geq 4$. □

Proposition 5.4. *If an element of R has a nontrivial action between more than one pair of $\langle M \rangle$-orbits, then $|R| \leq k - 3$.*

Proof. In this case $\bar{\mathcal{Y}}$ has at least three connected components, and still has no cycles. Thus $|R| \leq k - 3$. □

In what follows let L_a be the $\langle M \rangle$-orbit containing a and L_b the $\langle M \rangle$-orbit containing b.

Proposition 5.5. *If $|M| = m_1 + m_2 - 3$ then $|R| \leq k_1 + k_2 - 3$.*

Proof. As $\bar{\mathcal{Y}}$ is a forest and has at least two components, $|R| \leq k - 2$. If $k < k_1 + k_2$ then $|R| \leq k_1 + k_2 - 3$. Assume that $k = k_1 + k_2$ and that we have the equality $|R| = k_1 + k_2 - 2$. Then $\bar{\mathcal{Y}}$ has exactly two components and there are at least two $\langle M \rangle$-orbits in each Γ_i-orbit. As $|M| = m_1 + m_2 - 3$ we have, up to duality, one of the two possibilities for $\mathcal{X}$ given in Proposition 5.3.

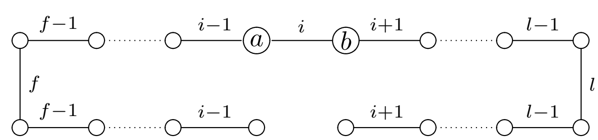
First suppose that $m_1 + m_2 > 4$. There are, up to duality, the two possibilities for $\mathcal{X}$ given in Proposition 5.3. In graph (1) there is only one possibility to connect L_a to another $\langle M \rangle$-orbit, that is using a pair of $(i + 1)$-edges. Then in both cases, (1) and (2), there is only one possibility to connect L_b to another $\langle M \rangle$-orbit, that is, with a single edge with label $l = i - m_2 - 2$ (between vertices of the last block of the second Γ_i-orbit). Furthermore, ρ_l swaps exactly one pair of vertices of these $\langle M \rangle$-orbits. Then as Γ is even ρ_l must swap another pair of $\langle M \rangle$-orbits, hence we have a contradiction with Proposition 5.4.

Now let $m_1 + m_2 = 4$. In this case $\mathcal{X}$ is as in figure (1) of Proposition 5.3 and $M = \{\rho_{i-1}\}$. To connect L_a to another $\langle M \rangle$-orbit, or L_b to another $\langle M \rangle$-orbit, there are only two possibilities for the labels, either $l = i - 2$ or $l = i + 1$. By Proposition 5.4 we may assume that either L_a or L_b is $(i - 2)$-adjacent to another $\langle M \rangle$-orbit. Then we use the fact that Γ is even and Proposition 5.4 to get a contradiction. □

Proposition 5.6. *If $|M| = m_1 + m_2 - 2$ then $|R| \leq k_1 + k_2 - 4$.*

Proof. Up to duality we may consider $\mathcal{X}$ as in Proposition 5.2. First the elements of R must fix all the blocks. Otherwise there is $\rho_j \in M$ such that Γ_j is transitive, a contradiction. Let C be the set of generators in R that commute with all elements of M. We have that $|R \setminus C| \leq 2$.

There is at most one $\langle M\rangle$-orbit adjacent to L_a in $\bar{\mathcal{Y}}$, and the label of the edge which might connect them is $\rho_f := \rho_{i-m_1}$. Similarly, there is at most one $\langle M\rangle$-orbit adjacent to L_b in $\bar{\mathcal{Y}}$, and the label of the edge which might connect them is $\rho_l := \rho_{i+m_2}$. We denote the $\langle M\rangle$-orbits adjacent to L_a and L_b, L'_a and L'_b resp. if they exist. Furthermore, both ρ_f and ρ_l, if they exist, swap a single pair of points in these $\langle M\rangle$-orbits.



Since Γ is even, ρ_f and ρ_l must both have nontrivial action on a point in another $\langle M\rangle$-orbit. We now consider separately the cases: $|R \setminus C| = 0$, $|R \setminus C| = 1$ and $|R \setminus C| = 2$.

If $R \setminus C = \emptyset$, then L_a and L_b are the unique $\langle M\rangle$-orbits, $R = \emptyset$ and $k = 2$. Thus $|R| = 0 \leq k_1 + k_2 - 4$.

Suppose that $R \setminus C = \{\rho_f\}$. In this case L_b coincide with the second Γ_i-orbit. As $\mathcal{Y}$ has at least two f-edges, by Proposition 5.4, $|R| \leq k - 3$. In addition, $k \leq k_1 + k_2 - 1$, thus $|R| \leq k - 3 \leq k_1 + k_2 - 4$.

Now let $R \setminus C = \{\rho_f, \rho_l\}$. We may assume $\bar{\mathcal{Y}}$ with an f-edge between L_a and L'_a, and with an l-edge between L_b and L'_b. Since Γ is even, both ρ_f and ρ_l act nontrivially on a point in a $\langle M\rangle$-orbit other than L_a, L_b, L'_a and L'_b. By Proposition 5.4, $\bar{\mathcal{Y}}$ has at least three components. Furthermore, to have three components there must exist a pair of $\langle M\rangle$-orbits $\{L, L'\}$ such that both ρ_l and ρ_f swap a point in L with a point in L'. Indeed, ρ_l acts trivially on the points not in L, L', L_b or L'_b and ρ_f acts trivially on the points not in L, L', L_a or L'_a. Let us assume that L and L' are both in the first Γ_i-orbit. Then ρ_l swaps L and L' entirely.

First suppose $L = L'_a$. Then, both ρ_f and ρ_l, act nontrivially on a point in L'_a. Thus there is an alternating square, with labels f and l, containing this point. Hence ρ_l acts nontrivially on a point in L_a, which we have shown is impossible. Consequently $L \neq L'_a$, as well as $L' \neq L'_a$.

Since Γ_i has only two orbits, there is a generator $\rho_j \in R$ that sends a point in either L or L' to a point not in these two $\langle M\rangle$-orbits; without loss of generality we may assume that ρ_j sends a point in L' to a point in a $\langle M\rangle$-orbit which we denote L''. Then the unique possibility is $j = l + 1$. However, then ρ_j commutes with all the elements of M which act between blocks in the first Γ_i-orbit. Thus ρ_j swaps L' and L''. This guarantees an alternating square with labels j and f, with ρ_f acting nontrivially on L''. This implies that $L'' = L'_a$. Moreover this forces ρ_j to have a nontrivial action on L_a, a contradiction. □

Proposition 5.7. *If A and B are both imprimitive then $r \leq \frac{n-1}{2}$.*

Proof. By Propositions 5.5 and 5.2, we have $r = |R|+|M|+1 \leq k_1+k_2+m_1+m_2-5$. As $k_1 + m_1 + k_2 + m_2 - 5 \leq \frac{k_1 m_1 + k_2 m_2 - 1}{2}$, we conclude that $r \leq \frac{n-1}{2}$. □

5.2. **Case 2: J_A and J_B are intervals and $i \notin \{0, r-1\}$.** We first recall two propositions on sggi that can be found in [14] that we use to deal with this case.

Proposition 5.8. [14, Proposition 3.3] *Let $\Phi = \langle \alpha_0, \ldots, \alpha_{d-1}\rangle$ be a transitive permutation group acting on the set of points $\{1, \ldots, n\}$ with $n \geq 5$, and let $\Phi^* = \langle \alpha_0, \ldots, \alpha_{d-1}, \alpha_d, \alpha_{d+1}\rangle$, where*

$$\begin{aligned} \alpha_r &= (i, n+1)(n+2, n+3) \text{ for some } i \in \{1, \ldots, n\} \\ \alpha_{r+1} &= (n+1, n+2)(n+3, n+4). \end{aligned}$$

Then Φ^* is isomorphic to S_{n+4} if it contains an odd permutation, and to A_{n+4} otherwise .

The term sesqui-extension was first introduced in [13]. Let us recall its meaning. Let $\Phi = \langle \alpha_0, \ldots, \alpha_{d-1}\rangle$ be a sggi, and let τ be an involution in a supergroup of Φ such that $\tau \notin \Phi$ and τ commutes with all of Φ. For fixed k, we define the group $\Phi^* = \langle \alpha_i \tau^{\eta_i} \mid i \in \{0, \ldots, d-1\}\rangle$ where $\eta_i = 1$ if $i = k$ and 0 otherwise, the *sesqui-extension* of Φ with respect to α_k and τ.

Proposition 5.9. [14, Proposition 5.4] *If $\Phi = \langle \alpha_i \mid i = 0, \ldots, d-1\rangle$ and $\Phi^* = \langle \alpha_i \tau^{\eta_i} \mid i \in \{0, \ldots, d-1\}\rangle$ is a sesqui-extension of Φ with respect to α_k, then:*

(a) $\Phi^* \cong \Phi$ *or* $\Phi^* \cong \Phi \times \langle \tau \rangle \cong \Phi \times 2$.
(b) *whenever $\tau \notin \Phi^*$, Φ is a string C-group if and only if Φ^* is a string C-group.*

In this case we may assume that $A = \Gamma_{<i}$ and $B = \Gamma_{>i}$. As A or B can have small degree ≤ 11, in what follows we list all primitive even string C-groups of small degree, with intransitive maximal parabolic subgroups, having rank $r \geq \frac{n-1}{2}$.

Proposition 5.10. *Let Φ be string C-group with a connected Coxeter diagram isomorphic to an even primitive group of degree $n \leq 11$. If Φ_j is intransitive for every $j \in \{0, \ldots, d-1\}$ then either $d \leq \frac{n-2}{2}$ or Φ has one of the permutation representation graphs given in Table 2.*

Proof. We used Magma to get this result. □

Proposition 5.11. *Let $j \in \{0, \ldots, r-1\}$. Suppose that $\Gamma_{<j}$ is transitive on m points and fixes the remaining $m - n$ points. If $\Gamma_{<j}$ is a primitive group of degree $m < 12$ and $j \geq \frac{m-1}{2}$ then it must have permutation representation graph (1), (2) or (3) of Table 2.*

The dual version of this proposition is also true.

Proof. By Proposition 5.10, $\Gamma_{<j}$ has one of the 16 permutation representation graphs given in Table 2. Let $X := \{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{<j})$ with $|X| = m$. Then there exists a generator ρ_k of Γ, with $k \geq j$, such that $c\rho_k = d$ with $c \in X$ and $d \notin X$. As $\Gamma_{<j}$ is primitive, k must be consecutive with $j-1$. Moreover, c has degree two with the edges incident to c having labels $j-1$ and j. In graphs (8), (9), (13) and (15) there is no such vertex c. Let us consider the remaining graphs.

Now let (4) be the permutation representation graph of $\Gamma_{<j}$. Then $\Gamma_{<j} = \Gamma_{<4} \cong A_9$ and $\Gamma_{>1} \cong A_{n-4}$ by Proposition 5.8 (note that $n - 4 \geq 5$), thus $\Gamma_{>1} \cap \Gamma_{<4} \cong A_5$. However, $\langle \rho_2, \rho_3\rangle$ is a dihedral group. Consequently Γ does not satisfy the intersection condition, a contradiction.

Only in cases (11) and (16) there are two possibilities for the vertex c, but as in case (4), case (16) is self-dual. Using similar arguments, summarised below, we

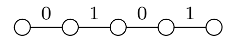
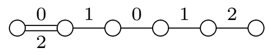
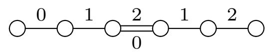
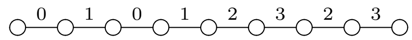
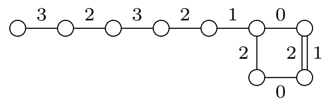
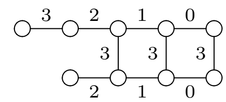
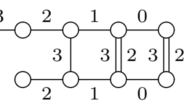
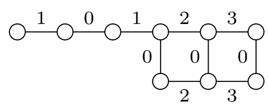
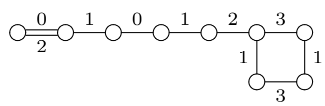
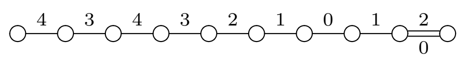
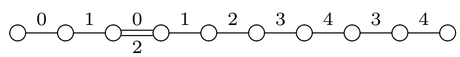
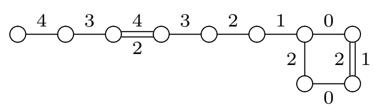
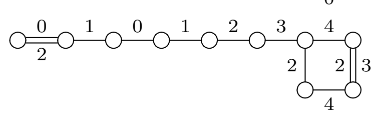
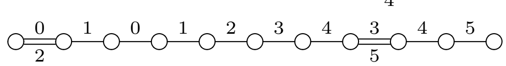
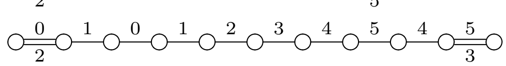
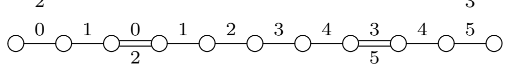
(1)	D_{10}	
(2)	$L_2(5)$	
(3)	$L_2(5)$	
(4)	A_9	
(5)	A_9	
(6)	A_9	
(7)	A_9	
(8)	A_9	
(9)	A_9	
(10)	A_{10}	
(11)	A_{10}	
(12)	A_{10}	
(13)	A_{10}	
(14)	A_{11}	
(15)	A_{11}	
(16)	A_{11}	

TABLE 2. Even transitive string C-groups of degree m with connected Coxeter diagram having intransitive maximal parabolic subgroups and rank $\geq \frac{m-1}{2}$.

conclude that $\Gamma_{<j}$ cannot be any of the graphs of Table 2 except graphs (1), (2) and (3). Let (11)* denote the dual of (11) with its labels interchanged by $k \leftrightarrow 4-k$.

(5), (6), (7): $\Gamma_{>1} \cong 2 \times A_{n-4}$ or $\Gamma_{>1} \cong A_{n-4}$ (by Proposition 5.9);
$\Gamma_{<4} \cong A_9$; $\Gamma_{>1} \cap \Gamma_{<4} \cong A_5 \ncong \langle \rho_2, \rho_3 \rangle$.

(10), (11), (11)*, (12): $\Gamma_{>2} \cong A_{n-5}$; $\Gamma_{<5} \cong A_{10}$; $\Gamma_{>2} \cap \Gamma_{<5} \cong A_5 \ncong \langle \rho_3, \rho_4 \rangle$.

(14), (16): $\Gamma_{>3} \cong A_{n-6}$; $\Gamma_{<6} \cong A_{11}$; $\Gamma_{>3} \cap \Gamma_{<6} \cong A_5 \ncong \langle \rho_4, \rho_5 \rangle$.

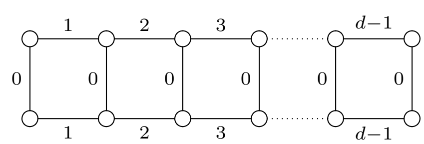
□

Proposition 5.12. *Let $\Phi = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle$ be a transitive sggi embedded into $S_a \wr S_b$ with X being the set of generators of Φ generating the block action independently. If Φ_j is intransitive for all $j \in \{0, \ldots, d-1\}$, then*

(a) $d \leq |X| + a - 1$;
(b) $d \leq \frac{ab-2}{2}$ *for* $a, b \neq 2$ *and* $ab \neq 9$;
(c) *if* $(a, d) = (2, b)$ *or* $(b, d) = (2, a)$ *or* $(a, b) = (3, 3)$ *then* Φ *either contains an odd permutation or has disconnected diagram.*

Proof. Consider a partition P of $\{1, \ldots, ab\}$ determined by the orbits of $\langle X \rangle$. Now let $\mathcal{X}$ be a graph with $|P|$ vertices corresponding the partitions of P and with exactly one j-edge for each generator of Φ that is not in X, connecting two partitions in different Φ_j-orbits. This graph has no cycles and $|P| \leq a$, thus $d - |X| \leq a - 1$. Hence, $d \leq |X| + a - 1$. Moreover as X generate the bock action independenty $|X| \leq b - 1$. Hence $d \leq\leq a + b - 2\frac{ab-2}{2}$ for $a, b \neq 2$ and $ab \neq 9$.

Now suppose that $(a, d) = (2, b)$ or $(b, d) = (2, a)$ and Φ is even, up to duality, Φ has the following permutation representation graph, and



Thus Φ has a disconnected diagram. □

Using induction over n and Proposition 5.12 we get the following result.

Proposition 5.13. *Suppose that Φ is string C-group generated by involutions of rank d, with connected diagram, having all maximal parabolic subgroups intransitive. If Φ is a transitive even group of degree m with $12 \leq m < n$ then, $d \leq \frac{m-1}{2}$. Moreover if $d = \frac{m-1}{2}$ then Φ is the alternating group A_m.*

Proof. If Φ is primitive and not the alternating group then, by Proposition 2.2, $d \leq \frac{m-2}{2}$. If Φ is imprimitive, then it is embedded into a group $S_a \wr S_b$ with $ab = m$ and with the block action being generated by at most $b - 1$ elements. As the maximal parabolic subgroups of Φ are intransitive we may use Proposition 5.12 to get

$$d \leq \frac{m-2}{2}.$$

for $a, b \neq 2$. If either $a = 2$ or $b = 2$ then, as Φ is even and has a connected diagram, $d < \frac{m}{2}$ with m even, hence $d \leq \frac{m}{2}$. Finally if $\Phi \cong A_m$ with $12 \leq m < n$, we then get the result by induction on n.

□

Proposition 5.14. *Suppose that $\Gamma_{<3}$ is not one of the string C-groups (2) or (3) Table 2. If A is the string C-group (1) of Table 2, then $r \leq \frac{n-1}{2}$. The same result holds for B.*

Proof. Consider first $n-5 < 12$. If $r-3 \geq \frac{(n-5)-1}{2}$ then, by Proposition 5.11, $B\,(=\Gamma_{>2})$ is one of the small examples (1), (2) or (3) of Table 2, thus $n < 12$, a contradiction. Hence $r-3 < \frac{(n-5)-1}{2}$. Let $n-5 \geq 12$.

Suppose first that ρ_2 has a trivial action on the first Γ_i- orbit. In this case $\Gamma_{>1}$ is an even transitive group on $n-4$ points. By Proposition 5.9 $\Gamma_1 \cong \langle\rho_0\rangle \times \Gamma_{>1}$. Now, by Proposition 5.13, we get that $r-2 \leq \frac{(n-4)-1}{2}$ and hence $r \leq \frac{n-1}{2}$.

Now suppose that ρ_2 has a nontrivial action on the first block. As $\Gamma_{<3}$ cannot be one of the string C-groups (2) or (3) of Table 2, it is a sesqui extension of it (with respecto to ρ_2). If B is not the alternating group, then by Proposition 5.13, we get that $r-3 \leq \frac{(n-5)-2}{2}$. We need only to consider the case $B = \Gamma_{>2} \cong A_{n_2}$. Let Φ be either the string C-group (2) or (3) of Table 2 and τ be the action of ρ_2 in the second Γ_2-orbit. As either $\tau = (\rho_1\rho_2)^3$ or $\tau = (\alpha_1\alpha_2)^5$ (according to each case (1) or (2)), $\Gamma_{<3}$ is isomorphic to $\langle\tau\rangle \times \Phi$. But then, as $\Gamma_{>2} \cong A_{n_2}$, $\tau \in \Gamma_{>2} \cap \Gamma_{<3}$, a contradiction. Hence $r \leq \frac{n-1}{2}$. □

Proposition 5.15. *Suppose neither $\Gamma_{<3}$ nor $\Gamma_{>r-4}$ is one of the string C-groups (2) or (3) of Table 2, or their duals. If A and B are not both the alternating groups, then $r \leq \frac{n-1}{2}$.*

Proof. By Proposition 5.14 it may be assumed that neither A nor B is the string C-group (1) of Table 2.

If A and B are not alternating groups then by Propositions 5.13 and 5.10, $i \leq \frac{n_1-2}{2}$ and $r-1-i \leq \frac{n_2-2}{2}$. If A is the alternating group, then either $n \leq 12$ and $i \leq \frac{n_1-2}{2}$ (by Proposition 5.10) or, $n_1 \geq 12$ and then, by Proposition 5.13 (induction) we conclude that $i \leq \frac{n_1-1}{2}$. Analogously if B is the alternating group then $r-1-i \leq \frac{n_2-1}{2}$. In any case if A and B are not both the alternating groups, $r \leq 1 + \frac{n_1+n_2-3}{2} = \frac{n-1}{2}$. □

Proposition 5.16. *Suppose neither $\Gamma_{<3}$ nor $\Gamma_{>r-4}$ is one of the string C-groups (2) or (3) of Table 2, or their duals. If A and B are both alternating groups then $r \leq \frac{n-1}{2}$.*

Proof. In this case $\Gamma_{<i+1}$ is a sesqui extension of a sggi Φ with respect to ρ_i, where Φ is a group of degree n_1+1. By Proposition 5.9 $\Gamma_{<i+1}$ is isomorphic either to $2\times\Phi$ or Φ. Suppose that $\Gamma_{<i+1} \cong 2\times\Phi$. In that case there is an even permutation τ on the second Γ_i-orbit that belongs to $\Gamma_{<i+1}$. As $\Gamma_{>i} \cong A_{n_2}$, $\tau \in \Gamma_{>i}$ and therefore $\tau \in \Gamma_{<i+1} \cap \Gamma_{>i}$, a contradiction. Hence $\Gamma_{<i+1} \cong \Phi$ and Φ is itself a string C-group. Using the same argument $\Gamma_{>i-1}$ is also isomorphic to a transitive group Ψ of degree n_2+1. Moreover either Φ or Ψ is a even group. Suppose Φ is even. Then, as A is not one of the string C-groups (1), (2) or (3), either $n_1 < 12$ and $i \leq \frac{n_1-2}{2}$, or $n_1+1 \geq 12$. In latest case, by Proposition 5.13 $i+1 \leq \frac{(n_1+1)-1}{2}$. In addition, $r-1-i \leq \frac{n_2-1}{2}$, hence $r \leq \frac{n-1}{2}$. □

Proposition 5.17. *Suppose neither $\Gamma_{<3}$ nor $\Gamma_{>r-4}$ is one of the string C-groups (2) or (3) of Table 2, or their duals. Let $i \notin \{0, r-1\}$. If $A \cong \Gamma_{<i}$ and $B = \Gamma_{>i}$ then $r \leq \frac{n-1}{2}$.*

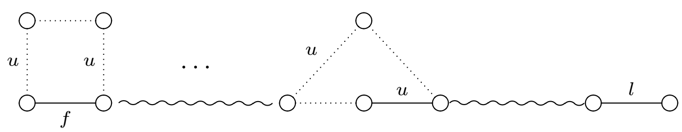
Proof. This is a consequence of Propositions 5.15 and 5.16. □

To complete this case we still need to deal with $\Gamma_{<3}$, $\Gamma_{>r-4}$, or both, being one of the string C-groups (2) or (3) of Table 2. This is included in Case 3, at the end.

5.3. **Case 3: The remaining cases.** Assume in this case that J_B is either empty or an interval. As before let $\mathcal{G}$ be the permutation representation graph of Γ.

Proposition 5.18. *If e is an f-edge of $\mathcal{G}$ not in an alternating square, then any path (not containing another f-edge) from e to an edge with label l, with $l < f$ (resp. $l > f$), contains all labels between l and f. Moreover, there exists a path from e to an l-edge, that is fixed by $\Gamma_{>l}$ (resp. $\Gamma_{<l}$).*

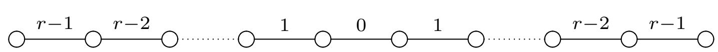
Proof. Consider a path starting in e and containing an l-edge. Let $f < l$. Suppose that none of the edges of the path has label k, for some $f < k < l$. Then in this path there is an edge with label $< k$ meeting an edge with label $u > k$. Suppose that this is the first time in the path that this happens. Then there is an alternating square, containing e and a u-edge, a contradiction, as shown in the following figure.



□

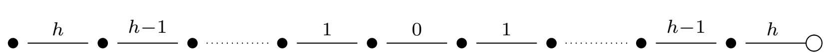
Proposition 5.19. *Let $i = 0$. If ρ_{r-1} acts non-trivially on both Γ_i-orbits, then $r \leq \frac{n-1}{2}$.*

Proof. By Proposition 5.18 there are two paths, one in the first and the other in the second Γ_i-orbit, each containing all labels from 1 to $r-1$. Thus $2(r-1)+1 \leq n-1$. Hence $r \leq \frac{n}{2}$. Suppose we have the equality. In that case the paths have consecutive labels, as in the following figure and n is precisely the number of vertices of the two paths.



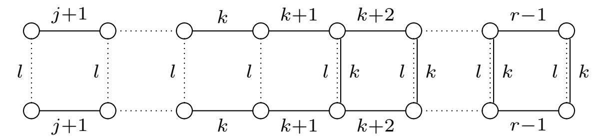
But then there is no place for extra i-edges unless $r = 3$ (which gives the bound trivially as $n \geq 12$). □

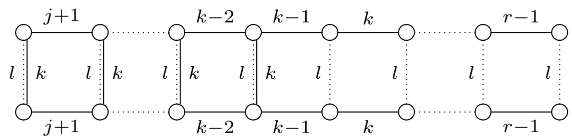
Proposition 5.20. *Let $i = 0$. Let $h \neq r-1$ be the maximal label such that ρ_h acts non-trivially on both Γ_i-orbits. There exists a set of vertices X, contained in the Γ_i-orbit fixed by ρ_{r-1}, such that $h \leq \frac{n-|X|-1}{2}$ and $\Gamma_{>h}$ fixes $\{1, \dots, n\} \setminus X$. Moreover if $h = \frac{n-|X|-1}{2}$ then $\Gamma_{<h}$ has the following permutation representation graph, where the black dots represent the vertices of $\{1, \dots, n\} \setminus X$.*



Proof. By Proposition 5.18 there exist a path $\mathcal{P}_1$ from the h-edge in the first Γ_i-orbit to the vertex a, and a path $\mathcal{P}_2$ from the h-edge in the second Γ_i-orbit to the vertex b, each of them containing all labels from 1 to $h-1$, and fixed by $\Gamma_{>h}$. In that case, the two paths give us $2h+1 \leq |\mathrm{Fix}(\Gamma_{>h})| = n-|X|$ with $X := \{1, \dots, n\} \setminus \mathrm{Fix}(\Gamma_{>h})$ as required. When equality holds, the permutation representation graph is the one given in this proposition. □

Proposition 5.21. *Let $X := \{1, \dots, n\} \setminus \mathrm{Fix}(\Gamma_{>j})$. If $\Gamma_{>j}$ is transitive on X and there exists a permutation ρ_l with $l < j$ acting non trivially on X, then $r - 1 - j \leq \frac{|X|}{2} - 1$. Moreover if $r - 1 - j = \frac{|X|}{2} - 1$ then, $\Gamma_{>j}$ has one of the following permutation representation graphs for some $k \in \{j+1, \dots, r-1\}$.*



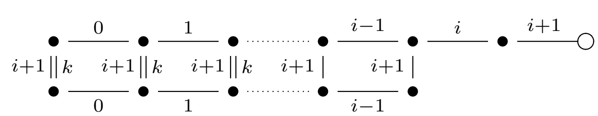


Proof. If ρ_l acts non trivially on X then $\Gamma_{>j}$ must be imprimitive with blocks of size two, with ρ_l swapping all pairs of vertices inside the blocks. Let M an independent generating set for the block action of $\Gamma_{>j}$ and suppose that the group generated by M is intransitive on X. Then there must be two orbits on X under the action of $\langle M \rangle$ and some $k > j$ such that ρ_k swaps points of distinct orbits. But then, as $\rho_l \in \Gamma_k$, Γ_k is transitive on X. Moreover, ρ_k only moves points in X which is the union of two Γ_k-orbits. These two orbits are already fused by $\rho_l \in \Gamma_k$. Therefore, Γ_k has to be transitive on $\{1, \dots, n\}$, a contradiction. So the action of $\langle M \rangle$ must be transitive on X. If $|M| = \frac{|X|}{2} - 1$ there are only the given possibilities for $\Gamma_{>j}$. □

In what follows we suppose that $i > 0$ and that all generators acting on the second Γ_i-orbit have labels $> i$. As J_A is not an interval, $\Gamma_{<i}$ can either transitive or intransitive on the first Γ_i-orbit. We consider these cases separately.

Proposition 5.22. *Let $r > \frac{n-1}{2}$. Let $h > i$ be the maximal label of a permutation acting non-trivially on both Γ_i-orbits. If $\Gamma_{<i}$ is transitive on the first Γ_i-orbit then $h = i + 1$, $h < r - 1$ and there exists a set of vertices X, contained in the second Γ_i-orbit, such that $h \leq \frac{n-|X|-1}{2}$ and $\Gamma_{>h}$ fixes $\{1, \dots, n\} \setminus X$.*

Moreover if $h = \frac{n-|X|-1}{2}$ then $\Gamma_{<h}$ (with $h = i+1$) has the following permutation representation graph for some $k \in \{2, \dots, i-1\}$ where the black dots represent the vertices of $\{1, \dots, n\} \setminus X$.



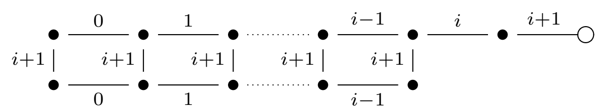
Proof. In this case, $\Gamma_{<i}$ is transitive on the first Γ_i-orbit O_1, which has size n_1. Moreover there exists $h > i$ such that ρ_h acts non-trivially on O_1. This action is fixed-point-free and hence ρ_h moves a and n_1 is even. But since the i-edge $\{a, b\}$ is not in a square by hypothesis, $h = i + 1$ and $\Gamma_{>h}$ acts trivially on O_1. Moreover by the dual of Proposition 5.21 we have that $i \leq \frac{n_1}{2} - 1$. Thus $h = i + 1 \leq \frac{n_1}{2}$. As J_B is an interval, ρ_{i+1} is the unique permutation acting nontrivially on b. If $h \neq r - 1$, $\Gamma_{>h}$ fixes O_1 and a vertex in the second Γ_i-orbit, so at least $n_1 + 1$ vertices altogether. Then $h \leq \frac{|\mathrm{Fix}(\Gamma_{>h})|-1}{2} \leq \frac{n-|X|-1}{2}$. The equality cannot occur as it would imply that $n - |X| = n_1 + 1$ is even, a contradiction. So $h \leq \frac{n-|X|-1}{2}$

as wanted. Moreover, when the equality holds, by the dual of Proposition 5.21, the only possibility is the permutation representation graph given in the statement of this proposition.

Now suppose that $h = r-1$. Then $r-1 \leq \frac{n_1}{2}$. As $n \geq n_1+2$, $r \leq \frac{n}{2}$. As by hypothesis $r > \frac{n-1}{2}$, we have the equality $r = \frac{n}{2}$. Then Γ has the permutation representation graph given in the statement of this proposition with $h = i+1 = r-1$, with some additional i-edges. As all Γ_j's must be intransitive, the extra i-edges must be vertical edges. Hence one of ρ_i and ρ_{i+1} has to be an odd permutation, a contradiction. □

Proposition 5.23. *Let $r > \frac{n-1}{2}$. Let $h > i$ be the maximal label of a permutation acting non-trivially on both Γ_i-orbits. If $\Gamma_{<i}$ is intransitive in the first Γ_i-orbit, then $h < r-1$ and there exists a set of vertices X, contained in the second Γ_i-orbit, such that $h \leq \frac{n-|X|-1}{2}$ and $\Gamma_{>h}$ fixes $\{1,\ldots,n\} \setminus X$.*

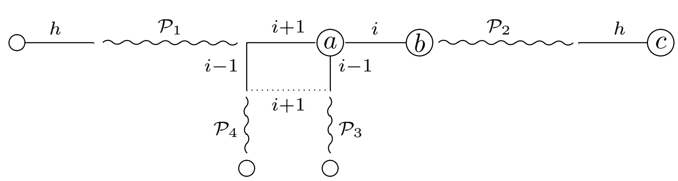
Moreover if $h = \frac{n-|X|-1}{2}$ then $h = i+1$ and $\Gamma_{<h+1}$ has the following permutation representation graph, where the black dots represent the vertices of X.



Proof. In this case J_A is not an interval thus, by Proposition 5.1, A is imprimitive embedded into $S_k \wr S_m$ and $\Gamma_{<i}$ is fixing all the blocks (with $k, m \geq 2$). By Proposition 5.18 there exist a path $\mathcal{P}_1$ from the h-edge in the first Γ_i-orbit to the vertex a, and a path $\mathcal{P}_2$ from the h-edge in the second Γ_i-orbit to the vertex b, each of them containing all labels from $i+1$ to $h-1$, and fixed by $\Gamma_{>h}$. In addition there is a path $\mathcal{P}_3$ in the block β containing the vertex a, and edges with all labels from 0 to $i-1$. Moreover, there is also a path $\mathcal{P}_4$ in the block $\beta\rho_{i+1}$ also containing edges with all labels from 0 to $i-1$.

Let us first assume that there is an edge with label $l > i$ inside one of the two blocks β or $\beta\rho_{i+1}$. Then the generator ρ_l is fixed-point-free on the block containing that edge. Suppose that block is β. Then $l = i+1$, a contradiction. Suppose then that the edge is in $\beta\rho_{i+1}$. Then $l = i+2 = h$ and there are only two blocks in the block system. Moreover, ρ_l acts inside $\beta\rho_{i+1}$ and provides an embedding of $\Gamma_{<i}$ into $S_2 \wr S_{k/2}$. Thus, since Γ_j is intransitive for every j, a similar argument to the one used in the proof of Proposition 5.21 shows that $i \leq k/2-1$. Hence $h-2 \leq k/2-1$. Now if $h = r-1$, as $n \geq 2k+2$ (the extra 2 coming from the h-edge in the second Γ_i-orbit), we get $r-1 \leq \frac{n+2}{4} < \frac{n-2}{2}$ (as n must be at least 7 in this case) giving a contradiction with the hypotheses of the proposition. Thus $h < r-1$ and $\Gamma_{>h}$ fixes a set P of size $2k+2$. Therefore, $h \leq \frac{|P|+2}{4} \leq \frac{|P|-1}{2}$ for every $|P| \geq 4$ which is obviously the case here as $k \geq 2$.

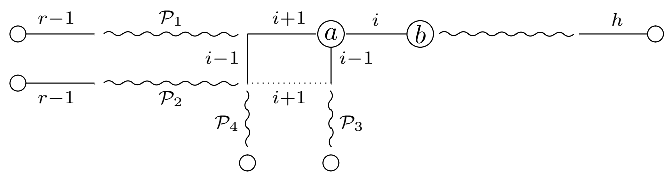
Let us now assume that all edges with label $l > i$ are not contained in β nor in $\beta\rho_{i+1}$. Then the paths $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ have no edge in common, as shown in the following figure.



Suppose that $h \neq r-1$ and that ρ_{h+1} acts trivially on the first Γ_i-orbit. Let P be the set of vertices of $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$ excluding the vertex c on the right hand side of the diagram above, and $X = \{1, \ldots, n\} \setminus P$. The set P is fixed by $\Gamma_{>h}$ and $2h+1 \leq |P| = n - |X|$. Moreover when $2h+1 = |P|$ the permutation representation of $\Gamma_{<h}$ is the one given in this proposition with $h = i+1$.

Now if $h = r-1$, we have $2(r-1)+1 \leq n-1$. Hence $r \leq \frac{n}{2}$, and by hypotheses we must have the equality. Therefore Γ has precisely the permutation representation graph given in the statement of this proposition with some additional i-edges. As all Γ_j's must be intransitive, the extra i-edges must be vertical. Hence one of ρ_i and ρ_{i+1} has to be an odd permutation, a contradiction.

It remains to prove that $\Gamma_{>h}$ fixes the first Γ_i-orbit. Suppose the contrary. Then ρ_{r-1} acts nontrivially on the first Γ_i-orbit. In this case consider the path $\mathcal{P}_1$ as before, with the label of its first edge being $r-1$ instead of h, and consider $\mathcal{P}_3$ and $\mathcal{P}_4$ has before. Now let $\mathcal{P}_2$ be a copy of $\mathcal{P}_1$ whose last vertex is $a\rho_{i-1}$, as in the following figure.



In this case $2(r-1)+1 \leq (n-1)-1$, a contradiction. □

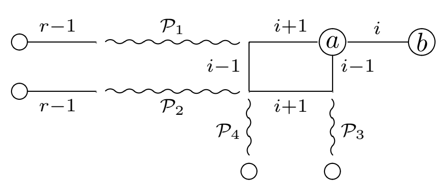
Proposition 5.24. *Let $r > \frac{n-1}{2}$. Let $i \neq 0$ and J_B be an interval with labels $> i$. If $h > i$ is the maximal label of a permutation acting non-trivially on both Γ_i-orbits, then $h < r-1$ and there exists a set of vertices X, contained in the second Γ_i-orbit, such that $h \leq \frac{n-|X|-1}{2}$ and $\Gamma_{>h}$ acts trivially on $\{1, \ldots, n\} \setminus X$.*

Proof. This is consequence of Propositions 5.22 and 5.23. □

Proposition 5.25. *Let $n \geq 8$. If $i \neq 0$ and $J_B = \emptyset$ then $r \leq \frac{n-1}{2}$.*

Proof. Similarly to the proof of Proposition 5.23, A is embedded into a wreath product $S_k \wr S_m$ with $n = km+1$ and $\Gamma_{<i}$ fixing the blocks. If there is a label $l > i$ such that ρ_l acts nontrivially inside a block then $m = 2$, $r-1 = l = i+2$ and $\Gamma_{<i}$ is embedded into $S_2 \wr S_{k/2}$, giving the inequality $i = r-3 \leq k/2 - 1 \leq \frac{n-1}{4} - 1$. Hence for $n \geq 8$, $r \leq \frac{n-1}{2}$.

Suppose then that every generator with label $> i$ acts trivially on the blocks it fixes. Thus there are four paths $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$, as in the following graph, containing all but one label twice and one cycle that is an alternating square.



Hence $2(r-1)+1 \leq n$. If equality holds, the paths contain exactly one edge for each label, but then as all Γ_j's are intransitive, ρ_i acts trivially on $\{1, \ldots, n\} \setminus \{a, b\}$, and thus is a transposition, a contradiction. Therefore $r \leq \frac{n}{2}$. If n is odd then $r \leq \frac{n-1}{2}$. If n is even then n_1 is odd, and neither $k = 2$ nor $m = 2$. Hence there at least two more vertices not in the paths, and then $2(r-1)+1 \leq n-2$. Therefore $r \leq \frac{n-1}{2}$. □

Proposition 5.26. *Let $n \geq 12$. Suppose that $h > i$ is the maximal label of a permutation acting non-trivially on both Γ_i-orbits. If $h \neq r-1$ and $\Gamma_{>h}$ is transitive on $X := \{1,\dots,n\} \setminus \mathrm{Fix}(\Gamma_{>h})$ then, either $r \leq \frac{n-1}{2}$, or one of the groups $\Gamma_{<3}$ or $\Gamma_{>r-4}$ is one the string C-groups (2) and (3) given in Table 2 or their duals.*

Proof. Suppose that $\Gamma_{>h}$ is neither the dual of the string C-group (2) nor the string C-group (3) given in Table 2. Then, by Propositions 5.11, 5.13 and 5.14, $r-1-h \leq \frac{|X|-1}{2}$. Moreover $r-1-h = \frac{|X|-1}{2}$ if $\Gamma_{>h}$ is the alternating group $A_{|X|}$.

By Proposition 5.24, $h \leq \frac{n-|X|-1}{2}$. Suppose that $\Gamma_{>h} \cong A_{|X|}$ and $h = \frac{n-|X|-1}{2}$. If ρ_i has a nontrivial action on X then, by Proposition 5.21, $\Gamma_{>h}$ must be imprimitive with blocks of size two, a contradiction. Thus we may assume that ρ_i fixes X pointwise and swaps a pair of vertices in $\{1,\dots,n\} \setminus X$.

If $i = 0$ then $\Gamma_{<h}$ has the permutation representation graph given in Proposition 5.20. The string condition implies that $h = 2$ and $\Gamma_{<3}$ must be the string C-group (2) given in Table 2, a contradiction.

When $i \neq 0$, either $r \leq \frac{n-1}{2}$ or $\Gamma_{<h}$ has either the permutation representation graph given in Proposition 5.22 or the one given in Proposition 5.23. As all Γ_j's must be intransitive, the extra i-edges must be vertical edges. Hence one of ρ_i and ρ_{i+1} has to be an odd permutation, a contradiction.

Hence either $h < \frac{n-|X|-1}{2}$ or $r-1-h < \frac{|X|-1}{2}$, which implies in both cases that $r \leq \frac{n-1}{2}$. □

In the previous proposition we considered the case where ρ_h acts nontrivially in both Γ_i-orbits and $\Gamma_{>h}$ is transitive on the points it does not fix. Let us now deal with the case where $\Gamma_{>h}$ is intransitive.

In what follows we use the results of the previous section on 2-fracture graphs. Observe that in the proofs of Propositions 4.10 and 4.12 none of the following three conditions on Γ were needed: intersection condition, connectedness of the diagram and the group being even. Indeed all we need is a string group G generated by a set $\{\delta_j \mid j \in \{0,\dots,d-1\}\}$ of involutions where δ_j swaps two pairs of vertices in different G_j-orbits for each $j \in \{0,\dots,d-1\}$.

Proposition 5.27. *Let $t \in \{0,\dots,r-2\}$ and $U := \{1,\dots,n\} \setminus \mathrm{Fix}(\Gamma_{>t})$. If t is such that*

- *$t \leq \frac{n-|U|-1}{2}$,*
- *$\Gamma_{>t}$ has a 2-fracture graph,*
- *$\Gamma_{>t}$ acts intransitively on U,*

then $r \leq \frac{n-1}{2}$.

Proof. Assume first that $t = r-2$. In this case, $|U| \geq 4$ for ρ_{r-2} to be an even permutation. Hence $r \leq \frac{n-1}{2}$. Let $t < r-2$ be the maximal label satisfying the conditions of this proposition.

Suppose that $\Gamma_{>t}$ has c nontrivial orbits $U_1,\dots,U_c$. For each set U_s with $s \in \{1,\dots,c\}$ denote by I_s the set of labels $l\,(>t)$ of edges in U_s. Consider the graph $\mathcal{C}$ whose vertices are the orbits $U_1,\dots,U_c$, and two orbits U_s and U_q are joined by an l-edge if there exist a point in U_s and a point in U_q that are swapped by ρ_l.

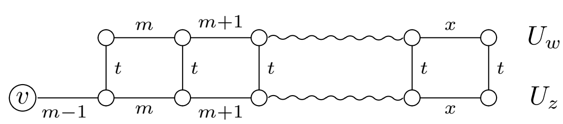
Consider a (simple) fracture graph $\mathcal{F}$ of $\Phi := \Gamma_{>t}$, that is, a graph with $|U|$ vertices and $r-1-t$ edges. Such a graph exists by the second hypothesis of the proposition. Each l-edge of $\mathcal{F}$ connects vertices in different Φ_l-orbits. Let $s \in \{1,\dots,c\}$ and F_s be the set of labels of edges of $\mathcal{F}$ within U_s. Clearly $F_s \subseteq I_s$. Choose $\mathcal{F}$ such that it satisfies the following property:

P1 if $l \in F_s$ is the label of the unique l-edge in one component swapping vertices in different Γ_l-orbits, then no other component has more than one pair of vertices in different Γ_l-orbits.

Let G_s be the group action of $\Gamma_{>t}$ in U_s. We have that G_s is generated by a set of involutions (not necessarily independent) with labels in I_s. The subset of involutions with labels in F_s is independent since F_s is the subset of labels of edges of $\mathcal{F}$. We denote by $(G_s)_j$ the group generated by all involutions of the generating set of G_s except the one with label j.

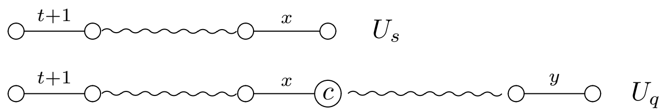
If G_s does not admit a 2-fracture graph with set of labels F_s, then there exists an edge e with label $l \in F_s$ with ρ_l swapping only one pair of vertices of U_s, in different Γ_l-orbits. Let m be the minimal label and x be the maximal label of an edge inside U_s. By Proposition 5.18 there exists a path $\mathcal{P}_1$ containing all labels from $l-1$ to m and another path $\mathcal{P}_2$ containing all labels from $l+1$ to x in U_s. Let $\mathcal{P} = \mathcal{P}_1 \cup \{e\} \cup \mathcal{P}_2$. Now we deal separately with the cases $m > t+1$ and $m = t+1$ and we conclude, in both cases, that $x \leq \frac{n-|X|-2}{2}$ where $X := \{1,\dots,n\}\setminus \mathrm{Fix}(\Gamma_{>x})$.

$\underline{m > t+1}$: If $m > t+1$ then a component U_q adjacent to U_s also contains all labels from m to x. Let U_z be a component containing an edge with label $m-1$. We can reach it from U_s with a shortest path in $\mathcal{C}$. The last component before U_z in this path, say U_w, contains a copy $\mathcal{P}'$ of $\mathcal{P}$. Thus there is a t-edge from U_w to U_z.



As m is the minimal label in U_w, $m-1 \notin U_w$, thus $m-1 = t+1$. Let P be the set of vertices of $\mathcal{P}\cup\mathcal{P}'$. We have $2(x-(t+1)) \leq |P|-2$, hence $x \leq t+\frac{|P|}{2} \leq \frac{n-|U|+|P|-1}{2}$. As $\Gamma_{>t}$ has a 2-fracture graph by hypothesis, there is at least one more edge with label $t+1$, so $|U| > |P|+1$ and $x \leq \frac{n-3}{2}$. If $x = r-1$ then $r \leq \frac{n-1}{2}$. Let $x \neq r-1$. The vertex v is fixed by $\Gamma_{>x}$ as well as all vertices of P. Let $X := \{1,\dots,n\}\setminus\mathrm{Fix}(\Gamma_{>x})$. Then we have $x \leq \frac{n-|X|-2}{2}$.

$\underline{m = t+1}$: Suppose that $m = t+1$. Furthermore assume that any component containing a unique l-edge between vertices in different Γ_l-orbits, has minimal label $t+1$. Let x be the maximal label in U_s and let $\mathcal{P}$ be as before. We now use the fact that $\Gamma_{>t}$ has a 2-fracture graph, and thus there exists another component U_q having one l-edge e' between vertices in different components. Moreover by Property (P1) this l-edge cannot be in an alternating square inside U_q. By assumption the minimal label in U_q is also $t+1$. Let y be the maximal label in U_q. Assume that $y \geq x$. Then, there exists a path $\mathcal{P}_1'$ containing all labels from $l-1$ to $t+1$ and another path $\mathcal{P}_2'$ containing all labels from $l+1$ to x, both in U_q. Consider $\mathcal{P}' = \mathcal{P}_1' \cup \{e'\} \cup \mathcal{P}_2'$. There is at most one vertex of $\mathcal{P}'$ that is not fixed by $\Gamma_{>x}$. Let Q be the set of vertices of $\mathcal{P} \cup \mathcal{P}'$.



In this case $\Gamma_{>x}$ fixes the set $Q \setminus \{c\}$. Hence $2(x-t) \leq |Q| - 2$ and we get

$$x \leq \frac{n - |X| - 2}{2}$$

where $X := \{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{>x})$.

In both cases, thanks to the maximality of t we conclude that either $x = r-1$ or $\Gamma_{>x}$ is transitive on X. If $x = r-1$, $r \leq \frac{n-2}{2}$. Suppose $\Gamma_{>x}$ is transitive on X. In this case X is a subset of U_s for some $s \in \{1, \ldots, c\}$. If $\Gamma_{>x}$ is fix-point-free on U_s, then ρ_t centralizes $\Gamma_{>x}$, thus, by Proposition 5.21, $r - 1 - x \leq \frac{|X|}{2} - 1$. Hence $r \leq \frac{n-1}{2}$. If $\Gamma_{>x}$ has fixed points in U_s, then $x \neq \frac{n-|X|-2}{2}$ and, as by Proposition 5.13 $r - 1 - x \leq \frac{X}{2}$, we get $r \leq \frac{n-1}{2}$. This concludes the case where G_s has no 2-fracture graph.

Therefore we may assume that G_s admits a 2-fracture graph with set of labels F_s and therefore $F_s \leq \frac{|U_s|}{2}$. We can use the results of Section 4.

Suppose that $\Gamma_{>t+1}$ is transitive on U_s. Take the closest orbit U_q to U_s in $\mathcal{C}$ such that $\Gamma_{>t+1}$ is intransitive on U_q and let P be a shortest path from U_s to U_q. We can use ρ_t modify $\mathcal{F}$ along this path, to concentrate all edges of the fracture graph in U_q. We obtain another fracture graph also satisfying P1. Hence, we may assume that F_x is empty for every orbit U_x of P except U_q. Therefore $|F_x| = 0 \leq \frac{|U_x|-2}{2}$. Thus in every component with $F_s \neq \emptyset$ there exists an edge with label $t+1$ between vertices in different Γ_{t+1}-orbits. Consequently there is at most one component having a connected 2-fracture graph, and $r - 1 - t \leq \frac{|U|-(c-1)}{2}$ where c is the number of $\Gamma_{>t}$-orbits of size at least 2. Whenever $c > 2$ or if $|F_x| = 0$ for some x, we get $r \leq \frac{n-1}{2}$. Let us now assume $c = 2$. In this case, if $r = \frac{n}{2}$, the 2-fracture graph on U_s is connected and has an alternating square (by Proposition 4.10), and the 2-fracture graph on U_q is disconnected, and has two components, one having an alternating square and the other being a tree. These two components in U_q must moreover be connected by a t-edge. These components satisfy the following equalities: $|F_s| = \frac{|U_s|}{2}$, $|F_q| = \frac{|U_q|-1}{2}$, $I_s = F_s$ and $I_q = F_q \cup \{t+1\}$. Let G_s be the group action in U_s and let G_q be the group action in U_q.

Recall that $t < r-2$. If $(G_q)_{t+2}$ is transitive on U_q then ρ_{t+1} centralizes G_q, thus the two components of the 2-fracture graph of G_q have the same shape, so they must both be trees, a contradiction. Thus $(G_q)_{t+2}$ is intransitive on U_q. Moreover $t+2 \in F_q$ as $I_q = F_q \cup \{t+1\}$. Therefore $t+2 \notin F_s = I_s$, hence ρ_{t+1} centralizes G_s. But then there exists an edge with label $l \neq t+1$ in U_s incident to the t-edge that connects U_s to U_q, thus $l \in I_s \cap I_q$, a contradiction. □

Proposition 5.28. *Let $n \geq 12$. Suppose that i is the maximal label such that*

- *Γ_i has two orbits with a single i-edge joining them;*
- *there exists $h > i$ such that ρ_h acts non-trivially on both Γ_i-orbits.*

If $r > \frac{n-1}{2}$, then one of the following occurs.

(a) $\Gamma_{>r-4}$ *is the dual of the string C-group (2), or the string C-group (3) given in Table 2;*

(b) Γ_{r-1} *has exactly two orbits, one being trivial.*

Proof. Let h be maximal. Let us choose the vertices a and b, and consequently the groups A and B, such that J_B is an interval with all labels in J_B being $> i$. If J_A is also an interval, then, as $h \in J_A \cap J_B$, $i = 0$; we assume without loss of generality that $r-1 \in J_A \setminus J_B$ (Recall that the case $h = r-1$ has been dealt with in Proposition 5.19).

Denote by O_1 and O_2 the orbits of A and B respectively. By Proposition 5.20 when $i = 0$, and Proposition 5.24 when $i \neq 0$, we have that $h < r-1$ and $h \leq \frac{n-|X|-1}{2}$ with $X := \{1,\dots,n\} \setminus \mathrm{Fix}(\Gamma_{>h}) \subseteq O_2$. If $\Gamma_{>h}$ is transitive, by Proposition 5.26, one of the groups $\Gamma_{<3}$ or $\Gamma_{>r-4}$ is one of the string C-groups (2) or (3) given in Table 2 or their duals. Thus we may assume that $\Gamma_{>h}$ is intransitive. Now if $\Gamma_{>h}$ has a 2-fracture graph, by Proposition 5.27, $r \leq \frac{n-1}{2}$, a contradiction. Hence $\Gamma_{>h}$ does not admit a 2-fracture graph. Then there exists $j > h$ such that ρ_j only swaps one pair of vertices in different Γ_j-orbits. Choose j minimal with this property. Then $\Gamma_{\{h+1,\dots j-1\}}$ has a 2-fracture graph. Let C and D be the group actions of Γ_j in each Γ_j-orbit with C acting on the orbit L_1 containing the i-edge $\{a,b\}$. Let L_2 denote the other Γ_j-orbit. If one of the groups is trivial then either we get case (b) of the statement of this proposition or $r \leq \frac{n-1}{2}$ by the dual of Proposition 5.25, a contradiction. Thus assume both C and D are nontrivial sggi's. Indeed let $\rho_j = \gamma_j\delta_j$ with γ_j and δ_j being the permutations in each Γ_j-orbit. Then $C = \langle \gamma_i \,|\, i \in \{0,\dots,r-1\}\rangle$ and $D := \langle \delta_i \,|\, i \in \{0,\dots,r-1\}\rangle$. Let $J_C := \{i \in \{0,\dots,r-1\} \,|\, \gamma_i \neq 1_C\}$ and $J_D := \{i \in \{0,\dots,r-1\} \,|\, \delta_i \neq 1_D\}$. By Propositions 5.1 and 5.7 either J_C or J_D is an interval. If $C = \Gamma_{<j}$ and $D = \Gamma_{>j}$ then by Proposition 5.17, $r \leq \frac{n-1}{2}$ or $\Gamma_{>r-4}$ is, up to duality, one of the string C-groups (2) or (3) given in Table 2. This gives case (a) of the statement.

It remains to consider the case where there exists a permutation ρ_g acting nontrivially on both Γ_j-orbits. Choose g minimal. Thanks to the maximality of i, $g < j$. As $g \in J_D$ we have that $g > i$. We now consider four cases:

(1) If $j = r-1$ and $g = 0$, then by the dual of Proposition 5.19, we get a contradiction.

(2) If $j = r-1$ and $g \neq 0$, then both J_C and J_D are intervals and g is the minimal label of a permutation acting nontrivially on L_2. Then we can use the dual of Proposition 5.20 to conclude that $r-g \leq \frac{n-|Y|-1}{2}$ with $Y := \{1,\dots,n\}\setminus\mathrm{Fix}(\Gamma_{<g}) \subseteq L_1$. If $g \leq h$ then, by Proposition 5.18, there is a path in X containing all labels from h to $r-2$ twice. Let V be the set of vertices of this path, then $r-h \leq \frac{|V|}{2}$. Hence $r \leq \frac{n-|X|-|V|-1}{2}$, giving a contradiction. Hence $g > h$ and $X \cap Y \neq \emptyset$.

(3) Let $j \neq r-1$ and J_D be an interval. In this case both ρ_0 and ρ_{r-1} act nontrivially on the first Γ_j-orbit. Hence we can use the dual argument to the one used in the last paragraph of the proof of Proposition 5.23, to conclude that $r \leq \frac{n-1}{2}$, a contradiction.

(4) Let $j \neq r-1$ and J_C be an interval. By the dual of Proposition 5.24, $g < 0$ and $r-g \leq \frac{n-|Y|-1}{2}$ with $Y := \{1,\dots,n\} \setminus \mathrm{Fix}(\Gamma_{<g}) \subseteq L_1$. By the same reasoning as in (2), $h > g$.

Only cases (2) and (4) are possible and both give the inequality $r-g \leq \frac{n-|Y|-1}{2}$ with $Y := \{1,\dots,n\} \setminus \mathrm{Fix}(\Gamma_{<g}) \subseteq L_1$ and $h > g$.

By choice of j, there is no other label l between h and g having only one pair of vertices in different Γ_l-orbits. Then $\Gamma_{\{h+1,\ldots,g-1\}}$ has a 2-fracture graph. Moreover $\Gamma_{>h} \cap \Gamma_{<g}$ is intransitive on $X \cap Y$ for if it were transitive, then $\Gamma_{>h}$ would be transitive on X, contradicting the assumptions of the proposition. Hence the 2-fracture graph for $\Gamma_{\{h+1,\ldots,g-1\}}$ is disconnected and $g-h-1 \leq \frac{|X\cap Y|-1}{2}$. We have $h \leq \frac{n-|X|-1}{2}$, $g-h \leq \frac{|X\cap Y|-1}{2}+1$, $r-g \leq \frac{n-|Y|-1}{2}$ and $n = |X|+|Y|-|X\cap Y|$, thus $r \leq \frac{n-1}{2}$, a contradiction. □

We now consider that A is trivial or has the permutation representation graph (2) or (3) of Table 2.

Proposition 5.29. *Let $i = 0$. If A is trivial and $\Gamma_{>1}$ is transitive on $\{1,\ldots,n\} \setminus \{a,b\}$, then $r \leq \frac{n-1}{2}$.*

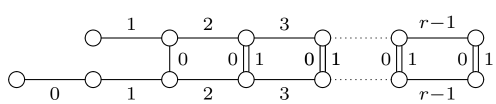
Proof. As ρ_0 acts nontrivially on $\{1,\ldots,n\} \setminus \{a,b\}$ and centralizes $\Gamma_{>1}$, by Proposition 5.21, $r-2 \leq \frac{n-2}{2}-1$ and therefore $r \leq n/2$. Suppose that we have the equality. Then $\Gamma_{>1}$ is as given in Proposition 5.21, and either ρ_0 or ρ_1 is odd, a contradiction. Hence $r \leq \frac{n-1}{2}$. □

Proposition 5.30. *Let A have the permutation representation (2) or (3) of Table 2 (this implies that $n \geq 7$). If $\Gamma_{>4}$ is transitive on $n-7$ vertices then $r \leq \frac{n-1}{2}$.*

Proof. In this case $i = 3$ and ρ_i has full support on the $n-7$ vertices that are not fixed by $\Gamma_{>4}$. So $n-7$ is even. By Proposition 5.21, $r-5 \leq \frac{n-7}{2}-1$. Moreover if we have the equality then either ρ_3 or ρ_4 is odd, a contradiction. Hence $r-5 < \frac{n-7}{2}-1$ and $r \leq \frac{n-1}{2}$. □

Proposition 5.31. *Suppose that A is trivial or has permutation representation graph (2) or (3) of Table 2 and B is the alternating group. If $\Gamma_{>i+1}$ is intransitive on the second Γ_i-orbit and $\Gamma_{>i}$ has a 2-fracture graph, then $r \leq \frac{n-1}{2}$. Moreover if $r > \frac{n-3}{2}$ then one of the following possibilities must occur.*

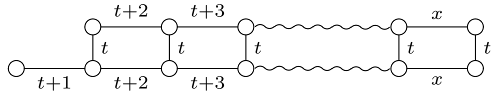
(a) *there exists $x \in \{0,\ldots,r-1\}$ such that $x \leq \frac{n-|X|-1}{2}$ with $X = \{1,\ldots,n\} \setminus \mathrm{Fix}(\Gamma_{>x})$.*
(b) *Γ has the following permutation representation graph.*



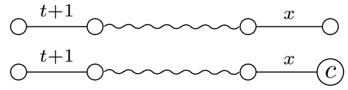
Proof. Consider the graph $\mathcal{C}$ as in the proof of Proposition 5.27, with $t = i+1$. Let U, U_s, G_s, F_s and I_s be as in Proposition 5.27.

Suppose there is a component U_s that does not have a 2-fracture graph. Let m and x be the minimal and the maximal label of that component respectively. We proved in Proposition 5.27 that $m \in \{t+1, t+2\}$ and accordantly to these possibilities for m the permutation representation graph of Γ contains one of the following graphs.

$m = t+2:$



$m = t+1:$



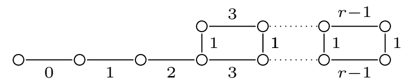
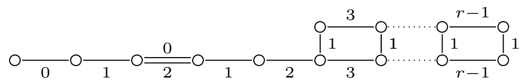
#	$\|P\|$	**Possibilities for Γ when A is trivial**
1	$n-2$	
2	$n-4$	

TABLE 3. Possibilities depending on $|P|$ for Γ when A is trivial

Let P be the set of vertices of the first graph and Q be the set of vertices of the second graph. If $x \neq r-1$, in the first case $\Gamma_{>x}$ fixes P, and in the second case $\Gamma_{>x}$ fixes $Q \setminus \{c\}$. If $i = 0$ and $t = 1$ we have that $x \leq \frac{|P|+1}{2}$ or $x \leq \frac{|Q|}{2}$. If $i = 3$ (and $t = 4$), $x \leq \frac{|P|+7}{2}$ or $x \leq \frac{|Q|+6}{2}$.

Suppose that $x = r-1$ and $r \geq \frac{n-2}{2}$. Consider first that A is trivial and $m = t+2$. In this case $n \geq |P|+2$ and $\frac{|P|+1}{2} \geq \frac{n-4}{2}$, thus $|P| \in \{n-5, n-4, n-3, n-2\}$. Then it is possible to determined the permutation representation graph of Γ according to the value of $|P|$. If $|P| = n-2$ then Γ is the graph (1) of Table 3 which is not an even group. It is also not possible to get an even group when $|P| = n-3$, as there must exist a $\Gamma_{>1}$-orbit with a 2-edge. For $|P| = n-4$ it is possible to create such a component and we get the graph (2) of Table 3. But as $\Gamma_{>1}$ does not have a 2-fracture graph we get a contradiction. If $|P| = n-5$, it is not possible to create a third $\Gamma_{>1}$-orbit, thus $\Gamma_{>1}$ has exactly two components, one containing P and the other having an even number of vertices swapped pairwise by ρ_0, a contradiction.

Now let A be trivial and $m = t+1$. In this case $|Q| \in \{n-4, n-3, n-2\}$. In Table 4 we list all possibilities for the permutation representation graph of Γ for each value of $|Q|$. If the permutation representation graph of Γ is one of the graphs (3), (5a), (5b) or (5c), then Γ is odd, a contradiction. Thus the only possibility is the permutation representation graph (4), giving the graph of the statement of this proposition.

Now let A be the permutation representation graph (2) of Table 2 and $m = t+2$. If $r \geq \frac{n-2}{2}$ and $x = r-1$ then $|P| \in \{n-11,\ n-10,\ n-9,\ n-8,\ n-7\}$. In Table 5 we list all possibilities for Γ according to $|P|$. For $|P| = n-7$ we get the permutation representation graph (1) and Γ is odd, a contradiction. Then $|P| < n-8$, as there must exist a $\Gamma_{>4}$-orbit containing a 5-edge and a 3-edge. Thus for $|P| = n-9$ we get the permutation representation graph (2), but $\Gamma_{>4}$ doesn't have a 2-fracture graph, a contradiction. Now suppose $|P| \leq n-10$. Either there are two $\Gamma_{>4}$-orbits, one having a set of vertices P and another having ρ_3 swapping all its vertices pairwise, or there exist a third $\Gamma_{>4}$-orbit containing a 5-edge. For this to happen at least two additional vertices are needed, thus $|P| = n-11$. This gives the permutation representation graph (3), which again corresponds to an odd group, a contradiction.

In Table 6, we list all possibilities for Γ when A has the permutation representation graph (2) of Table 2 and $m = t+1$. As before $r \geq \frac{n-2}{2}$ and $x = r-1$, hence $|Q| \in \{n-10,\ n-9,\ n-8,\ n-7\}$. In (4), (6a), (6b), (6c), (6d), (7b), (7c), (7d), (7e) and (7f) Γ is odd. In the remaining case the intersection condition fails.

If A has the permutation representation graph (3) of Table 2 we get the same contradictions as in Tables 5 and 6.

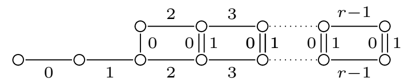
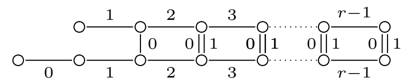
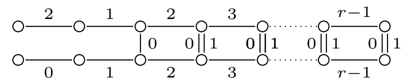
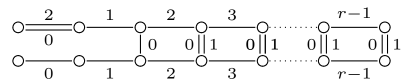
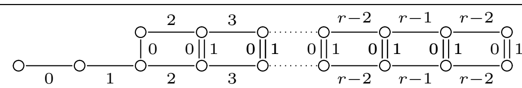
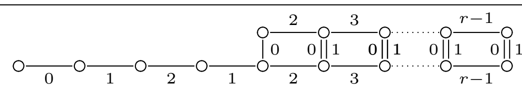
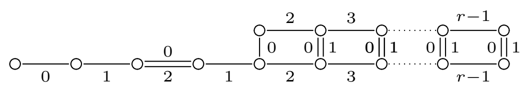
#	$\|Q\|$	**Possibilities for Γ when A is trivial**
3	$n-2$	
4	$n-3$	
5a	$n-4$	 or 
5b	$n-4$	
5c	$n-4$	 or 

Table 4. Possibilities depending on $|Q|$ for Γ when A is trivial

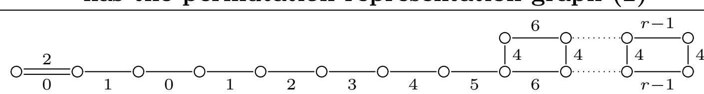
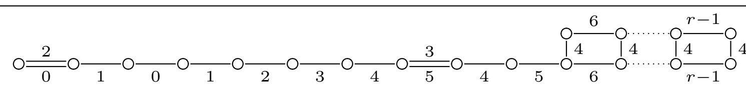
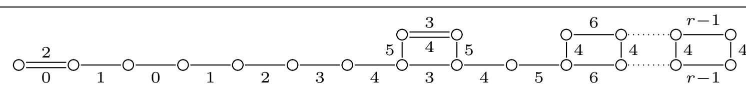
#	$\|P\|$	**Possibilities for Γ when A has the permutation representation graph (2)**
1	$n-7$	
2	$n-9$	
3	$n-11$	

Table 5. Possibilities depending on $|P|$ for Γ when A has permutation representation graph (2)

Now suppose that $x \neq r-1$. The group $\Gamma_{>x}$ fixes the first Γ_i-orbit and the vertex b. Let $X := \{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{>x})$. When $i = 0$ and $t = 1$ we have $|X| \leq n-(|P|+2)$ and $x \leq \frac{|P|+1}{2}$, or $|X| \leq n - (|Q \setminus \{c\}| + 2)$ and $x \leq \frac{|Q|}{2}$, giving in any case $x \leq \frac{n-|X|-1}{2}$. When $i = 3$ and $t = 4$, we have $|X| \leq n - (|P| + 7)$ and $x \leq \frac{|P|+7}{2}$, or $|X| \leq n-(|Q \setminus \{c\}|+7)$ and $x \leq \frac{|Q|+6}{2}$. Hence $x \leq \frac{n-|X|}{2}$. Suppose we have the equality. Then Γ contains one of following two graphs, having all vertices fixed by

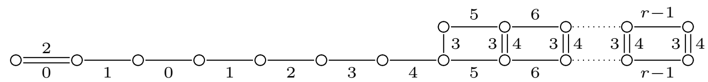
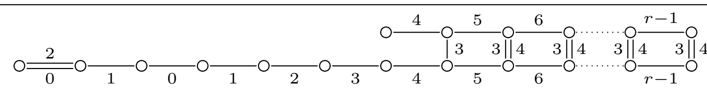
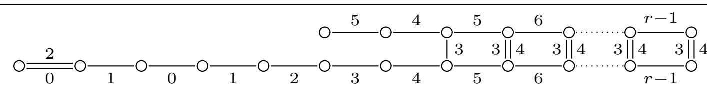
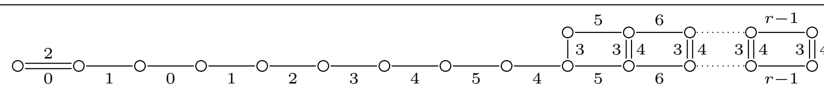
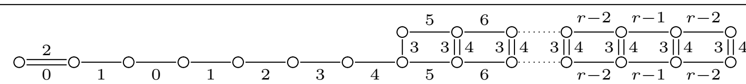
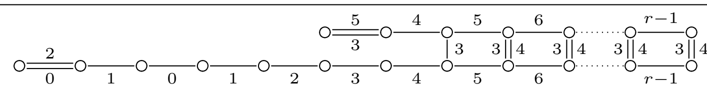
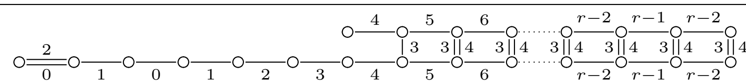
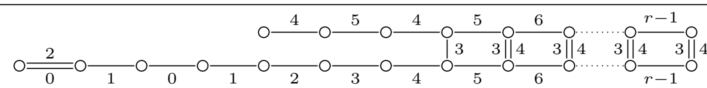
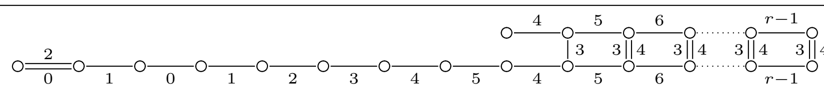
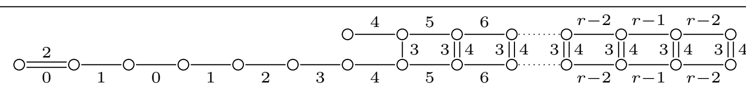
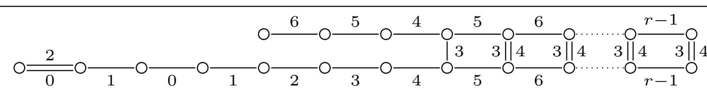
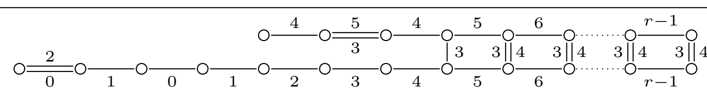
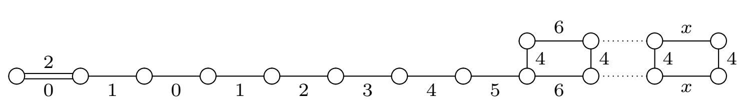
#	$\|Q\|$	**Possibilities for Γ when A has the permutation representation graph (2)**
4	$n-7$	
5	$n-8$	
6a	$n-9$	
6b	$n-9$	
6c	$n-9$	
6d	$n-9$	
7a	$n-10$	
7b	$n-10$	
7c	$n-10$	
7d	$n-10$	
7e	$n-10$	
7f	$n-10$	

TABLE 6. Possibilities depending on $|Q|$ for Γ when A has permutation representation graph (2)

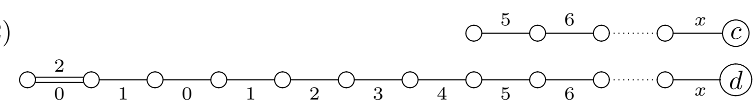
$\Gamma_{>x}$, except one of vertices c or d of graph (2).

(1)

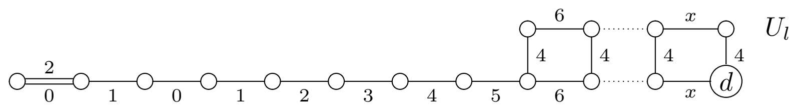


or

(2)



Then as $x \neq r-1$ there is another vertex v, not fixed by $\Gamma_{>x}$, incident to one of the vertices of one of the two graphs above. If Γ contains the graph (1), v must be attached to it by a 4-edge, which is not possible. This rules out graph (1). In the second graph, the two components of the figure are not adjacent, since otherwise c and d are both fixed by $\Gamma_{>x}$. Suppose first that c is the vertex not fixed by $\Gamma_{>x}$. Then there is another component U_l adjacent to the $\Gamma_{>4}$-orbit containing the vertex d, as in the following figure.



But $\Gamma_{>x}$ is fix-point-free in U_l, a contradiction. This shows that $x \leq \frac{n-|X|-1}{2}$. If d is the vertex not fixed by $\Gamma_{>x}$, then the component adjacent to the component having the vertex c must be fixed by $\Gamma_{>x}$, giving a contradiction as before.

This finishes the case where some component does not have a 2-fracture graph.

We now consider that each group G_s has a 2-fracture graph for F_s. Particularly each component U_s with $F_s \neq \emptyset$ has at least four vertices. In addition let $\mathcal{F}$ be a fracture graph satisfying the property (P1) and such that $E := \{F_s \,|\, F_s = \emptyset, s \in \{1, \dots, c\}\}$ has maximal size. Denote by S and δ_s the following numbers.

$$\delta_s = |F_s| - \frac{|U_s|}{2} \text{ and } S := \sum_{s=1}^{c} \delta_s.$$

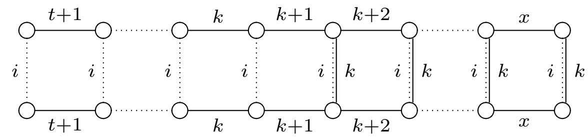
As G_s has a 2-fracture graph for F_s, $\delta_s \leq 0$ for all $s \in \{1, \dots, c\}$. Let U_p be the component such that $t+1 \in F_p$. In Proposition 5.27 we proved that $(G_s)_{t+1}$ cannot be transitive in U_s. Therefore a fracture graph for G_s with $s \neq p$ is disconnected. Hence $\delta_s \leq -0.5$ for $s \neq p$. If $S \leq -3$, then $r-1-t \leq \frac{|U|}{2} - 3$. For $(i,t) = (0,1)$, $|U| \leq n-2$ hence $r-2 \leq \frac{n-2}{2} - 3$. For $(i,t) = (3,4)$, $|U| \leq n-7$ hence $r-5 \leq \frac{n-7}{2} - 3$. In any case $r \leq \frac{n-3}{2}$. In what follows we prove that either $S \leq -3$ or we have (a) of the statement of this proposition.

Note that if $F_s \neq \emptyset$, then as G_s has a 2-fracture graph for F_s, $|U_s|$ has at least four vertices.

As ρ_i is an even permutation and fixes the first Γ_i-orbit except the vertex a, it must act nontrivially as an odd permutation in U. In what follows we consider separately the following cases: (1) ρ_i swaps an odd number of pairs of vertices (v, w) with $v \in U_s$ and $w \in U_x$ with $s \neq x$; (2) ρ_i acts as an odd permutation inside a component U_s with $F_s \neq \emptyset$ and $s \neq p$; (3) ρ_i acts as an odd permutation inside U_p; (4) ρ_i acts as an odd permutation inside a component U_s with $F_s = \emptyset$.

(1) If ρ_i swaps an odd number of pairs of vertices (v, w) with $v \in U_s$ and $w \in U_x$, then $|U_s| = |U_x|$ is odd. If $|U_x| = |U_s| = 3$ then $F_x = F_s = \emptyset$, hence $S \leq -3$. Consider $|U_x| = |U_s| \geq 5$. As $|E|$ is maximal, either $F_x = \emptyset$ or $F_s = \emptyset$. We may assume that $|F_x| = 0$. Then $|F_s| < \frac{5}{2}$. Thus $\delta_x \leq -2,5$ and $\delta_s \leq -0,5$. Hence $S \leq -3$.

(2) Consider now that ρ_i acts as an odd permutation inside a $\Gamma_{>t}$-orbit U_s with $F_s \neq \emptyset$ and $s \neq p$. In this case ρ_i centralizes G_s, therefore $|U_s|$ is even and $|U_s| \geq 6$. Moreover there is a 2-fracture graph for G_s with labels in $S := F_s \cup \{t+1\}$ being disconnected and having no cycles. Hence $|F_s| + 1 \leq \frac{|U_s|-2}{2}$, thus $\delta_s \leq -2$. Suppose $\delta_s = -2$. In that case the permutation representation graph of G_s is as follows, where x is the maximal label in U_s and $k \in \{t+1, \dots, x\}$.



Suppose that $x \neq r-1$. Then $\Gamma_{>x}$ fixes U_s, the vertex b and the first Γ_i-orbit. Hence there exists $x \in \{0, \ldots, r-1\}$ as in statement (a) of this proposition. Thus we may consider that $\delta_s < -2$. As in this case $|U_s|$ is even, δ_s is an integer thus $\delta_s \leq -3$ and $S \leq -3$.

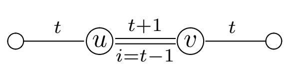
If $x = r-1$, we have $r - 1 - t \leq \frac{|U_s|}{2} - 1$. For $(i,t) = (0,1)$, $|U_s| \leq (n-2) - 4$ where 4 is the minimal size of U_p. For $(i,t) = (3,4)$, $|U_s| \leq (n-7) - 4$. In any case we get $r \leq \frac{n-3}{2}$.

(3) If ρ_i acts as an odd permutation inside U_p, then a 2-fracture graph of G_p is disconnected without cycles. Hence $|F_p| \leq \frac{|U_p|-2}{2}$, that is $\delta_p \leq -1$. Suppose we have the equality $|F_p| = \frac{|U_p|-2}{2}$. Then the permutation representation graph of G_s is as the permutation representation graph given in case (2). The only difference is that in this case $t+1 \in F_p$. In this case we get exactly the same result as before, that is statement (a) of this proposition. Assume now that $\delta_p < -1$. As in case (2), $|U_p|$ is even, thus δ_p is an integer. Then we may assume that $\delta_p \leq -2$.

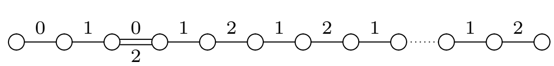
Suppose that $t+2 \notin F_p$. If $t+2 \notin I_p$ or $(G_p)_{t+2}$ is transitive in U_p, then ρ_{t+1} also centralizes G_p. But then there exists $l \in I_p \cap I_x$ with I_x being the set of labels of a component U_x adjacent to U_p. Then a 2-fracture graph of G_x has at least three components, hence $\delta_x \leq -1$. Thus $\delta_p + \delta_s \leq -3$ and $|S| \leq -3$. If $t+2 \in I_p$ and $(G_p)_{t+2}$ is intransitive in U_p, then G_p has a disconnected 2-fracture graph without cycles for $F_s \cup \{\rho_{t+2}\}$. Hence $2(|F_p|+1) \leq |U_p| - 2$. Moreover if we have the equality then G_s is as the permutation representation graph given in case (2). Hence $2(|F_p|+1) < |U_p| - 2$ and $\delta_p \leq -3$.

We now consider that $t+2 \in F_p$. Suppose $|S| = -2.5$. Then $c = 2$ and the second component U_s is such that $\delta_s = -0.5$. In particular, $F_s \neq \emptyset$, for otherwise $\delta_s \leq -1$. First suppose that $(G_s)_{t+2}$ is transitive in U_s. Then ρ_{t+1} centralizes G_s. Hence $2|F_s| \leq |U_s| - 2$ and $\delta_s \leq -1$, a contradiction. Thus G_{t+2} is intransitive in U_s. As $t+2 \notin F_s$, a 2-fracture graph for G_s, with labels in F_s, has at least 3 components with at most one cycle, hence $2|F_s| \leq |U_s| - 2$, as before, a contradiction.

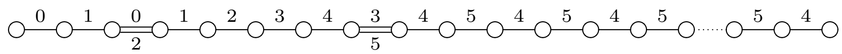
(4) Suppose that ρ_i acts as an odd permutation inside a component U_s with $F_s = \emptyset$. Either $|U_s| = 2$ or $|U_s| \geq 6$. But in the second case clearly $\delta_s \leq -3$, thus we consider $U_s = \{u, v\}$. Let l be the number of components of size two, whose vertices are swapped by ρ_i. We need to consider the case l odd. If $l \geq 3$ then $S \leq -3$, hence we assume $l = 1$. Moreover let ρ_i be a 2-transposition, $\rho_i = (a\,b)(u\,v)$. Then $I_s = \{t+1\}$.



If $t+1 = r-1$ then Γ has one of the following permutation representation graphs.
$(i,t) = (0,1)$:



$(i,t) = (3,4)$:

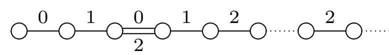


As by hypotheses $\Gamma_{>i}$ has a 2-fracture graph, for $(i,t) = (0,1)$, we have $n \geq 9$, and for $(i,t) = (3,4)$, we have $n \geq 15$. In both cases $r \leq \frac{n-3}{2}$.

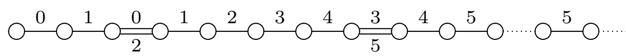
Assume $t+1 \neq r-1$. In this case $\Gamma_{>t+1}$ fixes the first Γ_i-orbit and $\{b, u, v, u\rho_t, v\rho_t\}$.

If $|\{b, u, v, u\rho_t, v\rho_t\}| = 5$, consider $x = t+1$. Then x satisfies (a) of this proposition. Otherwise, $|\{b, u, v, u\rho_t, v\rho_t\}| = 4$. Then Γ contains one of the following graphs.

For $(i,t) = (0,1)$:



For $(i,t) = (3,4)$:

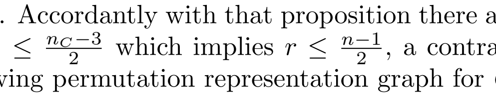


In the first case, $\Gamma_{>2}$ has at least five fixed points. Therefore $x = 2$ satisfies the statement (a) of this proposition. Consider the second case. As $B = \Gamma_{>3} \cong A_{n-6}$, $\Gamma_{>2} \cong A_{n-5}$. In addition $\Gamma_{<5} \cong A_{10}$ or $\Gamma_{<5} \cong A_{10} \times \langle\tau\rangle$ where τ is an even involution. Thus $\Gamma_{>2} \cap \Gamma_{<5}$ is either A_5 or $A_5 \times \langle\tau\rangle$. But $\langle\rho_3, \rho_4\rangle \cong D_5 \times \langle\tau\rangle$, contradicting the intersection condition. □

Proposition 5.32. *Let $n \geq 12$. If A is trivial or A has the permutation representation graph (2) or (3) of Table 2 and B is an alternating group, then $r \leq \frac{n-1}{2}$.*

Proof. Assume for contradiction that $r > \frac{n-1}{2}$. By the dual of Proposition 5.25 we may consider, up to duality, that when A is trivial, $i = 0$. By Propositions 5.29, 5.30 and 5.31 we may assume that $\Gamma_{>i+1}$ is intransitive on $\{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{>i+1})$, but $\Gamma_{>i}$ does not have a 2-fracture graph. In addition, by Propositions 5.17 and 5.28 we can consider that if Γ_j has exactly two orbits and a single j-edge connecting them, then the group orbits are either $C = \Gamma_{r-1} \cong A_{n-1}$ and D trivial, or $C = \Gamma_{<j} \cong A_{n-6}$ and $D = \Gamma_{>j} \cong A_5$, D having the permutation representation graph dual of (2) or the graph (3).

Consider the group C. Let r_C be the rank of C and n_C the degree of C. Thanks to the intersection condition $\Gamma_{i+1,\ldots,j-1}$ is the alternating group. In addition $C_{>i+1}$ is intransitive on the second C_i-orbit (the orbit containing the vertex b), for otherwise $\Gamma_{>i+1}$ is transitive on $\{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{>i+1})$. Thus C satisfies the condition of Proposition 5.31. Accordantly with that proposition there are three possibilities. The first one is $r_C \leq \frac{n_C - 3}{2}$ which implies $r \leq \frac{n-1}{2}$, a contradiction. The second one gives the following permutation representation graph for C.



But then, it is not possible to attach the permutation representation graph of D by a single j-edge, a contradiction. The last establishes that there exists $x > i$ such that $x \leq \frac{n-|X|-1}{2}$ with $X := \{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{>x})$. On the other hand the dual of Proposition 5.31 gives the same three possibilities for B and, as before, one of them gives $r \leq \frac{n-1}{2}$, and the second gives a contradiction. Thus it may be assumed that there exists $y > i$ such that $r - y \leq \frac{n-|Y|-1}{2}$ with $X := \{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{<y})$. If $\Gamma_{\{x+1,\ldots,y-1\}}$ is intransitive on $X \cap Y = \{1, \ldots, n\} \setminus \mathrm{Fix}(\Gamma_{\{x+1,\ldots,y-1\}})$ then it has a disconnected 2-fracture graph. Hence $y - 1 - x \leq \frac{|X \cap Y|-1}{2}$. Otherwise, as $\Gamma_{\{x+1,\ldots,y-1\}}$ has a 2-fracture graph it cannot be one of the graphs (2) or (3) of Table 2. Thus by Propositions 5.11 and 5.13, we also have $y - 1 - x \leq \frac{|X \cap Y|-1}{2}$. As $n = |X| + |Y| - |X \cap Y|$, we get $r \leq \frac{n-1}{2}$, a contradiction. □

The cases we have covered, that some Γ_i is primitive, or transitive imprimitive, or all Γ_i are intransitive and 2-fracture graphs do or do not exist, exhaust all possibilities; so Theorem 1.1 is proved.

6. Acknowledgements

This research was supported by a Marsden grant (UOA1218) of the Royal Society of New Zealand, and by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013. The authors would also like to thank an anonymous referee for useful comments on the paper.

References

1. L. Babai. On the order of uniprimitive permutation groups. *Ann. Math.* (2) 113:553–558, 1981.
2. W. Bosma, J. J. Cannon and C. Playoust. The Magma Algebra System. I. The User Language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational Algebra and Number Theory (London, 1993).
3. P. A. Brooksbank and D. Leemans. Polytopes of large rank for $\mathrm{PSL}(4, q)$. *J. Algebra* 452:390–400, 2016.
4. P. A. Brooksbank and D. A. Vicinsky. Three-dimensional classical groups acting on polytopes. *Discrete Comput. Geom.*, 44(3):654–659, 2010.
5. F. Buekenhout and D. Leemans. On the list of finite primitive permutation groups of degree ≤ 50. *J. Symbolic Comput.* 22:215–225, 1996.
6. P. J. Cameron and P. Cara. Independent generating sets and geometries for symmetric groups. *J. Algebra* 258(2):641–650, 2002.
7. P. J. Cameron, M. E. Fernandes, D. Leemans and M. Mixer. String C-groups as transitive subgroups of S_n. *J. Algebra* 447:468–478, 2016.
8. M. Conder. The smallest regular polytopes of given rank. *Adv. Math.* 236:92–110, 2013.
9. T. Connor, J. De Saedeleer and D. Leemans. Almost simple groups with socle $\mathrm{PSL}(2, q)$ acting on abstract regular polytopes. *J. Algebra* 423:550–558, 2015.
10. T. Connor, D. Leemans, and M. Mixer. Abstract regular polytopes for the O'Nan group. *Int. J. Alg. Comput.* 24(1):59–68, 2014.
11. M. E. Fernandes and D. Leemans, Polytopes of high rank for the symmetric groups, *Adv. Math.* 228:3207–3222, 2011.
12. M. E. Fernandes, D. Leemans and M. Mixer. *Corrigendum to "Polytopes of high rank for the symmetric groups"*. Adv. Math., 238:506–508, 2013.
13. M. E. Fernandes, D. Leemans, and M. Mixer. *Polytopes of high rank for the alternating groups. J. Combin. Theory Ser. A*, 119:42–56, 2012.
14. M. E. Fernandes, D. Leemans, and M. Mixer. All alternating groups A_n with $n \geq 12$ have polytopes of rank $\lfloor \frac{n-1}{2} \rfloor$. *SIAM J. Discrete Math.*, 26(2):482–498, 2012.
15. M. E. Fernandes, D. Leemans and M. Mixer. *An extension of the classification of high rank regular polytopes.* Preprint, 2015. arXiv:1509.01032
16. A. Maróti, On the orders of primitive groups. *J. Algebra* 258:631–640, 2002.
17. L. Pyber. On the orders of doubly transitive permutation groups, elementary estimates. *J. Combin. Theory Ser. A* 62:361–366, 1993.
18. M. I. Hartley. An atlas of small regular abstract polytopes. *Period. Math. Hungar.*, 53(1-2):149–156, 2006.

19. M. I. Hartley and A. Hulpke. Polytopes derived from sporadic simple groups. *Contrib. Discrete Math.*, 5(2):106–118, 2010.
20. D. Leemans. Almost simple groups of Suzuki type acting on polytopes. *Proc. Amer. Math. Soc.*, 134(12):3649–3651 (electronic), 2006.
21. D. Leemans and M. Mixer. Algorithms for classifying regular polytopes with a fixed automorphism group. *Contr. Discrete Math.*, 7(2):105–118, 2012.
22. D. Leemans and E. Schulte. Groups of type $L_2(q)$ acting on polytopes. *Adv. Geom.*, 7(4):529–539, 2007.
23. D. Leemans and E. Schulte. Polytopes with groups of type $\mathrm{PGL}_2(q)$. *Ars Math. Contemp.*, 2(2):163–171, 2009.
24. D. Leemans, E. Schulte and H. Van Maldeghem. Groups of Ree type in characteristic 3 acting on polytopes Preprint, 2015. arXiv:1501.01125
25. D. Leemans and L. Vauthier. An atlas of abstract regular polytopes for small groups. *Aequationes Math.*, 72(3):313–320, 2006.
26. P. McMullen and E. Schulte. *Abstract regular polytopes*, volume 92 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.
27. D. Pellicer. *CPR graphs and regular polytopes.* European J. Combin., 29(1):59–71, 2008.
28. J. Whiston. Maximal independent generating sets of the symmetric group, *J. Algebra* 232(1):255–268, 2000.

Peter J. Cameron, School of Mathematics and Statistics, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, UK
E-mail address: pjc20@st-andrews.ac.uk

Maria Elisa Fernandes, Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal
E-mail address: maria.elisa@ua.pt

Dimitri Leemans, Université Libre de Bruxelles, Département de Mathématique, C.P.216 - Algèbre et Combinatoire, Boulevard du Triomphe, 1050 Brussels, Belgium
E-mail address: dleemans@ulb.ac.be

Mark Mixer, Department of Applied Mathematics, Wentworth Institute of Technology, Boston, MA 02115, USA
E-mail address: mixerm@wit.edu