HIGHEST WEIGHT CATEGORIES ARISING FROM KHOVANOV'S DIAGRAM ALGEBRA III: CATEGORY $\mathcal{O}$

JONATHAN BRUNDAN AND CATHARINA STROPPEL

ABSTRACT. We prove that integral blocks of parabolic category $\mathcal{O}$ associated to the subalgebra $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$ of $\mathfrak{gl}_{m+n}(\mathbb{C})$ are Morita equivalent to quasi-hereditary covers of generalised Khovanov algebras. Although this result is in principle known, the existing proof is quite indirect, going via perverse sheaves on Grassmannians. Our new approach is completely algebraic, exploiting Schur-Weyl duality for higher levels. As a by-product we get a concrete combinatorial construction of 2-Kac-Moody representations in the sense of Rouquier corresponding to level two weights in finite type A.

CONTENTS

1. INTRODUCTION

The *generalised Khovanov algebra* H_m^n is a certain positively graded symmetric algebra defined via an explicit calculus of diagrams. It was introduced by Khovanov in the case $m = n$ as part of his ground-breaking work categorifying the Jones polynomial [K1, K2]. In [CK, S3], the definition was extended to obtain another algebra K_m^n, known as the *quasi-hereditary cover of H_m^n*. In [BS1, BS2], we undertook a systematic study of the representation theory of K_m^n, showing in particular that K_m^n is Koszul and computing the various natural bases for its graded Grothendieck group in an explicit combinatorial fashion using the diagram calculus. We also set up a general theory of projective functors for the algebras K_m^n, extending ideas of Khovanov from [K2].

The category $\mathrm{rep}(K_m^n)$ of finite dimensional left K_m^n-modules is equivalent to the category of perverse sheaves (constructible with respect to the Schubert stratification) on the Grassmannian $\mathrm{Gr}(m, m+n)$ of m-dimensional subspaces of an $(m+n)$-dimensional complex vector space. This statement is proved in the case $m = n$ in [S3, Theorem 5.8.1] as an application of work of Braden [B], and it should be possible to obtain the general case by similar arguments as indicated in [S3, Remark 5.8.2]. In turn, by the Beilinson-Bernstein localisation theorem and the Riemann-Hilbert correspondence, this category of perverse sheaves is equivalent to the principal block of the parabolic analogue of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ associated to the subalgebra $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$ of $\mathfrak{gl}_{m+n}(\mathbb{C})$.

The main goal of this article is to give a direct algebraic construction of an equivalence between $\mathrm{rep}(K_m^n)$ and the principal block of the parabolic category $\mathcal{O}$ just mentioned. Our new approach actually does rather more. For one thing, it applies to *all* integral blocks, not just the principal block; these are also equivalent to categories of K_m^n-modules but for possibly smaller m and n. We are also able to prove for the first time that the diagrammatically defined projective functors from [BS2, §4] correspond under the equivalence of categories to the projective functors from [BG] that arise by tensoring with finite dimensional irreducible modules. This is the key identification needed to verify [S2, Conjecture 2.9], which relates Khovanov's functorial tangle invariants from [K2] to the functorial tangle invariants defined in [S1].

To formulate some of the results in more detail, let $\mathfrak{g} := \mathfrak{gl}_{m+n}(\mathbb{C})$, let $\mathfrak{h}$ be the standard Cartan subalgebra consisting of diagonal matrices, and let $\mathfrak{b}$ be the standard Borel subalgebra of upper triangular matrices. The dual space $\mathfrak{h}^*$ has orthonormal basis $\varepsilon_1, \ldots, \varepsilon_{m+n}$ with respect to the trace form $(.,.)$, where ε_i is the weight picking out the ith diagonal entry of a diagonal matrix. Let $\mathfrak{l}$ denote the naturally embedded subalgebra $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$, and $\mathfrak{p} := \mathfrak{l} + \mathfrak{b}$ be the corresponding standard parabolic subalgebra. Let $\mathcal{O}(m, n)$ be the category of finitely generated $\mathfrak{g}$-modules that are locally finite over $\mathfrak{p}$, semisimple over $\mathfrak{h}$, and have all weights belonging to $\mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_{m+n}$. This is the sum of all integral blocks of the parabolic category $\mathcal{O}$ associated to $\mathfrak{p} \subseteq \mathfrak{g}$. A full set of representatives for the isomorphism classes of irreducible modules in $\mathcal{O}(m, n)$ is given by the modules $\{\mathcal{L}(\lambda) \mid \lambda \in \Lambda(m, n)\}$, where

$$\Lambda(m, n) := \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \begin{array}{l} (\lambda + \rho, \varepsilon_i) \in \mathbb{Z} \text{ for all } 1 \leq i \leq m + n, \\ (\lambda + \rho, \varepsilon_1) > \cdots > (\lambda + \rho, \varepsilon_m), \\ (\lambda + \rho, \varepsilon_{m+1}) > \cdots > (\lambda + \rho, \varepsilon_{m+n}) \end{array} \right\}, \qquad (1.1)$$

$$\rho := -\varepsilon_2 - 2\varepsilon_3 - \cdots - (m + n - 1)\varepsilon_{m+n} \in \mathfrak{h}^*, \qquad (1.2)$$

and $\mathcal{L}(\lambda)$ denotes the irreducible $\mathfrak{g}$-module of highest weight λ, i.e. the irreducible $\mathfrak{g}$-module generated by a highest weight vector v_+ such that $xv_+ = 0$ for all x in the nilradical of $\mathfrak{b}$ and $hv_+ = \lambda(h)v_+$ for all $h \in \mathfrak{h}$.

In order to be able to exploit the diagram calculus from [BS1, BS2], we need to identify weights in $\Lambda(m, n)$ with weights in the combinatorial sense of [BS1,

§2] via the following *weight dictionary.* Given $\lambda \in \Lambda(m,n)$ we define

$$I_\vee(\lambda) := \{(\lambda + \rho, \varepsilon_1), \ldots, (\lambda + \rho, \varepsilon_m)\}, \tag{1.3}$$

$$I_\wedge(\lambda) := \{(\lambda + \rho, \varepsilon_{m+1}), \ldots, (\lambda + \rho, \varepsilon_{m+n})\}. \tag{1.4}$$

Then we identify λ with the diagram consisting of a number line whose vertices are indexed by $\mathbb{Z}$ such that the ith vertex is labelled

$$\begin{cases} \circ & \text{if } i \text{ does not belong to either } I_\vee(\lambda) \text{ or } I_\wedge(\lambda), \\ \vee & \text{if } i \text{ belongs to } I_\vee(\lambda) \text{ but not to } I_\wedge(\lambda), \\ \wedge & \text{if } i \text{ belongs to } I_\wedge(\lambda) \text{ but not to } I_\vee(\lambda), \\ \times & \text{if } i \text{ belongs to both } I_\vee(\lambda) \text{ and } I_\wedge(\lambda). \end{cases} \tag{1.5}$$

For example, the zero weight (which parametrises the trivial module) is identified with

where the $\wedge$'s and $\vee$'s are on the vertices indexed $1 - m - n, \ldots, -1, 0$. Viewing $\lambda, \mu \in \Lambda(m,n)$ as diagrams in this way, the irreducible $\mathfrak{g}$-modules $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ have the same central character if and only if λ can be obtained from μ by permuting $\wedge$'s and $\vee$'s. This defines an equivalence relation $\sim$ on $\Lambda(m,n)$. We let $P(m,n)$ denote the set $\Lambda(m,n)/\sim$ of all $\sim$-equivalence classes, and refer to elements of $P(m,n)$ as *blocks.*

For each block $\Gamma \in P(m,n)$, there is a finite dimensional algebra K_Γ exactly as defined in [BS1, §4]. As a vector space, K_Γ has basis

$$\{(a\lambda b) \mid \text{for all oriented circle diagrams } a\lambda b \text{ with } \lambda \in \Gamma\},$$

and its multiplication is defined by an explicit combinatorial procedure in terms of such diagrams. In particular, if Γ is the principal block, that is, the block generated by the zero weight, then the algebra K_Γ is identified in an obvious way with the quasi-hereditary cover K_m^n of H_m^n. Consider the (locally unital) algebra

$$K(m,n) := \bigoplus_{\Gamma \in P(m,n)} K_\Gamma. \tag{1.6}$$

Let $\text{rep}(K(m,n))$ denote the category of (locally unital) finite dimensional left $K(m,n)$-modules. According to [BS1, §5], the irreducible $K(m,n)$-modules are all one dimensional and their isomorphism classes are indexed in a canonical way by the set $\Lambda(m,n)$. Let $L(\lambda)$ denote the irreducible $K(m,n)$-module associated to $\lambda \in \Lambda(m,n)$.

Theorem 1.1. *There is an equivalence of categories*

$$\mathbb{E} : \mathcal{O}(m,n) \xrightarrow{\sim} \text{rep}(K(m,n))$$

such that $\mathbb{E}(\mathcal{L}(\lambda)) \cong L(\lambda)$ *for every* $\lambda \in \Lambda(m,n)$. *In particular,* $\mathbb{E}$ *restricts to an equivalence between the principal block of* $\mathcal{O}(m,n)$ *and* $\text{rep}(K_m^n)$.

The next theorem is concerned with projective functors. On the diagram algebra side, these functors arise from crossingless matchings. More precisely, take $\Gamma, \Delta \in P(m,n)$ and let t be a proper $\Delta\Gamma$-matching in the sense of [BS2,

§2]. To this data there is associated a non-zero (K_Δ, K_Γ)-bimodule $K_{\Delta\Gamma}^t$; see [BS2, §3]. We view it as a $(K(m,n), K(m,n))$-bimodule by extending the left action of K_Δ (resp. the right action of K_Γ) to all of $K(m,n)$ by declaring that all the summands of (1.6) different from K_Δ (resp. K_Γ) act as zero. Tensoring with this bimodule defines an exact functor

$$K_{\Delta\Gamma}^t \otimes_{K(m,n)} ? : \mathrm{rep}(K(m,n)) \to \mathrm{rep}(K(m,n)).$$

By [BS2, Theorem 4.14], this functor is indecomposable. A *projective functor* on $\mathrm{rep}(K(m,n))$ simply means any endofunctor that is isomorphic to a finite direct sum of such functors.

On the category $\mathcal{O}$ side, following [BG], a *projective functor* means a functor that is isomorphic to a summand of one of the exact endofunctors of $\mathcal{O}(m,n)$ that arise by tensoring with a finite dimensional $\mathfrak{g}$-module. In order to classify such projective functors, it suffices by a variation on the Krull-Schmidt theorem to classify the *indecomposable* projective functors; see e.g. [S1, §3.1]. The classification of indecomposable projective functors on the principal block of $\mathcal{O}(m,n)$ can be deduced from [S1, Theorem 5.1] (using [S1, Proposition 4.2] to determine which projective functors have non-zero restrictions). This had been conjectured earlier by Bernstein, Frenkel and Khovanov [BFK, p.237]. The following theorem gives an alternative approach to this classification, and extends it to arbitrary integral blocks.

Theorem 1.2. *Given blocks $\Gamma, \Delta \in P(m,n)$ and a proper $\Delta\Gamma$-matching t, there is an indecomposable projective functor $\mathcal{G}_{\Delta\Gamma}^t : \mathcal{O}(m,n) \to \mathcal{O}(m,n)$ and an isomorphism of functors*

$$\mathbb{E} \circ \mathcal{G}_{\Delta\Gamma}^t \cong (K_{\Delta\Gamma}^t \otimes_{K(m,n)} ?) \circ \mathbb{E} : \mathcal{O}(m,n) \to \mathrm{rep}(K(m,n)).$$

Every indecomposable projective functor on $\mathcal{O}(m,n)$ is isomorphic to such a functor $\mathcal{G}_{\Delta\Gamma}^t$ for unique Γ, Δ and t.

One pleasant feature of the diagram algebra setup is that it makes some natural but hard-to-see gradings on the category $\mathcal{O}$ side absolutely explicit. Indeed, the algebra $K(m,n)$ carries an obvious grading with respect to which its diagram basis is homogeneous. This grading makes $K(m,n)$ into a Koszul algebra; see [BS2, Theorem 5.6]. In view of Theorem 1.1, the category $\mathrm{Rep}(K(m,n))$ of finite dimensional *graded* left $K(m,n)$-modules can be interpreted as a graded version of $\mathcal{O}(m,n)$. By the unicity of Koszul gradings [BGS, §2.5], this is equivalent to the graded version of $\mathcal{O}(m,n)$ constructed geometrically (and in far greater generality) in [BGS, §3.11] and [Ba]. The bimodules $K_{\Delta\Gamma}^t$ are also naturally graded, so in view of Theorem 1.2 they define explicit graded lifts of the indecomposable projective functors on $\mathcal{O}(m,n)$.

For the proofs, the basic idea is to exploit a special case of the Schur-Weyl duality for higher levels developed in [BK2]. Recall that classical Schur-Weyl duality relates polynomial representations of $\mathfrak{gl}_{m+n}(\mathbb{C})$ to the representation theory of symmetric groups. Schur-Weyl duality for level two relates the category $\mathcal{O}(m,n)$ to the representation theory of degenerate cyclotomic Hecke algebras of level two. By mimicking this Schur-Weyl duality on the diagram algebra side, we obtain a natural realization of another (at first sight quite

different) family of Hecke algebras, namely, level two cyclotomic quotients R_α^Λ of certain algebras introduced independently by Khovanov and Lauda in [KLa] and Rouquier in [R] (for Cartan matrices of finite type A). The bridge between parabolic category $\mathcal{O}$ and the diagram algebra side finally comes as an application of the main result of [BK3], which asserts that these two sorts of Hecke algebra arising from the two versions of Schur-Weyl duality are actually isomorphic algebras.

As a by-product of the argument, we also obtain an elementary proof of the categorification conjecture formulated by Khovanov and Lauda in [KLa, §3.4] for level two weights in finite type A; see also [BK5, VV] which treat much more general situations (but using geometric methods). At the same time we give a conceptual interpretation of the grading on R_α^Λ: for us this algebra arises naturally as the endomorphism algebra of a certain projective module in $\mathcal{O}(m,n) \cong \mathrm{rep}(K(m,n))$, and the $\mathbb{Z}$-grading on it is the grading induced by the Koszul grading on these projective modules suitably shifted in degree. Finally our methods yield a special *graded cellular basis* for R_α^Λ parametrised by some diagrams which are in bijection with certain Young tableaux; see [BKW, Remark 4.12] where the existence of such bases is predicted in more general situations. In particular we deduce from this a graded dimension formula for the irreducible R_α^Λ-modules (in level two for finite type A).

Notation. For the remainder of the article, we fix an index set I that is a nonempty, bounded-below set of consecutive integers and let m, n be integers with $0 \leq m, n \leq |I| + 1$. Set $I^+ := I \cup (I + 1)$ and $o := \min(I) - 1$. The reader will lose little in generality by assuming that $I = I^+ = \{1, 2, 3, \dots\}$ and $o = 0$.

2. Combinatorics of canonical bases

This and the next two sections are concerned with some essential combinatorial book-keeping at the level of Grothendieck groups. We begin in this section by introducing an auxiliary space $\bigwedge^m V \otimes \bigwedge^n V$ together with three natural bases, namely, the monomial basis, the dual-canonical basis and the quasi-canonical basis, following the setup of [BK4, section 2] closely.

The quantised enveloping algebra. We begin with some basic notions related to the general linear Lie algebra of $I^+ \times I^+$ matrices. The underlying *weight lattice* P is the free $\mathbb{Z}$-module on basis $\{\delta_i \mid i \in I^+\}$ equipped with a bilinear form $(.,.)$ such that $(\delta_i, \delta_j) = \delta_{i,j}$ (where $\delta_{i,j}$ is the usual Kronecker δ-function). For $i \in I$ and $j \in I^+$, let

$$\alpha_i := \delta_i - \delta_{i+1}, \qquad \Lambda_j := \sum_{I \ni i \leq j} \delta_i$$

denote the ith *simple root* and the jth *fundamental weight*, respectively; note that $(\alpha_i, \Lambda_j) = \delta_{i,j}$. The *root lattice* Q is the $\mathbb{Z}$-submodule of P generated by

the simple roots. Let Q_+ be the subset of Q consisting of all α that have non-negative coefficients when expressed in terms of the simple roots, and define the *height* $\mathrm{ht}(\alpha)$ to be the sum of these coefficients.

Let U denote the (generic) quantised enveloping algebra associated to this root datum. So U is the associative algebra over the field of rational functions $\mathbb{Q}(q)$ in an indeterminate q, with generators $\{E_i, F_i \mid i \in I\} \cup \{D_i, D_i^{-1} \mid i \in I^+\}$ subject to the following well-known relations:

$$D_i D_i^{-1} = D_i^{-1} D_i = 1, \qquad\qquad E_i E_j = E_j E_i \qquad\qquad \text{if } |i-j| > 1,$$
$$D_i D_j = D_j D_i, \qquad E_i^2 E_j + E_j E_i^2 = (q + q^{-1}) E_i E_j E_i \quad \text{if } |i-j| = 1,$$
$$D_i E_j D_i^{-1} = q^{(\delta_i, \alpha_j)} E_j, \qquad\qquad F_i F_j = F_j F_i \qquad\qquad \text{if } |i-j| > 1,$$
$$D_i F_j D_i^{-1} = q^{-(\delta_i, \alpha_j)} F_j, \quad F_i^2 F_j + F_j F_i^2 = (q + q^{-1}) F_i F_j F_i \quad \text{if } |i-j| = 1,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{D_i D_{i+1}^{-1} - D_{i+1} D_i^{-1}}{q - q^{-1}}.$$

We view U as a Hopf algebra with comultiplication Δ defined on generators by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes D_i D_{i+1}^{-1},$$
$$\Delta(F_i) = F_i \otimes 1 + D_i^{-1} D_{i+1} \otimes F_i,$$
$$\Delta(D_i^{\pm 1}) = D_i^{\pm 1} \otimes D_i^{\pm 1}.$$

The space $\bigwedge^m V \otimes \bigwedge^n V$. Let V denote the natural U-module with basis $\{v_i \mid i \in I^+\}$. The generators act on this basis by the rules

$$E_i v_j = \delta_{i+1, j} v_i, \qquad F_i v_j = \delta_{i, j} v_{i+1}, \qquad D_i^{\pm 1} v_j = q^{\pm \delta_{i,j}} v_j.$$

Following [B1, §5] (noting the roles of q and q^{-1} are switched there), we define the nth *quantum exterior power* $\bigwedge^n V$ to be the U-submodule of $\bigotimes^n V$ with basis given by the vectors

$$v_{i_1} \wedge \cdots \wedge v_{i_n} := \sum_{w \in S_n} (-q)^{\ell(w)} v_{i_{w(1)}} \otimes \cdots \otimes v_{i_{w(n)}} \tag{2.1}$$

for all $i_1 > \cdots > i_n$ from the index set I^+. Here, $\ell(w)$ denotes the usual length of a permutation w in the symmetric group S_n.

Consider the U-module $\bigwedge^m V \otimes \bigwedge^n V$. It has the obvious monomial basis

$$\left\{ (v_{i_1} \wedge \cdots \wedge v_{i_m}) \otimes (v_{j_1} \wedge \cdots \wedge v_{j_n}) \,\middle|\, \begin{array}{l} i_1, \ldots, i_m, j_1, \ldots, j_n \in I^+, \\ i_1 > \cdots > i_m, j_1 > \cdots > j_n \end{array} \right\}. \tag{2.2}$$

Each vector $(v_{i_1} \wedge \cdots \wedge v_{i_m}) \otimes (v_{j_1} \wedge \cdots \wedge v_{j_n})$ from this basis is of weight $(\delta_{i_1} + \cdots + \delta_{i_m}) + (\delta_{j_1} + \cdots + \delta_{j_n}) \in P$. Let $P(m, n; I)$ denote the set of all $\Gamma \in P$ such that

- $0 \leq (\Gamma, \delta_i) \leq 2$ for all $i \in I^+$;
- $\sum_{i \in I^+} (\Gamma, \delta_i) = m + n$;
- the number of $i \in I^+$ such that $(\Gamma, \delta_i) = 2$ is at most $\min(m, n)$.

In other words, $P(m, n; I)$ is the set of weights that arise with non-zero multiplicity in the module $\bigwedge^m V \otimes \bigwedge^n V$. We reserve the notation Λ from now on for the weight

$$\Lambda := \Lambda_{o+m} + \Lambda_{o+n}, \tag{2.3}$$

recalling that $o = \min(I) - 1$ and Λ_i denotes the ith fundamental weight. This is the unique maximal element of $P(m, n; I)$ in the dominance ordering, i.e. all elements of $P(m, n; I)$ are of the form $\Lambda - \alpha$ for $\alpha \in Q_+$.

Combinatorics of weights and blocks. Unfortunately the word "weight" gets over-used in this business. In the remainder of the article, the terminology *weight* will always refer to weights in the combinatorial sense of [BS1, §2], namely, diagrams consisting of a number line with vertices labelled by the symbols $\circ$, $\vee$, $\wedge$ or $\times$. Recall also from [BS1, §2] the Bruhat order $\leq$ on weights, which is generated by the basic operation of interchanging $\vee\wedge$ pairs of labels, and the equivalence relation $\sim$, which arises by permuting $\wedge$'s and $\vee$'s.

Let $\Lambda(m, n; I)$ denote the set of all weights drawn on a number line with vertices indexed by I^+ such that exactly m of the vertices are labelled $\vee$ or $\times$ and exactly n vertices are labelled $\wedge$ or $\times$. By a *block* we mean a $\sim$-equivalence class of weights from $\Lambda(m, n; I)$. It is often convenient to represent such a block Γ diagrammatically by replacing all the vertices labelled $\vee$ or $\wedge$ in the weights from Γ by the symbol $\bullet$. For example, taking $I = \{1, \ldots, 8\}$, $m = 5$ and $n = 4$, the block Γ generated by the weight

$$\lambda = \;\times \!\!-\!\!\circ\!\!-\!\!\wedge\!\!-\!\!\vee\!\!-\!\!\vee\!\!-\!\!\times\!\!-\!\!\wedge\!\!-\!\!\circ\!\!-\!\!\vee \tag{2.4}$$

is represented by the block diagram

$$\Gamma = \;\times \!\!-\!\!\circ\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\times\!\!-\!\!\bullet\!\!-\!\!\circ\!\!-\!\!\bullet \tag{2.5}$$

Abusing notation further, we identify the set $P(m, n; I)$ defined in the previous subsection with the set $\Lambda(m, n; I)/\sim$ of all blocks by identifying $\Gamma \in P(m, n; I)$ with the block diagram whose ith vertex is labelled $\circ, \bullet$ or $\times$ according to whether $(\Gamma, \delta_i) = 0$, 1 or 2. For example, the special element $\Lambda \in P(m, n; I)$ from (2.3) is identified in this way with the block consisting of just one weight, namely, the weight

$$\iota := \begin{cases} \overbrace{\times\;\;\times\;\;\times}^{n}\;\;\overbrace{\vee\;\;\vee\;\;\vee}^{m-n}\;\;\circ\;\;\circ\;\;\circ\;\;\cdots & \text{if } m \geq n, \\[2em] \underbrace{\times\;\;\times\;\;\times}_{m}\;\;\underbrace{\wedge\;\;\wedge\;\;\wedge}_{n-m}\;\;\circ\;\;\circ\;\;\circ\;\;\cdots & \text{if } m \leq n. \end{cases} \tag{2.6}$$

We refer to this special weight as the *ground-state*. Recall finally the notion of *defect* $\mathrm{def}(\Gamma)$ of a block Γ from [BS1, §2]. In our setting, we have simply that

$$\mathrm{def}(\Gamma) = \min(m, n) - \#\left(\begin{array}{c} \text{vertices labelled } \times \text{ in} \\ \text{the diagram for } \Gamma \end{array}\right). \tag{2.7}$$

For example, the block Γ from (2.5) is of defect 2. In Lie theoretic terms, we have equivalently that

$$\mathrm{def}(\Gamma) = ((\Lambda, \Lambda) - (\Gamma, \Gamma))/2. \tag{2.8}$$

This formula gives meaning to the notion of defect for more general $\Gamma \in P$ that do not necessarily belong to $P(m, n; I)$.

The monomial basis. With this combinatorial notation behind us, given $\lambda \in \Lambda(m, n; I)$, define

$$V_\lambda := (v_{i_1} \wedge \cdots \wedge v_{i_m}) \otimes (v_{j_1} \wedge \cdots \wedge v_{j_n})$$

where $i_1 > \cdots > i_m$ index the vertices of λ labelled $\vee$ or $\times$ and $j_1 > \cdots > j_n$ index the vertices of λ labelled $\wedge$ or $\times$. For example, if λ is as in (2.4) then $V_\lambda = (v_9 \wedge v_6 \wedge v_5 \wedge v_4 \wedge v_1) \otimes (v_7 \wedge v_6 \wedge v_3 \wedge v_1)$. The monomial basis for the space $\bigwedge^m V \otimes \bigwedge^n V$ from (2.2) is then the set

$$\{V_\lambda \mid \lambda \in \Lambda(m, n; I)\}.$$

Remark 2.1. To help the reader to make the connection with combinatorics elsewhere in the literature (e.g. as in [BKW, BK5]), we note that there is an inclusion $\Lambda(m, n; I) \hookrightarrow \mathscr{P}^2$, where $\mathscr{P}^2$ denotes the set of all *bipartitions*, meaning pairs $(\lambda^{(1)}, \lambda^{(2)})$ of partitions in the usual sense. To define this, take a weight $\lambda \in \Lambda(m, n; I)$ and read off the sequences $i_1 > \cdots > i_m$ and $j_1 > \cdots > j_n$ as above. Then we associate to λ the bipartition $(\lambda^{(1)}, \lambda^{(2)})$ where $\lambda^{(1)} = (\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \cdots)$ and $\lambda^{(2)} = (\lambda_1^{(2)} \geq \lambda_2^{(2)} \geq \cdots)$ are defined from

$$\lambda_r^{(1)} := i_r - o - m + r - 1, \qquad \lambda_s^{(2)} := j_s - o - n + s - 1$$

for $1 \leq r \leq m$ and $1 \leq s \leq n$, with all other parts of $\lambda^{(1)}$ and $\lambda^{(2)}$ being zero. Assuming I is not bounded above, this map gives a bijection between $\Lambda(m, n; I)$ and the set of all bipartitions $(\lambda^{(1)}, \lambda^{(2)})$ where $\lambda^{(1)}$ has at most m and $\lambda^{(2)}$ has at most n non-zero parts.

The dual-canonical basis. As well as the monomial basis, we need to introduce two other bases for the space $\bigwedge^m V \otimes \bigwedge^n V$. The first of these is the so-called *dual-canonical basis*

$$\{L_\lambda \mid \lambda \in \Lambda(m, n; I)\}.$$

To define L_λ formally following [BK4, §2.3], we need some bar-involutions. The *bar-involution* on U is the automorphism $- : U \to U$ that is anti-linear with respect to the field automorphism $\mathbb{Q}(q) \to \mathbb{Q}(q), f(q) \mapsto f(q^{-1})$ and satisfies

$$\overline{E_i} = E_i, \qquad \overline{F_i} = F_i, \qquad \overline{K_i} = K_i^{-1}. \tag{2.9}$$

By a *compatible bar-involution* on a U-module M we mean an anti-linear involution $- : M \to M$ such that $\overline{uv} = \overline{u}\,\overline{v}$ for each $u \in U, v \in M$. The next lemma shows that our module $\bigwedge^m V \otimes \bigwedge^n V$ possesses a compatible bar-involution.

Lemma 2.2. *There is a unique compatible bar-involution on $\bigwedge^m V \otimes \bigwedge^n V$ such that $\overline{V_\lambda} = V_\lambda$ for each weight $\lambda \in \Lambda(m, n; I)$ that is minimal with respect to the Bruhat order. Moreover:*

$$\overline{V_\lambda} = V_\lambda + \left(a \ \mathbb{Z}[q, q^{-1}]\text{-linear combination of } V_\mu\text{'s for } \mu < \lambda\right)$$

for any $\lambda \in \Lambda(m, n; I)$.

Proof. The module $\bigwedge^n V$ possesses a compatible bar-involution, namely, the unique anti-linear involution fixing the basis vectors of the form (2.1). Similarly so does $\bigwedge^m V$. Combining this with Lusztig's tensor product construction from [L, §27.3], we obtain a compatible bar-involution on $\bigwedge^m V \otimes \bigwedge^n V$ as in the statement of the lemma. Uniqueness can be checked by induction on the Bruhat ordering; see [CWZ, Proposition 4.5]. □

Now we can define the dual-canonical basis element $L_\lambda \in \bigwedge^m V \otimes \bigwedge^n V$ for any $\lambda \in \Lambda(m, n; I)$: it is the unique bar invariant vector such that

$$L_\lambda = V_\lambda + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } V_\mu\text{'s for } \mu \in \Lambda(m,n;I)).$$

The existence and uniqueness of L_λ follows from Lemma 2.2 by a general argument originating in [KL], sometimes known as Lusztig's lemma; see e.g. [Du, 1.2] for a concise formulation. The polynomials $d_{\lambda,\mu}(q), p_{\lambda,\mu}(q) \in \mathbb{Z}[q]$ defined from

$$V_\mu = \sum_{\lambda \in \Lambda(m,n;I)} d_{\lambda,\mu}(q) L_\lambda, \tag{2.10}$$

$$L_\mu = \sum_{\lambda \in \Lambda(m,n;I)} p_{\lambda,\mu}(-q) V_\lambda \tag{2.11}$$

satisfy $p_{\lambda,\lambda}(q) = d_{\lambda,\lambda}(q) = 1$ and $p_{\lambda,\mu}(q) = d_{\lambda,\mu}(q) = 0$ unless $\lambda \leq \mu$.

Remark 2.3. Although not needed explicitly here, it is important to note that the polynomials $d_{\lambda,\mu}(q)$ and $p_{\lambda,\mu}(q)$ can be expressed in terms of certain special Kazhdan-Lusztig polynomials associated to the symmetric group S_{m+n}; see [B1, Remark 14]. In particular, up to a trivial renormalisation, the $p_{\lambda,\mu}(q)$'s are Deodhar's parabolic Kazhdan-Lusztig polynomials associated to the subgroup $S_m \times S_n$ of S_{m+n}.

The quasi-canonical basis. There is another basis

$$\{P_\lambda \mid \lambda \in \Lambda(m, n; I)\}$$

for $\bigwedge^m V \otimes \bigwedge^n V$ such that

$$P_\lambda = \sum_{\mu \in \Lambda(m,n;I)} d_{\lambda,\mu}(q) V_\mu, \tag{2.12}$$

$$V_\lambda = \sum_{\mu \in \Lambda(m,n;I)} p_{\lambda,\mu}(-q) P_\mu. \tag{2.13}$$

(We have simply transposed the transition matrices (2.10)–(2.11)). Following [BK4, §2.6], we call this the *quasi-canonical basis* to emphasise that it is *not* the same as Lusztig's canonical basis from [L, §27.3]: our P_λ's are not in general invariant under the bar-involution. Nevertheless, our quasi-canonical and dual-canonical bases are dual to each other in a suitable sense.

To explain this, let $\langle., .\rangle$ be the sesquilinear form on $\bigwedge^m V \otimes \bigwedge^n V$ (anti-linear in the first argument, linear in the second argument) such that

$$\langle V_\lambda, \overline{V_\mu} \rangle = \delta_{\lambda,\mu} \tag{2.14}$$

for each $\lambda, \mu \in \Lambda(m, n; I)$. Then, by a straightforward computation using (2.12) and the formula obtained from (2.11) by applying the bar-involution, we get that

$$\langle P_\lambda, L_\mu \rangle = \delta_{\lambda, \mu} \qquad (2.15)$$

for $\lambda, \mu \in \Lambda(m, n; I)$. For this to be useful, we need to formulate one other basic property of the form $\langle ., . \rangle$. Let $\tau : U \to U$ be the anti-linear anti-automorphism such that

$$\tau(E_i) = qF_i D_i^{-1} D_{i+1}, \qquad \tau(F_i) = q^{-1} D_i D_{i+1}^{-1} E_i, \qquad \tau(D_i) = D_i^{-1}. \quad (2.16)$$

Then, as can be checked directly from (2.14), we have that

$$\langle ux, y \rangle = \langle x, \tau(u)y \rangle \qquad (2.17)$$

for all $x, y \in \bigwedge^m V \otimes \bigwedge^n V$ and $u \in U$.

Specialisation at $q = 1$. Let $\mathscr{A} := \mathbb{Z}[q, q^{-1}]$ and $U_\mathscr{A}$ denote Lusztig's $\mathscr{A}$-form for U. This is the $\mathscr{A}$-subalgebra of U generated by the quantum divided powers $E_i^{(r)}$, $F_i^{(r)}$, the elements D_i, D_i^{-1}, and the elements

$$\begin{bmatrix} D_i \\ r \end{bmatrix} := \prod_{s=1}^r \frac{D_i q^{1-s} - D_i^{-1} q^{s-1}}{q^s - q^{-s}}$$

for all i and $r \geq 0$. The Hopf algebra structure on U makes $U_\mathscr{A}$ into a Hopf algebra over $\mathscr{A}$. The maps (2.9) and (2.16) restrict to well-defined maps on $U_\mathscr{A}$ too. Let $V_\mathscr{A}$ denote the $U_\mathscr{A}$-submodule of V generated as a free $\mathscr{A}$-module by the basis vectors $\{v_i \mid i \in I^+\}$. Taking all tensor products over $\mathscr{A}$ instead of $\mathbb{Q}(q)$, we construct the $U_\mathscr{A}$-module $\bigwedge^m V_\mathscr{A} \otimes \bigwedge^n V_\mathscr{A}$ as above. It is a free $\mathscr{A}$-submodule of $\bigwedge^m V \otimes \bigwedge^n V$ with the three distinguished bases $\{V_\lambda\}$, $\{L_\lambda\}$ and $\{P_\lambda\}$, all indexed by the set $\Lambda(m, n; I)$.

Let $U_\mathbb{Z}$ and $\bigwedge^m V_\mathbb{Z} \otimes \bigwedge^n V_\mathbb{Z}$ denote specialisations of $U_\mathscr{A}$ and $\bigwedge^m V_\mathscr{A} \otimes \bigwedge^n V_\mathscr{A}$ at $q = 1$, i.e. we apply the base change functor $\mathbb{Z} \otimes_\mathscr{A} ?$ viewing $\mathbb{Z}$ as an $\mathscr{A}$-module so that q acts as 1. It causes no problems here to replace $U_\mathbb{Z}$ with the usual Kostant $\mathbb{Z}$-form for the universal enveloping algebra of the general linear Lie algebra of $I^+ \times I^+$ matrices, with Chevalley generators $\mathcal{E}_i$ and $\mathcal{F}_i$ for $i \in I$. The bases $\{V_\lambda\}, \{L_\lambda\}$ and $\{P_\lambda\}$ specialise to give bases $\{\mathcal{V}_\lambda\}, \{\mathcal{L}_\lambda\}$ and $\{\mathcal{P}_\lambda\}$ for $\bigwedge^m V_\mathbb{Z} \otimes \bigwedge^n V_\mathbb{Z}$ as a free $\mathbb{Z}$-module, with

$$\mathcal{V}_\mu = \sum_{\lambda \in \Lambda(m, n; I)} d_{\lambda, \mu}(1) \mathcal{L}_\lambda, \qquad \mathcal{P}_\lambda = \sum_{\mu \in \Lambda(m, n; I)} d_{\lambda, \mu}(1) \mathcal{V}_\mu. \qquad (2.18)$$

The form $\langle ., . \rangle$ specialises to a symmetric bilinear form on $\bigwedge^m V_\mathbb{Z} \otimes \bigwedge^n V_\mathbb{Z}$ with respect to which the basis $\{\mathcal{V}_\lambda\}$ is orthonormal and the bases $\{\mathcal{P}_\lambda\}$ and $\{\mathcal{L}_\lambda\}$ are dual. Moreover the Chevalley generators $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint with respect to this form, i.e. $\langle \mathcal{E}_i v, w \rangle = \langle v, \mathcal{F}_i w \rangle$ and $\langle \mathcal{F}_i v, w \rangle = \langle v, \mathcal{E}_i w \rangle$.

3. CATEGORIFICATION VIA DIAGRAM ALGEBRAS

Following the ideas of [HK, C], we next construct a graded diagram algebra $K(m, n; I)$, and show that the Grothendieck group of the category of graded $K(m, n; I)$-modules can be identified with $\bigwedge^m V_\mathscr{A} \otimes \bigwedge^n V_\mathscr{A}$ so that the standard

modules, irreducible modules and projective indecomposable modules correspond to the monomial, dual-canonical and quasi-canonical bases, respectively.

The category $\mathbf{Rep}(K(m,n;I))$. If A is any locally unital graded algebra, we write $\mathrm{rep}(A)$ for the category of all finite dimensional locally unital left A-modules and $\mathrm{Rep}(A)$ for the category of all finite dimensional *graded* locally unital left A-modules. There an obvious functor

$$\mathrm{f} : \mathrm{Rep}(A) \to \mathrm{rep}(A) \tag{3.1}$$

that forgets the grading on a module. Recalling that $\mathscr{A}$ denotes $\mathbb{Z}[q, q^{-1}]$, the Grothendieck group $[\mathrm{Rep}(A)]$ of the category $\mathrm{Rep}(A)$ is naturally an $\mathscr{A}$-module so that $q^i[M] = [M\langle i\rangle]$ for each $i \in \mathbb{Z}$ and $M \in \mathrm{Rep}(A)$, where $\langle i\rangle$ denotes the degree shift functor defined so that $M\langle i\rangle_j = M_{i-j}$. We refer to [BS1, §5] for other conventions regarding graded modules over locally unital algebras.

Introduce the locally unital graded algebra

$$K(m,n;I) := \bigoplus_{\Gamma \in P(m,n;I)} K_\Gamma, \tag{3.2}$$

where K_Γ is the finite dimensional graded algebra defined in [BS1, §4]. Here we are using the identification explained just after (2.5) of elements of $P(m,n;I)$ with blocks of weights from $\Lambda(m,n;I)$. For $\Gamma \in P(m,n;I)$ and $\lambda \in \Gamma$, we have the K_Γ-modules $L(\lambda), V(\lambda)$ and $P(\lambda)$, which are the irreducible, cell and projective indecomposable modules from [BS1, §5], respectively. We always view them as $K(m,n;I)$-modules by extending the K_Γ-action so that all the summands from (3.2) different from K_Γ act as zero.

The modules $L(\lambda)$, $V(\lambda)$ and $P(\lambda)$ are naturally graded so that $L(\lambda)$ is concentrated in degree zero and the canonical quotient maps $P(\lambda) \twoheadrightarrow V(\lambda) \twoheadrightarrow L(\lambda)$ are homogeneous of degree zero. By [BS1, Theorem 5.3], $\mathrm{Rep}(K(m,n;I))$ is a graded highest weight category and the cell modules are its standard modules in the general sense of [CPS]. Because of this, we refer to $V(\lambda)$ as a *standard module* from now on. The isomorphism classes $\{[L(\lambda)] \mid \lambda \in \Lambda(m,n;I)\}$, $\{[V(\lambda)] \mid \lambda \in \Lambda(m,n;I)\}$ and $\{[P(\lambda)] \mid \lambda \in \Lambda(m,n;I)\}$ give three distinguished bases for $[\mathrm{Rep}(K(m,n;I))]$ as a free $\mathscr{A}$-module.

There is also a duality $\circledast$ on $\mathrm{Rep}(K(m,n;I))$ which induces an anti-linear involution

$$\circledast : [\mathrm{Rep}(K(m,n;I))] \to [\mathrm{Rep}(K(m,n;I))]$$

on the Grothendieck group fixing the $[L(\lambda)]$'s; see [BS1, (5.4)].

Geometric bimodules and projective functors. Recall for blocks $\Gamma, \Delta \in P(m,n;I)$ and a $\Delta\Gamma$-matching t in the sense of [BS2, §2] that there is associated a graded (K_Δ, K_Γ)-bimodule $K_{\Delta\Gamma}^t$; see [BS2, §3]. This bimodule is non-zero if and only if t is a *proper* $\Delta\Gamma$-matching, i.e. at least one oriented $\Delta\Gamma$-matching $\delta t\gamma$ exists; here, δ and γ are weights from Δ and Γ, respectively. As in the introduction, we always view $K_{\Delta\Gamma}^t$ as a $(K(m,n;I), K(m,n;I))$-bimodule by extending the K_Δ- and K_Γ-actions to all of $K(m,n;I)$ in the obvious way. Writing $\mathrm{caps}(t)$ (resp. $\mathrm{cups}(t)$) for the number of caps (resp. cups) in the matching t, let

$$G_{\Delta\Gamma}^t := K_{\Delta\Gamma}^t\langle -\mathrm{caps}(t)\rangle \otimes_{K(m,n;I)} ? : \mathrm{Rep}(K(m,n;I)) \to \mathrm{Rep}(K(m,n;I)).$$

Assuming t is proper, $G_{\Delta\Gamma}^t$ is an indecomposable functor; see [BS2, Theorem 4.14]. A *graded projective functor* on $\mathrm{Rep}(K(m,n;I))$ means an endofunctor that is isomorphic to a finite direct sum of $G_{\Delta\Gamma}^t$'s, possibly shifted in degree. By [BS2, Theorem 4.10], each $G_{\Delta\Gamma}^t$ commutes with the duality $\circledast$, i.e. there is a canonical degree zero isomorphism

$$G_{\Delta\Gamma}^t \circ \circledast \cong \circledast \circ G_{\Delta\Gamma}^t. \tag{3.3}$$

Let t^* denote the mirror image of t in a horizontal axis. By [BS2, Corollary 4.9], there is a canonical degree zero adjunction making

$$\left(G_{\Gamma\Delta}^{t^*} \langle \mathrm{def}(\Gamma) - \mathrm{def}(\Delta)\rangle, G_{\Delta\Gamma}^t \right) \tag{3.4}$$

into an adjoint pair. In particular, this implies that each $G_{\Delta\Gamma}^t$ is exact.

More generally, suppose that $\boldsymbol{\Gamma} = \Gamma_d \cdots \Gamma_0$ is any sequence of blocks in $P(m,n;I)$ and $\boldsymbol{t} = t_d \cdots t_1$ is a $\boldsymbol{\Gamma}$-matching in the sense of [BS2, §2]. We write $\boldsymbol{\Gamma}^*$ for the opposite block sequence $\Gamma_0 \cdots \Gamma_d$ and $\boldsymbol{t}^*$ for $t_1^* \cdots t_d^*$, which is a $\boldsymbol{\Gamma}^*$-matching. To this data, there is associated a graded $(K(m,n;I), K(m,n;I))$-bimodule $K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$; see [BS2, §3]. By its definition, it is non-zero if and only if $\boldsymbol{t}$ is a proper $\boldsymbol{\Gamma}$-matching, i.e. at least one oriented $\boldsymbol{\Gamma}$-matching

$$\boldsymbol{t}[\boldsymbol{\gamma}] = \gamma_d t_d \gamma_{d-1} \cdots \gamma_1 t_1 \gamma_0 \tag{3.5}$$

exists, where $\boldsymbol{\gamma} = \gamma_d \cdots \gamma_0$ is a sequence of weights with $\gamma_r \in \Gamma_r$ for each r. Moreover, $K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$ has an explicit homogeneous basis

$$\{(a\,\boldsymbol{t}[\boldsymbol{\gamma}]\,b) \mid \text{for all oriented } \boldsymbol{\Gamma}\text{-circle diagrams } a\,\boldsymbol{t}[\boldsymbol{\gamma}]\,b\}, \tag{3.6}$$

in which the degree of $(a\,\boldsymbol{t}[\boldsymbol{\gamma}]\,b)$ is equal to the total number of clockwise cups and caps in the diagram. Let

$$G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} := K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} \langle -\mathrm{caps}(\boldsymbol{t})\rangle \otimes_{K(m,n;I)} ? : \mathrm{Rep}(K(m,n;I)) \to \mathrm{Rep}(K(m,n;I)).$$

By [BS2, Theorem 3.5(iii)], the associative multiplication from [BS2, (3.12)] defines a canonical graded bimodule isomorphism

$$K_{\Gamma_d\Gamma_{d-1}}^{t_d} \otimes_{K(m,n;I)} \cdots \otimes_{K(m,n;I)} K_{\Gamma_1\Gamma_0}^{t_1} \xrightarrow{\sim} K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}. \tag{3.7}$$

This induces a canonical isomorphism of functors

$$G_{\Gamma_d\Gamma_{d-1}}^{t_d} \circ \cdots \circ G_{\Gamma_1\Gamma_0}^{t_1} \xrightarrow{\sim} G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}. \tag{3.8}$$

Hence $G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$ is exact and commutes with duality, as that is true for each individual $G_{\Gamma_r\Gamma_{r-1}}^{t_r}$. By [BS2, Theorem 3.6] we have for proper $\boldsymbol{t}$ that

$$K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} \langle -\mathrm{caps}(\boldsymbol{t})\rangle \cong K_{\Gamma_d\Gamma_0}^s \langle -\mathrm{caps}(s)\rangle \otimes R^{\otimes\,\mathrm{circles}(\boldsymbol{t})} \tag{3.9}$$

where s denotes the reduction of $\boldsymbol{t}$ in the sense of [BS2, §2], $\mathrm{circles}(\boldsymbol{t})$ is the number of internal circles in $\boldsymbol{t}$, and R denotes $\mathbb{C}[x]/(x^2)$ graded by declaring that 1 in degree -1 and x in degree 1. This induces an isomorphism of functors

$$G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} \cong (? \otimes R^{\otimes\,\mathrm{circles}(\boldsymbol{t})}) \circ G_{\Gamma_d\Gamma_0}^s \tag{3.10}$$

for proper $\boldsymbol{t}$. This explains how to decompose $G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$ as a direct sum of indecomposable projective functors.

Special projective functors. Now we define some important functors F_i and E_i for each $i \in I$. Given a block $\Gamma \in P(m, n; I)$, we say that $i \in I$ is Γ-*admissible* if $\Gamma - \alpha_i$ belongs to $P(m, n; I)$. Viewing blocks diagrammatically like in (2.5), this means that the ith and $(i+1)$th vertices of Γ match the top number line of a unique one of the following diagrams, and $\operatorname{def}(\Gamma)$ is as indicated:

$$F_i$$

$$\operatorname{def}(\Gamma) \geq 1 \qquad \operatorname{def}(\Gamma) \geq 0 \qquad \operatorname{def}(\Gamma) \geq 0 \qquad \operatorname{def}(\Gamma) \geq 0$$

$$\Gamma$$
$$t_i(\Gamma) \qquad (3.11)$$
$$\Gamma - \alpha_i$$

"cup"   "cap"   "right-shift"   "left-shift"

Also define a $(\Gamma - \alpha_i)\Gamma$-matching $t_i(\Gamma)$ so that the strip between the ith and $(i+1)$th vertices is as in the diagram, and there are only vertical "identity" line segments elsewhere. The *special projective functors* are the functors

$$F_i := \bigoplus_\Gamma G_{(\Gamma - \alpha_i)\Gamma}^{t_i(\Gamma)}, \qquad\qquad E_i := \bigoplus_\Gamma G_{\Gamma(\Gamma - \alpha_i)}^{t_i(\Gamma)^*}, \qquad (3.12)$$

where the direct sums are over all $\Gamma \in P(m, n; I)$ such that i is Γ-admissible. The following lemma makes precise the sense in which these functors generate all other projective functors on $\operatorname{Rep}(K(m, n; I))$.

Lemma 3.1. *Suppose we are given blocks $\Gamma, \Delta \in P(m, n; I)$ and a proper $\Gamma\Delta$-matching t. Up to a degree shift, the indecomposable projective functor $G_{\Gamma\Delta}^t$ is a summand of a composition of finitely many special projective functors.*

Proof. The key point is that t can be obtained as the reduction of a composite matching built from diagrams of the form (3.11) and their duals; we omit some combinatorial details here. Given this, the lemma follows from (3.8) and (3.10). $\square$

Properties of special projective functors. We proceed to record some other basic properties of the functors F_i and E_i.

Lemma 3.2. *The functors F_i and E_i commute with the duality $\circledast$, i.e. there are canonical degree zero isomorphisms $F_i \circ \circledast \cong \circledast \circ F_i$ and $E_i \circ \circledast \cong \circledast \circ E_i$ of functors on $\operatorname{Rep}(K(m, n; I))$.*

Proof. This is immediate from (3.3). $\square$

For $i \in I^+$, we let

$$D_i^{\pm 1} : \operatorname{Rep}(K(m, n; I)) \to \operatorname{Rep}(K(m, n; I)) \qquad (3.13)$$

be the degree shift functor mapping a module $M \in \operatorname{Rep}(K_\Gamma)$ to $M\langle \pm(\Gamma, \delta_i)\rangle$. The following lemma should be compared with (2.16)–(2.17) and [FKS, Proposition 4.2].

Lemma 3.3. *There are degree zero adjunctions making $(F_i \circ D_i D_{i+1}^{-1}\langle -1\rangle, E_i)$ and $(E_i, F_i \circ D_i^{-1} D_{i+1}\langle 1\rangle)$ into adjoint pairs of functors on $\operatorname{Rep}(K(m, n; I))$.*

Proof. We just derive the adjunction for $(F_i \circ D_i D_{i+1}^{-1}\langle -1 \rangle, E_i)$, the other being similar. Let $\Gamma \in P(m, n; I)$ such that i is Γ-admissible. By looking at the diagrams in (3.11), we have that

$$\mathrm{def}(\Gamma - \alpha_i) - \mathrm{def}(\Gamma) = (\Gamma, \delta_i - \delta_{i+1}) - 1. \qquad (3.14)$$

Hence applying (3.4), we get a canonical degree zero adjunction making

$$\left(G^{t_i(\Gamma)}_{(\Gamma - \alpha_i)\Gamma} \langle (\Gamma, \delta_i - \delta_{i+1}) - 1 \rangle, G^{t_i(\Gamma)^*}_{\Gamma(\Gamma - \alpha_i)} \right)$$

into an adjoint pair. Now use the definitions (3.12) and (3.13). $\qquad \square$

Lemma 3.4. *Let $\lambda \in \Lambda(m, n; I)$ and $i \in I$. For symbols $x, y \in \{\circ, \wedge, \vee, \times\}$ we write λ_{xy} for the diagram obtained from λ by relabelling the ith and $(i+1)$th vertices by x and y, respectively.*

   (i) *If $\lambda = \lambda_{\vee\circ}$ then $F_i P(\lambda) \cong P(\lambda_{\circ\vee})$, $F_i V(\lambda) \cong V(\lambda_{\circ\vee})$, $F_i L(\lambda) \cong L(\lambda_{\circ\vee})$.*
   (ii) *If $\lambda = \lambda_{\wedge\circ}$ then $F_i P(\lambda) \cong P(\lambda_{\circ\wedge})$, $F_i V(\lambda) \cong V(\lambda_{\circ\wedge})$, $F_i L(\lambda) \cong L(\lambda_{\circ\wedge})$.*
   (iii) *If $\lambda = \lambda_{\times\vee}$ then $F_i P(\lambda) \cong P(\lambda_{\vee\times})$, $F_i V(\lambda) \cong V(\lambda_{\vee\times})$, $F_i L(\lambda) \cong L(\lambda_{\vee\times})$.*
   (iv) *If $\lambda = \lambda_{\times\wedge}$ then $F_i P(\lambda) \cong P(\lambda_{\wedge\times})$, $F_i V(\lambda) \cong V(\lambda_{\wedge\times})$, $F_i L(\lambda) \cong L(\lambda_{\wedge\times})$.*
   (v) *If $\lambda = \lambda_{\times\circ}$ then:*
      (a) $F_i P(\lambda) \cong P(\lambda_{\vee\wedge})\langle -1 \rangle$;
      (b) *there is a short exact sequence*

$$0 \to V(\lambda_{\wedge\vee}) \to F_i V(\lambda) \to V(\lambda_{\vee\wedge})\langle -1 \rangle \to 0;$$

      (c) *$F_i L(\lambda)$ has irreducible socle $L(\lambda_{\vee\wedge})\langle 1 \rangle$ and head $L(\lambda_{\vee\wedge})\langle -1 \rangle$, and all other composition factors are of the form $L(\mu)$ for $\mu \in \Lambda(m, n; I)$ such that $\mu = \mu_{\vee\vee}$, $\mu = \mu_{\wedge\wedge}$ or $\mu = \mu_{\wedge\vee}$.*
   (vi) *If $\lambda = \lambda_{\vee\wedge}$ then $F_i P(\lambda) \cong P(\lambda_{\circ\times}) \oplus P(\lambda_{\circ\times})\langle 2 \rangle$, $F_i V(\lambda) \cong V(\lambda_{\circ\times})$ and $F_i L(\lambda) \cong L(\lambda_{\circ\times})$.*
   (vii) *If $\lambda = \lambda_{\wedge\vee}$ then $F_i V(\lambda) \cong V(\lambda_{\circ\times})\langle 1 \rangle$ and $F_i L(\lambda) = \{0\}$.*
   (viii) *If $\lambda = \lambda_{\vee\vee}$ then $F_i V(\lambda) = F_i L(\lambda) = \{0\}$.*
   (ix) *If $\lambda = \lambda_{\wedge\wedge}$ then $F_i V(\lambda) = F_i L(\lambda) = \{0\}$.*
   (x) *For all other λ we have that $F_i P(\lambda) = F_i V(\lambda) = F_i L(\lambda) = \{0\}$.*

For the dual statement about E_i, interchange all occurrences of $\circ$ and $\times$.

Proof. Apply [BS2, Theorem 4.2] for $P(\lambda)$, [BS2, Theorem 4.5] for $V(\lambda)$, and [BS2, Theorem 4.11] for $L(\lambda)$. $\qquad \square$

The first categorification theorem. The following theorem explains the connection between the Grothendieck group of $\mathrm{Rep}(K(m, n; I))$ and the $\mathscr{A}$-form of the module $\bigwedge^m V \otimes \bigwedge^n V$ from §2.

Theorem 3.5. *Identify the Grothendieck group $[\mathrm{Rep}(K(m, n; I))]$ with the $U_{\mathscr{A}}$-module $\bigwedge^m V_{\mathscr{A}} \otimes \bigwedge^n V_{\mathscr{A}}$ by identifying $[V(\lambda)]$ with V_λ for each $\lambda \in \Lambda(m, n; I)$.*

   (i) *We have that $[L(\lambda)] = L_\lambda$ and $[P(\lambda)] = P_\lambda$ for each $\lambda \in \Lambda(m, n; I)$.*
   (ii) *The endomorphisms of the Grothendieck group induced by the exact functors E_i, F_i and $D_i^{\pm 1}$ coincide with the action of the generators E_i, F_i and $D_i^{\pm 1}$ of $U_{\mathscr{A}}$ for each $i \in I$.*

(iii) *We have that*
$$\sum_{j \in \mathbb{Z}} q^j \dim \operatorname{Hom}_{K(m,n;I)}(P, M)_j = \langle [P], [M] \rangle$$

for $M, P \in \operatorname{Rep}(K(m, n; I))$ with P projective.
(iv) *We have that $[M^{\circledast}] = \overline{[M]}$ for each $M \in \operatorname{Rep}(K(m, n; I))$.*

Proof. We first check (ii), explaining the argument just in the case of F_i; a similar argument establishes the statement for E_i and the statement for $D_i^{\pm 1}$ is obvious. By the definition of the action of F_i on $\bigwedge^m V \otimes \bigwedge^n V$ it maps

$$(-v_i-) \otimes (-) \mapsto (-v_{i+1}-) \otimes (-),$$
$$(-) \otimes (-v_i-) \mapsto (-) \otimes (-v_{i+1}-),$$
$$(-v_{i+1} \wedge v_i-) \otimes (-v_i-) \mapsto (-v_{i+1} \wedge v_i-) \otimes (-v_{i+1}-),$$
$$(-v_i-) \otimes (-v_{i+1} \wedge v_i-) \mapsto (-v_{i+1}-) \otimes (-v_{i+1} \wedge v_i-),$$
$$(-v_i-) \otimes (-v_i-) \mapsto (-v_{i+1}-) \otimes (-v_i-) + q^{-1}(-v_i-) \otimes (-v_{i+1}-),$$
$$(-v_i-) \otimes (-v_{i+1}-) \mapsto (-v_{i+1}-) \otimes (-v_{i+1}-),$$
$$(-v_{i+1}-) \otimes (-v_i-) \mapsto q(-v_{i+1}-) \otimes (-v_{i+1}-),$$

where $-$ denotes a wedge product of basis vectors v_j for $j \neq i, i+1$. Moreover F_i acts as zero on all other V_λ's. Comparing with Lemma 3.4, this is the same as the action of the functor F_i on the basis for the Grothendieck group coming from the standard modules.

Next we consider (iv). If λ is minimal in the Bruhat order then $V(\lambda) = L(\lambda)$ by [BS1, Theorem 5.2], so $V(\lambda)^{\circledast} \cong V(\lambda)$. Thus, $\circledast$ induces an anti-linear endomorphism of the Grothendieck group that fixes V_λ for each minimal λ. Moreover by (ii) and Lemma 3.2 this induced endomorphism commutes with the actions of E_i, F_i for all $i \in I$. It follows easily that the induced endomorphism is a compatible bar-involution. Hence it coincides with the bar-involution from Lemma 2.2 by the uniqueness from that lemma.

Using (ii) and (iv), we can now establish (i). As $L(\lambda)^{\circledast} \cong L(\lambda)$, we get from (iv) that the vector $[L(\lambda)]$ is bar invariant for each $\lambda \in \Lambda(m, n; I)$. Also the inverse of the q-decomposition matrix from [BS1, (5.14)] has 1's on the diagonal and all other entries belong to $q\mathbb{Z}[q]$. So:

$$[L(\lambda)] = [V(\lambda)] + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } [V(\mu)]\text{'s}).$$

This verifies that $[L(\lambda)]$ satisfies the defining properties of the dual-canonical basis vector L_λ. Hence $[L(\lambda)] = L_\lambda$. Then the fact that $[P(\lambda)] = P_\lambda$ follows on comparing (2.12) and [BS1, (5.15)].

Finally, (iii) is clear from (2.15), (i) and the sesquilinearity of the form $\langle ., . \rangle$, since $\dim \operatorname{Hom}_{K(m,n;I)}(P(\lambda), L(\mu))_j = \delta_{\lambda,\mu} \delta_{j,0}$ for each $\lambda, \mu \in \Lambda(m, n; I)$. $\square$

Remark 3.6. In view of Theorem 3.5, the polynomials $d_{\lambda,\mu}(q)$ and $p_{\lambda,\mu}(q)$ from (2.10)–(2.11) are the same as the polynomials defined by explicit closed formulae in [BS1, (5.12)] and [BS2, (5.3)], the latter going back to [LS]. Hence the quasi-canonical and dual-canonical bases for $\bigwedge^m V \otimes \bigwedge^n V$ are known exactly. Moreover [BS2, Theorems 4.2 and 4.11] give explicit formulae for the action of F_i

and E_i on these bases; in almost all cases this is described already by Lemma 3.4. This gives an explicit diagram calculus for working with the various bases of $\bigwedge^m V \otimes \bigwedge^n V$, which is closely related to the diagram calculus developed by Frenkel and Khovanov in [FK].

The crystal graph. Given $i \in I$ and $\lambda, \mu \in \Lambda(m, n; I)$, we write $\lambda = \tilde{f}_i(\mu)$ if the ith and $(i+1)$th vertices of λ and μ are labelled according to one of the six cases in the following table, and all other vertices of λ and μ are labelled in the same way:

$$
\begin{array}{c|c|c|c|c|c|c}
\mu & \vee\,\circ & \wedge\,\circ & \times\,\vee & \times\,\wedge & \times\,\circ & \vee\,\wedge \\
\hline
\lambda & \circ\,\vee & \circ\,\wedge & \vee\,\times & \wedge\,\times & \vee\,\wedge & \circ\,\times
\end{array}
\tag{3.15}
$$

Define the *crystal graph* to be the directed coloured graph with vertex set equal to $\Lambda(m, n; I)$ and a directed edge $\mu \xrightarrow{i} \lambda$ of colour $i \in I$ whenever $\lambda = \tilde{f}_i(\mu)$. This graph is isomorphic to the crystal graph that is the tensor product of the crystal graphs associated to the irreducible U-modules $\bigwedge^m V$ and $\bigwedge^n V$ in the sense of Kashiwara. The representation theoretic significance of the crystal graph is clear from Lemma 3.4: $\lambda = \tilde{f}_i(\mu)$ if and only if $L(\lambda)$ is a quotient of $F_i L(\mu)$ (possibly shifted in degree).

Lemma 3.7. *Let $\lambda \in \Lambda(m, n; I)$. Then there exists $\mu \in \Lambda(m, n; I)$ that is maximal in the Bruhat ordering and a sequence $i_1, \dots, i_k \in I$ for some $k \geq 0$ such that $\lambda = \tilde{f}_{i_k} \cdots \tilde{f}_{i_1}(\mu)$.*

Proof. If λ is maximal in the Bruhat ordering, there is nothing to do. So assume it is not maximal. Then we can find $i < j$ such that the ith vertex of λ is labelled $\vee$, the jth vertex of λ is labelled $\wedge$, and all vertices of λ in between are labelled $\circ$ or $\times$. By using crystal graph edges of the form $\times\circ \to \vee\wedge \to \circ\times$, we reduce to the situation that the vertices $i + 1, \dots, j - 1$ are labelled so all the $\circ$'s are to the right of the $\times$'s. Then by using edges of the form $\wedge\circ \to \circ\wedge$ and $\times\vee \to \vee\times$ we reduce to the situation that the vertices $\vee$ and $\wedge$ are neighbours. Finally use an edge of the form $\times\circ \to \vee\wedge$ to eliminate this $\vee\wedge$ pair altogether. Then iterate. $\square$

Theorem 3.8. *Take $\lambda \in \Lambda(m, n; I)$ and let $\mu, i_1, \dots, i_k$ be as in Lemma 3.7. Then*
$$
F_{i_k} \cdots F_{i_1} V_\mu = (q + q^{-1})^r q^{r-s} P_\lambda,
$$
where r (resp. s) is the number of crystal graph edges in the path $\mu \xrightarrow{i_1} \cdots \xrightarrow{i_d} \lambda$ that are of the form $\vee\wedge \to \circ\times$ (resp. $\times\circ \to \vee\wedge$).

Proof. By applying Lemma 3.4(vi) a total of r times and Lemma 3.4(v) a total of s times, we get that $[F_{i_k} \circ \cdots \circ F_{i_1}(P(\mu))] = (q + q^{-1})^r q^{r-s}[P(\lambda)]$ in the Grothendieck group $[\mathrm{Rep}(K(m, n; I))]$. As μ is maximal in the Bruhat ordering we have by [BS1, Theorem 5.1] that $P(\mu) \cong V(\mu)$. Hence
$$
F_{i_k} \cdots F_{i_1} V_\mu = [F_{i_k} \circ \cdots \circ F_{i_1}(V(\mu))] = [F_{i_k} \circ \cdots \circ F_{i_1}(P(\mu))]
$$
$$
= (q + q^{-1})^r q^{r-s}[P(\lambda)] = (q + q^{-1})^r q^{r-s} P_\lambda,
$$
using Theorem 3.5(i)–(ii). $\square$

As in [BS1, (6.7)], given a block $\Gamma \in P(m, n; I)$, we let Γ° denote the set of all weights $\gamma \in \Gamma$ that are of maximal defect, i.e. the associated cup diagram $\underline{\gamma}$ from [BS1, §2] has $\operatorname{def}(\Gamma)$ cups. By a *prinjective module* we mean a module that is both projective and injective.

Lemma 3.9. *Up to shifts in degree, the modules $\{P(\lambda)\,|\,\lambda \in \Gamma^\circ\}$ give a complete set of representatives for the isomorphism classes of prinjective indecomposable modules in $\operatorname{Rep}(K_\Gamma)$. Moreover for any $\lambda \in \Gamma^\circ$ the module $P(\lambda)\langle -\operatorname{def}(\Gamma)\rangle$ is self-dual.*

Proof. This follows from [BS2, Theorem 6.1]. $\qquad\square$

Our final lemma gives an alternative description of the set
$$\Lambda(m, n; I)^\circ := \bigcup_{\Gamma \in P(m,n;I)} \Gamma^\circ. \tag{3.16}$$
in terms of the crystal graph: it is the connected component of the crystal graph generated by the ground-state ι from (2.6).

Lemma 3.10. *For $\lambda \in \Lambda(m, n; I)$, we have that $\lambda \in \Lambda(m, n; I)^\circ$ if and only if there exists a sequence $i_1, \ldots, i_d \in I$ such that $\lambda = \tilde{f}_{i_d} \cdots \tilde{f}_{i_1}(\iota)$.*

Proof. Suppose first that there exists $\mu \in \Lambda(m, n; I)$ such that $\lambda = \tilde{f}_i(\mu)$ for some $i \in I$. By inspecting (3.15), we have that λ is of maximal defect in its block if and only if μ is of maximal defect in its block. Hence we are reduced to the case that λ is extremal in the crystal graph in the sense that it cannot be written as $\tilde{f}_i(\mu)$ for any $\mu \in \Lambda(m, n; I)$ or $i \in I$. Then by (3.15) again the weight λ consists of $\times$'s then $\wedge$'s then $\vee$'s then $\circ$'s. For such a weight it is clear that λ is of maximal defect in its block if and only if $\lambda = \iota$. $\qquad\square$

4. Categorification via parabolic category $\mathcal{O}$

In this section, we give a self-contained account of another known categorification theorem, this time categorifying $\bigwedge^m V_\mathbb{Z} \otimes \bigwedge^n V_\mathbb{Z}$ (the same space as in the previous section but specialised at $q = 1$) using a certain sum of blocks of the parabolic category $\mathcal{O}$ corresponding to the Grassmannian $\operatorname{Gr}(m, m + n)$. The arguments in this section provide an elementary proof of the Kazhdan-Lusztig conjecture in this very special case; the possibility of doing this goes back to work of Enright and Shelton [ES] though they used a different strategy.

The category $\boldsymbol{\mathcal{O}(m, n; I)}$. Let $\mathfrak{g} := \mathfrak{gl}_{m+n}(\mathbb{C})$ with its standard Cartan subalgebra $\mathfrak{h}$ of diagonal matrices and its standard Borel subalgebra $\mathfrak{b}$ of upper triangular matrices. We define the standard coordinates $\varepsilon_1, \ldots, \varepsilon_{m+n}$ for $\mathfrak{h}^*$, the weight ρ, the subalgebras $\mathfrak{l}$ and $\mathfrak{p}$, and the category $\mathcal{O}(m, n)$ as in the introduction. We refer the reader to [H, Chapter 9] for a detailed treatment of the basic properties of parabolic category $\mathcal{O}$ (for any semisimple Lie algebra).

The category $\mathcal{O}(m, n)$ is a highest weight category in the sense of [CPS] with irreducible modules $\{\mathcal{L}(\lambda)\,|\,\lambda \in \Lambda(m, n)\}$, standard modules $\{\mathcal{V}(\lambda)\,|\,\lambda \in \Lambda(m, n)\}$ and projective indecomposable modules $\{\mathcal{P}(\lambda)\,|\,\lambda \in \Lambda(m, n)\}$, where $\Lambda(m, n)$ is

the subset of $\mathfrak{h}^*$ defined by (1.1). The standard module $\mathcal{V}(\lambda)$ can be constructed explicitly as a *parabolic Verma module*:

$$\mathcal{V}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathcal{S}(\lambda) \tag{4.1}$$

where $\mathcal{S}(\lambda)$ is the finite dimensional irreducible $\mathfrak{l}$-module of highest weight λ, viewed as a $\mathfrak{p}$-module via the natural projection $\mathfrak{p} \twoheadrightarrow \mathfrak{l}$. The irreducible module $\mathcal{L}(\lambda)$ is the unique irreducible quotient of $\mathcal{V}(\lambda)$, and the projective indecomposable module $\mathcal{P}(\lambda)$ is its projective cover in $\mathcal{O}(m, n)$.

Let $\circledast$ denote the standard duality on $\mathcal{O}(m, n)$, namely, $M^{\circledast}$ is the direct sum of the duals of all the weight spaces of M, with $x \in \mathfrak{g}$ acting on $f \in M^{\circledast}$ by $(xf)(v) := f(x^T v)$ (matrix transposition). This duality fixes irreducible modules, i.e. $\mathcal{L}(\lambda)^{\circledast} \cong \mathcal{L}(\lambda)$ for each $\lambda \in \Lambda(m, n)$.

Two irreducible modules $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ have the same central character if and only if $\lambda + \rho$ and $\mu + \rho$ lie in the same orbit under the natural action of the symmetric group S_{m+n} permuting the ε_i's. We denote this equivalence relation on $\Lambda(m, n)$ by $\sim$, and let $P(m, n)$ denote the set $\Lambda(m, n)/\sim$ of equivalence classes. The category $\mathcal{O}(m, n)$ decomposes according to generalised central characters as

$$\mathcal{O}(m, n) = \bigoplus_{\Gamma \in P(m, n)} \mathcal{O}_{\Gamma} \tag{4.2}$$

where $\mathcal{O}_{\Gamma}$ denotes the Serre subcategory of $\mathcal{O}(m, n)$ generated by the irreducible objects $\mathcal{L}(\lambda)$ with $\lambda \in \Gamma$. For $\Gamma \in P(m, n)$, we let

$$\mathrm{pr}_{\Gamma} : \mathcal{O}(m, n) \to \mathcal{O}(m, n)$$

denote the projection onto the summand $\mathcal{O}_{\Gamma}$ along (4.2).

Remark 4.1. In fact it is known by a special case of [B2, Theorem 2] (see also ([BN, 2.4.4 Corollary B and 2.9 Proposition B]) that each $\mathcal{O}_{\Gamma}$ is a single block of $\mathcal{O}(m, n)$, i.e. it is an indecomposable category, though we will not need to use this.

Now recall the set $\Lambda(m, n; I)$ of weights defined in diagrammatic terms in §2. Using the weight dictionary from (1.5), we can identify $\Lambda(m, n; I)$ with the following subset of $\Lambda(m, n)$:

$$\Lambda(m, n; I) = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \begin{array}{l} (\lambda + \rho, \varepsilon_i) \in I^+ \text{ for all } 1 \le i \le m + n, \\ (\lambda + \rho, \varepsilon_1) > \cdots > (\lambda + \rho, \varepsilon_m), \\ (\lambda + \rho, \varepsilon_{m+1}) > \cdots > (\lambda + \rho, \varepsilon_{m+n}) \end{array} \right\}. \tag{4.3}$$

For example, taking $I = \{1, \ldots, 8\}, m = 5$ and $n = 4$, the weight

$$\lambda = 9\varepsilon_1 + 7\varepsilon_2 + 7\varepsilon_3 + 7\varepsilon_4 + 5\varepsilon_5 + 12\varepsilon_6 + 12\varepsilon_7 + 10\varepsilon_8 + 9\varepsilon_9 \in \mathfrak{h}^*$$

is an element of $\Lambda(m, n; I)$. The corresponding sets $I_{\vee}(\lambda)$ and $I_{\wedge}(\lambda)$ from (1.3)–(1.4) are $\{9, 6, 5, 4, 1\}$ and $\{7, 6, 3, 1\}$, respectively. Hence via the weight dictionary λ is identified with the weight displayed in (2.4).

Recall also that $P(m, n; I)$ denotes the $\sim$-equivalence classes in $\Lambda(m, n; I)$. Under the identification just made, $P(m, n; I)$ becomes a subset of the set

$P(m, n)$ appearing in (4.2). So it makes sense to consider the following sum of blocks in $\mathcal{O}(m, n)$:

$$\mathcal{O}(m, n; I) := \bigoplus_{\Gamma \in P(m,n;I)} \mathcal{O}_\Gamma. \tag{4.4}$$

Equivalently, this is the category of all $\mathfrak{g}$-modules that are semisimple over $\mathfrak{h}$ and possess a composition series with composition factors of the form $\mathcal{L}(\lambda)$ for $\lambda \in \Lambda(m, n; I)$. The irreducible, standard and projective indecomposable modules in $\mathcal{O}(m, n; I)$ are the modules $\mathcal{L}(\lambda), \mathcal{V}(\lambda)$ and $\mathcal{P}(\lambda)$ for $\lambda \in \Lambda(m, n; I)$. Their isomorphism classes $\{[\mathcal{L}(\lambda)]\}$, $\{[\mathcal{V}(\lambda)]\}$ and $\{[\mathcal{P}(\lambda)]\}$ give three natural bases for the Grothendieck group $[\mathcal{O}(m, n; I)]$.

The following lemma originates in work of Irving [I].

Lemma 4.2. *Recalling (3.16), the modules $\{\mathcal{P}(\lambda) \mid \lambda \in \Lambda(m, n; I)^\circ\}$ give a complete set of representatives for the isomorphism classes of prinjective indecomposable modules in $\mathcal{O}(m, n; I)$.*

Proof. If $m \geq n$ then this follows by a special case of [BK2, Theorem 4.8]. A similar argument establishes the result if $m < n$ too. $\square$

Special projective functors. Now we introduce the special projective functors on $\mathcal{O}(m, n; I)$ following [BK1, §4.4] and [CR, §7.4]. It is convenient to work first on all of $\mathcal{O}(m, n)$, defining functors $\mathcal{F}_i$ and $\mathcal{E}_i$ for all $i \in \mathbb{Z}$, before restricting attention to $\mathcal{O}(m, n; I)$.

Given $\Gamma \in P(m, n)$ and $i \in \mathbb{Z}$, we say that i is Γ-*admissible* if there exists $\lambda \in \Gamma$ and $1 \leq j \leq m + n$ such that $\lambda + \varepsilon_j \in \Lambda(m, n)$ and $(\lambda + \rho, \varepsilon_j) = i$. In that case, we let $\Gamma - \alpha_i \in P(m, n)$ denote the $\sim$-equivalence class generated by the weight $\lambda + \varepsilon_j$, for any λ and j as in the previous sentence. If $\Gamma \in P(m, n; I)$ and $i \in I$ then these notions agree with the ones introduced in diagrammatic terms in the preceeding sections.

Let $\mathcal{V}$ be the natural $\mathfrak{g}$-module of column vectors and $\mathcal{V}^*$ be its dual in the usual sense of Lie algebras. The *special projective functors* on $\mathcal{O}(m, n)$ are the endofunctors $\mathcal{F}_i$ and $\mathcal{E}_i$ defined for each $i \in \mathbb{Z}$ by

$$\mathcal{F}_i := \bigoplus_\Gamma \mathrm{pr}_{\Gamma - \alpha_i} \circ (? \otimes \mathcal{V}) \circ \mathrm{pr}_\Gamma, \qquad \mathcal{E}_i := \bigoplus_\Gamma \mathrm{pr}_\Gamma \circ (? \otimes \mathcal{V}^*) \circ \mathrm{pr}_{\Gamma - \alpha_i}, \tag{4.5}$$

where the direct sums are over all $\Gamma \in P(m, n)$ such that i is Γ-admissible. Because the functors $? \otimes \mathcal{V}$ and $? \otimes \mathcal{V}^*$ commute with the duality $\circledast$, so do the functors $\mathcal{F}_i$ and $\mathcal{E}_i$. Moreover $\mathcal{F}_i$ and $\mathcal{E}_i$ are biadjoint, hence they are both exact and send projectives to projectives.

Lemma 4.3. *For $\lambda \in \Lambda(m, n)$, $\mathcal{V}(\lambda) \otimes \mathcal{V}$ has a filtration with sections isomorphic to $\mathcal{V}(\lambda + \varepsilon_j)$ for all $j = 1, \ldots, m + n$ such that $\lambda + \varepsilon_j \in \Lambda(m, n)$, arranged in order from bottom to top. Dually, $\mathcal{V}(\lambda) \otimes \mathcal{V}^*$ has a filtration with sections isomorphic to $\mathcal{V}(\lambda - \varepsilon_j)$ for all $j = 1, \ldots, m + n$ such that $\lambda - \varepsilon_j \in \Lambda(m, n)$, arranged in order from top to bottom.*

Proof. This is a standard consequence of the definition (4.1) and the tensor identity; see e.g. [H, Theorem 3.6]. $\square$

Corollary 4.4. *For $\lambda \in \Lambda(m,n)$ and $i \in \mathbb{Z}$, $\mathcal{F}_i \mathcal{V}(\lambda)$ has a filtration with sections isomorphic to $\mathcal{V}(\lambda + \varepsilon_j)$ for all $j = 1, \ldots, m+n$ such that $\lambda + \varepsilon_j \in \Lambda(m,n)$ and $(\lambda + \rho, \varepsilon_j) = i$, arranged in order from bottom to top. Dually, $\mathcal{V}(\lambda) \otimes \mathcal{V}^*$ has a filtration with sections isomorphic to $\mathcal{V}(\lambda - \varepsilon_j)$ for all $j = 1, \ldots, m+n$ such that $\lambda - \varepsilon_j \in \Lambda(m,n)$ and $(\lambda + \rho, \varepsilon_j) = i + 1$, arranged in order from top to bottom.*

Corollary 4.5. *The functors $? \otimes \mathcal{V}$ and $? \otimes \mathcal{V}^*$ on $\mathcal{O}(m,n)$ decompose as*

$$? \otimes \mathcal{V} = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i, \qquad ? \otimes \mathcal{V}^* = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i.$$

Much later on we will also need the following lemma first observed in [CR, §7.4] which gives an alternative description of the functors $\mathcal{F}_i$ and $\mathcal{E}_i$. Let

$$\Omega := \sum_{j,k=1}^{m+n} e_{j,k} \otimes e_{k,j} \in \mathfrak{g} \otimes \mathfrak{g}. \tag{4.6}$$

This corresponds to the (invariant) trace form on $\mathfrak{g}$.

Lemma 4.6. *For any $M \in \mathcal{O}(m,n)$, $\mathcal{F}_i M$ (resp. $\mathcal{E}_i M$) is the generalised i-eigenspace (resp. the generalised $-(m+n+i)$-eigenspace) of the operator Ω acting on $M \otimes \mathcal{V}$ (resp. $M \otimes \mathcal{V}^*$).*

Proof. We just prove the statement about $\mathcal{F}_i$, a similar argument treating $\mathcal{E}_i$. By classical theory, the center of $U(\mathfrak{g})$ is a free polynomial algebra with generators $z_1, \ldots, z_{m+n}$, where z_r is the central element determined uniquely by the property that it acts on all highest weight modules of highest weight $\lambda \in \mathfrak{h}^*$ by multiplication by the scalar

$$e_r(\lambda) := \sum_{1 \leq i_1 < \cdots < i_r \leq m+n} (\lambda + \rho, \varepsilon_{i_1})(\lambda + \rho, \varepsilon_{i_2}) \cdots (\lambda + \rho, \varepsilon_{i_r}).$$

Now fix $\lambda \in \Lambda(m,n)$. If we can check that the statement of the lemma holds in the special case that $M = \mathcal{V}(\lambda)$, then it follows at once that it is true on every irreducible module in $\mathcal{O}(m,n)$, hence it is true on any module. By [B2, Lemma 5.1], Ω acts on $\mathcal{V}(\lambda) \otimes \mathcal{V}$ in the same way as the central element

$$e_1(\lambda) + e_2(\lambda) - z_2.$$

Hence, fixing a filtration of $\mathcal{V}(\lambda) \otimes \mathcal{V}$ as in Lemma 4.3, Ω respects the filtration and the induced action on the section isomorphic to $\mathcal{V}(\lambda + \varepsilon_j)$ is by multiplication by the scalar

$$e_1(\lambda) + e_2(\lambda) - e_2(\lambda + \varepsilon_j) = (\lambda + \rho, \varepsilon_j).$$

Comparing with Corollary 4.4, we deduce that $\mathcal{F}_i \mathcal{V}(\lambda)$ is the generalised i-eigenspace of Ω, as required. $\qquad \square$

Following [BG], a *projective functor* on $\mathcal{O}(m,n)$ means any endofunctor that is isomorphic to a summand of a functor arising from tensoring with a finite dimensional rational $\mathfrak{g}$-module.

Lemma 4.7. *Any indecomposable projective functor on $\mathcal{O}(m,n)$ is a summand of a composition of finitely many special projective functors.*

Proof. Observe that any irreducible rational $\mathfrak{g}$-module is a summand of a tensor product of finitely many copies of $\mathcal{V}$ and $\mathcal{V}^*$. Given this, the lemma follows from Corollary 4.5. $\qquad\square$

For $i \in I$, the functors $\mathcal{F}_i$ and $\mathcal{E}_i$ restrict to well-defined endofunctors of the subcategory $\mathcal{O}(m, n; I)$; the resulting restrictions are given explicitly by the same formulae as (4.5), but summing now only over $\Gamma \in P(m, n; I)$ such that i is Γ-admissible. We call these the *special projective functors* on $\mathcal{O}(m, n; I)$.

Properties of special projective functors. The goal in the remainder of the section is to prove an analogue of Theorem 3.5 in the present setting. Our approach is entirely elementary, based just on Corollary 4.4 and the following technical result; we include a simple computational proof in order to make the exposition self-contained.

Lemma 4.8. *Let $i \in I$ and $\lambda \in \Lambda(m, n; I)$ be a weight such that the ith and $(i + 1)$th vertices of λ are labelled $\wedge$ and $\vee$, respectively. Let μ be the weight obtained from λ by interchanging the labels on these two vertices. Then $\mathcal{L}(\mu)$ is a composition factor of $\mathcal{V}(\lambda)$.*

Proof. Let $a_j := (\lambda + \rho, \varepsilon_j)$ and v_+ be a non-zero highest weight vector in $\mathcal{V}(\lambda)$. For $1 \leq j < k \leq m + n$, let

$$
s_{k,j} := \mathrm{cdet}
\begin{pmatrix}
e_{j+1,j} & a_{j+1} - a_j & 0 & \ldots & 0 \\
e_{j+2,j} & e_{j+2,j+1} & a_{j+2} - a_j & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
e_{k-2,j} & e_{k-2,j+1} & \ldots & a_{k-2} - a_j & 0 \\
e_{k-1,j} & e_{k-1,j+1} & \ldots & e_{k-1,k-2} & a_{k-1} - a_j \\
e_{k,j} & e_{k,j+1} & \ldots & e_{k,k-2} & e_{k,k-1}
\end{pmatrix}
\in U(\mathfrak{g})
$$

where cdet means the usual Laplace expansion of determinant, ordering monomials in column order. These lowering operators were introduced originally (in a slightly different form) in [NM]. The following key properties are easily checked by direct calculation from the above matrix:

- $e_{r,r+1}s_{k,j}v_+ = 0$ for $1 \leq r < m + n$ with $r \neq k - 1$;
- $e_{k-1,k}s_{k,j}v_+ = (a_j - a_k - 1)s_{k-1,j}v_+$ (interpreting $s_{j,j}$ as 1).

Now, the assumptions on λ mean that there are integers $1 \leq j \leq m$ and $m + 1 \leq k \leq m + n$ such that $a_j = i + 1, a_k = i$ and none of the numbers $a_{j+1}, \ldots, a_{k-1}$ are equal to i or $i + 1$. The assumptions on μ mean that μ is the weight obtained from λ by subtracting the positive root $\varepsilon_j - \varepsilon_k$. We claim that the vector $s_{k,j}v_+$ is a non-zero highest weight vector in $\mathcal{V}(\lambda)$. Since it has weight μ this claim proves the lemma.

For the claim, the two properties from the previous paragraph and the fact that $a_j - a_k - 1 = 0$ give at once that $s_{k,j}v_+$ is a highest weight vector. The problem is to show that it is non-zero. For this, we can expand

$$
s_{k,j}v_+ = \sum_{\substack{j \leq q \leq m \\ m+1 \leq p \leq k}} e_{p,q} \otimes s_{k,j}^{p,q}v_+ \in U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathcal{S}(\lambda)
$$

for unique vectors $s_{k,j}^{p,q} v_+ \in \mathcal{S}(\lambda)$. In particular:

$$s_{k,j}^{k,m} v_+ = (a_{m+1} - a_j)(a_{m+2} - a_j) \cdots (a_{k-1} - a_j) s_{m,j} v_+.$$

To complete the proof we show that $s_{k,j}^{k,m} v_+ \neq 0$. Since none of $a_{m+1}, \ldots, a_{k-1}$ equal $a_j = i + 1$, we just need to show that $s_{m,j} v_+ \neq 0$. For this we apply the operator $e_{j,j+1} e_{j+1,j+2} \cdots e_{m-1,m}$ using the second property from the previous paragraph to get $(a_j - a_{j+1} - 1) \cdots (a_j - a_m - 1) v_+$, which is non-zero as none of $a_{j+1}, \ldots, a_m$ equal $a_j - 1 = i$. $\qquad\square$

The next lemma is obviously the same statement as Lemma 3.4, except that there are no degree shifts to keep track of in the present ungraded setting.

Lemma 4.9. *Let $\lambda \in \Lambda(m, n; I)$ and $i \in I$. For symbols $x, y \in \{\circ, \wedge, \vee, \times\}$ we write λ_{xy} for the diagram obtained from λ by relabelling the ith and $(i+1)$th vertices by x and y, respectively.*

(i) *If $\lambda = \lambda_{\vee\circ}$ then $\mathcal{F}_i \mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\circ\vee})$, $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\circ\vee})$, $\mathcal{F}_i \mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\circ\vee})$.*

(ii) *If $\lambda = \lambda_{\wedge\circ}$ then $\mathcal{F}_i \mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\circ\wedge})$, $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\circ\wedge})$, $\mathcal{F}_i \mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\circ\wedge})$.*

(iii) *If $\lambda = \lambda_{\times\vee}$ then $\mathcal{F}_i \mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\vee\times})$, $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\vee\times})$, $\mathcal{F}_i \mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\vee\times})$.*

(iv) *If $\lambda = \lambda_{\times\wedge}$ then $\mathcal{F}_i \mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\wedge\times})$, $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\wedge\times})$, $\mathcal{F}_i \mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\wedge\times})$.*

(v) *If $\lambda = \lambda_{\times\circ}$ then:*

   (a) $\mathcal{F}_i \mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\vee\wedge})$;

   (b) *there is a short exact sequence*

$$0 \to \mathcal{V}(\lambda_{\wedge\vee}) \to \mathcal{F}_i \mathcal{V}(\lambda) \to \mathcal{V}(\lambda_{\vee\wedge}) \to 0;$$

   (c) $[\mathcal{F}_i \mathcal{L}(\lambda) : \mathcal{L}(\lambda_{\vee\wedge})] = 2$ *and all other composition factors are of the form $\mathcal{L}(\mu)$ for μ with $\mu = \mu_{\vee\vee}$, $\mu = \mu_{\wedge\wedge}$ or $\mu = \mu_{\wedge\vee}$;*

   (d) $\mathcal{F}_i \mathcal{L}(\lambda)$ *has irreducible socle and head isomorphic to $\mathcal{L}(\lambda_{\vee\wedge})$.*

(vi) *If $\lambda = \lambda_{\vee\wedge}$ then $\mathcal{F}_i \mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\circ\times}) \oplus \mathcal{P}(\lambda_{\circ\times})$, $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\circ\times})$ and $\mathcal{F}_i \mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\circ\times})$.*

(vii) *If $\lambda = \lambda_{\wedge\vee}$ then $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\circ\times})$ and $\mathcal{F}_i \mathcal{L}(\lambda) = \{0\}$.*

(viii) *If $\lambda = \lambda_{\vee\vee}$ then $\mathcal{F}_i \mathcal{V}(\lambda) = \mathcal{F}_i \mathcal{L}(\lambda) = \{0\}$.*

(ix) *If $\lambda = \lambda_{\wedge\wedge}$ then $\mathcal{F}_i \mathcal{V}(\lambda) = \mathcal{F}_i \mathcal{L}(\lambda) = \{0\}$.*

(x) *For all other λ we have that $\mathcal{F}_i \mathcal{P}(\lambda) = \mathcal{F}_i \mathcal{V}(\lambda) = \mathcal{F}_i \mathcal{L}(\lambda) = \{0\}$.*

For the dual statement about $\mathcal{E}_i$, interchange all occurrences of $\circ$ and $\times$.

Proof. The statements (i)–(x) for $\mathcal{V}(\lambda)$ follow directly from Corollary 4.4 on translating into the diagrammatic language.

We next check (viii), (ix) and (x) for $\mathcal{L}(\lambda)$. In all these cases, we know already that $\mathcal{F}_i \mathcal{V}(\lambda) = \{0\}$. As $\mathcal{L}(\lambda)$ is a quotient of $\mathcal{V}(\lambda)$ and $\mathcal{F}_i$ is exact, it follows immediately that $\mathcal{F}_i \mathcal{L}(\lambda) = \{0\}$ as required.

The proofs of (i), (ii), (iii) and (iv) for $\mathcal{L}(\lambda)$ are not much harder. For example, if $\lambda = \lambda_{\vee\circ}$ as in (i), then $\mathcal{F}_i \mathcal{L}(\lambda)$ is a quotient of $\mathcal{F}_i \mathcal{V}(\lambda) \cong \mathcal{V}(\lambda_{\circ\vee})$. Moreover it is self-dual as $\mathcal{L}(\lambda)$ is self-dual and $\mathcal{F}_i$ commutes with duality. Hence we either have that $\mathcal{F}_i \mathcal{L}(\lambda) = \{0\}$ or $\mathcal{F}_i \mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\circ\vee})$, as these are the only self-dual quotients of $\mathcal{V}(\lambda_{\circ\vee})$. To rule out the possibility that it is zero, let Γ be the block generated by λ, and note that $\mathcal{F}_i$ maps $\mathcal{O}_\Gamma$ to $\mathcal{O}_{\Gamma - \alpha_i}$. Moreover it induces a $\mathbb{Z}$-module isomorphism $[\mathcal{O}_\Gamma] \xrightarrow{\sim} [\mathcal{O}_{\Gamma - \alpha_i}]$ because it defines a bijection between the bases of these Grothendieck groups arising from the

standard modules. Hence $\mathcal{F}_i$ is non-zero on every non-zero module in $\mathcal{O}_\Gamma$. This proves (i) for $\mathcal{L}(\lambda)$, and the proofs of (ii), (iii) and (iv) are similar.

Next we check (vi) and (vii) for $\mathcal{L}(\lambda)$, i.e. we show that $\mathcal{F}_i\mathcal{L}(\lambda_{\vee\wedge}) \cong \mathcal{L}(\lambda_{\circ\times})$ and $\mathcal{F}_i\mathcal{L}(\lambda_{\wedge\vee}) = \{0\}$. We know that $\mathcal{F}_i\mathcal{V}(\lambda_{\vee\wedge}) \cong \mathcal{F}_i\mathcal{V}(\lambda_{\wedge\vee}) \cong \mathcal{V}(\lambda_{\circ\times})$. So by an argument from the previous paragraph, we either have that $\mathcal{F}_i\mathcal{L}(\lambda_{\vee\wedge}) \cong \mathcal{L}(\lambda_{\circ\times})$ or $\mathcal{F}_i\mathcal{L}(\lambda_{\vee\wedge}) = \{0\}$. Similarly, either $\mathcal{F}_i\mathcal{L}(\lambda_{\wedge\vee}) \cong \mathcal{L}(\lambda_{\circ\times})$ or $\mathcal{F}_i\mathcal{L}(\lambda_{\wedge\vee}) = \{0\}$. As $[\mathcal{F}_i\mathcal{V}(\lambda_{\vee\wedge}) : \mathcal{L}(\lambda_{\circ\times})] = 1$, there must be some composition factor $\mathcal{L}(\mu)$ of $\mathcal{V}(\lambda_{\vee\wedge})$ such that $[\mathcal{F}_i\mathcal{L}(\mu) : \mathcal{L}(\lambda_{\circ\times})] = 1$. The facts proved so far imply either that $\mu = \lambda_{\vee\wedge}$ or that $\mu = \lambda_{\wedge\vee}$. But the latter case cannot occur as $\lambda_{\wedge\vee}$ is strictly bigger than $\lambda_{\vee\wedge}$ in the Bruhat ordering. Hence $\mu = \lambda_{\vee\wedge}$ and we have proved that $[\mathcal{F}_i\mathcal{L}(\lambda_{\vee\wedge}) : \mathcal{L}(\lambda_{\circ\times})] = 1$. This gives $\mathcal{F}_i\mathcal{L}(\lambda_{\vee\wedge}) \cong \mathcal{L}(\lambda_{\circ\times})$ as required for (vi). It remains for (vii) to show that $\mathcal{F}_i\mathcal{L}(\lambda_{\wedge\vee}) = \{0\}$. Suppose for a contradiction that it is non-zero, hence $\mathcal{F}_i\mathcal{L}(\lambda_{\wedge\vee}) \cong \mathcal{L}(\lambda_{\circ\times})$. By Lemma 4.8, $\mathcal{V}(\lambda_{\wedge\vee})$ has both $\mathcal{L}(\lambda_{\wedge\vee})$ and $\mathcal{L}(\lambda_{\vee\wedge})$ as composition factors, so we deduce that $[\mathcal{F}_i\mathcal{V}(\lambda_{\wedge\vee}) : \mathcal{L}(\lambda_{\circ\times})] \geq 2$, which is the desired contradiction.

In this paragraph, we check (v)(c). Take λ with $\lambda = \lambda_{\times\circ}$. Let Γ be the block generated by λ. Note $\mathcal{F}_i$ maps $\mathcal{O}_\Gamma$ to $\mathcal{O}_{\Gamma-\alpha_i}$ and $\mathcal{O}_{\Gamma-\alpha_i}$ to $\mathcal{O}_{\Gamma-2\alpha_i}$. We know for any $\nu \in \Gamma$ that $\mathcal{F}_i^2\mathcal{V}(\nu) \cong \mathcal{V}(\nu_{\circ\times}) \oplus \mathcal{V}(\nu_{\circ\times})$. Hence $\mathcal{F}_i^2$ induces a $\mathbb{Z}$-module isomorphism between $[\mathcal{O}_\Gamma]$ and $2[\mathcal{O}_{\Gamma-2\alpha_i}]$. We deduce for any non-zero module $M \in \mathcal{O}_\Gamma$ that $\mathcal{F}_i^2M$ is non-zero and its class is divisible by two in $[\mathcal{O}_{\Gamma-2\alpha_i}]$. In particular, $\mathcal{F}_i^2\mathcal{L}(\lambda)$ is a non-zero self-dual quotient of $\mathcal{V}(\lambda_{\circ\times}) \oplus \mathcal{V}(\lambda_{\circ\times})$ whose class is divisible by two. This implies that

$$\mathcal{F}_i^2\mathcal{L}(\lambda) \cong \mathcal{L}(\lambda_{\circ\times}) \oplus \mathcal{L}(\lambda_{\circ\times}). \tag{4.7}$$

Now take any $\mu \in \Gamma - \alpha_i$. We know already that $\mathcal{F}_i\mathcal{L}(\mu) \cong \mathcal{L}(\mu_{\circ\times})$ if $\mu = \mu_{\vee\wedge}$, and $\mathcal{F}_i\mathcal{L}(\mu) = \{0\}$ otherwise. Assuming now that $\mu = \mu_{\vee\wedge}$, we deduce from this that $[\mathcal{F}_i\mathcal{L}(\lambda) : \mathcal{L}(\mu)] = [\mathcal{F}_i^2\mathcal{L}(\lambda) : \mathcal{L}(\mu_{\circ\times})]$. Using (4.7), we conclude for $\mu = \mu_{\vee\wedge}$ that $[\mathcal{F}_i\mathcal{L}(\lambda) : \mathcal{L}(\mu)] = 0$ unless $\mu = \lambda_{\vee\wedge}$, and $[\mathcal{F}_i\mathcal{L}(\lambda) : \mathcal{L}(\lambda_{\vee\wedge})] = 2$.

Now we deduce all the statements (i)–(x) for $\mathcal{P}(\lambda)$ by using the fact that $(\mathcal{F}_i, \mathcal{E}_i)$ is an adjoint pair of functors. We just explain the argument in case (vi), since the other cases are similar (actually, easier). As $\mathcal{F}_i$ sends projectives to projectives, $\mathcal{F}_i\mathcal{P}(\lambda)$ is a direct sum of projective indecomposables. To compute the multiplicity of $\mathcal{P}(\mu)$ in this decomposition we calculate

$$\mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}_i\mathcal{P}(\lambda), \mathcal{L}(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}(\lambda), \mathcal{E}_i\mathcal{L}(\mu)) = [\mathcal{E}_i\mathcal{L}(\mu) : \mathcal{L}(\lambda)].$$

By (v)(c) (or rather, its analogue for $\mathcal{E}_i$) this multiplicity is zero unless $\mu = \lambda_{\circ\times}$, when it is two. Hence $\mathcal{F}_i\mathcal{P}(\lambda) \cong \mathcal{P}(\lambda_{\circ\times}) \oplus \mathcal{P}(\lambda_{\circ\times})$.

It just remains to deduce (v)(d). By (v)(a), (v)(c) and exactness of $\mathcal{F}_i$, we get that $\mathcal{F}_i\mathcal{L}(\lambda)$ is a non-zero quotient of $\mathcal{P}(\lambda_{\vee\wedge})$, hence it has irreducible head isomorphic to $\mathcal{L}(\lambda_{\vee\wedge})$. Since it is self-dual it also has irreducible socle isomorphic to $\mathcal{L}(\lambda_{\vee\wedge})$. $\square$

The second categorification theorem. The following theorem should be compared with Theorem 3.5.

Theorem 4.10. *Identify $[\mathcal{O}(m, n; I)]$ with $\bigwedge^m V_\mathbb{Z} \otimes \bigwedge^n V_\mathbb{Z}$ by identifying $[\mathcal{V}(\lambda)]$ with $\mathcal{V}_\lambda$ for each $\lambda \in \Lambda(m, n; I)$.*

(i) We have that $[\mathcal{L}(\lambda)] = \mathcal{L}_\lambda$ and $[\mathcal{P}(\lambda)] = \mathcal{P}_\lambda$ for each $\lambda \in \Lambda(m, n; I)$.

   (ii) *The endomorphisms of the Grothendieck group induced by the exact*
        functors $\mathcal{E}_i$ and $\mathcal{F}_i$ coincide with the action of the Chevalley genera-
        tors $\mathcal{E}_i$ and $\mathcal{F}_i$ of $U_{\mathbb{Z}}$ for each $i \in I$.
  (iii) *We have that*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(P, M) = \langle [P], [M] \rangle$$

   for $M, P \in \mathcal{O}(m, n; I)$ with P projective.

Proof. In view of Theorem 3.5, Lemma 3.4 can be re-interpreted as describing
how the generators E_i and F_i of $U_{\mathscr{A}}$ act on the basis elements V_λ, L_λ and P_λ of
$\bigwedge^m V_{\mathscr{A}} \otimes \bigwedge^n V_{\mathscr{A}}$. Specializing at $q = 1$, we get analogous descriptions of how
the generators $\mathcal{E}_i$ and $\mathcal{F}_i$ of $U_{\mathbb{Z}}$ act on $\mathcal{V}_\lambda, \mathcal{L}_\lambda$ and $\mathcal{P}_\lambda$. In particular, we see that
the Chevalley generators $\mathcal{E}_i$ and $\mathcal{F}_i$ act on $\mathcal{V}_\lambda$ in exactly the same way as the
functors $\mathcal{E}_i$ and $\mathcal{F}_i$ act on $[\mathcal{V}(\lambda)]$ as described by Lemma 4.9. This proves (ii).

   To deduce (i), take any $\lambda \in \Lambda(m, n; I)$ and let $\mu, i_1, \dots, i_k$ and r be as in
Lemma 3.7. In view of Theorem 3.8 specialised at $q = 1$, we know already that

$$\mathcal{F}_{i_k} \cdots \mathcal{F}_{i_1} \mathcal{V}_\mu = 2^r \mathcal{P}_\lambda.$$

On the other hand by Lemma 4.9 we have that

$$[\mathcal{F}_{i_k} \circ \cdots \circ \mathcal{F}_{i_1}(\mathcal{P}(\mu))] = 2^r [\mathcal{P}(\lambda)].$$

As μ is maximal in the Bruhat ordering, the parabolic Verma module $\mathcal{V}(\mu)$
is projective, i.e. $[\mathcal{P}(\mu)] = [\mathcal{V}(\mu)] = \mathcal{V}_\mu$. Hence combining the above two
equations, we get that $[\mathcal{P}(\lambda)] = \mathcal{P}_\lambda$. It then follows that $[\mathcal{L}(\lambda)] = \mathcal{L}_\lambda$ too, by
(2.18) and the usual BGG reciprocity in the highest weight category $\mathcal{O}(m, n; I)$.

   Finally (iii) follows because the bases $\{\mathcal{P}_\lambda\}$ and $\{\mathcal{L}_\lambda\}$ are dual with respect
to the form $\langle ., . \rangle$ and also $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}(\lambda), \mathcal{L}(\mu)) = \delta_{\lambda, \mu}$. $\square$

Remark 4.11. Theorem 4.10 is certainly not new; for example, essentially
this theorem appears already in [CWZ, Theorem 5.5]. Its generalisation from
2-block to k-block parabolic subalgebras in type A is recorded in [BK1, Theorem
4.5], where it is deduced from the Kazhdan-Lusztig conjecture; see also [BK4,
§3.1]. The graded version of this result gives a categorification of a k-fold tensor
product of quantum exterior powers, which has been used in [S] and [MS] to
define functorial knot and tangle invariants which decategorify to the $\mathfrak{sl}_k$-version
of the HOMFLY-PT polynomial [MOY].

5. LOCAL ANALYSIS OF SPECIAL PROJECTIVE FUNCTORS

   The goal in the remainder of the article is to explain the combinatorial
coincidence between Theorems 3.5 and 4.10 by showing that the categories
$\mathrm{rep}(K(m, n; I))$ and $\mathcal{O}(m, n; I)$ are equivalent. Most of the new work needed to
establish this takes place on the diagram algebra side. We begin in this section
by studying some locally-defined natural transformations between compositions
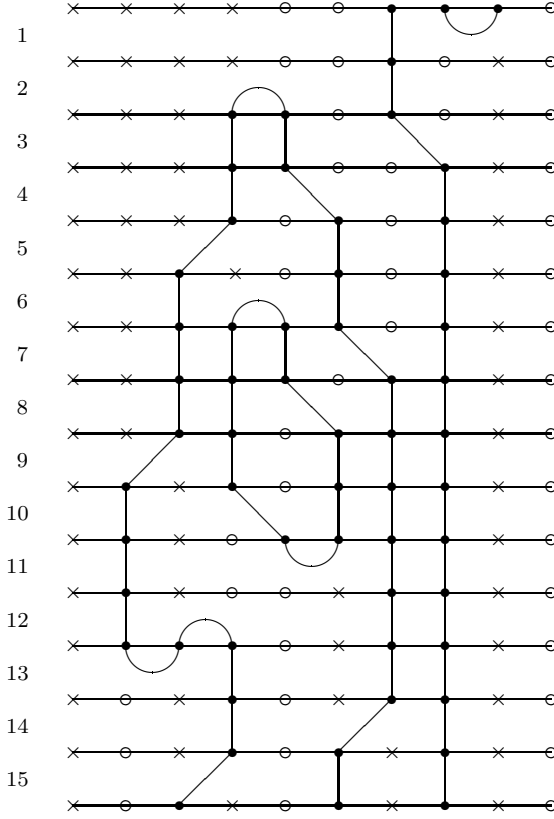of special projective functors.

FIGURE 5.1. An associated composite matching

Admissible sequences and associated composite matchings. For any sequence $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^d$, we can consider the compositions

$$F_{\boldsymbol{i}} := F_{i_d} \circ \cdots \circ F_{i_1} : \operatorname{Rep}(K(m, n; I)) \to \operatorname{Rep}(K(m, n; I)), \qquad (5.1)$$

$$E_{\boldsymbol{i}} := E_{i_d} \circ \cdots \circ E_{i_1} : \operatorname{Rep}(K(m, n; I)) \to \operatorname{Rep}(K(m, n; I)). \qquad (5.2)$$

We are mainly going to be interested here in the properties of the first of these.

Suppose we are given a block $\Gamma \in P(m, n; I)$. Generalising the definition made just before (3.11), we say that $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^d$ is a Γ-*admissible sequence* if $\Gamma - \alpha_{i_1} - \cdots - \alpha_{i_r} \in P(m, n; I)$ for each $r = 1, \ldots, d$. The restriction $F_{\boldsymbol{i}}|_{\operatorname{Rep}(K_\Gamma)}$ of the functor $F_{\boldsymbol{i}}$ to the subcategory $\operatorname{Rep}(K_\Gamma)$ is obviously zero unless $\boldsymbol{i}$ is a Γ-admissible sequence.

Given a Γ-admissible sequence $\boldsymbol{i} \in I^d$, we define the *associated block sequence* $\boldsymbol{\Gamma} = \Gamma_d \cdots \Gamma_0$ by setting $\Gamma_r := \Gamma - \alpha_{i_1} - \cdots - \alpha_{i_r}$ for each $r = 0, \ldots, d$. Then define the *associated composite matching* $\boldsymbol{t} = t_d \cdots t_1$ by setting $t_r := t_{i_r}(\Gamma_{r-1})$ for each $r = 1, \ldots, d$; see Figure 5.1 for an example. We say that $\boldsymbol{i}$ is a *proper* Γ-admissible sequence if $\boldsymbol{t}$ is a proper $\boldsymbol{\Gamma}$-matching in the sense of [BS2, §2].

Lemma 5.1. *Let $\boldsymbol{i} \in I^d$ be a Γ-admissible sequence, and $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ be the associated block sequence and composite matching. There is a canonical isomorphism*

between $F_{\boldsymbol{i}}|_{\mathrm{Rep}(K_\Gamma)}$ and the functor $G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$ that arises by tensoring with the bimodule $K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}\langle -\operatorname{caps}(\boldsymbol{t})\rangle$.

Proof. By the definition (3.12), the restriction of $F_{\boldsymbol{i}}$ to $\mathrm{Rep}(K_\Gamma)$ is equal to the composition $G_{\Gamma_d\Gamma_{d-1}}^{t_d} \circ \cdots \circ G_{\Gamma_1\Gamma_0}^{t_1}$. Now apply (3.8). $\qquad\square$

Corollary 5.2. *The restriction $F_{\boldsymbol{i}}|_{\mathrm{Rep}(K_\Gamma)}$ is non-zero if and only if $\boldsymbol{i}$ is a proper Γ-admissible sequence.*

Proof. This follows from Lemma 5.1 and the fact that $G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$ is non-zero if and only if $\boldsymbol{t}$ is a proper $\boldsymbol{\Gamma}$-matching (recall the discussion immediately before (3.5)). $\qquad\square$

Suppose that $\boldsymbol{t} = t_d \cdots t_1$ is the composite matching associated to some Γ-admissible sequence $\boldsymbol{i} \in I^d$. The diagram $\boldsymbol{t}$ involves various different connected components. We refer to such a component C as a *generalised cap* if it is connected to the bottom number line but not the top number line, a *generalised cup* if it is connected to the top number line but not the bottom number line, a *propagating line* if it is connected to both the bottom and the top number lines, and an *internal circle* if it is not connected to either the top or the bottom number lines. The example in Figure 5.1 contains one of each of these sorts of components. An internal circle is called a *small circle* if it consists just of a single cap and a single cup.

Recalling (3.11), each level t_r of $\boldsymbol{t}$ contains exactly one of the following: a cup, a cap, a right-shift or a left-shift. We say that a component C of $\boldsymbol{t}$ has a cup at level r, a cap at level r, a right-shift at level r or a left-shift at level r if there is such a cup, cap, right-shift or left-shift in t_r that lies on the component C. We say that C is *non-trivial* at level r if one of these four things occurs. Define the *height* of C to be the number of $r = 1, \ldots, d$ such that C is non-trivial at level r. The sum of the heights of all the components of $\boldsymbol{t}$ is equal to the height d of the composite matching $\boldsymbol{t}$ itself. For example, if C is the internal circle in Figure 5.1 then C is non-trivial at levels 6, 8, 10 and 11, hence it is of height four, and it has a cap at level 6.

Lemma 5.3. *Let $\Gamma \in P(m, n; I)$.*

(i) *(i, i) is a Γ-admissible sequence if and only if the ith and $(i+1)$th vertices of Γ are labelled $\times$ and $\circ$, respectively. The asssociated composite matching contains a small internal circle:*
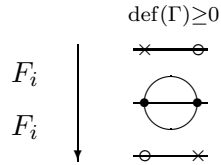
(ii) *$(i, i+1)$ and $(i+1, i)$ are both Γ-admissible sequences if and only if both of the following conditions hold:*
   (a) *the $(i+1)$th vertex of Γ is labelled $\bullet$;*
   (b) *if the ith or the $(i+2)$th vertex of Γ is labelled $\bullet$ then $\mathrm{def}(\Gamma) \geq 1$.*

The associated composite matchings fall into the following four families (displaying only the strip between i and $i + 2$):
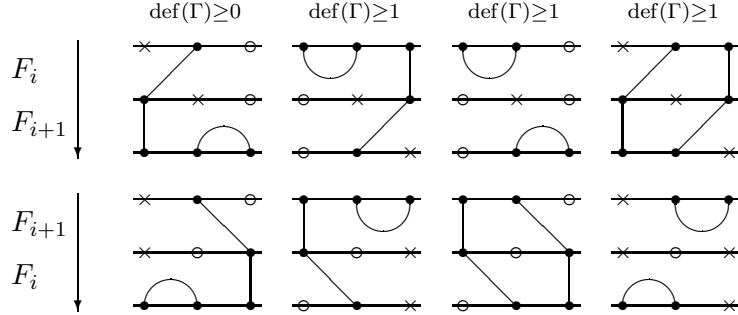
(iii) $(i, i+1)$ *is a Γ-admissible sequence but $(i+1, i)$ is not if and only if exactly one of the following conditions holds:*
  (a) *the $(i+1)$th vertex of Γ is labelled $\circ$, and moreover if the ith and $(i+2)$th vertices are labelled $\bullet$ then $\operatorname{def}(\Gamma) \geq 1$;*
  (b) *the ith, $(i+1)$th and $(i+2)$th vertices are labelled $\times, \bullet$ and $\bullet$, respectively, and $\operatorname{def}(\Gamma) = 0$.*

The associated composite matchings are as follows:
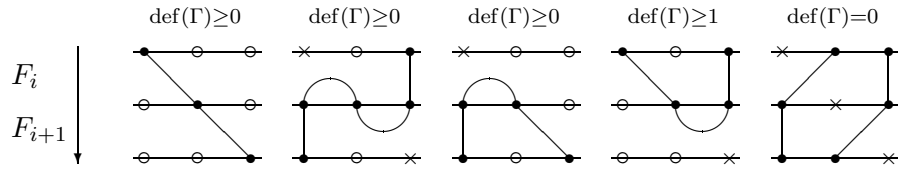
(iv) $(i+1, i)$ *is a Γ-admissible sequence but $(i, i+1)$ is not if and only exactly one of the following conditions holds:*
  (a) *the $(i+1)$th vertex of Γ is labelled $\times$, and moreover if the ith and $(i+2)$th vertices are labelled $\bullet$ then $\operatorname{def}(\Gamma) \geq 1$;*
  (b) *the ith, $(i+1)$th and $(i+2)$th vertices are labelled $\bullet, \bullet$ and $\circ$, respectively, and $\operatorname{def}(\Gamma) = 0$.*

The associated composite matchings are as follows:
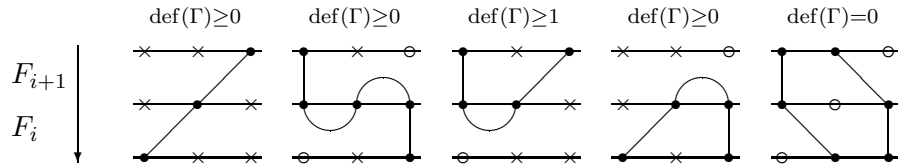
(v) *For $|i - j| > 1$, (i, j) and (j, i) are both Γ-admissible sequences if and only if all of the following conditions hold:*
  (a) *the ith vertex of Γ is labelled $\bullet$ or $\times$, the $(i+1)$th vertex of Γ is labelled $\circ$ or $\bullet$, and if both are labelled $\bullet$ then $\operatorname{def}(\Gamma) \geq 1$;*
  (b) *the jth vertex of Γ is labelled $\bullet$ or $\times$, the $(j+1)$th vertex of Γ is labelled $\circ$ or $\bullet$, and if both are labelled $\bullet$ then $\operatorname{def}(\Gamma) \geq 1$;*
  (c) *if the ith, $(i+1)$th, jth and $(j+1)$th vertices of Γ are all labelled $\bullet$ then $\operatorname{def}(\Gamma) \geq 2$.*

In all cases, (i, j) and (j, i) are proper Γ-admissible sequences, and the associated composite matchings have the same reductions and contain no internal circles.

(vi) *For $|i - j| > 1$, (i, j) is a Γ-admissible sequence but (j, i) is not if and only if the ith, $(i+1)$th, jth and $(j+1)$th vertices of Γ are labelled $\times, \circ, \bullet$ and $\bullet$, respectively, and $\mathrm{def}(\Gamma) = 0$. In this case (i, j) is not a proper Γ-admissible sequence:*
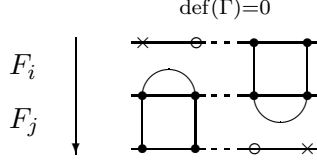
Proof. This follows from the definitions. □

The natural transformations $y(i)$ and $\psi(ij)$. In this subsection, we construct natural transformations

$$y(i): \qquad F_i \to F_i, \tag{5.3}$$

$$\psi(ij): \quad F_j \circ F_i \to F_i \circ F_j, \tag{5.4}$$

for each $i, j \in I$. To do this, it suffices by additivity to define natural transformations

$$y(i)_\Gamma: \qquad F_i|_{\mathrm{Rep}(K_\Gamma)} \to F_i|_{\mathrm{Rep}(K_\Gamma)},$$

$$\psi(ij)_\Gamma: (F_j \circ F_i)|_{\mathrm{Rep}(K_\Gamma)} \to (F_i \circ F_j)|_{\mathrm{Rep}(K_\Gamma)},$$

for each block $\Gamma \in P(m, n; I)$.

The definition of $y(i)_\Gamma$. If i is not Γ-admissible, then $F_i|_{\mathrm{Rep}(K_\Gamma)}$ is the zero functor, and we have to take $y(i)_\Gamma := 0$. Now assume that i is Γ-admissible. By the definition (3.12), $F_i|_{\mathrm{Rep}(K_\Gamma)}$ is the functor defined by tensoring with the bimodule $K_{\Delta\Gamma}^t\langle -\operatorname{caps}(t)\rangle$, where $t := t_i(\Gamma)$ and $\Delta := \Gamma - \alpha_i$. In the next paragraph, we define a bimodule endomorphism

$$\overline{y} = \overline{y}(i)_\Gamma: K_{\Delta\Gamma}^t\langle -\operatorname{caps}(t)\rangle \to K_{\Delta\Gamma}^t\langle -\operatorname{caps}(t)\rangle. \tag{5.5}$$

Given this, we let the desired natural transformation $y(i)_\Gamma$ on $M \in \mathrm{Rep}(K_\Gamma)$ be the homomorphism $y(i)_M: F_i M \to F_i M$ defined by

$$y(i)_M := (-1)^{(\Gamma, \Lambda_i)} \overline{y}(i)_\Gamma \otimes \mathrm{id}_M. \tag{5.6}$$

The sign $(-1)^{(\Gamma, \Lambda_i)}$ here may be computed in practise by counting the number of vertices to the left or equal to the ith vertex that are labelled $\bullet$ in the block diagram for Γ. We point out according to the definition in the next paragraph that the bimodule endomorphism $\overline{y}(i)_\Gamma$ is homogeneous of degree two, hence each $y(i)_M$ is of degree two as well.

It remains to define the bimodule endomorphism $\overline{y}(i)_\Gamma$, which we often refer to as a *positive circle move* ("positive" because it is of positive degree). Recall that $K_{\Delta\Gamma}^t\langle -\operatorname{caps}(t)\rangle$ has a basis consisting of vectors $(a\delta t\gamma b)$ for each oriented circle diagram $a\delta t\gamma b$ with $\delta \in \Delta$ and $\gamma \in \Gamma$. Define $\overline{y}(i)_\Gamma$ to be the linear map sending the basis vector $(a\delta t\gamma b)$ to $(a\delta' t\gamma' b)$ if the component of t between

vertices i and $i + 1$ lies on an anti-clockwise circle in $a\delta t\gamma b$, or to 0 otherwise; here δ' and γ' are the weights obtained on switching the orientation of this circle so that it becomes a clockwise circle.

Unfortunately, it is not obvious from the definition in the previous paragraph that $\overline{y}(i)_\Gamma$ is actually a bimodule homomorphism. To see this, we reinterpret the map $\overline{y}(i)_\Gamma$ in terms of the surgery procedure from [BS1, §3]: roughly, it is "multiplication by a clockwise circle" at the component of t between vertices i and $i + 1$. We explain precisely what we mean by this just in the case that t is a right-shift; the other three cases from (3.11) are interpreted in a similar way. In this case, the definition of $\overline{y}(i)_\Gamma$ is summarised by the following diagram (we display only the strip between vertices i and $i + 1$):
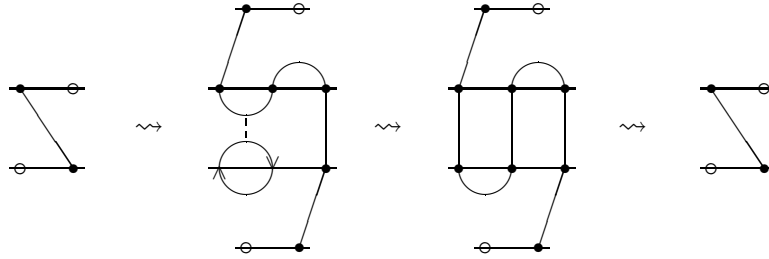
Formally, we take a basis vector $(a\delta t\gamma b) \in K_{\Delta\Gamma}^t$ and proceed as follows:

▶ apply the closure operation from [BS2, §3] to convert $a\delta t\gamma b$ into a closed oriented circle diagram;

▶ extend the diagram by inserting additional number lines and an internal clockwise circle as indicated in the above diagram;

▶ apply the surgery procedure from [BS1, §3] at the position indicated by the dotted line in the diagram;

▶ reduce the result by removing the additional internal number lines;

▶ finally apply the inverse of the closure operation to get back to an element of $K_{\Delta\Gamma}^t$.

This procedure gives the same linear map $\overline{y}(i)_\Gamma$ as defined in the previous paragraph. Moreover the new description implies that $\overline{y}(i)_\Gamma$ is a bimodule homomorphism, because we know that any sequence of surgery procedures produces the same result independent of the order chosen. This is the same observation as used to justify that the algebra multiplication is well defined and associative in [BS1, §3]; its proof involves reformulating the surgery procedure in the language of TQFT's.

The definition of $\psi(ij)_\Gamma$. If (i, j) (resp. (j, i)) is not a Γ-admissible sequence then the functor $(F_j \circ F_i)|_{\mathrm{Rep}(K_\Gamma)}$ (resp. $(F_i \circ F_j)|_{\mathrm{Rep}(K_\Gamma)}$) is the zero functor, in which case we have to take $\psi(ij)_\Gamma := 0$. Now assume that both (i, j) and (j, i) are Γ-admissible sequences. Lemma 5.1 gives us canonical isomorphisms

$$c' : (F_j \circ F_i)|_{\mathrm{Rep}(K_\Gamma)} \xrightarrow{\sim} G_\Gamma^t, \qquad c : (F_i \circ F_j)|_{\mathrm{Rep}(K_\Gamma)} \xrightarrow{\sim} G_\Delta^u,$$

where $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ (resp. $\boldsymbol{\Delta}$ and $\boldsymbol{u}$) denote the block sequence and composite matching associated to (i, j) (resp. (j, i)). Recalling G_Γ^t (resp. G_Δ^u) is the functor defined by tensoring with the bimodule $K_\Gamma^t\langle - \operatorname{caps}(\boldsymbol{t}) \rangle$ (resp. $K_\Delta^u\langle - \operatorname{caps}(\boldsymbol{u}) \rangle$),

the plan is to define a bimodule homomorphism

$$\overline{\psi} = \overline{\psi}(ij)_\Gamma : K_{\mathbf{\Gamma}}^{\mathbf{t}}\langle -\operatorname{caps}(\mathbf{t})\rangle \to K_{\mathbf{\Delta}}^{\mathbf{u}}\langle -\operatorname{caps}(\mathbf{u})\rangle. \qquad (5.7)$$

Given this, we let the desired natural transformation $\psi(ij)_\Gamma$ on $M \in \operatorname{Rep}(K_\Gamma)$ be the homomorphism $\psi(ij)_M : F_j F_i M \to F_i F_j M$ defined by

$$\psi(ij)_M := \begin{cases} -(-1)^{(\Gamma,\Lambda_i)} c_M^{-1} \circ (\overline{\psi}(ij)_\Gamma \otimes \operatorname{id}_M) \circ c_M' & \text{if } j = i \text{ or } i+1, \\ c_M^{-1} \circ (\overline{\psi}(ij)_\Gamma \otimes \operatorname{id}_M) \circ c_M' & \text{otherwise.} \end{cases} \qquad (5.8)$$

To define the bimodule homomorphism $\overline{\psi}(ij)_\Gamma$, we split into cases according to whether $i = j$, $|i - j| = 1$ or $|i - j| > 1$. We will refer to $\overline{\psi}(ij)_\Gamma$ in these three cases as a *negative circle move*, a *crossing move* or a *height move*, respectively. It will turn out that negative circle moves are homogeneous of degree -2, crossing moves are homogeneous of degree 1, and height moves are homogeneous of degree 0.

Negative circle moves. Suppose first that $i = j$. By Lemma 5.3(i) the matching $\mathbf{t} = \mathbf{u}$ contains a small internal circle. We define $\overline{\psi}(ii)_\Gamma$ on a basis vector $(a\,\mathbf{t}[\boldsymbol{\gamma}]\,b) \in K_{\mathbf{\Gamma}}^{\mathbf{t}}\langle -1\rangle$ as follows. If the internal circle in $a\,\mathbf{t}[\boldsymbol{\gamma}]\,b$ is anti-clockwise then we map $(a\,\mathbf{t}[\boldsymbol{\gamma}]\,b)$ to zero; if it is clockwise then we map $(a\,\mathbf{t}[\boldsymbol{\gamma}]\,b)$ to the basis vector $(a\,\mathbf{t}[\boldsymbol{\gamma}']\,b)$ where $\boldsymbol{\gamma}'$ is obtained by switching the orientation of this circle to anti-clockwise.

Crossing moves. Next suppose that $|i - j| = 1$. The eight possibilities for $\mathbf{t}$ and the corresponding possibilities for $\mathbf{u}$ are listed in Lemma 5.3(ii). In all cases, the part of the matching displayed involves two distinct components. Define $\overline{\psi}(ij)_\Gamma$ by applying the surgery procedure to cut these two components in $\mathbf{t}$ and rejoin them as in $\mathbf{u}$. We again make this precise just in one case, in which the definition of $\overline{\psi}(ij)_\Gamma$ is as summarised by the following diagram:
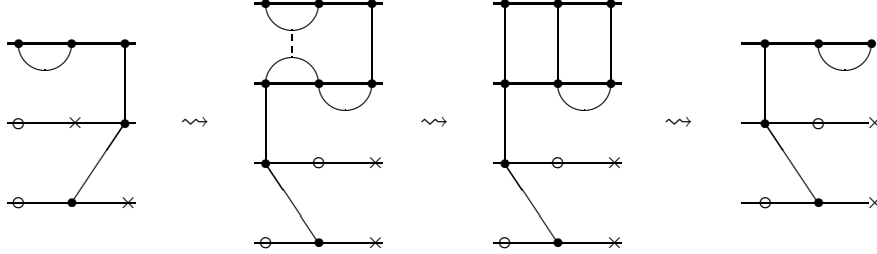
The interpretation of this follows the same steps as in the earlier definition of $\overline{y}(i)_\Gamma$. Note in this case that the map $\overline{\psi}(ij)_\Gamma$ is homogeneous of degree 1: the first map in the above diagram is of degree 1 (as one additional clockwise cap or cup gets added), the second map is of degree 0 (as the surgery procedure preserves the number of clockwise cups and caps), and the final map is obviously of degree 0 too. In fact in all eight of these cases, $\overline{\psi}(ij)_\Gamma$ is homogeneous of degree 1.

Height moves. Finally assume that $|i - j| > 1$. By Lemma 5.3(v), the composite matchings $\mathbf{t}$ and $\mathbf{u}$ are proper, have the same reductions and no internal circles. Hence we get an *isomorphism* $\overline{\psi}(ij)_\Gamma : K_{\mathbf{\Gamma}}^{\mathbf{t}}\langle -\operatorname{caps}(\mathbf{t})\rangle \xrightarrow{\sim} K_{\mathbf{\Delta}}^{\mathbf{u}}\langle -\operatorname{caps}(\mathbf{u})\rangle$ by composing two isomorphisms of the form (3.9). In all these cases, $\overline{\psi}(ij)_\Gamma$ is of degree 0.

Composite natural transformations. Suppose now we are given a d-tuple $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^d$, and recall the composite functor $F_{\boldsymbol{i}}$ from (5.1). We will often use the natural left action of the symmetric group S_d on I^d by permuting the entries. Let $e(\boldsymbol{i}) : F_{\boldsymbol{i}} \to F_{\boldsymbol{i}}$ denote the identity endomorphism of the functor $F_{\boldsymbol{i}}$. Note that

$$e(\boldsymbol{i}) = e(i_d) \cdots e(i_1).$$

The multiplication being used in this expression is the "horizontal" composition of natural transformations from [M, §II.5]; we reserve the notation $\circ$ for the "vertical" composition from [M, §II.4].

The natural transformation $y(i) : F_i \to F_i$ from (5.3) induces

$$y_r(\boldsymbol{i}) := e(i_d) \cdots e(i_{r+1}) y(i_r) e(i_{r-1}) \cdots e(i_1) : F_{\boldsymbol{i}} \to F_{\boldsymbol{i}} \tag{5.9}$$

for $r = 1, \ldots, d$. In other words, for a module M, $y_r(\boldsymbol{i})_M : F_{\boldsymbol{i}}M \to F_{\boldsymbol{i}}M$ is the homomorphism $F_{i_d} \cdots F_{i_{r+1}} y(i_r)_{F_{i_{r-1}} \cdots F_{i_1}M}$.

Similarly the natural transformation $\psi(ij) : F_j \circ F_i \to F_i \circ F_j$ from (5.4) induces

$$\psi_r(\boldsymbol{i}) := e(i_d) \cdots e(i_{r+2}) \psi(i_r i_{r+1}) e(i_{r-1}) \cdots e(i_1) : F_{\boldsymbol{i}} \to F_{s_r \cdot \boldsymbol{i}} \tag{5.10}$$

for $r = 1, \ldots, d-1$. In other words, for a module M, $\psi_r(\boldsymbol{i})_M : F_{\boldsymbol{i}}M \to F_{s_r \cdot \boldsymbol{i}}M$ is the homomorphism $F_{i_d} \cdots F_{i_{r+2}} \psi(i_r i_{r+1})_{F_{i_{r-1}} \cdots F_{i_1}M}$.

Given a block $\Gamma \in P(m, n; I)$, we also use the notation $e(\boldsymbol{i})_\Gamma, y_r(\boldsymbol{i})_\Gamma$ and $\psi_r(\boldsymbol{i})_\Gamma$ for the natural transformations obtained from $e(\boldsymbol{i})$, $y_r(\boldsymbol{i})$ and $\psi_r(\boldsymbol{i})$ by restricting to $\mathrm{Rep}(K_\Gamma)$. For computational purposes, it is important to have available a more concrete description of $y_r(\boldsymbol{i})_\Gamma$ and $\psi_r(\boldsymbol{i})_\Gamma$ in terms of bimodule homomorphisms.

To explain this for $y_r(\boldsymbol{i})_\Gamma$, we may assume that $\boldsymbol{i}$ is a Γ-admissible sequence, so that according to Lemma 5.1 there is a canonical isomorphism

$$c : F_{\boldsymbol{i}}|_{\mathrm{Rep}(K_\Gamma)} \xrightarrow{\sim} G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}$$

where $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ are the block sequence and composite matching associated to $\boldsymbol{i}$. We define a bimodule endomorphism

$$\overline{y}_r = \overline{y}_r(\boldsymbol{i})_\Gamma : K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} \langle - \operatorname{caps}(\boldsymbol{t}) \rangle \to K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} \langle - \operatorname{caps}(\boldsymbol{t}) \rangle \tag{5.11}$$

exactly like in (5.5), but performing the positive circle move to the component of $\boldsymbol{t}$ that is non-trivial at level r.

Lemma 5.4. *With the above notation, we have that*

$$y_r(\boldsymbol{i})_M = (-1)^{(\Gamma_{r-1}, \Lambda_{i_r})} c_M^{-1} \circ (\overline{y}_r(\boldsymbol{i})_\Gamma \otimes \mathrm{id}_M) \circ c_M,$$

for any $M \in \mathrm{Rep}(K_\Gamma)$.

Proof. There is nothing to prove if $d = 1$, as this is just the original definition of $y(i)_M$ from (5.6) in this case. If $d > 1$ then it suffices to show that

$$c_M \circ y_r(\boldsymbol{i})_M = (-1)^{(\Gamma_{r-1}, \Lambda_{i_r})} (\overline{y}_r(\boldsymbol{i})_\Gamma \otimes \mathrm{id}_M) \circ c_M.$$

Recalling the definition of c_M which goes back to (3.7), both sides amount to applying the same two sequences of surgery procedures, but in different orders, then multiplying by the same sign. So they are equal because the order does not matter when applying sequences of surgery procedures. $\square$

The bimodule interpretation of $\psi_r(\boldsymbol{i})_\Gamma$ is similar. We may assume that both $\boldsymbol{i}$ and $s_r \cdot \boldsymbol{i}$ are Γ-admissible sequences, and let $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ (resp. $\boldsymbol{\Delta}$ and $\boldsymbol{u}$) denote the block sequence and composite matching associated to $\boldsymbol{i}$ (resp. $s_r \cdot \boldsymbol{i}$). Lemma 5.1 gives us canonical isomorphisms

$$c' : F_{\boldsymbol{i}}|_{\text{Rep}(K_\Gamma)} \overset{\sim}{\to} G_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}, \qquad c : F_{s_r \cdot \boldsymbol{i}}|_{\text{Rep}(K_\Gamma)} \overset{\sim}{\to} G_{\boldsymbol{\Delta}}^{\boldsymbol{u}}.$$

We define a bimodule homomorphism

$$\overline{\psi}_r = \overline{\psi}_r(\boldsymbol{i})_\Gamma : K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}\langle -\operatorname{caps}(\boldsymbol{t})\rangle \to K_{\boldsymbol{\Delta}}^{\boldsymbol{u}}\langle -\operatorname{caps}(\boldsymbol{u})\rangle \tag{5.12}$$

in similar fashion to (5.7), making either a negative circle move, a crossing move or a height move at levels r and $(r+1)$ of the matching $\boldsymbol{t}$ according to whether $i_r = i_{r+1}$, $|i_r - i_{r+1}| = 1$ or $|i_r - i_{r+1}| > 1$.

Lemma 5.5. *With the above notation, we have that*

$$\psi_r(\boldsymbol{i})_M = \begin{cases} -(-1)^{(\Gamma_{r-1}, \Lambda_{i_r})} c_M^{-1} \circ (\overline{\psi}_r(\boldsymbol{i})_\Gamma \otimes \operatorname{id}_M) \circ c'_M & \text{if } i_{r+1} = i_r \text{ or } i_r + 1, \\ c_M^{-1} \circ (\overline{\psi}_r(\boldsymbol{i})_\Gamma \otimes \operatorname{id}_M) \circ c'_M & \text{otherwise,} \end{cases}$$

for any $M \in \text{Rep}(K_\Gamma)$.

Proof. This follows by a similar argument to the proof of Lemma 5.4. $\square$

Local relations. Now we can prove the following key result, which describes the relations that hold betwen the natural transformations $y(i)_\Gamma$ and $\psi(ij)_\Gamma$.

Theorem 5.6. *The following hold for $\Gamma \in P(m,n;I)$ and $i,j,k \in I$.*

   (i)   (a) $y(i)_\Gamma \circ y(i)_\Gamma = 0$;
        (b) $y(i)_\Gamma = 0$ *if* $\operatorname{def}(\Gamma) = 0$ *and either the ith or the $(i+1)$th vertex of Γ is labelled $\bullet$.*

  (ii)   (a) $\psi(ii)_\Gamma \circ \psi(ii)_\Gamma = 0$;
        (b) $\psi(ii)_\Gamma \circ y_2(ii)_\Gamma = y_1(ii)_\Gamma \circ \psi(ii)_\Gamma + e(ii)_\Gamma$;
        (c) $y_2(ii)_\Gamma \circ \psi(ii)_\Gamma = \psi(ii)_\Gamma \circ y_1(ii)_\Gamma + e(ii)_\Gamma$;
        (d) $y_1(ii)_\Gamma + y_2(ii)_\Gamma = 0$.

 (iii) *If $i \neq j$ then*
        (a) $\psi(ij)_\Gamma \circ y_2(ij)_\Gamma = y_1(ji)_\Gamma \circ \psi(ij)_\Gamma$;
        (b) $y_2(ji)_\Gamma \circ \psi(ij)_\Gamma = \psi(ij)_\Gamma \circ y_1(ij)_\Gamma$.

 (iv) *If $|i - j| > 1$ then $\psi(ji)_\Gamma \circ \psi(ij)_\Gamma = e(ij)_\Gamma$.*

  (v) *If $|i - j| = 1$ and the $\max(i,j)$th vertex of Γ is $\bullet$ then*
        (a) $\psi(ji)_\Gamma \circ \psi(ij)_\Gamma = (i-j)y_1(ij)_\Gamma + (j-i)y_2(ij)_\Gamma$;
        (b) $\psi(ij)_\Gamma \circ y_1(ij)_\Gamma + \psi(ij)_\Gamma \circ y_2(ij)_\Gamma = 0$;
        (c) $\psi_1(jii)_\Gamma \circ \psi_2(jii)_\Gamma \circ \psi_1(iji)_\Gamma = (j-i)e(iji)_\Gamma$;
        (d) $\psi_2(iij)_\Gamma \circ \psi_1(iij)_\Gamma \circ \psi_2(iji)_\Gamma = 0$.

 (vi) *If $|i - j| = 1$ and the $\max(i,j)$th vertex of Γ is $\circ$ or $\times$ then*
        (a) $\psi(ji)_\Gamma \circ \psi(ij)_\Gamma = 0$;
        (b) $y_1(ij)_\Gamma = y_2(ij)_\Gamma$.
        (c) $\psi_1(jii)_\Gamma \circ \psi_2(jii)_\Gamma \circ \psi_1(iji)_\Gamma = 0$;
        (d) $\psi_2(iij)_\Gamma \circ \psi_1(iij)_\Gamma \circ \psi_2(iji)_\Gamma = (i-j)e(iji)_\Gamma$.

 (vii) $\psi_1(jki)_\Gamma \circ \psi_2(jik)_\Gamma \circ \psi_1(ijk)_\Gamma = \psi_2(kij)_\Gamma \circ \psi_1(ikj)_\Gamma \circ \psi_2(ijk)_\Gamma$ *either if $i \neq k$ or if $|i - j| \neq 1$.*

Proof. In all cases, the strategy is to translate into a statement about bimodule homomorphisms using Lemmas 5.4 and 5.5, then to verify that statement by direct computations with the diagram bases. To get started, consider (i). We trivially have that $y(i)_\Gamma = 0$ unless i is Γ-admissible. So assume that i is Γ-admissible. By the definition of the bimodule homomorphism (5.5), it is clear that $\overline{y}(i)_\Gamma^2 = 0$, and moreover $\overline{y}(i)_\Gamma = 0$ if $\mathrm{def}(\Gamma) = 0$ and either the ith or $(i+1)$th vertex of Γ is labelled $\bullet$ (see the last two diagrams from (3.11)). In view of (5.6), this implies the desired statement about the natural transformation $y(i)_\Gamma$.

Next consider (ii). The desired relations are all trivially true if (i, i) is not a Γ-admissible sequence. So assume that (i, i) is Γ-admissible. Then we are in the situation of Lemma 5.3(i). By (3.10), the functor $(F_i \circ F_i)|_{\mathrm{Rep}(K_\Gamma)}$ can be identified with the functor defined by tensoring with the vector space R (in which 1 corresponds to an anti-clockwise circle and x corresponds to a clockwise circle). The bimodule endomorphisms $\overline{y}_1(ii)_\Gamma$ and $\overline{y}_2(ii)_\Gamma$ are both equal to the same positive circle move coming from the map $R \to R, 1 \mapsto x, x \mapsto 0$, and the endomorphism $\overline{\psi}(ii)_\Gamma$ is the negative circle move coming from the map $R \to R, x \mapsto 1, 1 \mapsto 0$. Using this it is trivial to check that

$$\overline{\psi}(ii)_\Gamma \circ \overline{\psi}(ii)_\Gamma = 0,\ \overline{\psi}(ii)_\Gamma \circ \overline{y}_1(ii)_\Gamma + \overline{y}_2(ii)_\Gamma \circ \overline{\psi}(ii)_\Gamma = e(ii)_\Gamma,\ \overline{y}_1(ii)_\Gamma = \overline{y}_2(ii)_\Gamma.$$

Incorporating the signs from Lemma 5.4 and (5.8), these equations imply the desired identities (a), (c) and (d). Then (b) follows from (c) and (d).

For (iii), we may assume that both (i, j) and (j, i) are Λ-admissible sequences, as both sides of the desired identities are trivially zero if they are not. Adopting the same notation as in (5.7), and noting that the additional signs coming from (5.8) and Lemma 5.4 are the same on both sides, it suffices to check that

$$\overline{\psi}(ij)_\Gamma \circ \overline{y}_2(ij)_\Gamma = \overline{y}_1(ji)_\Gamma \circ \overline{\psi}(ij)_\Gamma, \qquad \overline{\psi}(ij)_\Gamma \circ \overline{y}_2(ij)_\Gamma = \overline{y}_1(ji)_\Gamma \circ \overline{\psi}(ij)_\Gamma,$$

as bimodule homomorphisms from $K_\Gamma^t \langle - \mathrm{caps}(t) \rangle$ to $K_\Delta^u \langle - \mathrm{caps}(u) \rangle$. These identities are obvious if $|i - j| > 1$. If $|i - j| = 1$ then the possibilities for t and u are listed in Lemma 5.3(ii). In (the closure of) a diagram basis vector from $K_\Gamma^t \langle - \mathrm{caps}(t) \rangle$ (resp. $K_\Delta^u \langle - \mathrm{caps}(u) \rangle$) the two components of t (resp. u) from the diagrams in Lemma 5.3(ii) could either be joined into one component in the big picture or they could remain as two distinct components in the big picture. In the former case we denote the basis vector by 1 or x according to whether this single component is anti-clockwise or clockwise; in the latter case we denote the basis vector by $1 \otimes 1$, $1 \otimes x$, $x \otimes 1$ or $x \otimes x$ according to the orientations of the two components. With this notation, we can represent our bimodule homomorphisms as

$$\overline{\psi} : 1 \mapsto 1 \otimes x + x \otimes 1,\ x \mapsto x \otimes x,$$
$$\overline{y}_1 : 1 \mapsto x,\ x \mapsto 0,$$
$$\overline{y}_2 : 1 \mapsto x,\ x \mapsto 0.$$

in the one component case, or

$$\overline{\psi} : 1 \otimes 1 \mapsto 1, \ x \otimes 1 \mapsto x, \ 1 \otimes x \mapsto x, \ x \otimes x \mapsto 0,$$
$$\overline{y}_1 : 1 \otimes 1 \mapsto x \otimes 1, \ 1 \otimes x \mapsto x \otimes x, \ x \otimes 1 \mapsto 0, \ x \otimes x \mapsto 0,$$
$$\overline{y}_2 : 1 \otimes 1 \mapsto 1 \otimes x, \ 1 \otimes x \mapsto 0, \ x \otimes 1 \mapsto x \otimes x, \ x \otimes x \mapsto 0$$

in the two component case. Now it is easy to check that $\overline{\psi} \circ \overline{y}_1 = \overline{y}_2 \circ \overline{\psi}$ and $\overline{\psi} \circ \overline{y}_2 = \overline{y}_1 \circ \overline{\psi}$, as required.

For (iv), we may assume (i, j) is a proper Γ-admissible sequence. In view of Lemma 5.3(vi), we get that (j, i) is admissible too, and we are in the situation of Lemma 5.3(v). By the definition (5.7), the height moves $\overline{\psi}(ij)_\Gamma$ and $\overline{\psi}(ji)_\Gamma$ are inverses of each other, and there are no additional signs, so we are done.

Next consider (v)(a), (b). We may assume that (i, j) is a Γ-admissible sequence. Then we are either in the situation of Lemma 5.3(ii) or the fifth cases from Lemma 5.3(iii),(iv). In the latter two cases all of $\psi(ij)_\Gamma, y_1(ij)_\Gamma$ and $y_2(ij)_\Gamma$ are zero, so we are done. In the former case, (j, i) is also Γ-admissible and, noting that $(\Gamma, \Lambda_i) = -(\Gamma, \Lambda_j)$, we reduce to checking the following identities

$$\overline{\psi}(ji)_\Gamma \circ \overline{\psi}(ij)_\Gamma = \overline{y}_1(ij)_\Gamma + \overline{y}_2(ij)_\Gamma, \qquad \overline{\psi}(ij)_\Gamma \circ \overline{y}_1(ij)_\Gamma = \overline{\psi}(ij)_\Gamma \circ \overline{y}_2(ij)_\Gamma$$

at the level of bimodules. This is easy to do using the formulae for $\overline{\psi}, \overline{y}_1$ and $\overline{y}_2$ from the previous paragraph, considering the one component and two component cases separately.

Next consider (v)(c), (d). We explain just the situation when $j = i + 1$, the case $i = j + 1$ being entirely similar. We may assume that (i, j, i) is a proper Γ-admissible sequence, and deduce by Lemma 5.3 that there are only two possibilities for the associated matching $\boldsymbol{t}$. The two possibilities are as displayed on the left hand side of the following (the second possibility arises only for $\mathrm{def}(\Gamma) > 0$):
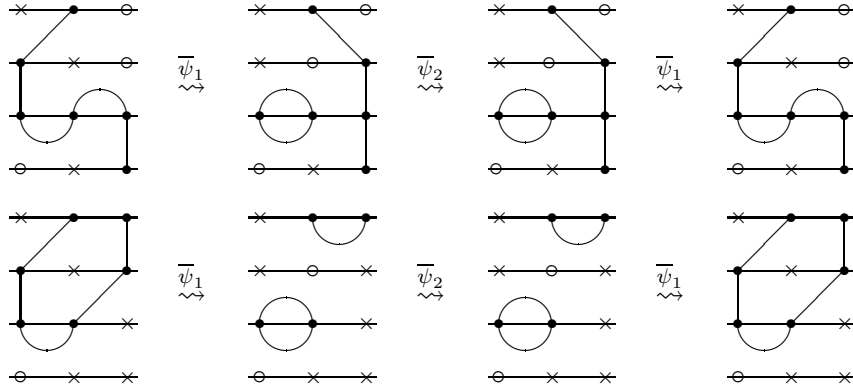
In either case it is clear that (i, i, j) is not Γ-admissible, hence $\psi_2(iji)_\Gamma = 0$ proving (d). For (c), the usual consideration of signs reduces to checking at the level of bimodule homomorphisms that $\overline{\psi}_1 \circ \overline{\psi}_2 \circ \overline{\psi}_1 = 1$; we have indicated the maps involved in the above diagram. To check this identity, we write 1 for an anti-clockwise circle and x for a clockwise circle as usual, and get that the

composition $\overline{\psi}_1 \circ \overline{\psi}_2 \circ \overline{\psi}_1$ is

$$1 \mapsto 1 \otimes x + x \otimes 1 \mapsto 1 \otimes 1 \mapsto 1, \qquad x \mapsto x \otimes x \mapsto 1 \otimes x \mapsto x,$$

i.e. it is the identity map.

The proof of (vi) is similar to (v) so we omit it. Finally (vii) can be checked by analogous techniques, though it is somewhat more lengthy. We omit the details.   □

Remark 5.7. The functors F_i and E_i for $i \in I$ together with the natural transformations (5.3)–(5.4) and the adjunction $(F_i, E_i \circ D_i^{-1} D_{i+1}\langle 1 \rangle)$ from Lemma 3.3 make the graded abelian category $\mathrm{Rep}(K(m, n; I))$ into an integrable representation of the 2-Kac-Moody algebra $\mathfrak{A}(\mathfrak{sl}_{I+})$ in the sense of [R] (except that we have interchanged the roles of E_i and F_i). This is easy to verify using Theorem 5.6, Theorem 3.5(ii), Lemma 3.3 and [R, Theorem 5.27]. We will not pursue this connection further here.

6. Homogeneous Schur-Weyl duality and a graded cellular basis

It is time to give a rough sketch of the strategy in the remainder of the article. For each block $\Gamma = \Lambda - \alpha \in P(m, n; I)$, we will define modules $T_\alpha^\Lambda \in \mathrm{Rep}(K_\Gamma)$ on the diagram algebra side and $\mathcal{T}_\alpha^\Lambda \in \mathcal{O}_\Gamma$ on the category $\mathcal{O}$ side, both of which are built using the respective special projective functors; see (6.1) and (8.3). These modules both satisfy a *double centralizer property* which ensures that the categories $\mathrm{Rep}(K_\Gamma)$ and $\mathcal{O}_\Gamma$ can be reconstructed from the endomorphism algebras $\mathrm{End}_{K_\Gamma}(T_\alpha^\Lambda)^{\mathrm{op}}$ and $\mathrm{End}_\mathfrak{g}(\mathcal{T}_\alpha^\Lambda)^{\mathrm{op}}$, respectively; see Corollaries 8.15 and 8.10. As we will explain in detail later on, this reduces the problem of proving our two categories are equivalent to showing that

$$\mathrm{End}_{K_\Gamma}(T_\alpha^\Lambda)^{\mathrm{op}} \cong \mathrm{End}_\mathfrak{g}(\mathcal{T}_\alpha^\Lambda)^{\mathrm{op}}$$

as algebras. The right hand endomorphism algebra (from the category $\mathcal{O}$ side) is already well understood thanks to a special case of the *Schur-Weyl duality for higher levels* from [BK2]: it is a certain block of a degenerate cyclotomic Hecke algebra of level two.

In this section we are going to focus instead on the left hand endomorphism algebra (from the diagram algebra side), which we denote by E_α^Λ. The most important result of the section gives an explicit graded cellular basis for E_α^Λ parametrized by certain diagrams called *oriented stretched circle diagrams*; see Theorem 6.9. We loosely refer to all this as *homogeneous Schur-Weyl duality*, as it gives rise to a naturally graded analogue on the diagram algebra side of the Schur-Weyl duality from [BK2].

The other important ingredient in the section is the definition of another algebra R_α^Λ by certain generators and relations originating in [KLa, R], which is already known by [BK3] to be isomorphic to the Hecke algebra block mentioned in the opening paragraph; see Theorem 8.5. As an application of the local relations from Theorem 5.6, we will construct a homomorphism

$$\omega : R_\alpha^\Lambda \to E_\alpha^\Lambda;$$

see Theorem 6.3. This homomorphism will eventually turn out to be an isomorphism; see Corollary 8.6. Hence we will have proved that the above two

endomorphism algebras are isomorphic, and the all-important link between the two sides will be made.

The prinjective generator T_α^Λ. Given $\alpha \in Q_+$ of height d, let

$$I^\alpha := \{\boldsymbol{i} = (i_1, \ldots, i_d) \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha\}.$$

This is a single orbit under the action of the symmetric group S_d on I^d. Recall also the special block $\Lambda \in P(m,n;I)$ fixed in (2.3); it is a block of defect zero containing only one weight ι, namely, the ground-state from (2.6). Moreover every block $\Gamma \in P(m,n;I)$ can be written as $\Gamma = \Lambda - \alpha$ for some $\alpha \in Q_+$. It is convenient at this point to set

$$K_\alpha^\Lambda := \left\{ \begin{array}{ll} K_{\Lambda-\alpha} & \text{if } \Lambda - \alpha \in P(m,n;I), \\ \boldsymbol{0} \text{ (the zero algebra)} & \text{if } \Lambda - \alpha \notin P(m,n;I). \end{array} \right.$$

Let

$$T_\alpha^\Lambda := \bigoplus_{\boldsymbol{i} \in I^\alpha} F_{\boldsymbol{i}} L(\iota) \langle \mathrm{def}(\Lambda - \alpha) \rangle. \tag{6.1}$$

If $\Lambda - \alpha \notin P(m,n;I)$, there are no Λ-admissible sequences $\boldsymbol{i} \in I^\alpha$, so T_α^Λ is the zero module. If $\Lambda - \alpha \in P(m,n;I)$ then T_α^Λ is a $K_{\Lambda-\alpha}$-module. Hence, in all cases, it makes sense to regard T_α^Λ as a K_α^Λ-module.

To justify the importance of this module, recall a prinjective module is a module that is both projective and injective. A *prinjective generator* for $\mathrm{Rep}(K_\alpha^\Lambda)$ means a finite dimensional prinjective K_α^Λ-module such that every prinjective indecomposable module from Lemma 3.9 appears as a summand (possibly shifted in degree).

Lemma 6.1. *The module T_α^Λ is a prinjective generator for $\mathrm{Rep}(K_\alpha^\Lambda)$. It is non-zero if and only if $\Lambda - \alpha \in P(m,n;I)$.*

Proof. The fact that T_α^Λ is a prinjective module follows easily because $L(\iota)$ is and projective functors send prinjectives to prinjectives. To see it is a generator, we may assume that $\Gamma := \Lambda - \alpha \in P(m,n;I)$ and take any $\lambda \in \Gamma^\circ$. By Lemma 3.10 we can write $\lambda = \tilde{f}_{i_d} \cdots \tilde{f}_{i_1}(\iota)$. Applying Lemma 3.4, we deduce that $L(\lambda)\langle j \rangle$ appears in the head of $F_{\boldsymbol{i}} L(\iota)$ for $\boldsymbol{i} = (i_1, \ldots, i_d)$ and some $j \in \mathbb{Z}$. Hence $P(\lambda)\langle k \rangle$ is a summand of T_α^Λ for some $k \in \mathbb{Z}$. For the final statement about the non-zeroness of T_α^Λ, it just remains to observe that Γ° is non-empty for $\Gamma \in P(m,n;I)$. $\square$

The endomorphism algebra E_α^Λ. We are interested in the remainder of this section in the (naturally graded) endomorphism algebra

$$E_\alpha^\Lambda := \mathrm{End}_{K_\alpha^\Lambda}(T_\alpha^\Lambda)^{\mathrm{op}}. \tag{6.2}$$

The op here indicates that we view T_α^Λ as a *right* E_α^Λ-module, i.e. it is a graded $(K_\alpha^\Lambda, E_\alpha^\Lambda)$-bimodule.

The algebra E_α^Λ is Morita equivalent to a generalised Khovanov algebra. To explain this precisely, introduce the idempotent

$$e := \left\{ \begin{array}{ll} \sum_{\lambda \in (\Lambda-\alpha)^\circ} e_\lambda & \text{if } \Lambda - \alpha \in P(m,n;I), \\ 0 & \text{if } \Lambda - \alpha \notin P(m,n;I). \end{array} \right. \tag{6.3}$$

As in [BS1, (6.8)], the *generalised Khovanov algebra* is the subalgebra

$$H_\alpha^\Lambda := eK_\alpha^\Lambda e. \tag{6.4}$$

Moreover by [BS2, Corollary 6.3], the familiar truncation functor

$$e : \mathrm{Mod}_{lf}(K_\alpha^\Lambda) \to \mathrm{Mod}_{lf}(H_\alpha^\Lambda) \tag{6.5}$$

defined by left multiplication by the idempotent e is fully faithful on projectives.

Theorem 6.2. *The module eT_α^Λ is a projective generator for H_α^Λ. Moreover the natural restriction map gives an isomorphism between E_α^Λ and the endomorphism algebra $\mathrm{End}_{H_\alpha^\Lambda}(eT_\alpha^\Lambda)^{\mathrm{op}}$. Hence E_α^Λ is the endomorphism algebra of a projective generator for H_α^Λ, so E_α^Λ and H_α^Λ are Morita equivalent.*

Proof. The fact that eT_α^Λ is a projective generator for H_α^Λ follows from Lemmas 6.1 and 3.9, together with standard facts about truncation functors of the form (6.5). The fact that $E_\alpha^\Lambda \cong \mathrm{End}_{H_\alpha^\Lambda}(eT_\alpha^\Lambda)^{\mathrm{op}}$ is a consequence of the definition (6.2) and the fact that the functor e is fully faithful on projectives. $\square$

The cyclotomic Khovanov-Lauda-Rouquier algebra R_α^Λ. Introduce another algebra R_α^Λ defined by generators

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^\alpha\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\}, \tag{6.6}$$

where $d = \mathrm{ht}(\alpha)$ as before, subject to the following relations for $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$ and all admissible r, s:

$$y_1^{(\alpha_{i_1}, \Lambda)} e(\boldsymbol{i}) = 0; \tag{6.7}$$

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}}e(\boldsymbol{i}); \qquad \textstyle\sum_{\boldsymbol{i} \in I^\alpha} e(\boldsymbol{i}) = 1; \tag{6.8}$$

$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i})y_r; \qquad \psi_r e(\boldsymbol{i}) = e(s_r \cdot \boldsymbol{i})\psi_r; \tag{6.9}$$

$$y_r y_s = y_s y_r; \tag{6.10}$$

$$\psi_r y_s = y_s \psi_r \qquad \text{if } s \neq r, r+1; \tag{6.11}$$

$$\psi_r \psi_s = \psi_s \psi_r \qquad \text{if } |r - s| > 1; \tag{6.12}$$

$$\psi_r y_{r+1} e(\boldsymbol{i}) = \begin{cases} (y_r \psi_r + 1)e(\boldsymbol{i}) & \text{if } i_r = i_{r+1}, \\ y_r \psi_r e(\boldsymbol{i}) & \text{if } i_r \neq i_{r+1}; \end{cases} \tag{6.13}$$

$$y_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_r y_r + 1)e(\boldsymbol{i}) & \text{if } i_r = i_{r+1}, \\ \psi_r y_r e(\boldsymbol{i}) & \text{if } i_r \neq i_{r+1}; \end{cases} \tag{6.14}$$

$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ (i_{r+1} - i_r)(y_{r+1} - y_r)e(\boldsymbol{i}) & \text{if } i_r = i_{r+1} \pm 1, \\ e(\boldsymbol{i}) & \text{otherwise}; \end{cases} \tag{6.15}$$

$$\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + (i_{r+1} - i_r))e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1} \pm 1, \\ \psi_{r+1} \psi_r \psi_{r+1} e(\boldsymbol{i}) & \text{otherwise.} \end{cases} \tag{6.16}$$

This algebra is isomorphic in an obvious way to the algebra denoted $R(\alpha; \Lambda)$ in [KLa, §3.4], which is a level two cyclotomic quotient of the Khovanov-Lauda-Rouquier algebra associated to the Dynkin diagram of type A with vertices

indexed by I. By inspecting the relations, it follows that there is a $\mathbb{Z}$-grading on R_α^Λ defined by declaring that each $e(\boldsymbol{i})$ is of degree 0, each y_r is of degree 2, and $\psi_r e(\boldsymbol{i})$ is of degree $-2, 1$ or 0 according to whether $i_r = i_{r+1}$, $|i_r - i_{r+1}| = 1$ or $|i_r - i_{r+1}| > 1$.

The connection between the endomorphism algebra E_α^Λ and the Khovanov-Lauda-Rouquier algebra R_α^Λ is suggested by the following theorem, which makes T_α^Λ into a right R_α^Λ-module.

Theorem 6.3. *For any $\alpha \in Q_+$, there is a homomorphism of graded algebras*

$$\omega : R_\alpha^\Lambda \to E_\alpha^\Lambda$$

mapping $e(\boldsymbol{i})$ to the projection of T_α^Λ onto the summand $F_{\boldsymbol{i}} L(\iota)\langle \mathrm{def}(\Lambda - \alpha)\rangle$ along the decomposition (6.1), y_r to the endomorphism $\sum_{\boldsymbol{i} \in I^\alpha} y_r(\boldsymbol{i})_{L(\iota)}$, and ψ_r to the endomorphism $\sum_{\boldsymbol{i} \in I^\alpha} \psi_r(\boldsymbol{i})_{L(\iota)}$.

Proof. The relations (6.8) are immediate, while (6.9) follows because y_r leaves each summand $F_{\boldsymbol{i}} L(\iota)\langle \mathrm{def}(\Lambda - \alpha)\rangle$ invariant and ψ_r maps $F_{\boldsymbol{i}} L(\iota)\langle \mathrm{def}(\Lambda - \alpha)\rangle$ to $F_{s_r \cdot \boldsymbol{i}} L(\iota)\langle \mathrm{def}(\Lambda - \alpha)\rangle$ according to (5.9) and (5.10). Also the commuting relations (6.10), (6.11) and (6.12) follow from the local nature of the definitions (5.9) and (5.10). All the remaining relations follow from Theorem 5.6. $\square$

Stretched cap, cup and circle diagrams. Continue with $\alpha \in Q_+$ of height d. In the subsequent two subsections, we are going to introduce explicit diagram bases for T_α^Λ and its endomorphism algebra E_α^Λ. In this subsection we define the various sorts of diagrams needed to do this.

We refer to the composite matchings $\boldsymbol{t} = t_d \cdots t_1$ that are associated to Λ-admissible sequences $\boldsymbol{i} \in I^\alpha$ as *stretched cap diagrams* of type α, calling $\boldsymbol{t}$ *proper* if $\boldsymbol{i}$ is a proper Λ-admissible sequence. Here are some examples, the last of which is not proper:
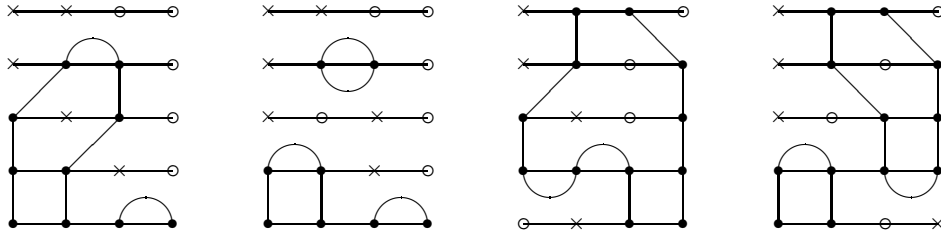
If $\boldsymbol{t}$ is a stretched cap diagram, its *upper reduction* means the (ordinary) cap diagram obtained by removing all its number lines except for the bottom one together with any internal circles and generalised cups it may contain; propagating lines become rays up to infinity. A *(proper) stretched cup diagram* means the mirror image $\boldsymbol{t}^* = t_1^* \cdots t_d^*$ of a (proper) stretched cap diagram $\boldsymbol{t} = t_d \cdots t_1$. Its *lower reduction* is the mirror image of the upper reduction of $\boldsymbol{t}$.

Given a stretched cap diagram $\boldsymbol{t}$ and a cup diagram a whose top number line matches the bottom number line of $\boldsymbol{t}$ (i.e. their free vertices in all the same positions) we can glue a under $\boldsymbol{t}$ to obtain a new diagram $a\boldsymbol{t}$. We call this an *upper-stretched circle diagram*. The mirror image of an upper-stretched circle diagram is a *lower-stretched circle diagram*. Given a pair of stretched cap diagrams $\boldsymbol{t}$ and $\boldsymbol{u}$ of type α, we can glue $\boldsymbol{u}^*$ under $\boldsymbol{t}$ to obtain a *stretched circle*

diagram $\boldsymbol{u}^*\boldsymbol{t}$ of type α. Note such a diagram has a distinguished number line in the middle, which we call the *boundary line*. We refer to the internal circles of $\boldsymbol{t}$ (resp. $\boldsymbol{u}^*$) as the *upper circles* (resp. *lower circles*) of $\boldsymbol{u}^*\boldsymbol{t}$. All remaining internal circles in $\boldsymbol{u}^*\boldsymbol{t}$ cross the boundary line, so we call them *boundary circles*.

All the diagrams so far have oriented versions too. Given a stretched cap diagram $\boldsymbol{t}$ of type α we can uniquely recover the underlying Λ-admissible sequence $\boldsymbol{i} \in I^\alpha$ from it, hence $\boldsymbol{t}$ also determines the *associated block sequence* $\boldsymbol{\Gamma} = \Gamma_d \cdots \Gamma_0$ in which $\Gamma_r = \Lambda - \alpha_{i_1} - \cdots - \alpha_{i_r}$. For $\boldsymbol{\Gamma}$ arising from $\boldsymbol{t}$ in this way, we refer to an oriented $\boldsymbol{\Gamma}$-matching $\boldsymbol{t}[\boldsymbol{\gamma}]$ as in (3.5) as an *oriented stretched cap diagram* of type α. Note it is necessarily the case that $\gamma_0 = \iota$, the ground-state. An *oriented stretched cup diagram* means the mirror image $\boldsymbol{t}^*[\boldsymbol{\gamma}^*]$ of an oriented stretched cup diagram; here, for $\boldsymbol{\gamma} = \gamma_d \cdots \gamma_0$ we write $\boldsymbol{\gamma}^*$ for the opposite weight sequence $\gamma_0 \cdots \gamma_d$.

An *oriented upper-stretched circle diagram* means a diagram of the form $a\boldsymbol{t}[\boldsymbol{\gamma}]$ where $a\gamma_d$ is an oriented cup diagram and $\boldsymbol{t}[\boldsymbol{\gamma}]$ is an oriented stretched cap diagram. The mirror image $\boldsymbol{t}^*[\boldsymbol{\gamma}^*] a^*$ of such a diagram is an *oriented lower-stretched circle diagram*. An *oriented stretched circle diagram* is a diagram obtained by gluing an oriented stretched cup diagram $\boldsymbol{u}^*[\boldsymbol{\delta}^*]$ underneath an oriented stretched cap diagram $\boldsymbol{t}[\boldsymbol{\gamma}]$ assuming that $\gamma_d = \delta_d$; we denote this composite diagram by

$$\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}] = \delta_0 u_1^* \delta_1 \cdots \delta_{d-1} u_d^* \gamma_d t_d \gamma_{d-1} \cdots \gamma_1 t_1 \gamma_0.$$

Here is an example of an oriented stretched circle diagram, taking $m = n = 2$ and $I = \{1, 2\}$, together with its lower and upper reductions:

$$(6.17)$$

We say a stretched circle diagram $\boldsymbol{u}^*\boldsymbol{t}$ is *proper* if oriented diagrams of the form $\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ exist; this implies that both $\boldsymbol{u}$ and $\boldsymbol{t}$ are proper too.

Finally we introduce some degrees. If $\boldsymbol{t}[\boldsymbol{\gamma}]$, $\boldsymbol{u}[\boldsymbol{\delta}^*]$ and $\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ are oriented stretched cap, cup and circle diagrams, respectively, we define their degrees from

$$\deg(\boldsymbol{t}[\boldsymbol{\gamma}]) := \#(\text{clockwise caps}) - \#(\text{anti-clockwise cups}), \qquad (6.18)$$

$$\deg(\boldsymbol{u}^*[\boldsymbol{\delta}^*]) := \#(\text{clockwise cups}) - \#(\text{anti-clockwise caps}), \qquad (6.19)$$

$$\deg(\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]) := \deg(\boldsymbol{u}^*[\boldsymbol{\delta}^*]) + \deg(\boldsymbol{t}[\boldsymbol{\gamma}]) = \deg(\boldsymbol{u}[\boldsymbol{\delta}]) + \deg(\boldsymbol{t}[\boldsymbol{\gamma}]). \quad (6.20)$$

The following lemma gives an alternative description of the last one of these; for instance, the oriented stretched circle diagram in (6.17) is of degree 1.

Lemma 6.4. *The degree of the oriented stretched circle diagram $\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ is equal to $\mathrm{def}(\Lambda - \alpha) + \#(clockwise\ circles) - \#(anti\text{-}clockwise\ circles)$.*

Proof. Using the observation that $\mathrm{def}(\Lambda - \alpha) = \mathrm{caps}(\boldsymbol{t}) - \mathrm{cups}(\boldsymbol{t})$, it is easy to see from the definition (6.20) that $\deg(\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}])$ is equal to $\mathrm{def}(\Lambda - \alpha)$ plus the number of clockwise cups and caps in the diagram $\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ minus the total number of caps in $\boldsymbol{u}^*\boldsymbol{t}$. Now apply [BS2, Lemma 2.2]. $\qquad\square$

Remark 6.5. There is a natural bijection between the set of oriented stretched cap diagrams and a special case of the standard tableaux from [BKW, §3.2] (taking $e := 0$, $l := 2$ and $(k_1, k_2) := (o + m, o + n)$). To make this precise, define a *standard bitableau* to be a pair $\mathtt{T} = (\mathtt{T}^{(1)}, \mathtt{T}^{(2)})$ of Young tableaux in the usual sense, with boxes filled with the distinct integers $1, \ldots, d$ so that within each $\mathtt{T}^{(i)}$ the entries are strictly increasing along rows and down columns, like in the following example:

$$\mathtt{T} = \left(\begin{array}{|c|} \hline 5 \\ \hline 8 \\ \hline \end{array} , \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 6 & 7 \\ \hline \end{array} \right). \tag{6.21}$$

In such a diagram, we define the *residue* of the node in row r and column c of $\mathtt{T}^{(1)}$ (resp. $T^{(2)}$) to be the integer $(o + m + c - r)$ (resp. $(o + n + c - r)$). The *residue sequence* $\boldsymbol{i}^{\mathtt{T}}$ of $\mathtt{T}$ means the sequence $(i_1, \ldots, i_d)$ where i_k is the residue of the node with entry k, and the *type* of $\mathtt{T}$ is $\alpha_{i_1} + \cdots + \alpha_{i_d} \in Q_+$; for instance, taking $o = 0$ and $m = n = 2$, the tableau in (6.21) has $\boldsymbol{i}^{\mathtt{T}} = (2, 3, 4, 1, 2, 2, 3, 1)$ and is of type $2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$.

Now suppose we are given an oriented stretched cap diagram $\boldsymbol{t}[\boldsymbol{\gamma}]$ with underlying admissible sequence $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^\alpha$. We let $\mathtt{T}$ be the standard bitableau obtained by starting from the empty bitableau, then adding a node of residue i_k labelled by the entry k for each $k = 1, \ldots, d$, so that the diagram obtained after each step is itself a standard bitableau. At the kth step, there is a unique way to add such a node with one exception: when the kth level t_k of $\boldsymbol{t}$ is a cap there are two possible places to add a node of residue i_k; in that case we add the node into $\mathtt{T}^{(1)}$ if the cap is clockwise or into $\mathtt{T}^{(2)}$ if the cap is anti-clockwise. In this way, we obtain a well-defined map from oriented stretched cap diagrams $\boldsymbol{t}[\boldsymbol{\gamma}]$ of type α to standard bitableaux of type α. Moreover the underlying shape of the bitableau $\mathtt{T}$ is the bipartition $(\lambda^{(1)}, \lambda^{(2)})$ associated to the bottom weight $\lambda := \gamma_d$ according to the map from Remark 2.1.

It is quite easy to reverse the construction, hence the map $\boldsymbol{t}[\boldsymbol{\gamma}] \mapsto \mathtt{T}$ just defined is actually a bijection. Moreover, the notion of degree from (6.18) coincides under this bijection to the notion of degree of a standard bitableau from [BKW, (3.5)]. For example, taking $m = n = 2$ again, the tableau (6.21)

arises via our bijection from the following oriented stretched cap diagram

$$(6.22)$$
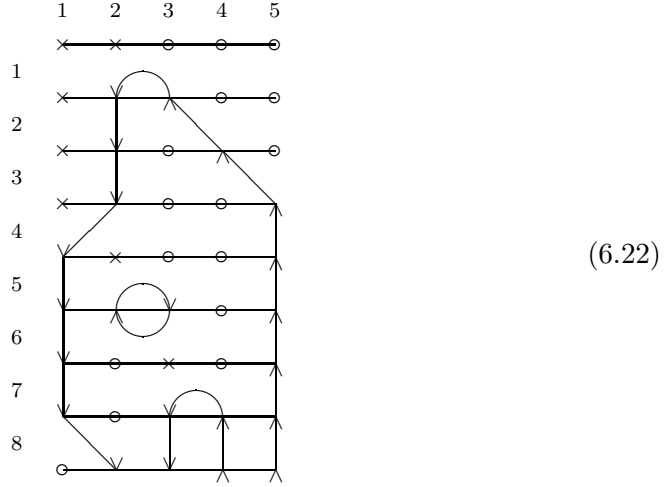
which is of degree 1.

In a similar way, there is a bijection from stretched oriented cup diagrams of type α to standard bitableaux of type α. Putting the two bijections together, we obtain a bijection from the set of stretched oriented circle diagrams of type α to the set of all pairs (S, T) of standard bitableaux of type α such that S and T have the same underlying shape.

Stretched circle diagram bases for T_α^Λ and its dual. As Λ is a block of defect zero, the algebra K_Λ is trivial, so the irreducible module $L(\iota)$ is just a copy of the regular module K_Λ. Hence we can rewrite the definition (6.1) as

$$T_\alpha^\Lambda := \bigoplus_{\boldsymbol{i} \in I^\alpha} F_{\boldsymbol{i}} K_\Lambda \langle \operatorname{def}(\Lambda - \alpha) \rangle.$$

Also introduce a dual object

$$\widehat{T}_\alpha^\Lambda := \bigoplus_{\boldsymbol{i} \in I^\alpha} E_{\boldsymbol{i}^*} K_\alpha^\Lambda$$

where for $\boldsymbol{i} = (i_1, \ldots, i_d)$ we set $\boldsymbol{i}^* := (i_d, \ldots, i_1)$. Note T_α^Λ is a graded left K_α^Λ-module, and $\widehat{T}_\alpha^\Lambda$ is a graded right K_α^Λ-module (arising from the right regular action of K_α^Λ on itself).

Using Lemma 5.1, noting also that $\operatorname{def}(\Lambda - \alpha) = \operatorname{caps}(\boldsymbol{t}) - \operatorname{cups}(\boldsymbol{t})$, we get a canonical isomorphism

$$T_\alpha^\Lambda \xrightarrow{\sim} \bigoplus K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}} \langle -\operatorname{cups}(\boldsymbol{t}) \rangle \tag{6.23}$$

of graded left K_α^Λ-modules, where the direct sum is over all proper Λ-admissible sequences $\boldsymbol{i} \in I^\alpha$, and $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ denote the associated block sequence and composite matching, respectively. Similarly, using $\operatorname{caps}(\boldsymbol{t}^*) = \operatorname{cups}(\boldsymbol{t})$, there is a canonical isomorphism

$$\widehat{T}_\alpha^\Lambda \xrightarrow{\sim} \bigoplus K_{\boldsymbol{\Gamma}^*}^{\boldsymbol{t}^*} \langle -\operatorname{cups}(\boldsymbol{t}) \rangle, \tag{6.24}$$

where the direct sum is over the same collection of tuples as in (6.23). Pulling the diagram bases from (3.6) back through these canonical isomorphisms, we deduce that T_α^Λ and $\widehat{T}_\alpha^\Lambda$ have distinguished homogeneous bases denoted

$$\left\{ (a\, \boldsymbol{t}[\boldsymbol{\gamma}]\,|\ \middle|\ \begin{array}{l} \text{for all oriented upper-stretched} \\ \text{circle diagrams } a\, \boldsymbol{t}[\boldsymbol{\gamma}] \text{ of type } \alpha \end{array} \right\}, \qquad (6.25)$$

$$\left\{ |\, \boldsymbol{u}^*[\boldsymbol{\delta}^*]\, b)\ \middle|\ \begin{array}{l} \text{for all oriented lower-stretched} \\ \text{circle diagrams } \boldsymbol{u}^*[\boldsymbol{\delta}^*]\, b \text{ of type } \alpha \end{array} \right\}. \qquad (6.26)$$

The bases (6.25)–(6.26) are quite convenient for computations. For example, the natural right action of the generators y_r and ψ_r of R_α^Λ on a basis element $(a\, \boldsymbol{t}[\boldsymbol{\gamma}]\,|$ of T_α^Λ can be computed quickly by making the appropriate circle move, crossing move or height move to the diagram $a\, \boldsymbol{t}[\boldsymbol{\gamma}]$ as in the definitions (5.11) and (5.12) (remembering also the signs from Lemmas 5.4 and 5.5). The left K_α^Λ-module structure on T_α^Λ can be computed in terms of the basis as follows. Take $(a\lambda b) \in K_\alpha^\Lambda$ and $(c\, \boldsymbol{t}[\boldsymbol{\gamma}]\,| \in T_\alpha^\Lambda$. If $b^* \neq c$, we have that $(a\lambda b)(c\, \boldsymbol{t}[\boldsymbol{\gamma}]\,| = 0$. If $b^* = c$, proceed as follows:

▶ draw $a\lambda b$ underneath $c\, \boldsymbol{t}[\boldsymbol{\gamma}]$;
▶ iterate the generalised surgery procedure from [BS1, §6] to the symmetric section of the diagram containing b under c to eliminate all cap-cup pairs;
▶ finally remove the bottom number line to obtain a linear combination of basis vectors of T_α^Λ.

The right K_α^Λ-module structure on $\widehat{T}_\alpha^\Lambda$ can be computed similarly.

There is a precise sense in which $\widehat{T}_\alpha^\Lambda$ is dual to T_α^Λ. To explain this, define another multiplication map

$$\varphi : T_\alpha^\Lambda \otimes \widehat{T}_\alpha^\Lambda \to K_\alpha^\Lambda \qquad (6.27)$$

as follows. Take basis vectors $(a\, \boldsymbol{t}[\boldsymbol{\gamma}]\,| \in T_\alpha^\Lambda$ and $|\, \boldsymbol{u}^*[\boldsymbol{\delta}^*]\, b) \in \widehat{T}_\alpha^\Lambda$. If $\boldsymbol{t} = \boldsymbol{u}$ and all mirror image pairs of internal circles in $\boldsymbol{t}[\boldsymbol{\gamma}]$ and $\boldsymbol{u}^*[\boldsymbol{\delta}^*]$ are oriented so that one is clockwise, the other anti-clockwise, then we define

$$\varphi((a\, \boldsymbol{t}[\boldsymbol{\gamma}]\,| \otimes |\, \boldsymbol{u}^*[\boldsymbol{\delta}^*]\, b)) := (a\, \gamma_d\, c)(c^*\delta_d\, b) \in K_\alpha^\Lambda,$$

where c is the upper reduction of $\boldsymbol{t} = \boldsymbol{u}$. Otherwise, we set

$$\varphi((a\, \boldsymbol{t}[\boldsymbol{\gamma}]\,| \otimes |\, \boldsymbol{u}^*[\boldsymbol{\delta}^*]\, b)) := 0.$$

This definition is a slight variation on the definition of the adjunction morphism φ from [BS2, (4.3)]. The following lemma establishes in particular that φ is a degree zero homomorphism of graded $(K_\alpha^\Lambda, K_\alpha^\Lambda)$-bimodules.

Lemma 6.6. *There is a degree zero isomorphism of graded right K_α^Λ-modules*

$$\widehat{T}_\alpha^\Lambda \xrightarrow{\sim} \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda)$$

mapping $y \in \widehat{T}_\alpha^\Lambda$ to the homomorphism $T_\alpha^\Lambda \to K_\alpha^\Lambda$, $x \mapsto \varphi(x \otimes y)$.

Proof. It suffices to show that the restriction of the given map is a degree zero K_α^Λ-module isomorphism

$$K_{\boldsymbol{\Gamma}^*}^{\boldsymbol{t}^*}\langle -\operatorname{cups}(\boldsymbol{t})\rangle \xrightarrow{\sim} \mathrm{Hom}_{K_\alpha^\Lambda}(K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}\langle -\operatorname{cups}(\boldsymbol{t})\rangle, K_\alpha^\Lambda)$$

for $\mathbf{\Gamma}$ and $\boldsymbol{t}$ associated to a fixed proper Λ-admissible sequence $\boldsymbol{i} \in I^\alpha$. Let s be the reduction of $\boldsymbol{t}$, and note that $\mathrm{caps}(s) - \mathrm{cups}(s) = \mathrm{caps}(\boldsymbol{t}) - \mathrm{cups}(\boldsymbol{t})$. Using (3.9) and [BS2, Theorem 4.7], we have canonical isomorphisms

$$
\begin{aligned}
K_{\mathbf{\Gamma}^*}^{\boldsymbol{t}^*}\langle -\mathrm{cups}(\boldsymbol{t})\rangle &\cong K_{\Lambda(\Lambda-\alpha)}^{s^*}\langle -\mathrm{cups}(s)\rangle \otimes R^{\mathrm{circles}(\boldsymbol{t})} \\
&\cong \mathrm{Hom}_{K_\alpha^\Lambda}(K_{(\Lambda-\alpha)\Lambda}^s, K_\alpha^\Lambda)\langle\mathrm{cups}(s)\rangle \otimes R^{\mathrm{circles}(\boldsymbol{t})} \\
&\cong \mathrm{Hom}_{K_\alpha^\Lambda}(K_{(\Lambda-\alpha)\Lambda}^s\langle -\mathrm{cups}(s)\rangle \otimes R^{\mathrm{circles}(\boldsymbol{t})}, K_\alpha^\Lambda) \\
&\cong \mathrm{Hom}_{K_\alpha^\Lambda}(K_{\mathbf{\Gamma}}^{\boldsymbol{t}}\langle -\mathrm{cups}(\boldsymbol{t})\rangle, K_\alpha^\Lambda).
\end{aligned}
$$

The composite of all of these isomorphisms is exactly the map defined in the statement of the lemma. $\qquad\square$

Note via the isomorphism from Lemma 6.6 that $\widehat{T}_\alpha^\Lambda$ becomes a graded left E_α^Λ-module, because $\mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda)$ is naturally such. Applying the homomorphism ω from Theorem 6.3, we deduce $\widehat{T}_\alpha^\Lambda$ is a graded left R_α^Λ-module too.

Remark 6.7. Let e be the truncation functor from (6.5). The isomorphism in Lemma 6.6 restricts to an isomorphism

$$
\widehat{T}_\alpha^\Lambda e \xrightarrow{\sim} \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda)e = \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda e).
$$

Since the truncation functor is fully faithful on projectives, it defines an isomorphism $\mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda e) \cong \mathrm{Hom}_{H_\alpha^\Lambda}(eT_\alpha^\Lambda, H_\alpha^\Lambda)$. In this way, we deduce also the existence of an isomorphism

$$
\widehat{T}_\alpha^\Lambda e \xrightarrow{\sim} \mathrm{Hom}_{H_\alpha^\Lambda}(eT_\alpha^\Lambda, H_\alpha^\Lambda)
$$

as graded $(E_\alpha^\Lambda, H_\alpha^\Lambda)$-bimodules.

The stretched circle diagram basis for E_α^Λ. Now we turn our attention to the endomorphism algebra E_α^Λ from (6.2), still assuming we are given $\alpha \in Q_+$ of height d. The following theorem gives an explicit description of this algebra.

Theorem 6.8. *There is a graded vector space isomorphism*

$$
\widehat{T}_\alpha^\Lambda \otimes_{K_\alpha^\Lambda} T_\alpha^\Lambda \xrightarrow{\sim} E_\alpha^\Lambda
$$

mapping $y \otimes z$ to the endomorphism $T_\alpha^\Lambda \to T_\alpha^\Lambda$, $x \mapsto \varphi(x \otimes y)z$. Using this isomorphism, the multiplication on E_α^Λ lifts to define a graded multiplication on $\widehat{T}_\alpha^\Lambda \otimes_{K_\alpha^\Lambda} T_\alpha^\Lambda$ with $(u \otimes x)(y \otimes z) = u\varphi(x \otimes y) \otimes z = u \otimes \varphi(x \otimes y)z$.

Proof. The map in the statement of the theorem is the composite of the canonical isomorphisms

$$
\widehat{T}_\alpha^\Lambda \otimes_{K_\alpha^\Lambda} T_\alpha^\Lambda \xrightarrow{\sim} \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda) \otimes_{K_\alpha^\Lambda} T_\alpha^\Lambda \xrightarrow{\sim} \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, T_\alpha^\Lambda).
$$

The first of these arises from Lemma 6.6, and the second is the map sending $f \otimes t$ to the map $x \mapsto f(x)t$ (which is an isomorphism because T_α^Λ is a projective left K_α^Λ-module by [BS2, Corollary 4.4]). $\qquad\square$

Finally, we need one more canonical isomorphism arising from the bimodule multiplication from (3.7):

$$\widehat{T}_\alpha^\Lambda \otimes_{K_\alpha^\Lambda} T_\alpha^\Lambda \xrightarrow{\sim} \bigoplus K_{\boldsymbol{\Delta}^*\wr\boldsymbol{\Gamma}}^{\boldsymbol{u}^*\boldsymbol{t}}\langle -\operatorname{cups}(\boldsymbol{u}) - \operatorname{cups}(\boldsymbol{t})\rangle,$$

where the direct sum is over all proper Λ-admissible sequences $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$, and $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ (resp. $\boldsymbol{\Delta}$ and $\boldsymbol{u}$) denote the block sequence and composite matching associated to $\boldsymbol{i}$ (resp. $\boldsymbol{j}$). Pulling back the diagram basis for each $K_{\boldsymbol{\Delta}^*\wr\boldsymbol{\Gamma}}^{\boldsymbol{u}^*\boldsymbol{t}}$ from (3.6) through this isomorphism, then pushing forward from there to E_α^Λ using the isomorphism from Theorem 6.8, we deduce that E_α^Λ has a distinguished homogeneous basis

$$\left\{\, |\,\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]\,|\ \middle|\ \begin{array}{l}\text{for all oriented stretched circle}\\ \text{diagrams } \boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}] \text{ of type } \alpha.\end{array}\right\} \qquad (6.28)$$

The degree of the vector $|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]\,|$ is given by the formula (6.20).

Algorithms for computing with stretched circle diagrams. We next spell out how to compute the various sorts of products in terms of the diagram basis (6.28). These explicit descriptions all follow from Theorem 6.8 and the appropriate associativity.

First of all, given two basis vectors $|\,\boldsymbol{s}^*[\boldsymbol{\tau}^*] \wr \boldsymbol{r}[\boldsymbol{\sigma}]\,|$ and $|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]\,|$ in E_α^Λ, their product $|\,\boldsymbol{s}^*[\boldsymbol{\tau}^*] \wr \boldsymbol{r}[\boldsymbol{\sigma}]\,| \cdot |\,\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]\,|$ is zero unless $\boldsymbol{r} = \boldsymbol{u}$ and all mirror image pairs of internal circles in $\boldsymbol{r}[\boldsymbol{\sigma}]$ and $\boldsymbol{u}^*[\boldsymbol{\delta}^*]$ are oriented so that one is clockwise, the other anti-clockwise. Assuming these conditions hold, we let b be the upper reduction of $\boldsymbol{r} = \boldsymbol{u}$ and proceed as follows:

- ▸ draw $\boldsymbol{s}^*[\boldsymbol{\tau}^*]\, b$ under $b^*\, \boldsymbol{t}[\boldsymbol{\gamma}]$;
- ▸ iterate the generalised surgery procedure from [BS1, §6] to eliminate all cap-cup pairs in the symmetric middle section of the diagram;
- ▸ collapse the middle section to obtained the desired linear combination of basis vectors of E_α^Λ.

The right action of E_α^Λ on T_α^Λ and the left action of E_α^Λ on $\widehat{T}_\alpha^\Lambda$ are both computed on basis vectors by analogous procedures.

Let us describe explicitly the image of the idempotent $e(\boldsymbol{i}) \in R_\alpha^\Lambda$ under the homomorphism ω from Theorem 6.3. If $\boldsymbol{i}$ is not Λ-admissible then $\omega(e(\boldsymbol{i})) = 0$. If $\boldsymbol{i}$ is Λ-admissible, we let $\boldsymbol{\Gamma}$ and $\boldsymbol{t}$ be the associated block sequence and composite matching. Then

$$\omega(e(\boldsymbol{i})) = \sum |\,\boldsymbol{t}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]\,| \qquad (6.29)$$

where the sum is over all $\boldsymbol{\gamma} = \gamma_d \cdots \gamma_0$ and $\boldsymbol{\delta} = \delta_d \cdots \delta_0$ such that

- ▸ $\gamma_r, \delta_r \in \Gamma_r$ for all $r = 0, \ldots, d$, and $\gamma_d = \delta_d$;
- ▸ $\boldsymbol{t}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ is an oriented stretched circle diagram;
- ▸ every boundary circle in $\boldsymbol{t}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ is anti-clockwise;
- ▸ every mirror image pair of upper and lower circles in $\boldsymbol{t}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ is oriented so one is clockwise, the other anti-clockwise.

Note in particular that this element is homogeneous of degree zero. The set $\{\omega(e(\boldsymbol{i})) \,|\, \boldsymbol{i} \in I^\alpha\}$ is a system of mutually orthogonal idempotents in E_α^Λ whose sum is the identity endomorphism.

Via the homomorphism from Theorem 6.3 we get right and left actions of the generators y_r and ψ_r of R_α^Λ on E_α^Λ. These can be computed directly on the basis vectors of E_α^Λ by mimicking the definitions of (5.11) and (5.12) (and incorporating a sign as in Lemmas 5.4 and 5.5). So multiplying a basis vector $|\, u^*[\delta^*] \wr t[\gamma]\,|$ on the right (resp. left) by y_r involves making a signed positive circle move at the rth level of t (resp. u). Similarly multiplying a basis vector $|\, u^*[\delta^*] \wr t[\gamma]\,|$ on the right (resp. left) by ψ_r involves making a signed negative circle move, crossing move or height move at the rth and $(r+1)$th levels of t (resp. u).

The image of $y_r e(i) = e(i) y_r$ under the homomorphism ω can be obtained explicitly in terms of the diagram basis by starting from $\omega(e(i))$ as in (6.29) then acting by y_r, either on the left or the right. Similarly the image of $\psi_r e(i) = e(s_r \cdot i)\psi_r$ can be obtained either by starting from $\omega(e(i))$ and acting on the left by ψ_r, or by starting with $e(s_r \cdot i)$ and acting on the right by ψ_r.

Cellularity of E_α^Λ. There are involutory graded algebra antiautomorphisms

$$* : R_\alpha^\Lambda \to R_\alpha^\Lambda, \qquad * : E_\alpha^\Lambda \to E_\alpha^\Lambda. \tag{6.30}$$

On R_α^Λ, $*$ is the antiautomorphism with $e(i)^* = e(i), y_r^* = y_r$ and $\psi_r^* = \psi_r$ for each i and r. On E_α^Λ, $*$ is the linear map sending $|\, u^*[\delta^*] \wr t[\gamma]\,|$ to its mirror image $|\, t^*[\gamma^*] \wr u[\delta]\,|$. Recalling the homomorphism ω from Theorem 6.3, the following diagram commutes:

$$\begin{array}{ccc} R_\alpha^\Lambda & \xrightarrow{\;*\;} & R_\alpha^\Lambda \\ \omega \downarrow & & \downarrow \omega \\ E_\alpha^\Lambda & \xrightarrow{\;*\;} & E_\alpha^\Lambda \end{array} \tag{6.31}$$

Theorem 6.9. *Assume that $\Gamma := \Lambda - \alpha \in P(m,n;I)$. Then the algebra E_α^Λ is a cellular algebra in the sense of Graham and Lehrer [GL] with cell datum $(\Gamma, M, C, *)$ where*

  (i) *Γ is viewed as a poset with respect to the Bruhat order;*
  (ii) *$M(\lambda)$ for each $\lambda \in \Gamma$ denotes the set of all oriented upper-stretched cap diagrams $t[\gamma]$ of type α such that $\gamma_d = \lambda$;*
  (iii) *C is defined by setting $C^\lambda_{u[\delta], t[\gamma]} := |\, u^*[\delta^*] \wr t[\gamma]\,|$ for each $\lambda \in \Gamma$ and $u[\delta], t[\gamma] \in M(\lambda)$;*
  (iv) *$* : E_\alpha^\Lambda \to E_\alpha^\Lambda$ is the algebra antiautomorphism from (6.30).*

(For our conventions regarding cellular algebras, see the paragraph after the statement of [BS1, Corollary 3.3].)

Proof. This follows by similar arguments to [BS1, Corollary 3.3].   □

Remark 6.10. (1) In fact our cellular basis makes E_α^Λ into *graded* cellular algebra in the obvious sense, with the degree of $t[\gamma] \in M(\lambda)$ being defined by (6.18). We show in Corollary 8.6 below that $E_\alpha^\Lambda \cong R_\alpha^\Lambda$, so that E_α^Λ is a particular example of a cyclotomic Khovanov-Lauda-Rouquier algebra of type A. The latter algebras were conjectured to be graded cellular algebras in [BKW, Remark 4.12], with a graded cellular basis parametrized by pairs of bitableaux of the

same shape and type α. Combined with the observations made in Remark 6.5, Theorem 6.9 verifies a particular instance of this conjecture.

(2) By arguments completely analogous to [BS1, Theorem 6.3], we get also that E_α^Λ is a *graded symmetric algebra*: it possesses a homogeneous symmetrizing form $\tau : E_\alpha^\Lambda \to \mathbb{C}$ of degree $-2\operatorname{def}(\Lambda - \alpha)$. Explicitly, τ is the linear map defined on a basis vector $|\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]|$ by declaring that it is zero unless $\boldsymbol{u} = \boldsymbol{t}$, all boundary circles are clockwise, and all mirror image pairs of internal circles are oppositely oriented, in which case it is 1.

(3) For $\Gamma := \Lambda - \alpha \in P(m, n; I)$ as in Theorem 6.9, we showed already in Theorem 6.2 that E_α^Λ is Morita equivalent to the generalised Khovanov algebra H_Γ from [BS1, (6.2)]. The latter algebra was already shown to be both cellular and symmetric in [BS1, Theorems 6.2–6.3].

Addition of a simple root. Suppose we are given $\alpha \in Q_+$ of height d as before, and also $i \in I$. For $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^\alpha$, we write $\boldsymbol{i} + i$ for the tuple $(i_1, \ldots, i_d, i) \in I^{\alpha + \alpha_i}$. By the relations, there is a (non-unital) algebra homomorphism

$$\theta_i : R_\alpha^\Lambda \to R_{\alpha + \alpha_i}^\Lambda \tag{6.32}$$

such that $\theta_i(e(\boldsymbol{i})) = e(\boldsymbol{i} + i)$, $\theta_i(y_r e(\boldsymbol{i})) = y_r e(\boldsymbol{i} + i)$ and $\theta_i(\psi_r e(\boldsymbol{i})) = \psi_r e(\boldsymbol{i} + i)$. The image of the identity element of R_α^Λ under this homomorphism is the idempotent

$$\sum_{\substack{\boldsymbol{i} \in I^{\alpha + \alpha_i} \\ i_{d+1} = i}} e(\boldsymbol{i}) \in R_{\alpha + \alpha_i}^\Lambda. \tag{6.33}$$

We want to construct a corresponding homomorphism at the level of the endomorphism algebra E_α^Λ, which will be a key tool for inductive arguments.

We begin by defining a degree-preserving linear map

$$\theta_i : E_\alpha^\Lambda \to E_{\alpha + \alpha_i}^\Lambda. \tag{6.34}$$

Take a basis vector $|\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]| \in E_\alpha^\Lambda$ as in (6.28). If i is not $(\Lambda - \alpha)$-admissible then we set $\theta_i(|\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]|) := 0$. If i is $(\Lambda - \alpha)$-admissible, there are various cases according to whether $s := t_i(\Lambda - \alpha)$ is a cup, a cap, a right-shift or a left-shift. In all cases the basic idea is to replace the matching $\boldsymbol{u}^* \boldsymbol{t}$ in the diagram by $\boldsymbol{u}^* s^* s \boldsymbol{t}$, i.e. we want to insert two extra levels at the boundary line:

$$\tag{6.35}$$

Here, as in (3.11), we display only the strip between the ith and $(i + 1)$th vertices, all other strips being the "identity". Somewhat informally, we proceed as follows in the various cases:

Case one: s is a right-shift or a left-shift. We compute $\theta_i(|\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]|)$ simply by inserting the matching $s^* s$ into the middle of the diagram $\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ then

orienting the new boundary line in the only sensible way, leaving all other orientations unchanged.

Case two: s is a cap. This time, s^*s contains a small circle. Again we simply insert s^*s into the diagram, orienting the new boundary line so that the new small circle is anti-clockwise and leaving all other orientations unchanged.

Case three: s is a cup. If s is a cup, then inserting s^*s into the diagram either splits one component into two or joins two components into one. In this case we define $\theta_i(\,|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|\,)$ to be a sum of zero, one or two basis vectors of $E^\Lambda_{\alpha+\alpha_i}$, with the new component(s) oriented following exactly the same rules as in the generalised surgery procedure from [BS1, §6].

Put more formally, this means in all cases that

$$\theta_i(\,|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|\,) = \sum\,|\,\boldsymbol{u}^*[\hat{\boldsymbol{\delta}}^*]s^*\lambda s\boldsymbol{t}[\hat{\boldsymbol{\gamma}}]\,| \tag{6.36}$$

where the sum is over all $\hat{\boldsymbol{\delta}} = \hat{\delta}_d\cdots\hat{\delta}_0, \hat{\boldsymbol{\gamma}} = \hat{\gamma}_d\cdots\hat{\gamma}_0$ and $\lambda\in\Lambda-\alpha-\alpha_i$ such that

- ▸ each $\hat{\gamma}_r$ (resp. $\hat{\delta}_r$) lies in the same block as γ_r (resp. δ_r);
- ▸ $\boldsymbol{u}^*[\hat{\boldsymbol{\delta}}^*]s^*\lambda s\boldsymbol{t}[\hat{\boldsymbol{\gamma}}]$ is an oriented stretched circle diagram of the same degree as $\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]$;
- ▸ the components of $\boldsymbol{u}^*[\hat{\boldsymbol{\delta}}^*]s^*\lambda s\boldsymbol{t}[\hat{\boldsymbol{\gamma}}]$ that do not pass through the ith or $(i+1)$th vertices at the top or bottom of s or s^* are oriented in the same way as the corresponding components of $\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]$.

For example, writing e for the identity element of the one-dimensional algebra K_Λ, we have according to this definition that

$$\theta_{i_d}\circ\cdots\circ\theta_{i_1}(e) = \omega(e(\boldsymbol{i})) \tag{6.37}$$

as computed by (6.29), for any $\boldsymbol{i}\in I^\alpha$.

Theorem 6.11. *The linear map θ_i just defined is equal to the composition of the following maps:*

$$E^\Lambda_\alpha = \mathrm{End}_{K^\Lambda_\alpha}\left(\bigoplus_{\boldsymbol{i}\in I^\alpha} F_{\boldsymbol{i}}L(\iota)\right)^{\mathrm{op}} \qquad \to \mathrm{End}_{K^\Lambda_{\alpha+\alpha_i}}\left(\bigoplus_{\boldsymbol{i}\in I^\alpha} F_{\boldsymbol{i}+i}L(\iota)\right)^{\mathrm{op}}$$

$$= \mathrm{End}_{K^\Lambda_{\alpha+\alpha_i}}\left(\bigoplus_{\substack{\boldsymbol{i}\in I^{\alpha+\alpha_i}\\ \alpha_{d+1}=i}} F_{\boldsymbol{i}}L(\iota)\right)^{\mathrm{op}} \hookrightarrow \mathrm{End}_{K^\Lambda_{\alpha+\alpha_i}}\left(\bigoplus_{\boldsymbol{i}\in I^{\alpha+\alpha_i}} F_{\boldsymbol{i}}L(\iota)\right)^{\mathrm{op}} = E^\Lambda_{\alpha+\alpha_i},$$

where the first map is defined by applying the functor F_i and the second map is the natural inclusion. In particular, θ_i is a (non-unital) graded algebra homomorphism.

Proof. We may assume that i is $(\Lambda-\alpha)$-admissible and set $s := t_i(\Lambda-\alpha)$. Let $\bar{\theta}_i$ denote the composition of the maps from the statement of the theorem. Take any basis vector $|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|\in R^\Lambda_\alpha$. We want to show that

$$\theta_i(\,|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|\,) = \bar{\theta}_i(\,|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|\,)$$

as endomorphisms of $T^\Lambda_{\alpha+\alpha_i}$. It is clear that both sides are zero on all basis vectors of $T^\Lambda_{\alpha+\alpha_i}$ that are *not* of the special form $(a\mu s\boldsymbol{u}[\boldsymbol{\epsilon}]\,|$, so take such a

special basis vector. Then we need to show that

$$(a\mu s\boldsymbol{u}[\boldsymbol{\epsilon}]\,|\cdot\theta_i(|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|) = (a\mu s\boldsymbol{u}[\boldsymbol{\epsilon}]\,|\cdot\bar{\theta}_i(|\,\boldsymbol{u}^*[\boldsymbol{\delta}^*]\wr\boldsymbol{t}[\boldsymbol{\gamma}]\,|). \qquad (6.38)$$

The vector on the right hand side of (6.38) can be computed explicitly as follows. It is zero unless all mirror image pairs of internal circles in $\boldsymbol{u}[\boldsymbol{\epsilon}]$ and $\boldsymbol{u}^*[\boldsymbol{\delta}^*]$ are oppositely oriented. In that case, the product is obtained by applying a sequence of generalised surgery procedures to remove all cap-cup pairs from the bb^*-part of the diagram

$$a\mu s\varepsilon_d bb^*\boldsymbol{t}[\boldsymbol{\gamma}], \qquad (6.39)$$

where b is the upper reduction of $\boldsymbol{u}$. On the other hand, the vector on the left hand side of (6.38) is equal to

$$\sum(a\mu s\boldsymbol{u}[\boldsymbol{\epsilon}]\,|\cdot|\,[\boldsymbol{u}^*[\hat{\boldsymbol{\delta}}^*]s^*\lambda s\boldsymbol{t}[\hat{\boldsymbol{\gamma}}]\,| \qquad (6.40)$$

summing over the same tuples as in (6.36). Recall moreover to compute each product in (6.40), the product is zero unless mirror image pairs of internal circles in $s\boldsymbol{u}[\boldsymbol{\epsilon}]$ and $\boldsymbol{u}^*[\hat{\boldsymbol{\delta}}^*]s^*$ are oppositely oriented. In that case, the product is obtained by applying a sequence of generalised surgery procedure to remove all cap-cup pairs from the cc^*-part of the diagram

$$a\mu cc^*\lambda s\boldsymbol{t}[\hat{\boldsymbol{\gamma}}], \qquad (6.41)$$

where c is the upper reduction of $s\boldsymbol{u}$ (= the upper reduction of sb).

Comparing (6.41) and (6.39), it follows easily that (6.38) holds if s is a right-shift or a left-shift: both procedures amount to performing exactly the same sequence of surgery procedures. The case when s is a cap is not much harder: this time the computation of (6.41) involves one more surgery procedure than (6.39), but this extra surgery procedure corresponds to "multiplication by an anti-clockwise circle" which is the identity map. Finally we are left with the case when s is a cup, which is more difficult to explain. Recall from [BS1, §3] that our surgery procedures are associated to the TQFT defined by the commutative Frobenius algebra $R = \mathbb{C}[x]/(x^2)$, which has multiplication m, comultiplication δ and counit ε. We split into three sub-cases.

Sub-case one: b^*s^*st has one less component than b^*t. We only need to consider components that interact with s or s^*, as the surgery procedures on all other components clearly correspond in (6.39) and (6.41). We will assume moreover that the relevant components are all circles rather than propagating lines, leaving the appropriate modifications when propagating lines are involved to the reader. With these reductions, there is just one basic configuration to be considered, represented by the following diagram (in which for simplicity we have

displayed only the lower reduction d of $\boldsymbol{t}$ rather than $\boldsymbol{t}$ itself):
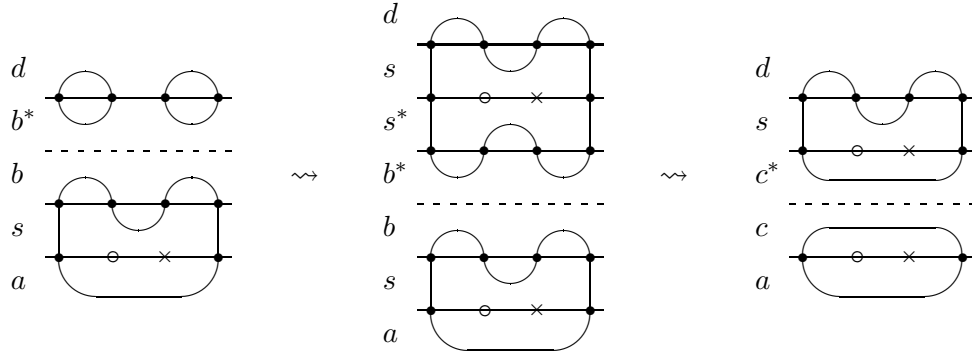
The point now is that to compute (6.39) involves two surgery procedures to the bb^*-part of the leftmost diagram. Each of these involves two circles combining into one circle, represented by the multiplication m of the Frobenius algebra R. So the computation of (6.39) is a map represented by $m \circ (m \otimes 1)$. On the other hand the passage from (6.39) to (6.40) involves applying the map $1 \otimes m$ to join the top two circles, then one surgery procedure gets performed to the cc^*-part of the rightmost diagram, which is the map m. Now we are done because $m \circ (m \otimes 1) = m \circ (1 \otimes m)$.

Sub-case two: $b^* s^* s \boldsymbol{t}$ has one more component than $b^* \boldsymbol{t}$ and sb does not have an internal circle. Making the same reductions as before, there is essentially only one basic configuration to consider, represented by the following diagram:
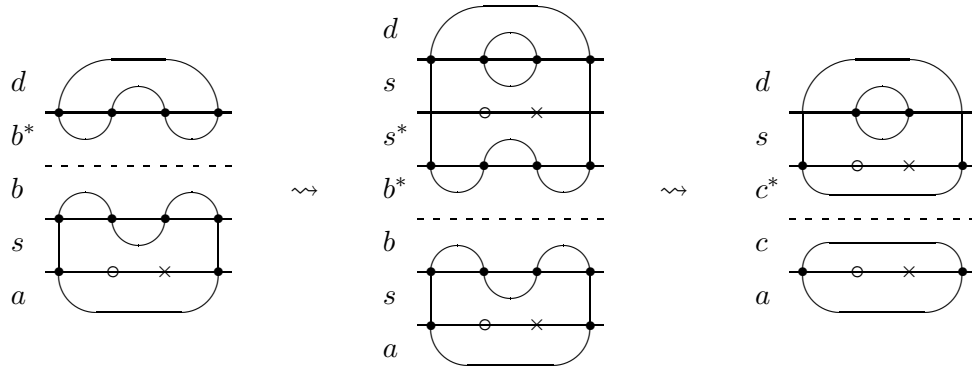
The direct computation of (6.39) from the leftmost diagram involves two surgery procedures encoded by the map $\delta \circ m$. On the other hand the passage from (6.39) to (6.41) is encoded by the map $1 \otimes \delta$, then the computation of (6.41) is encoded by $m \otimes 1$. So we are done by the identity $(m \otimes 1) \circ (1 \otimes \delta) = \delta \circ m$, which is the basic defining relation of a Frobenius algebra.

Sub-case three: b^*s^*st has one more component than b^*t and sb has an internal circle. The basic configuration to be considered is as follows:
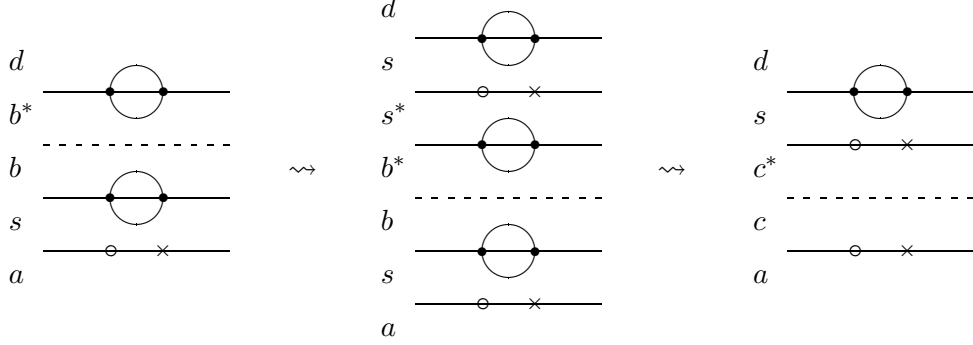
Here, the direct computation of (6.39) corresponds to the map m. On the other hand passing from (6.39) to (6.40) corresponds to the map $1 \otimes \delta$, then passing from (6.40) to (6.41) involves removing a mirror image pair of internal circles, which is encoded by the map $(\varepsilon \circ m) \otimes 1$, where $(\varepsilon \circ m)$ is the map $R \otimes R \to \mathbb{C}, 1 \otimes 1 \mapsto 0, 1 \otimes x \mapsto 1, x \otimes 1 \mapsto 1, x \otimes x \mapsto 0$. Again we are done since $((\varepsilon \circ m) \bar{\otimes} 1) \circ (1 \otimes \delta) = m$. $\qquad\square$

Corollary 6.12. *The following diagram commutes:*

$$
\begin{array}{ccc}
R_\alpha^\Lambda & \xrightarrow{\theta_i} & R_{\alpha+\alpha_i}^\Lambda \\
\omega \downarrow & & \downarrow \omega \\
E_\alpha^\Lambda & \xrightarrow{\theta_i} & E_{\alpha+\alpha_i}^\Lambda.
\end{array}
$$

Proof. In view of the theorem, it suffices to check the commutativity on the generators of R_α^Λ. For $e(\boldsymbol{i})$ this follows from (6.37). It then suffices to check that $\theta_i(y_r\omega(e(\boldsymbol{i}))) = y_r\omega(e(\boldsymbol{i}+i))$ and $\theta_i(\psi_r\omega(e(\boldsymbol{i}))) = \psi_r\omega(e(\boldsymbol{i}+i))$. This is routine for circle moves or height moves. Finally if $\overline{\psi}_r(\boldsymbol{i})_\Lambda$ is a crossing move there are various cases to be considered, all of which reduce to the associativity/coassociativity of the multiplication/comultiplication in R or to the identity $(m \otimes 1) \circ (1 \otimes \delta) = \delta \circ m$ like in the proof of Theorem 6.11. $\qquad\square$

7. SURJECTIVITY

In this section we prove that the homomorphism $\omega : R_\alpha^\Lambda \to E_\alpha^\Lambda$ from Theorem 6.3 is surjective.

Raising and lowering moves. Assume in this subsection that we are given a block $\Gamma \in P(m, n; I)$ and a proper Γ-admissible sequence $\boldsymbol{i} = (i_1, \ldots, i_d)$, and let $\boldsymbol{\Gamma} = \Gamma_d \cdots \Gamma_0$ and $\boldsymbol{t} = t_d \cdots t_1$ be the associated block sequence and composite matching. By composing sequences of the local circle moves, crossing moves and height moves from (5.11) and (5.12), we can define various "global" bimodule homomorphisms $K_{\boldsymbol{\Gamma}}^{\boldsymbol{t}}\langle -\operatorname{caps}(\boldsymbol{t})\rangle \to K_{\boldsymbol{\Delta}}^{\boldsymbol{u}}\langle -\operatorname{caps}(\boldsymbol{u})\rangle$ for other block sequences and composite matchings $\boldsymbol{\Delta}$ and $\boldsymbol{u}$. The following lemmas describe some useful raising and lowering moves which can be obtained in this way. The proofs of all of these are straightforward consequences of Lemma 5.3. The reader may find

Figure 5.1 helpful when trying to visualise the statements; for example consider applying Lemma 7.3 to raise the left-shift at level 14 of this diagram.

Lemma 7.1 (Raising caps). *Let C be a component of $\boldsymbol{t}$ and $1 \leq q \leq d$ such that C has a cap at level q. Exactly one of the following two statements holds:*

(i) *$|i_p - i_q| > 1$ for all $1 \leq p < q$;*

(ii) *there exists $1 \leq p < q$ such that $|i_p - i_q| = 1$ and $|i_s - i_q| > 1$ for all $p < s < q$.*

In case (i), *the composition $\overline{\psi}_1 \circ \cdots \circ \overline{\psi}_{q-1}$ is a sequence of height moves raising the cap to level 1. In case* (ii), *$\overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$ is a sequence of height moves raising the cap to level $p+1$, and after that $\overline{\psi}_p$ is a crossing move of one of the following four types:*
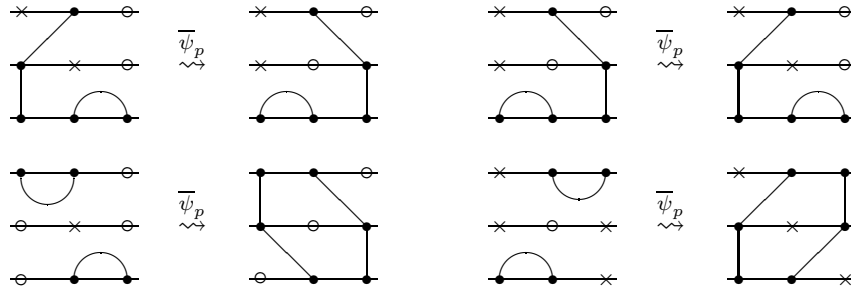
Lemma 7.2 (Lowering cups). *Let C be a component of $\boldsymbol{t}$ and $1 \leq p \leq d$ such that C has a cup at level p. Exactly one of the following two statements holds:*

(i) *$|i_p - i_q| > 1$ for all $p < q \leq d$;*

(ii) *there exists $p < q \leq d$ such that $|i_p - i_q| = 1$ and $|i_p - i_s| > 1$ for all $p < s < q$.*

In case (i), *the composition $\overline{\psi}_{d-1} \circ \cdots \circ \overline{\psi}_p$ is a sequence of height moves lowering the cup to level d. In case* (ii), *$\overline{\psi}_{q-2} \circ \cdots \circ \overline{\psi}_p$ is a sequence of height moves lowering the cup to level $q-1$, and after that $\overline{\psi}_{q-1}$ is a crossing move of one of the following four types:*

Lemma 7.3 (Raising shifts). *Let C be a component of $\boldsymbol{t}$ and $i \in I$ be minimal (resp. maximal) such that C passes through the ith (resp. $(i+1)$th) vertex of some number line. Suppose we are given $1 \leq q \leq d$ such that $i_q = i$ and C has a right-shift (resp. left-shift) at level q. Exactly one of the following two statements hold:*

(i) $|i_p - i_q| > 1$ for all $1 \leq p < q$;

(ii) there exists $1 \leq p < q$ such that $i_p = i_q + 1$ (resp. $i_p = i_q - 1$) and $|i_s - i_q| > 1$ for all $p < s < q$.
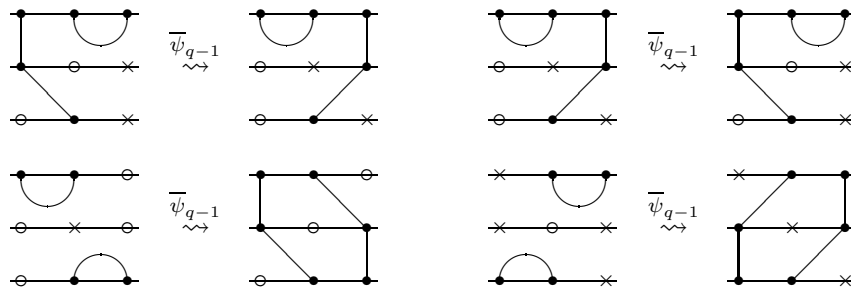
In case (i), the composition $\overline{\psi}_1 \circ \cdots \circ \overline{\psi}_{q-1}$ is a sequence of height moves raising the right-shift (resp. left-shift) to level 1. In case (ii), the composition $\overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$ is a sequence of height moves raising the right-shift (resp. left-shift) to level $p + 1$. After that, with one exception, $\overline{\psi}_p$ is a crossing move of one of the following types:
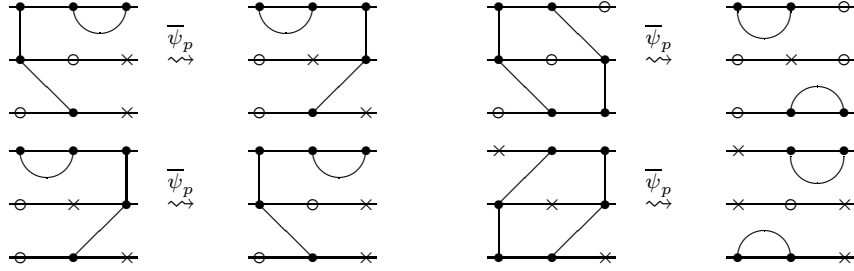
The exception is if Γ_q is a block of defect zero and there is a right-shift (resp. left-shift) at level p; this can happen only if the components of $\boldsymbol{t}$ that are non-trivial at levels p and q are two different propagating lines.

Lemma 7.4 (Lowering shifts). *Let C be a component of $\boldsymbol{t}$ and $i \in I$ be minimal (resp. maximal) such that C passes through the ith (resp. $(i+1)$th) vertex of some number line. Suppose we are given $1 \leq p \leq d$ such that $i_p = i$ and C has a left-shift (resp. right-shift) at level p. Exactly one of the following two statements hold:*

(i) $|i_p - i_q| > 1$ for all $p < q \leq d$;

(ii) there exists $p < q \leq d$ such that $i_q = i_p + 1$ (resp. $i_q = i_p - 1$) and $|i_p - i_s| > 1$ for all $p < s < q$.

In case (i), the composition $\overline{\psi}_{d-1} \circ \cdots \circ \overline{\psi}_p$ is a sequence of height moves lowering the left-shift (resp. right-shift) to level d. In case (ii), the composition $\overline{\psi}_{q-2} \circ \cdots \circ \overline{\psi}_p$ is a sequence of height moves lowering the left-shift (resp. right-shift) to level $q - 1$. After that, with one exception, $\overline{\psi}_{q-1}$ is a crossing move of one of the following types:

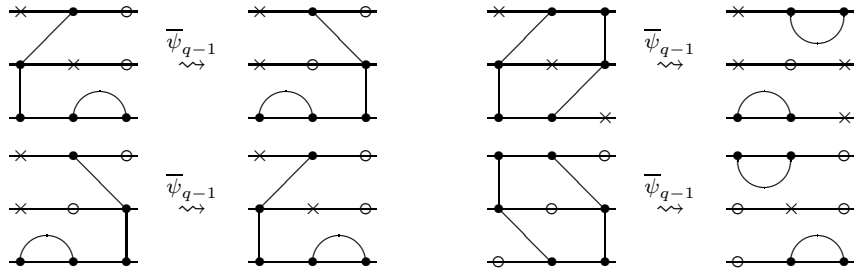The exception is if Γ_{p-1} is a block of defect zero and there is a left-shift (resp. right-shift) at level q; this can happen only if the components of $\boldsymbol{t}$ that are non-trivial at levels p and q are two different propagating lines.

Lemma 7.5 (Small circles)**.** *Let C be a component of $\boldsymbol{t}$ and $i \in I$ be minimal (resp. maximal) such that C passes through the ith (resp. $(i+1)$th) vertex of some number line. Suppose we are given $1 \le p < q \le d$ such that $i_p = i_q = i$, C has a cap at level p and a cup at level q, and $i_s \ne i$ for $p < s < q$. Then $\overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$ is a sequence of height moves transforming C into a small circle at levels p and $p+1$.*

Negative circle moves. Suppose again that we are given a block Γ and a proper Γ-admissible sequence $\boldsymbol{i} = (i_1, \ldots, i_d)$, with associated block sequence $\boldsymbol{\Gamma}$ and composite matching $\boldsymbol{t}$. Let C be an internal circle in $\boldsymbol{t}$. Define

$$z_C : K_\Lambda^{\boldsymbol{t}}\langle -\operatorname{caps}(\boldsymbol{t})\rangle \to K_\Lambda^{\boldsymbol{t}}\langle -\operatorname{caps}(\boldsymbol{t})\rangle \tag{7.1}$$

to be the endomorphism mapping $(a\, \boldsymbol{t}[\boldsymbol{\gamma}]\, b)$ to $(a\, \boldsymbol{t}[\boldsymbol{\gamma}']\, b)$ if C is clockwise in $\boldsymbol{t}[\boldsymbol{\gamma}]$ or to zero otherwise; here, $\boldsymbol{\gamma}'$ is the weight sequence obtained from $\boldsymbol{\gamma}$ on re-orienting C so it is anti-clockwise. It is obvious that this is a well-defined bimodule endomorphism. We call it a *negative circle move* ("negative" because it is of degree -2). Note if C is a small circle that is non-trivial at levels r and $r+1$ then z_C coincides with the local negative circle move $\overline{\psi}_r$ introduced already in (5.12).

Lemma 7.6. *Suppose that C is an internal circle of $\boldsymbol{t}$ containing no other nested circles. Then the negative circle move z_C can be expressed as a composition of local moves of the form $\overline{\psi}_r$ for various r.*

Proof. We proceed by induction on the height of C. In the base case, C has height 2. Suppose it has a cap at level p and a cup at level q. By Lemma 7.5, $\overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$ is a sequence of height moves transforming C into a small circle at levels p and $p+1$. After that $\overline{\psi}_p$ is an negative circle move. Then $\overline{\psi}_{q-1} \circ \cdots \circ \overline{\psi}_{p+1}$ is another sequence of height moves stretching the small circle back to C. Putting it together, we have that

$$z_C = \overline{\psi}_{q-1} \circ \cdots \circ \overline{\psi}_{p+1} \circ \overline{\psi}_p \circ \overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1},$$

hence the base of the induction is checked.

Now suppose for the induction step that C has height greater than 2. Let $i \in I$ be minimal such that C passes through the ith vertex of some number line. Let $1 \le p < d$ be minimal such that C is non-trivial at level p and $i_p = i$. Let $p < s \le d$ be minimal such that C is non-trivial at level s and $i_s = i$. Obviously C must either have a cap or a left-shift at level p and either a cup or a right-shift at level s. In view of Lemma 7.5 and the assumption that C has height greater than 2, it cannot have both a cap at level p and a cup at level s. Hence either C has a left-shift at level p or a right-shift at level s.

Suppose in this paragraph that C has a left-shift at level p. Because $i_p = i_s$ the hypotheses of Lemma 7.4(i) are not satisfied. Hence as in Lemma 7.4(ii) there exists $p < q < s$ such that $i_q = i_p + 1$ and $|i_p - i_k| > 1$ for $p < k < q$. Then

$$\delta := \overline{\psi}_{q-1} \circ \overline{\psi}_{q-2} \circ \cdots \circ \overline{\psi}_p$$

is a sequence of height moves lowering the left-shift to level $q-1$ followed by one crossing move. Because of the minimality of the choice of p and the fact

that C contains no nested circles, δ must cut C into two circles C' and C''. Both of these are of strictly smaller height than C and neither contains any nested circles. In the reverse direction,

$$m := \overline{\psi}_p \circ \cdots \circ \overline{\psi}_{q-2} \circ \overline{\psi}_{q-1}$$

joins C' and C'' back together to recover the original circle C. By induction, the negative circle moves $z_{C'}$ and $z_{C''}$ can both be written as compositions of local moves of the desired form. Now we claim that

$$z_C = m \circ z_{C'} \circ z_{C''} \circ \delta.$$

To see this, take a basis vector $(a \, \boldsymbol{t}[\boldsymbol{\lambda}] \, b) \in K_{\Lambda}^{\boldsymbol{t}} \langle -\operatorname{caps}(\boldsymbol{t}) \rangle$. If the circle C is anti-clockwise in $a \, \boldsymbol{t}[\boldsymbol{\lambda}] \, b$ then δ maps it to a sum of two basis vectors in which one of C' and C'' is anti-clockwise and the other is clockwise. Then $z_{C'} \circ z_{C''}$ maps that to zero, as required. If the circle C is clockwise then δ maps $(a \, \boldsymbol{t}[\boldsymbol{\lambda}] \, b)$ to a basis vector in which both C' and C'' are clockwise. Then $z_{C'} \circ z_{C''}$ converts both C' and C'' to anti-clockwise circles. Finally m returns us to our original basis vector but with C re-oriented so that it is anti-clockwise. This checks the claim.

Finally suppose that C has a right-shift at level s. Then we argue in a similar way to the previous paragraph, this time raising the right-shift using Lemma 7.3 instead of lowering the left-shift with Lemma 7.4. $\qquad\square$

Lemma 7.7. *For any internal circle C of $\boldsymbol{t}$, the negative circle move z_C can be written as a linear combination of compositions of local moves of the form $\overline{y}_r$ and $\overline{\psi}_r$ for various r.*

Proof. Proceed by induction on the number of nested circles contained in C. The base of the induction is Lemma 7.6. For the induction step suppose that C contains at least one nested circle. Let q be minimal such that some circle C' contained in C has a cap at level q. As C' is contained in C, C' and q do not satisfy the hypotheses of Lemma 7.1(i), hence as in Lemma 7.1(ii) there exists $1 \le p < q$ such that

$$m := \overline{\psi}_p \circ \overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$$

is a sequence of height moves raising the cap from level q to level $p+1$ followed by one crossing move. By the minimality of q, this composition m must join the circles C and C' into one circle C'' containing fewer nested internal circles than C. In the other direction,

$$\delta := \overline{\psi}_{q-1} \circ \cdots \circ \overline{\psi}_{p+1} \circ \overline{\psi}_p$$

cuts C'' to recover the two circles C and C' back again.

By induction, we already have $z_{C'}$ and $z_{C''}$ available. Take a basis vector in $K_{\Lambda}^{\boldsymbol{t}} \langle -\operatorname{caps}(\boldsymbol{t}) \rangle$. Let $1 \otimes 1, 1 \otimes x, x \otimes 1$ and $x \otimes x$ denote this vector re-oriented so that C (resp. C') is anti-clockwise or clockwise according to whether the first (resp. second) tensor is 1 or x. Similarly, represent the basis vector obtained from one of these by applying m by 1 or x according to whether C'' is anti-clockwise or clockwise. With this notation $\delta \circ z_{C''} \circ m$ is the map

$$\delta \circ z_{C''} \circ m : 1 \otimes 1 \mapsto 0, 1 \otimes x \mapsto 1 \otimes x + x \otimes 1, x \otimes 1 \mapsto 1 \otimes x + x \otimes 1, x \otimes x \mapsto 0.$$

Setting $y_{C'} := \overline{y}_q$ (remembering it is the circle C' that is non-trivial at level q), we also have

$$y_{C'} : 1 \otimes 1 \mapsto 1 \otimes x, x \otimes 1 \mapsto x \otimes x, 1 \otimes x \mapsto 0, x \otimes x \mapsto 0,$$
$$z_{C'} : 1 \otimes 1 \mapsto 0, x \otimes 1 \mapsto 0, 1 \otimes x \mapsto 1 \otimes 1, x \otimes x \mapsto x \otimes 1.$$

Putting these things together, we get that

$$z_C = z_{C'} \circ (\delta \circ z_{C''} \circ m) \circ z_{C'} \circ y_{C'} + y_{C'} \circ z_{C'} \circ (\delta \circ z_{C''} \circ m) \circ z_{C'}$$

and we are done. $\square$

Proof of surjectivity. Now we are ready to prove the main result of the section.

Theorem 7.8. *The algebra homomorphism $\omega : R_\alpha^\Lambda \to E_\alpha^\Lambda$ from Theorem 6.3 is surjective.*

Proof. We proceed by induction on $m + n + d$ where $d := \mathrm{ht}(\alpha)$, the case $d = 0$ being trivial. For the induction step, we assume that we are given $\alpha \in Q_+$ of height $d > 0$ and that the theorem has been proved for all smaller $m + n + d$. If $\boldsymbol{u}^* \boldsymbol{t}$ is a proper stretched circle diagram of type α, there is a unique oriented stretched circle diagram of the form $\boldsymbol{u}^*[\boldsymbol{\delta}^*] \wr \boldsymbol{t}[\boldsymbol{\gamma}]$ of minimal degree; all of its internal circles are anti-clockwise. Denote the corresponding basis vector of E_α^Λ by $e(\boldsymbol{u}^* \boldsymbol{t})$. Clearly every basis vector of E_α^Λ from (6.28) can be obtained from $e(\boldsymbol{u}^* \boldsymbol{t})$ by applying positive circle moves, i.e. by multiplying on the left and/or right by $\pm \omega(y_r)$ for various r. Hence it suffices to show that $e(\boldsymbol{u}^* \boldsymbol{t}) \in \mathrm{Im}\,\omega$ for every proper stretched circle diagram $\boldsymbol{u}^* \boldsymbol{t}$ of type α. We divide the proof of this statement into nine different cases, the last of which is the general case.

Case one: $\boldsymbol{u}^* \boldsymbol{t}$ has a boundary circle C containing no nested circles, such that for some $i \in I$ the ith and $(i+1)$th vertices of the boundary line are in the interior of C and are labelled $\circ \times$ in the block diagram for $\Lambda - \alpha$.

Let $\boldsymbol{i} \in I^\alpha$ denote the admissible sequence underlying the matching $\boldsymbol{t}$. By inspecting (3.11), the assumption that the ith and $(i+1)$th vertices of $\Lambda - \alpha$ are labelled $\circ \times$ implies that there exists $1 \le p \le d$ such that $i_p = i$, $\boldsymbol{t}$ has a cup at level p, and $|i_p - i_q| > 1$ for all $p < q \le d$. Hence we are in the situation of Lemma 7.2(i) and $\overline{\psi}_{d-1} \circ \cdots \circ \overline{\psi}_p$ is a sequence of height moves lowering the cup to level d. Let $\hat{\boldsymbol{t}}$ be the proper stretched cap diagram obtained from $\boldsymbol{t}$ by applying this sequence of height moves. Then

$$e(\boldsymbol{u}^* \boldsymbol{t}) = \overline{\psi}_p \circ \cdots \circ \overline{\psi}_{d-1}(e(\boldsymbol{u}^* \hat{\boldsymbol{t}})).$$

Since each of the maps $\overline{\psi}_r$ here is given by multiplying on the right by $\pm \omega(\psi_r)$, we can deduce from this that $e(\boldsymbol{u}^* \boldsymbol{t}) \in \mathrm{Im}\,\omega$ if we show that $e(\boldsymbol{u}^* \hat{\boldsymbol{t}}) \in \mathrm{Im}\,\omega$. Replacing $\boldsymbol{t}$ by $\hat{\boldsymbol{t}}$, this reduces to the situation that $\boldsymbol{t}$ has a cup at level d in the strip between the ith and $(i+1)$th vertices. A similar argument involving multiplying on the left by various $\pm \omega(\psi_r)$ reduces further to the situation that $\boldsymbol{u}$ also has a cup at level d in the same strip.

Now let $\bar{\boldsymbol{t}} := t_{d-1} \cdots t_0$ and $\bar{\boldsymbol{u}} := u_{d-1} \cdots u_0$. Then $\bar{\boldsymbol{u}}^* \bar{\boldsymbol{t}}$ is a proper stretched circle diagram of type $\alpha - \alpha_i$. By the main induction hypothesis, we know that

$e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}) \in \operatorname{Im}\omega$. The map $\theta_i : E^\Lambda_{\alpha-\alpha_i} \to E^\Lambda_\alpha$ from (6.34) in this situation joins two boundary circles from $\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}$ together to form the given boundary circle C, so that

$$\theta_i(e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}})) = e(\boldsymbol{u}^*\boldsymbol{t}).$$

Applying Corollary 6.12 we deduce that $e(\boldsymbol{u}^*\boldsymbol{t}) \in \operatorname{Im}\omega$.

Case two: $\boldsymbol{u}^*\boldsymbol{t}$ has a boundary circle C crossing the boundary line exactly twice at vertices $j < k$, there are no nested circles inside C, and the height of C is equal to $2(k-j)$.

Each of the vertices $j+1, \ldots, k-1$ of $\Lambda - \alpha$ must be labelled either $\times$ or $\circ$. In view of case one we may assume further that the vertices $j+1, \ldots, i$ are labelled $\times$ and the vertices $i+1, \ldots, k-1$ are labelled $\circ$, for some $j \le i < k$. Of course C consists of a generalised cap T from the matching $\boldsymbol{t}$ on top of a generalised cup B from $\boldsymbol{u}^*$. The smallest possible height of a generalised cap passing through the jth and kth vertices of the boundary line is $(k-j)$. Hence both T and B are of height exactly $(k-j)$. Considering (3.11), T must involve $(i-j)$ right shifts, $(k-1-i)$ left shifts, and one cap at the top. For example, here are the possibilities for T in case $k - j = 3$ (omitting trivial levels):
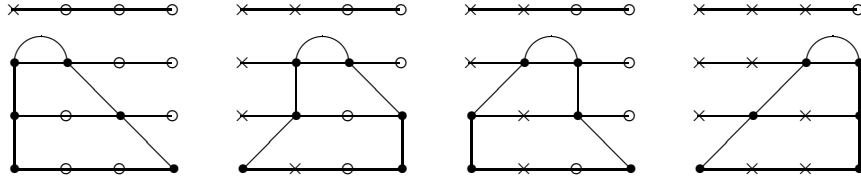
By applying height moves like we did in case one, this time using Lemma 7.4, we reduce to the case that the $(i-j)$ right shifts occur in the bottom $(i-j)$ levels of $\boldsymbol{t}$, then the $(k-1-i)$ left shifts appear in the next $(k-1-i)$ levels up, and finally the cap at the top of T occurs at the $(d+1-k+j)$th level of $\boldsymbol{t}$. A similar argument applied to B reduces further to the situation that B is the mirror image of T. Finally in this special situation, we have that

$$e(\boldsymbol{u}^*\boldsymbol{t}) = \theta_i(e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}))$$

where $\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}$ is the proper stretched circle diagram of type $(\alpha - \alpha_i)$ with $\bar{\boldsymbol{t}} := t_{d-1} \cdots t_0$ and $\bar{\boldsymbol{u}} := u_{d-1} \cdots u_0$. By induction $e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}) \in \operatorname{Im}\omega$. Hence applying Corollary 6.12 as in case one, we deduce that $e(\boldsymbol{u}^*\boldsymbol{t}) \in \operatorname{Im}\omega$ too.

Case three: $\boldsymbol{u}^*\boldsymbol{t}$ has a boundary circle C containing no other nested circles.

Proceed by induction on the height of C. The base case when C is of height 2 follows by case two. Now assume that C is of height greater than 2. Suppose first that C has a *concave cap*, i.e. a cap such that the region immediately above the cap is the interior of C. Applying Lemma 7.1 to $\boldsymbol{t}$ (resp. Lemma 7.2 to $\boldsymbol{u}$) if the cap is above (resp. below) the boundary line, we get a sequence of moves which either raises the cap until it reaches the boundary line from below or until it collides with another part of the same circle C. In the former case, we deduce that there exists $i \in I$ such that the ith and $(i+1)$th vertices of the boundary line are in the interior of C and are labelled $\circ\times$ in the block diagram for $\Lambda - \alpha$, so we are done by case one. In the latter case the given sequence of moves cuts the boundary circle C into another boundary circle C' of strictly smaller

height than C, together with another circle C'' which is definitely not nested inside C'. Let $\hat{\boldsymbol{u}}^*\hat{\boldsymbol{t}}$ be the proper stretched circle diagram obtained from $\boldsymbol{u}^*\boldsymbol{t}$ by applying these moves. By the induction hypothesis, we get that $e(\hat{\boldsymbol{u}}^*\hat{\boldsymbol{t}}) \in \mathrm{Im}\,\omega$. As $e(\boldsymbol{u}^*\boldsymbol{t})$ is obtained from this by multiplying on the left or right by the same moves taken in reverse order, we deduce that $e(\boldsymbol{u}^*\boldsymbol{t}) \in \mathrm{Im}\,\omega$ too, as required. A similar argument applies if C has a *concave cup*, i.e. a cup such that the region immediately below the cup is the interior of C.

We have now reduced to the situation that C has no concave cap or cup on its circumference. It follows that C actually has just one cap and just one cup. In particular, C crosses the boundary line exactly twice, say at vertices $j < k$. If the height of C is equal to $2(k-j)$ then we are done by case two. Hence either the top half T of C (which is a single generalised cap) or the bottom half B of C (which is a single generalised cup) must have height strictly greater than $(k-j)$. Let us assume it is T that is of height greater than $(k-j)$, a similar argument applying if it is B. We know already that T has exactly one cap and no cups. This means that T consists just of left-shifts, right-shifts and one cap at the top. Since the height of T is strictly greater than $(k-j)$, there is either a right-shift on the part of the T that is between the jth vertex of the boundary line and the cap at the top, or there is a left-shift on the part of T that is between the cap at the top and the kth vertex of the boundary line. We just explain the argument now in the former case, since the latter case is similar. Let q be minimal such that T has a right-shift at level q between the jth vertex of the boundary line and the cap at the top. Applying Lemma 7.3 to the component of $t_1 \cdots t_q$ containing this right-shift, we get a sequence $\overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$ of moves raising the right-shift. The last of these moves is a crossing move cutting C into a smaller boundary circle C' plus one new circle C'' that is not nested inside C'. Now we complete the proof as before by using the induction hypothesis.

Case four: $\boldsymbol{u}^*\boldsymbol{t}$ has a boundary circle.

Choose a boundary circle C in $\boldsymbol{u}^*\boldsymbol{t}$ that is minimal in the sense that it contains no nested boundary circles. Proceed by induction on the number of nested circles contained in C. The base case, when C contains no nested circles, follows by case three. For the induction step, assume C contains at least one nested circle. Since it contains no nested boundary circles by assumption, it must contain a nested upper or lower circle. We explain the argument in the upper case, since the lower case is entirely similar. Let q be minimal such that the component C' of $\boldsymbol{u}^*\boldsymbol{t}$ that is non-trivial at level q is an upper circle contained in C. Applying Lemma 7.1(ii), we get a sequence $\overline{\psi}_p \circ \cdots \circ \overline{\psi}_{q-1}$ of local moves raising the cap at level q. Let $\boldsymbol{u}^*\hat{\boldsymbol{t}}$ be the proper stretched circle diagram obtained by applying this sequence of moves to $\boldsymbol{u}^*\boldsymbol{t}$. By the minimality of q, the last move $\overline{\psi}_p$ in this sequence is a crossing move joining C and C' into one circle C''. The circle C'' is a boundary circle in $\boldsymbol{u}^*\hat{\boldsymbol{t}}$ containing no nested boundary circles and one less nested circle than $\boldsymbol{u}^*\boldsymbol{t}$. Hence by induction $e(\boldsymbol{u}^*\hat{\boldsymbol{t}}) \in \mathrm{Im}\,\omega$. Moreover,
$$e(\boldsymbol{u}^*\boldsymbol{t}) = z_{C'} \circ \overline{\psi}_{q-1} \circ \cdots \circ \overline{\psi}_p(e(\boldsymbol{u}^*\hat{\boldsymbol{t}})).$$

In view of Lemma 7.7, the map $z_{C'} \circ \overline{\psi}_{q-1} \circ \cdots \circ \overline{\psi}_p$ can be implemented by right multiplication by an element of R_α^Λ, so we deduce that $e(\boldsymbol{u}^*\boldsymbol{t}) \in \operatorname{Im} \omega$.

Case five: $\boldsymbol{u}^*\boldsymbol{t}$ has a propagating line L with no other circles or propagating lines to its left, such that for some $i \in I$ the ith and $(i+1)$th vertices of the boundary line lie strictly to the left of L and are labelled $\circ\times$ in the block diagram for $\Lambda - \alpha$.

This follows by a very similar argument to case one.

Case six: $\boldsymbol{u}^*\boldsymbol{t}$ has a propagating line L crossing the top and bottom number lines at vertex k and crossing the boundary line exactly once at vertex $j \leq k$, such that there are no other circles or propagating lines to the left of L and the height of L is equal to $2(k-j)$.

The propagating line L consists of a line segment T from $\boldsymbol{t}$ on top of another line segment B from $\boldsymbol{u}^*$. The assumptions on L imply that T has $(k-j)$ left-shifts and B has $(k-j)$ right-shifts. By applying height moves, we can move the left-shifts up so that they occur in the top $(k-j)$ levels of T, and similarly we move the right-shifts down so that they occur in the bottom $(k-j)$ levels of B. Then we simply erase the leftmost propagating line and the top and bottom $(k-j)$ levels from the diagram $\boldsymbol{u}^*\boldsymbol{t}$ altogether, removing one vertex from each number line in the process, to obtain a new proper stretched circle diagram $\hat{\boldsymbol{u}}^*\hat{\boldsymbol{t}}$ parametrizing a basis vector in a smaller case in the sense that for the new picture the number $m + n + d$ is strictly smaller than before. By the induction hypothesis, $e(\hat{\boldsymbol{u}}^*\hat{\boldsymbol{t}}) \in \operatorname{Im} \omega$. Restoring the parts of the diagram that were erased gives that $e(\boldsymbol{u}^*\boldsymbol{t}) \in \operatorname{Im} \omega$ too.

Case seven: $\boldsymbol{u}^*\boldsymbol{t}$ has a propagating line L with no other circles or propagating lines to its left.

Proceed by induction on the height of L. The base case when L is of height zero follows by case six. Now suppose L is of strictly positive height. If L has a cap such that the region immediately above the cap is the region to the left of L, then we raise this cap as usual, either until it reaches the boundary line from below or until it collides with another part of the line L. In the former case we are done by case five. In the latter case the given sequence of moves cuts L into a new propagating line L' of strictly smaller height together with a circle that necessarily lies to the right of L. Hence we are done by induction. An entirely similar argument treats the situation that L has a cup such that the region immediately below the cup is the region to the left of L.

This reduces to the situation that L involves only left-shifts and right-shifts. In particular, it crosses the boundary line exactly once, say at vertex k. Let T (resp. B) be the top (resp. bottom) half of L. If T involves no right-shifts and B involves no left-shifts, then we are done by case six. Hence either T involves right-shifts or B involves left-shifts. In fact, each time there is a right-shift in T, a vertex labelled $\circ$ gets added to the left hand side of L, so the number of right-shifts in T is equal to the number of vertices of $\Lambda - \alpha$ to the left of vertex k that are labelled $\circ$. By similar considerations this is the same as the number of left-shifts in B. Finally, considering (3.11), all the right-shifts in T must below

all its left-shifts, hence we can use Lemma 7.4(i) to move all the right-shifts in T to the bottom. Similarly we can move all the left-shifts in B to the top. We have now reduced to the situation that both $\boldsymbol{t}$ and $\boldsymbol{u}$ have right-shifts at level d in the strip between the $(k-1)$th and kth vertices. It is then the case that

$$e(\boldsymbol{u}^*\boldsymbol{t}) = \theta_{k-1}(e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}))$$

where $\bar{\boldsymbol{u}} := u_{d-1} \cdots u_0$ and $\bar{\boldsymbol{t}} := t_{d-1} \cdots t_0$, and we are done by the main induction hypothesis as usual.

Case eight: $\boldsymbol{u}^*\boldsymbol{t}$ has a propagating line but no boundary circles.

Let L be the leftmost propagating line and proceed by induction on the number of internal circles to the left of L. The base of the induction follows from case seven. For the induction step, let C be an internal circle to the left of L. We just explain the argument if C is an upper circle, since lower circles are treated similarly. Let q be minimal such that C has a cap at level q. Apply Lemma 7.1, we get a sequence $\overline{\psi}_{p+1} \circ \cdots \circ \overline{\psi}_{q-1}$ of moves raising the cap until either it collides with another upper circle or it collides with the line L. Let $\hat{\boldsymbol{t}}$ be the matching obtained from $\boldsymbol{t}$ by applying this sequence of moves. By the induction hypothesis, we have that $e(\boldsymbol{u}^*\hat{\boldsymbol{t}}) \in \mathrm{Im}\,\omega$. As

$$e(\boldsymbol{u}^*\boldsymbol{t}) = z_C \circ \overline{\psi}_{q-1} \circ \cdots \circ \overline{\psi}_{p+1}(e(\boldsymbol{u}^*\hat{\boldsymbol{t}})),$$

we deduce that $e(\boldsymbol{u}^*\boldsymbol{t}) \in \mathrm{Im}\,\omega$.

Case nine: the general case.

If $\boldsymbol{u}^*\boldsymbol{t}$ has a boundary circle or a propagating line we are done by cases five and eight. Hence $\boldsymbol{u}^*\boldsymbol{t}$ contains only upper and lower circles. Moreover as $d > 0$ it contains at least one upper circle and one lower circle. This means that $\boldsymbol{t}$ must have a cup at level d which is part of some upper circle C. Say this cup is in the strip between the ith and $(i+1)$th vertices. The ith and $(i+1)$th vertices of $\Lambda - \alpha$ are labelled $\circ\times$, hence like in case one $\boldsymbol{u}^*$ must have a cap in the same strip which can be raised by a sequence of height moves so that it is immediately below the boundary line. In this way we have reduced to the situation that both $\boldsymbol{t}$ and $\boldsymbol{u}$ have a cup at level d in the strip between the ith and $(i+1)$th vertices. Let $\bar{\boldsymbol{t}} := t_{d-1} \circ \cdots t_1$ and $\bar{\boldsymbol{u}} := u_{d-1} \circ \cdots u_1$. By induction we have that $e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}) \in \mathrm{Im}\,\omega$. It remains to observe that

$$e(\boldsymbol{u}^*\boldsymbol{t}) = z_C \circ \theta_i(e(\bar{\boldsymbol{u}}^*\bar{\boldsymbol{t}}))$$

as θ_i here cuts one circle into two. This completes the proof of case nine, hence the theorem. $\qquad\square$

8. Equivalence of categorifications

In this section we prove the main results of the article by comparing the homogeneous Schur-Weyl duality from section 6 with the known Schur-Weyl duality for level two on the category $\mathcal{O}$ side from [BK2].

The prinjective generator $\mathcal{T}_{\alpha}^{\Lambda}$ and its endomorphism algebra. For $\boldsymbol{i} \in \mathbb{Z}^d$, let $\mathcal{F}_{\boldsymbol{i}}$ denote the composition $\mathcal{F}_{i_d} \circ \cdots \circ \mathcal{F}_{i_1}$ of the special projective

functors from (4.5). Let $\mathcal{L}(\iota)$ denote the irreducible $\mathfrak{g}$-module of highest weight ι, where

$$\iota = (o + m)(\varepsilon_1 + \cdots + \varepsilon_m) + (o + m + n)(\varepsilon_{m+1} + \cdots + \varepsilon_{m+n}) \qquad (8.1)$$

is the element of $\mathfrak{h}^*$ corresponding to the ground-state (2.6) under the weight dictionary. In view of Corollary 4.5, we have that

$$\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d} = \bigoplus_{\boldsymbol{i} \in \mathbb{Z}^d} \mathcal{F}_{\boldsymbol{i}} \mathcal{L}(\iota). \qquad (8.2)$$

The summands $\mathcal{F}_{\boldsymbol{i}} \mathcal{L}(\iota)$ and $\mathcal{F}_{\boldsymbol{j}} \mathcal{L}(\iota)$ here belong to the same block according the decomposition (4.2) if and only if $\boldsymbol{i}$ and $\boldsymbol{j}$ lie in the same S_d-orbit.

Now fix $\alpha \in Q_+$ of height d and set

$$\mathcal{O}_\alpha^\Lambda := \begin{cases} \mathcal{O}_{\Lambda - \alpha} & \text{if } \Lambda - \alpha \in P(m, n; I), \\ \mathbf{0} \text{ (the zero category)} & \text{if } \Lambda - \alpha \notin P(m, n; I), \end{cases}$$

for short. Paralleling (6.1), we define

$$\mathcal{T}_\alpha^\Lambda := \bigoplus_{\boldsymbol{i} \in I^\alpha} \mathcal{F}_{\boldsymbol{i}} \mathcal{L}(\iota). \qquad (8.3)$$

This space is zero unless $\Lambda - \alpha \in P(m, n; I)$, in which case

$$\mathcal{T}_\alpha^\Lambda = \mathrm{pr}_{\Lambda - \alpha}(\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}), \qquad (8.4)$$

i.e. in all cases it is the largest submodule of $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$ that belongs to the subcategory $\mathcal{O}_\alpha^\Lambda$. By a *prinjective generator* for $\mathcal{O}_\alpha^\Lambda$, we mean a prinjective object of $\mathcal{O}_\alpha^\Lambda$ that involves each of the prinjective indecomposable modules from Lemma 4.2 as a summand.

Lemma 8.1. *The module $\mathcal{T}_\alpha^\Lambda$ is a prinjective generator for $\mathcal{O}_\alpha^\Lambda$. It is non-zero if and only if $\Lambda - \alpha \in P(m, n; I)$.*

Proof. This is proved in exactly the same way as Lemma 6.1, using Lemma 4.9 in place of Lemma 3.4. □

Next let H_d denote the degenerate affine Hecke algebra from [D]. So H_d is the associative algebra equal as a vector space to the tensor product $\mathbb{C}[x_1, \ldots, x_d] \otimes \mathbb{C}S_d$ of a polynomial algebra and the group algebra of the symmetric group S_d. Multiplication is defined so that $\mathbb{C}[x_1, \ldots, x_d] \equiv \mathbb{C}[x_1, \ldots, x_d] \otimes 1$ and $\mathbb{C}S_d \equiv 1 \otimes \mathbb{C}S_d$ are subalgebras of H_d, and also

$$s_r x_{r'} = x_{r'} s_r \text{ if } r' \neq r, r+1, \qquad s_r x_{r+1} = x_r s_r + 1.$$

By [AS, §2.2], there is a natural right action of H_d on $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$ commuting with the left action of $\mathfrak{g}$, such that the elements of S_d act by permuting tensors in $\mathcal{V}^{\otimes d}$ like in classical Schur-Weyl duality, and the remaining generator x_1 acts as $\Omega \otimes 1^{\otimes (d-1)}$, where Ω is as in (4.6).

Lemma 8.2. *The element $(x_1 - o - m)(x_1 - o - n) \in H_d$ acts as zero on $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$.*

Proof. It suffices to check this in the special case $d = 1$. By Lemma 4.3, $\mathcal{L}(\iota) \otimes \mathcal{V}$ has a two step filtration whose sections are highest weight modules of highest weights $\iota + \varepsilon_1$ and $\iota + \varepsilon_{m+1}$, respectively. As explained in the proof of Lemma 4.6, the action of $x_1 = \Omega$ respects this filtration, and the induced action on the two sections are by multiplication by the scalars $(\iota + \rho, \varepsilon_1) = o + m$ and $(\iota + \rho, \varepsilon_{m+1}) = o + n$, respectively. Hence $(x_1 - o - m)(x_1 - o - n)$ acts as zero. $\square$

In view of Lemma 8.2, the right action of H_d on $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$ factors to give an action of the quotient algebra

$$H_d^{o+m,o+n} := H_d / \langle (x_1 - o - m)(x_1 - o - n) \rangle, \tag{8.5}$$

which is a *degenerate cyclotomic Hecke algebra* of level two. As in [BK3, §3.1], this finite dimensional algebra possesses a natural system of mutually orthogonal *weight idempotents* $\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in \mathbb{Z}^d\}$ summing to 1, which are characterised by the property that

$$e(\boldsymbol{i})(x_r - i_r)^N = (x_i - i_r)^N e(\boldsymbol{i}) = 0$$

for each $r = 1, \ldots, d$ and $N \gg 0$. Multiplication by $e(\boldsymbol{i})$ projects any $H_d^{o+m,o+n}$-module onto its $\boldsymbol{i}$-*weight space*, that is, the simultaneous generalised eigenspace for the commuting operators $x_1, \ldots, x_r$ with respective eigenvalues $i_1, \ldots, i_r$.

Lemma 8.3. *The idempotent* $e(\boldsymbol{i}) \in H_d^{o+m,o+n}$ *acts on* $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$ *as the projection onto the summand* $\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota)$ *along the decomposition (8.2).*

Proof. By Lemma 4.6, $\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota)$ is $\boldsymbol{i}$-weight space of $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$ with respect to the right action of $H_d^{o+m,o+n}$. $\square$

It is clear from (8.4) that $\mathcal{T}_\alpha^\Lambda$ is invariant under the right action of $H_d^{o+m,o+n}$. By Lemma 8.3, the idempotent

$$e_\alpha := \sum_{\boldsymbol{i} \in I^\alpha} e(\boldsymbol{i}) \in H_d^{o+m,o+n} \tag{8.6}$$

acts on $\mathcal{L}(\iota) \otimes \mathcal{V}^{\otimes d}$ as the projection onto the summand $\mathcal{T}_\alpha^\Lambda$. Since this projection clearly commutes with the action of $H_d^{o+m,o+n}$, we deduce that e_α is actually a *central* idempotent in $H_d^{o+m,o+n}$. Hence $e_\alpha H_d^{o+m,o+n}$ is a subalgebra of $H_d^{o+m,o+n}$ with identity e_α, and $\mathcal{T}_\alpha^\Lambda$ is a unital right $e_\alpha H_d^{o+m,o+n}$-module.

Theorem 8.4. *The right action of H_d on $\mathcal{T}_\alpha^\Lambda$ induces an algebra isomorphism*

$$e_\alpha H_d^{o+m,o+n} \xrightarrow{\sim} \mathrm{End}_\mathfrak{g}(\mathcal{T}_\alpha^\Lambda)^{\mathrm{op}}.$$

Proof. This is a consequence of [BK2, Theorem 5.13] and [BK2, Corollary 6.7]. It is formulated in exactly this way in [BK4, Theorem 3.6] in the case $m \geq n$ or [BK4, Theorem 4.13] in the case $m \leq n$. $\square$

The isomorphism theorem. Continue working with a fixed $\alpha \in Q_+$ of height d. The following theorem is the key to our approach to the equivalence between $\mathcal{O}_\alpha^\Lambda$ and $\mathrm{rep}(K_\alpha^\Lambda)$: combined with Theorems 8.4 and 7.8 it implies at long last that the prinjective generators $\mathcal{T}_\alpha^\Lambda$ and T_α^Λ on the two sides have the same endomorphism algebras.

Theorem 8.5. *There is an algebra isomorphism*

$$\sigma : e_\alpha H_d^{o+m,o+n} \xrightarrow{\sim} R_\alpha^\Lambda$$

such that $e(\boldsymbol{i}) \mapsto e(\boldsymbol{i})$, $x_r e(\boldsymbol{i}) \mapsto (y_r + i_r)e(\boldsymbol{i})$ *and* $s_r e(\boldsymbol{i}) \mapsto (\psi_r q_r(\boldsymbol{i}) - p_r(\boldsymbol{i}))e(\boldsymbol{i})$
for each r *and* $\boldsymbol{i} \in I^\alpha$, *where* $p_r(\boldsymbol{i}), q_r(\boldsymbol{i}) \in R_\alpha^\Lambda$ *are chosen as in* [BK3, §3.3],
e.g. one could take

$$p_r(\boldsymbol{i}) := \begin{cases} 1 & \text{if } i_r = i_{r+1}, \\ -(i_{r+1} - i_r + y_{r+1} - y_r)^{-1} & \text{if } i_r \neq i_{r+1}; \end{cases}$$

$$q_r(\boldsymbol{i}) := \begin{cases} 1 + y_{r+1} - y_r & \text{if } i_r = i_{r+1}, \\ (2 + y_{r+1} - y_r)(1 + y_{r+1} - y_r)^{-1} & \text{if } i_{r+1} = i_r + 1, \\ 1 & \text{if } i_{r+1} = i_r - 1, \\ 1 + (i_{r+1} - i_r + y_{r+1} - y_r)^{-1} & \text{if } |i_r - i_{r+1}| > 1. \end{cases}$$

(The inverses on the right hand side of these formulae make sense because each $y_{r+1} - y_r$ *is nilpotent; see* [BK3, Lemma 2.1].*)*

Proof. This is a special case of the main theorem of [BK3]. $\qquad\square$

Corollary 8.6. *The homomorphism* $\omega : R_\alpha^\Lambda \to E_\alpha^\Lambda$ *from Theorem 6.3 is an isomorphism.*

Proof. By Theorem 7.8, ω is surjective, so it suffices to show that R_α^Λ and E_α^Λ have the same dimensions. According to Theorems 8.4 and 8.5 and the definition (8.3), we have that

$$\dim R_\alpha^\Lambda = \sum_{\boldsymbol{i},\boldsymbol{j} \in I^\alpha} \dim \operatorname{Hom}_{\mathfrak{g}}(\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota), \mathcal{F}_{\boldsymbol{j}}\mathcal{L}(\iota)).$$

On the other hand by the definition (6.1),

$$\dim E_\alpha^\Lambda = \sum_{\boldsymbol{i},\boldsymbol{j} \in I^\alpha} \dim \operatorname{Hom}_{K_\alpha^\Lambda}(F_{\boldsymbol{i}}L(\iota), F_{\boldsymbol{j}}L(\iota)).$$

Therefore we are done if we can show that

$$\dim \operatorname{Hom}_{\mathfrak{g}}(\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota), \mathcal{F}_{\boldsymbol{j}}\mathcal{L}(\iota)) = \dim \operatorname{Hom}_{K_\alpha^\Lambda}(F_{\boldsymbol{i}}L(\iota), F_{\boldsymbol{j}}L(\iota)) \qquad (8.7)$$

for each $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$. As ι is the only weight in its block, $\mathcal{L}(\iota)$ is projective, hence so is $\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota)$. Applying Theorem 4.10 we deduce that

$$\dim \operatorname{Hom}_{\mathfrak{g}}(\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota), \mathcal{F}_{\boldsymbol{j}}\mathcal{L}(\iota)) = \langle [\mathcal{F}_{\boldsymbol{i}}\mathcal{L}(\iota)], [\mathcal{F}_{\boldsymbol{j}}\mathcal{L}(\iota)] \rangle = \langle \mathcal{F}_{\boldsymbol{i}}\mathcal{L}_\iota, \mathcal{F}_{\boldsymbol{j}}\mathcal{L}_\iota \rangle.$$

A similar application of Theorem 3.5, setting $q = 1$ at the end, gives that

$$\langle \mathcal{F}_{\boldsymbol{i}}\mathcal{L}_\iota, \mathcal{F}_{\boldsymbol{j}}\mathcal{L}_\iota \rangle = \dim \operatorname{Hom}_{K_\alpha^\Lambda}(F_{\boldsymbol{i}}L(\iota), F_{\boldsymbol{j}}L(\iota)).$$

This establishes (8.7). $\qquad\square$

Remark 8.7. The equality $\dim E_\alpha^\Lambda = \dim R_\alpha^\Lambda$ in the above proof of Corollary 8.6 was deduced using Theorem 8.4, hence the argument relies ultimately on properties of parabolic category $\mathcal{O}$. Alternatively, this equality of dimensions can be proved by using the observation made in the last sentence of Remark 6.5 together with [BK5, Theorem 4.20], which gives a combinatorial formula for $\dim R_\alpha^\Lambda$ in terms of standard tableaux. One advantage of this alternate argument is that it is valid over an arbitrary ground field (including positive characteristic), not just over $\mathbb{C}$.

Comparison of embeddings. There is one more important identification to be made. Consider the obvious embedding $H_d \hookrightarrow H_{d+1}$. This factors through the quotients to induce an embedding $\theta : H_d^{o+m,o+n} \hookrightarrow H_{d+1}^{o+m,o+n}$. Composing θ on one side with the embedding $e_\alpha H_d^{o+m,o+n} \hookrightarrow H_d^{o+m,o+n}$ and on the other side with the projection $H_{d+1}^{o+m,o+n} \twoheadrightarrow e_{\alpha+\alpha_i} H_{d+1}^{o+m,o+n}$ defined by multiplication by the central idempotent $e_{\alpha+\alpha_i}$, we obtain a non-unital algebra homomorphism

$$\theta_i : e_\alpha H_d^{o+m,o+n} \to e_{\alpha+\alpha_i} H_{d+1}^{o+m,o+n}. \tag{8.8}$$

This maps the identity element e_α to the idempotent

$$e_{\alpha,\alpha_i} := \sum_{\substack{i \in I^{\alpha+\alpha_i} \\ i_{d+1}=i}} e(\boldsymbol{i}), \tag{8.9}$$

just like in (6.33). Recall also the homomorphism θ_i from (6.32).

Lemma 8.8. *The following diagram commutes:*

$$
\begin{array}{ccc}
e_\alpha H_d^{o+m,o+n} & \xrightarrow{\ \theta_i\ } & e_{\alpha+\alpha_i} H_{d+1}^{o+m,o+n} \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\
R_\alpha^\Lambda & \xrightarrow[\theta_i]{} & R_{\alpha+\alpha_i}^\Lambda.
\end{array}
$$

Proof. This is clear from the explicit form of the isomorphism in Theorem 8.5 together with the definitions of (6.32) and (8.8). $\qquad\square$

For the remainder of the article we will simply *identify* the following four algebras according to the isomorphisms from Theorems 8.4 and 8.5 and Corollary 8.6:

$$E_\alpha^\Lambda \equiv R_\alpha^\Lambda \equiv e_\alpha H_d^{o+m,o+n} \equiv \mathrm{End}_{\mathfrak{g}}(\mathcal{T}_\alpha^\Lambda)^{\mathrm{op}}. \tag{8.10}$$

Under the first of these identifications, the non-unital algebra homomorphism θ_i from Theorem 6.11 coincides with the map θ_i from (6.32), as follows from the commutative diagram in Corollary 6.12. Under the second of these identifications, the map θ_i from (6.32) coincides with the map θ_i from (8.8), as follows from the commutative diagram in Lemma 8.8. This justifies our use of the same notation θ_i in all the settings.

Schur functors: the category $\mathcal{O}$ side. For $\alpha \in Q_+$, consider the category $\mathrm{rep}(R_\alpha^\Lambda)$ of (ungraded) finite dimensional left R_α^Λ-modules. As we noted in (8.9), the homomorphism $\theta_i : R_\alpha^\Lambda \to R_{\alpha+\alpha_i}^\Lambda$ maps the identity element e_α of R_α^Λ to the idempotent e_{α,α_i}. Via this homomorphism, if M is any left (resp. right) $R_{\alpha+\alpha_i}^\Lambda$-module then $e_{\alpha,\alpha_i}M$ (resp. Me_{α,α_i}) is naturally an R_α^Λ-module. Following [BK4, §3.3], we introduce the *i-induction and i-restriction functors*

$$\mathcal{F}_i : \mathrm{rep}(R_\alpha^\Lambda) \to \mathrm{rep}(R_{\alpha+\alpha_i}^\Lambda), \qquad \mathcal{E}_i : \mathrm{rep}(R_{\alpha+\alpha_i}^\Lambda) \to \mathrm{rep}(R_\alpha^\Lambda),$$

by defining $\mathcal{F}_i$ to be the right exact functor $R_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i} \otimes_{R_\alpha^\Lambda} ?$ and $\mathcal{E}_i$ to be the exact functor defined by left multiplication by the idempotent e_{α,α_i}. By adjointness of tensor and hom, $(\mathcal{F}_i, \mathcal{E}_i)$ is an adjoint pair of functors. In fact, it is known that $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint, hence in particular $\mathcal{F}_i$ is actually exact;

see [K, Lemma 8.2.2] (the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are the same as the induction and restriction functors there restricted to particular blocks).

Recalling that we have identified R_α^Λ with $\mathrm{End}_{\mathfrak{g}}(\mathcal{T}_\alpha^\Lambda)^{\mathrm{op}}$ in (8.10), we introduce the *Schur functor* (or *quotient functor* in the sense of [G, §III.1]):

$$\pi := \mathrm{Hom}_{\mathfrak{g}}(\mathcal{T}_\alpha^\Lambda, ?) : \mathcal{O}_\alpha^\Lambda \to \mathrm{rep}(R_\alpha^\Lambda). \tag{8.11}$$

In view of Lemma 8.1 we are in a well-studied situation. Assuming that $\Lambda - \alpha \in P(m, n; I)$, define

$$\mathcal{D}(\lambda) := \pi(\mathcal{L}(\lambda)), \qquad \mathcal{Y}(\lambda) := \pi(\mathcal{P}(\lambda)). \tag{8.12}$$

for any $\lambda \in \Lambda - \alpha$. The following lemma collects some basic facts about this situation; see [BK4, §3.5] for a more detailed account.

Lemma 8.9. *Assume that $\Gamma := \Lambda - \alpha \in P(m, n; I)$. For $\lambda \in \Gamma$, the module $\mathcal{D}(\lambda)$ is non-zero if and only if $\lambda \in \Gamma^\circ$. The modules $\{\mathcal{D}(\lambda) \mid \lambda \in \Gamma^\circ\}$ give a complete set of representatives for the isomorphism classes of irreducible R_α^Λ-modules. Moreover:*

(i) *For $\lambda \in \Gamma^\circ$, $\mathcal{Y}(\lambda)$ is the projective cover of $\mathcal{D}(\lambda)$.*
(ii) *The functor π is fully faithful on projectives.*
(iii) *There is an isomorphism $\eta : \mathcal{F}_i \circ \pi \xrightarrow{\sim} \pi \circ \mathcal{F}_i$ of functors from $\mathrm{rep}(K_\alpha^\Lambda)$ to $\mathrm{rep}(R_{\alpha+\alpha_i}^\Lambda)$ for each $i \in I$.*

Proof. If $m \geq n$ this follows by a special case of [BK4, Theorem 3.7]. The case $m < n$ can be treated similarly; see also [BK4, Theorem 4.15]. $\square$

For any $\alpha \in Q_+$, we now define $\mathcal{P}_\alpha^\Lambda \in \mathcal{O}_\alpha^\Lambda$ and $\mathcal{Y}_\alpha^\Lambda \in \mathrm{rep}(R_\alpha^\Lambda)$ by

$$\mathcal{P}_\alpha^\Lambda := \bigoplus_{\lambda \in \Gamma} \mathcal{P}(\lambda), \qquad \mathcal{Y}_\alpha^\Lambda := \bigoplus_{\lambda \in \Gamma} \mathcal{Y}(\lambda) \tag{8.13}$$

if $\Gamma := \Lambda - \alpha \in P(m, n; I)$, or $\mathcal{P}_\alpha^\Lambda = \mathcal{Y}_\alpha^\Lambda := \{0\}$ if $\Lambda - \alpha \notin P(m, n; I)$. So $\mathcal{P}_\alpha^\Lambda$ is a *minimal projective generator* for $\mathcal{O}_\alpha^\Lambda$, and $\mathcal{Y}_\alpha^\Lambda$ is its image under the Schur functor π. Theorem 8.9(ii) immediately implies the following corollary.

Corollary 8.10. *The functor π defines an algebra isomorphism*

$$i : \mathrm{End}_{\mathfrak{g}}\big(\mathcal{P}_\alpha^\Lambda\big)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{R_\alpha^\Lambda}\big(\mathcal{Y}_\alpha^\Lambda\big)^{\mathrm{op}}.$$

Schur functors: the diagram algebra side. Now we go back to the diagram algebra side of the story. Let $\mathrm{Rep}(R_\alpha^\Lambda)$ denote the category of finite dimensional *graded* left R_α^Λ-modules, recalling the grading on R_α^Λ introduced immediately after (6.16). Let

$$f : \mathrm{Rep}(R_\alpha^\Lambda) \to \mathrm{rep}(R_\alpha^\Lambda) \tag{8.14}$$

be the forgetful functor here, just like in (3.1). There are graded versions of the *i*-induction and *i*-restriction functors

$$F_i : \mathrm{Rep}(R_\alpha^\Lambda) \to \mathrm{Rep}(R_{\alpha+\alpha_i}^\Lambda), \qquad E_i : \mathrm{Rep}(R_{\alpha+\alpha}^\Lambda) \to \mathrm{Rep}(R_\alpha^\Lambda),$$

where F_i is the functor $R_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i} \otimes_{R_\alpha^\Lambda} ?\langle 1 - (\Lambda - \alpha, \delta_i - \delta_{i+1})\rangle$ and E_i is defined by left multiplication by the idempotent e_{α,α_i} as before. It is obvious from these definitions that

$$f \circ F_i \cong \mathcal{F}_i \circ f, \qquad f \circ E_i \cong \mathcal{E}_i \circ f. \tag{8.15}$$

Hence F_i and E_i are exact, since we already know that about $\mathcal{F}_i$ and $\mathcal{E}_i$. For the next lemma, we let

$$D_i^{\pm 1} : \operatorname{Rep}(R_\alpha^\Lambda) \to \operatorname{Rep}(R_\alpha^\Lambda) \qquad (8.16)$$

be the degree shift functor mapping a module M to $M\langle \pm (\Lambda - \alpha, \delta_i) \rangle$, as in (3.13).

Lemma 8.11. *There exists a canonical adjunction of degree zero which makes $(F_i D_i D_{i+1}^{-1} \langle -1 \rangle, E_i)$ into an adjoint pair of functors.*

Proof. By the preceeding definition of F_i, we have that

$$F_i D_i D_{i+1}^{-1} \langle -1 \rangle = R_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i} \otimes_{R_\alpha^\Lambda} ?.$$

Given this, the lemma reduces to the usual adjointness of tensor and hom. $\square$

Remark 8.12. Although not needed here, there is another degree zero adjunction making $(E_i, F_i D_i^{-1} D_{i+1} \langle 1 \rangle)$ into an adjoint pair of functors too. This can be derived from Lemma 3.3 using Lemma 8.13(iii) and Remark 8.14 below.

Recalling that we have identified R_α^Λ with $\operatorname{End}_{K_\alpha^\Lambda}(T_\alpha^\Lambda)^{\mathrm{op}}$ in (8.10), we also have the *graded Schur functor*

$$\pi := \operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, ?) : \operatorname{Rep}(K_\alpha^\Lambda) \to \operatorname{Rep}(R_\alpha^\Lambda). \qquad (8.17)$$

We have used the same notation for this as in (8.11), hoping that it is always clear from context which we mean. Assuming that $\Lambda - \alpha \in P(m, n; I)$, define

$$D(\lambda) := \pi(L(\lambda)), \qquad Y(\lambda) := \pi(P(\lambda)) \qquad (8.18)$$

for any $\lambda \in \Lambda - \alpha$. The following gives a graded version of Lemma 8.9.

Lemma 8.13. *Assume that $\Gamma := \Lambda - \alpha \in P(m, n; I)$. For $\lambda \in \Gamma$, the module $D(\lambda)$ is non-zero if and only if $\lambda \in \Gamma^\circ$. The modules $\{D(\lambda)\langle j \rangle \mid \lambda \in \Gamma^\circ, j \in \mathbb{Z}\}$ give a complete set of representatives for the isomorphism classes of irreducible graded R_α^Λ-modules. Moreover:*
  (i) *For $\lambda \in \Gamma^\circ$, $Y(\lambda)$ is the projective cover of $D(\lambda)$.*
  (ii) *The functor π is fully faithful on projectives.*
  (iii) *There is an isomorphism $\eta : F_i \circ \pi \xrightarrow{\sim} \pi \circ F_i$ of functors from $\operatorname{Rep}(K_\alpha^\Lambda)$ to $\operatorname{Rep}(R_{\alpha+\alpha_i}^\Lambda)$ for each $i \in I$.*

Proof. The first statement and (i) follow from Lemma 6.1 by the same general argument as in the proof of Lemma 8.9(i). For (ii), see [BS2, Corollary 6.3]. Finally to prove (iii), we first construct a natural transformation

$$\eta : F_i \circ \pi \to \pi \circ F_i.$$

Take a graded R_α^Λ-module M. The functor F_i defines a natural degree-preserving linear map

$$\bar{\eta}_M : \operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, M)\langle 1 - (\Lambda - \alpha, \delta_i - \delta_{i+1}) \rangle$$
$$\longrightarrow \operatorname{Hom}_{K_{\alpha+\alpha_i}^\Lambda}(F_i T_\alpha^\Lambda \langle (\Lambda - \alpha, \delta_i - \delta_{i+1}) - 1 \rangle, F_i M).$$

In view of Theorem 6.11, this map is an R_α^Λ-module homomorphism. Note next by (3.14) that $(\Lambda - \alpha, \delta_i - \delta_{i+1}) - 1 = \operatorname{def}(\Lambda - \alpha - \alpha_i) - \operatorname{def}(\Lambda - \alpha)$. Recalling also

the definition (6.1), it follows that $F_i T_\alpha^\Lambda \langle (\Lambda - \alpha, \delta_i - \delta_{i+1}) - 1 \rangle = T_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i}$. So we have that

$$\operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, M)\langle 1 - (\Lambda - \alpha, \delta_i - \delta_{i+1}) \rangle = D_i^{-1} D_{i+1} \langle 1 \rangle \circ \pi(M),$$

$$\operatorname{Hom}_{K_{\alpha+\alpha_i}^\Lambda}(F_i T_\alpha^\Lambda \langle (\Lambda - \alpha, \delta_i - \delta_{i+1}) - 1 \rangle, F_i M) = \operatorname{Hom}_{K_{\alpha+\alpha_i}^\Lambda}(T_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i}, F_i M)$$

$$= e_{\alpha,\alpha_i} \operatorname{Hom}_{K_{\alpha+\alpha_i}^\Lambda}(T_{\alpha+\alpha_i}^\Lambda, F_i M)$$

$$= E_i \circ \pi \circ F_i(M).$$

This means that the maps $\bar\eta_M$ actually define a natural transformation

$$\bar\eta : D_i^{-1} D_{i+1} \langle 1 \rangle \circ \pi \to E_i \circ \pi \circ F_i.$$

Now we use the adjunction from Lemma 8.11, to get from this the natural transformation $\eta : F_i \circ \pi \to \pi \circ F_i$ that we wanted.

Now we need to show that η is an isomorphism. We first check that it gives an isomorphism on the module $M = T_\alpha^\Lambda$. In that case by Corollary 8.6 and Theorem 6.11, $\bar\eta_M$ can be identified with the map

$$\theta_i : R_\alpha^\Lambda \to e_{\alpha,\alpha_i} R_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i}.$$

Pushing through the adjunction from Lemma 8.11, this induces the identity map $R_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i} \to R_{\alpha+\alpha_i}^\Lambda e_{\alpha,\alpha_i}$, hence η_M is an isomorphism. From this, Lemma 6.1 and naturality, it follows that η gives an isomorphism on every prinjective indecomposable K_α^Λ-module. By [BS2, Theorem 6.2], every projective indecomposable K_α^Λ-module P has a two step resolution $0 \to P \to P^0 \to P^1$ where P^0 and P^1 are prinjective. From this we deduce by the five lemma (and the exactness of the functors involved) that η gives an isomorphism on every projective indecomposable module. Finally we take an arbitrary M and consider a two step projective resolution $P_1 \to P_0 \to M \to 0$ and use the five lemma again to complete the proof. $\qquad\square$

Remark 8.14. There is also an isomorphism $E_i \circ \pi \xrightarrow{\sim} \pi \circ E_i$ of functors from $\operatorname{Rep}(K_{\alpha+\alpha_i}^\Lambda)$ to $\operatorname{Rep}(R_\alpha^\Lambda)$. Since this does not play a central role in the rest of the article, we leave the details of its construction to the reader.

Corollary 8.15. *Identifying $\widehat{T}_\alpha^\Lambda$ with $\pi(K_\alpha^\Lambda)$ according to the isomorphism from Lemma 6.6, the functor π induces a graded algebra isomorphism*

$$j : K_\alpha^\Lambda \equiv \operatorname{End}_{K_\alpha^\Lambda}(K_\alpha^\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}_{R_\alpha^\Lambda}(\widehat{T}_\alpha^\Lambda)^{\mathrm{op}}.$$

The main equivalence of categories. In this subsection, we are going to ignore all gradings. We still write $P(\lambda), L(\lambda), Y(\lambda), T_\alpha^\Lambda, \widehat{T}_\alpha^\Lambda, \ldots$ for the various modules introduced earlier but view them as ordinary ungraded modules, i.e. we implicitly apply the functor f everywhere. Recall also the endofunctors E_i and F_i of $\operatorname{Rep}(K(m,n;I))$ from (3.12), which are defined by tensoring by certain graded bimodules. Tensoring with the same underlying bimodules but forgetting the grading gives endofunctors E_i and F_i of $\operatorname{rep}(K(m,n;I))$. The next important lemma identifies the modules from (8.12) and (8.18).

Lemma 8.16. *Assume that $\Gamma := \Lambda - \alpha \in P(m,n;I)$.*

(i) *We have that $\mathcal{D}(\lambda) \cong D(\lambda)$ for any $\lambda \in \Gamma^\circ$.*

(ii) *We have that $\mathcal{Y}(\lambda) \cong Y(\lambda)$ for any $\lambda \in \Gamma$.*

Proof. (i) It is obvious that $\mathcal{D}(\iota) \cong D(\iota)$ where ι is the ground-state, because the algebra R_0^Λ is just a copy of the ground field $\mathbb{C}$. Now take any $\lambda \in \Gamma^\circ$ with $\lambda \neq \iota$. By Lemma 3.10, we can write $\lambda = \tilde{f}_{i_d} \cdots \tilde{f}_{i_1}(\iota)$ for $d > 0$. Set $\mu := \tilde{f}_{i_{d-1}} \cdots \tilde{f}_{i_1}(\iota)$. Proceeding by induction on d, we may assume we have shown already that $\mathcal{D}(\mu) \cong D(\mu)$. By Lemma 4.9, $\mathcal{L}(\lambda)$ is the irreducible head of $\mathcal{F}_{i_d}(\mathcal{L}(\mu))$. Applying the Schur functor exactly as in the proof of [BK4, Theorem 3.10(vi)], we deduce that $\mathcal{D}(\lambda)$ is the irreducible head of $\pi \circ \mathcal{F}_{i_d}(\mathcal{L}(\mu))$. Equivalently by Lemma 8.9(iii), $\mathcal{D}(\lambda)$ is isomorphic to the irreducible head of $\mathcal{F}_{i_d} \circ \pi(\mathcal{L}(\mu)) = \mathcal{F}_{i_d}(\mathcal{D}(\mu))$. A similar argument on the diagram algebra side using Lemma 3.4, Lemma 8.13(iii) and (8.15) gives that $D(\lambda)$ is isomorphic to the irreducible head of $\mathcal{F}_{i_d}(D(\mu)) \cong \mathcal{F}_{i_d}(\mathcal{D}(\mu))$. Hence $\mathcal{D}(\lambda) \cong D(\lambda)$.

(ii) To start with suppose that $\lambda \in \Gamma$ is maximal in the Bruhat order. Then we have that $P(\lambda) \cong V(\lambda)$ by [BS1, Theorem 5.1]. Consider the possible $\mu \in \Gamma^\circ$ such that $L(\mu)$ appears as a composition factor of $V(\lambda)$ (possibly shifted in degree). As λ is Bruhat maximal, its diagram involves p $\wedge$'s followed by q $\vee$'s. By [BS1, Theorem 5.2], it must be the case that $\underline{\mu}\lambda$ is an oriented cup diagram. Since μ is of maximal defect it follows that there is only one possibility for μ: it is the weight obtained from λ by switching the rightmost $|p-q|$ vertices labelled $\wedge$ with the leftmost $|p-q|$ vertices labelled $\vee$. Applying the graded Schur functor using Lemma 8.13, we deduce that $Y(\lambda) = \pi(P(\lambda)) \cong \pi(V(\lambda)) \cong \pi(L(\mu)) \cong D(\mu)$ (ignoring the grading). Since the combinatorics is exactly the same on the category $\mathcal{O}$ side (compare Theorems 3.5 and 4.10), the same argument gives that $\mathcal{Y}(\lambda) \cong \mathcal{D}(\mu)$. In view of (i), we deduce that $\mathcal{Y}(\lambda) \cong Y(\lambda)$.

Now suppose that $\lambda \in \Gamma$ is not maximal in the Bruhat order. Applying Lemma 3.7, we get $\mu \in \Gamma$ that is maximal and $i_1, \ldots, i_k \in I$ such that $\lambda = \tilde{f}_{i_k} \cdots \tilde{f}_{i_1}(\mu)$. Applying Theorem 3.8 combined with Theorems 3.5 and 4.10, we deduce for some $N > 0$ that

$$\mathcal{F}_{i_k} \circ \cdots \circ \mathcal{F}_{i_1}(\mathcal{P}(\mu)) \cong \mathcal{P}(\lambda)^{\oplus N},$$
$$F_{i_k} \circ \cdots \circ F_{i_1}(P(\mu)) \cong P(\lambda)^{\oplus N}.$$

Now apply the Schur functors on both sides and use Lemmas 8.9(iii), 8.13(iii) and (8.15) to deduce that

$$\mathcal{F}_{i_k} \circ \cdots \circ \mathcal{F}_{i_1}(\mathcal{Y}(\mu)) \cong \mathcal{Y}(\lambda)^{\oplus N},$$
$$\mathcal{F}_{i_k} \circ \cdots \circ \mathcal{F}_{i_1}(Y(\mu)) \cong Y(\lambda)^{\oplus N}.$$

The modules on the left hand side here are isomorphic by the previous paragraph. Hence applying the Krull-Schmidt theorem we get that $\mathcal{Y}(\lambda) \cong Y(\lambda)$ as required. $\square$

Recall for the next lemma that K_Γ possesses a system $\{e_\lambda | \lambda \in \Gamma\}$ of mutually orthogonal idempotents such that the projective indecomposable K_Γ-module $P(\lambda)$ is just the left ideal $K_\Gamma e_\lambda$; see [BS1, §5].

Lemma 8.17. *Recalling the definition (8.13), there exists an isomorphism of R_α^Λ-modules $h : \mathcal{Y}_\alpha^\Lambda \xrightarrow{\sim} \widehat{T}_\alpha^\Lambda$ mapping each summand $\mathcal{Y}(\lambda)$ of $\mathcal{Y}_\alpha^\Lambda$ to the summand $\widehat{T}_\alpha^\Lambda e_\lambda \cong Y(\lambda)$ of $\widehat{T}_\alpha^\Lambda$.*

Proof. We may assume that $\Gamma := \Lambda - \alpha \in P(m, n; I)$. Recalling Lemma 6.6, we have that $\widehat{T}_\alpha^\Lambda = \bigoplus_{\lambda \in \Gamma} \widehat{T}_\alpha^\Lambda e_\lambda$, and each $\widehat{T}_\alpha^\Lambda e_\lambda$ is isomorphic to $Y(\lambda)$. Now apply Lemma 8.16(ii). $\square$

Fix once and for all a choice of isomorphism h as in Lemma 8.17 for each $\alpha \in Q_+$. Now we come to what is really the main theorem of the article.

Theorem 8.18. *For each $\alpha \in Q_+$, there is an algebra isomorphism*

$$\mathrm{End}_\mathfrak{g}\big(\mathcal{P}_\alpha^\Lambda\big)^{\mathrm{op}} \xrightarrow{\sim} K_\alpha^\Lambda, \qquad \theta \mapsto j^{-1}(h \circ i(\theta) \circ h^{-1}),$$

where i, j and h are the maps from Corollaries 8.10 and 8.15 and Lemma 8.17.

Proof. This follows because all of i, j and h are already known to be isomorphisms. $\square$

Corollary 8.19. *Viewing $\mathcal{P}_\alpha^\Lambda$ as a right K_α^Λ-module via the isomorphism from Theorem 8.18, the functor*

$$\mathbb{E}_\alpha := \mathrm{Hom}_\mathfrak{g}\big(\mathcal{P}_\alpha^\Lambda, ?\big) : \mathcal{O}_\alpha^\Lambda \to \mathrm{rep}(K_\alpha^\Lambda)$$

is an equivalence of categories. Moreover, if $\Gamma := \Lambda - \alpha \in P(m, n; I)$ then $\mathbb{E}_\alpha(\mathcal{L}(\lambda)) \cong L(\lambda), \mathbb{E}_\alpha(\mathcal{V}(\lambda)) \cong V(\lambda)$ and $\mathbb{E}_\alpha(\mathcal{P}(\lambda)) \cong P(\lambda)$ for each $\lambda \in \Gamma$.

Proof. For $\lambda \in \Gamma$ let $p_\lambda \in \mathrm{End}_\mathfrak{g}\big(\bigoplus_{\lambda \in \Gamma} \mathcal{P}(\lambda)\big)^{\mathrm{op}}$ be the projection onto the summand $\mathcal{P}(\lambda)$. Then by the definitions of i, j and h we have that $c(p_\lambda) = e_\lambda$. It follows easily that $\mathbb{E}_\alpha(\mathcal{P}(\lambda)) \cong P(\lambda)$, hence also $\mathbb{E}_\alpha(\mathcal{L}(\lambda)) \cong L(\lambda)$. Finally we get that $\mathbb{E}_\alpha(\mathcal{V}(\lambda)) \cong V(\lambda)$ because these are the standard modules of the highest weight categories on the two sides. $\square$

Corollary 8.20. *Let $\mathbb{E}_\alpha$ be as in Corollary 8.19 and π denote the respective Schur functors from (8.11) and (8.17). Then the following diagram*

$$\begin{array}{ccc} \mathcal{O}_\alpha^\Lambda & \xrightarrow{\ \ \mathbb{E}_\alpha\ \ } & \mathrm{rep}(K_\alpha^\Lambda) \\ {\scriptstyle \pi} \searrow & & \swarrow {\scriptstyle \pi} \\ & \mathrm{rep}(R_\alpha^\Lambda) & \end{array}$$

commutes in the sense that there is an isomorphism of functors $\pi \circ \mathbb{E}_\alpha \cong \pi$.

Proof. Regard $\mathcal{Y}_\alpha^\Lambda$ as an $(R_\alpha^\Lambda, K_\alpha^\Lambda)$-bimodule by defining the right K_α^Λ-action to be the one obtained by lifting its natural action on $\widehat{T}_\alpha^\Lambda$ through the isomorphism h from Lemma 8.17. Then the map

$$\mathrm{Hom}_{R_\alpha^\Lambda}(\widehat{T}_\alpha^\Lambda, R_\alpha^\Lambda) \to \mathrm{Hom}_{R_\alpha^\Lambda}(\mathcal{Y}_\alpha^\Lambda, R_\alpha^\Lambda), \qquad f \mapsto f \circ h$$

becomes an isomorphism of $(K_\alpha^\Lambda, R_\alpha^\Lambda)$-bimodules. Recall also from Lemma 6.6 that $\widehat{T}_\alpha^\Lambda \cong \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda)$ as $(R_\alpha^\Lambda, K_\alpha^\Lambda)$-bimodules. Combined with Lemmas 8.9(ii) and 8.13(ii) ("Schur functors are fully faithful on projectives")

and the definitions (8.12)–(8.13), we get the following sequence of $(K_\alpha^\Lambda, R_\alpha^\Lambda)$-bimodule isomorphisms:

$$T_\alpha^\Lambda \equiv \mathrm{Hom}_{K_\alpha^\Lambda}(K_\alpha^\Lambda, T_\alpha^\Lambda) \cong \mathrm{Hom}_{R_\alpha^\Lambda}(\mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda), \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, T_\alpha^\Lambda))$$

$$\cong \mathrm{Hom}_{R_\alpha^\Lambda}(\widehat{T}_\alpha^\Lambda, R_\alpha^\Lambda) \cong \mathrm{Hom}_{R_\alpha^\Lambda}(\mathcal{Y}_\alpha^\Lambda, R_\alpha^\Lambda)$$

$$\cong \mathrm{Hom}_{R_\alpha^\Lambda}(\mathrm{Hom}_{\mathfrak{g}}(\mathcal{T}_\alpha^\Lambda, \mathcal{P}_\alpha^\Lambda), \mathrm{Hom}_{\mathfrak{g}}(\mathcal{T}_\alpha^\Lambda, \mathcal{T}_\alpha^\Lambda))$$

$$\cong \mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}_\alpha^\Lambda, \mathcal{T}_\alpha^\Lambda).$$

Now to prove the lemma, take $M \in \mathcal{O}_\alpha^\Lambda$. Applying the equivalence of categories $\mathbb{E}_\alpha = \mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}_\alpha^\Lambda, ?)$ and using the bimodule isomorphism just constructed, we get natural R_α^Λ-module isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}}(\mathcal{T}_\alpha^\Lambda, M) \cong \mathrm{Hom}_{K_\alpha^\Lambda}(\mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}_\alpha^\Lambda, \mathcal{T}_\alpha^\Lambda), \mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}_\alpha^\Lambda, M))$$

$$\cong \mathrm{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, \mathbb{E}_\alpha(M)).$$

This shows that $\pi \cong \pi \circ \mathbb{E}_\alpha$. $\qquad\square$

Corollary 8.21. *The functor* $\mathbb{E} := \bigoplus_{\alpha \in Q_+} \mathbb{E}_\alpha$ *is an equivalence of categories*

$$\mathbb{E} : \mathcal{O}(m, n; I) \to \mathrm{rep}(K(m, n; I))$$

such that $\mathbb{E}(\mathcal{L}(\lambda)) \cong L(\lambda)$, $\mathbb{E}(\mathcal{V}(\lambda)) \cong V(\lambda)$ *and* $\mathbb{E}(\mathcal{P}(\lambda)) \cong P(\lambda)$ *for each* $\lambda \in \Lambda(m, n; I)$.

Corollary 8.22. *The parabolic Verma modules* $\mathcal{V}(\lambda)$ *in* $\mathcal{O}(m, n)$ *are rigid for all* $\lambda \in \Lambda(m, n; I)$.

Proof. This is immediate from Corollary 8.21 and [BS2, Corollary 6.7]. $\qquad\square$

For further discussion of rigidity in the context of parabolic category $\mathcal{O}$, we refer the reader to [H, §9.17].

Identification of special projective functors. We continue to ignore all gradings throughout the subsection. Let $\mathbb{E}$ be as in Corollary 8.21.

Theorem 8.23. *There are isomorphisms* $\mathbb{E} \circ \mathcal{E}_i \cong E_i \circ \mathbb{E}$ *and* $\mathbb{E} \circ \mathcal{F}_i \cong F_i \circ \mathbb{E}$.

Proof. It suffices to prove the second isomorphism; the first then follows as $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint, as are E_i and F_i. For the second isomorphism, it is enough to prove for each $\alpha \in Q_+$ that $\mathbb{E}_{\alpha+\alpha_i} \circ \mathcal{F}_i \circ \mathbb{E}_\alpha^* \cong F_i$ as functors from $\mathcal{O}_\alpha^\Lambda$ to $\mathcal{O}_{\alpha+\alpha_i}^\Lambda$, where we write $\mathbb{E}_\alpha^*$ for the functor $\mathcal{P}_\alpha^\Lambda \otimes_{K_\alpha^\Lambda} ?$ that is a quasi-inverse equivalence to $\mathbb{E}_\alpha$.

Observe to start with that $F_i : \mathrm{Rep}(K_\alpha^\Lambda) \to \mathrm{Rep}(K_{\alpha+\alpha_i}^\Lambda)$ is the functor defined by tensoring with the $(K_{\alpha+\alpha_i}^\Lambda, K_\alpha^\Lambda)$-bimodule $\mathrm{Hom}_{K_{\alpha+\alpha_i}^\Lambda}(K_{\alpha+\alpha_i}^\Lambda, F_i K_\alpha^\Lambda)$. Let us identify $\pi(K_\alpha^\Lambda)$ (resp. $\pi(K_{\alpha+\alpha_i}^\Lambda)$) with $\widehat{T}_\alpha^\Lambda$ (resp. $\widehat{T}_{\alpha+\alpha_i}^\Lambda$) according to Lemma 6.6, and identify $K_{\alpha+\alpha_i}^\Lambda$ with $\mathrm{End}_{R_{\alpha+\alpha_i}^\Lambda}(\widehat{T}_{\alpha+\alpha_i}^\Lambda)^{\mathrm{op}}$ via the isomorphism from Corollary 8.15. Then Lemma 8.13(ii)–(iii) gives us a $(K_{\alpha+\alpha_i}^\Lambda, K_\alpha^\Lambda)$-bimodule isomorphism

$$\mathrm{Hom}_{K_{\alpha+\alpha_i}^\Lambda}(K_{\alpha+\alpha_i}^\Lambda, F_i K_\alpha^\Lambda) \xrightarrow{\sim} \mathrm{Hom}_{R_{\alpha+\alpha_i}^\Lambda}(\widehat{T}_{\alpha+\alpha_i}^\Lambda, F_i \widehat{T}_\alpha^\Lambda), \quad \theta \mapsto \eta_{K_\alpha^\Lambda}^{-1} \circ \pi(\theta).$$

Similarly, $\mathbb{E}_{\alpha+\alpha_i} \circ \mathcal{F}_i \circ \mathbb{E}_\alpha^*$ is defined by tensoring with the $(K_{\alpha+\alpha_i}^\Lambda, K_\alpha^\Lambda)$-bimodule $\mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}_{\alpha+\alpha_i}^\Lambda, \mathcal{F}_i \mathcal{P}_\alpha^\Lambda)$ (where $\mathcal{P}_{\alpha+\alpha_i}^\Lambda$ and $\mathcal{P}_\alpha^\Lambda$ are viewed as right modules via the isomorphism from Theorem 8.18). Lemma 8.9(ii)–(iii) gives us a bimodule isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(\mathcal{P}_{\alpha+\alpha_i}^\Lambda, \mathcal{F}_i \mathcal{P}_\alpha^\Lambda) \xrightarrow{\sim} \mathrm{Hom}_{R_{\alpha+\alpha_i}^\Lambda}(\mathcal{Y}_{\alpha+\alpha_i}^\Lambda, \mathcal{F}_i \mathcal{Y}_\alpha^\Lambda), \quad \theta \mapsto \eta_{\mathcal{P}_\alpha^\Lambda}^{-1} \circ \pi(\theta).$$

So now we are reduced to checking that

$$\mathrm{Hom}_{R_{\alpha+\alpha_i}^\Lambda}(\mathcal{Y}_{\alpha+\alpha_i}^\Lambda, \mathcal{F}_i \mathcal{Y}_\alpha^\Lambda) \cong \mathrm{Hom}_{R_{\alpha+\alpha_i}^\Lambda}(\widehat{T}_{\alpha+\alpha_i}^\Lambda, F_i \widehat{T}_\alpha^\Lambda)$$

as bimodules. This follows from Lemma 8.17 and (8.15). $\square$

Proof of Theorems 1.1 and 1.2. With notation as in Theorem 1.1 from the introduction, let $\Gamma \in P(m, n)$. Choose $o \in \mathbb{Z}$ so that the oth vertex and all vertices to the left of that are labelled $\circ$ in all the weights from Γ. Then clearly the algebra K_Γ from the introduction can be identified with the algebra K_Γ from (3.2) for the index set $I := \{o+1, o+2, \dots\}$. Letting $\Lambda := \Lambda_{o+m} + \Lambda_{o+n}$ as usual and $\alpha := \Lambda - \Gamma$, the equivalence $\mathbb{E}_\alpha$ from Corollary 8.19 gives us an equivalence between $\mathcal{O}_\Gamma \to \mathrm{rep}(K_\Gamma)$. Taking the direct sum of these equivalences over all $\Gamma \in P(m, n)$ gives the equivalence $\mathbb{E}$ required to prove Theorem 1.1.

Now consider Theorem 1.2. The equivalence $\mathbb{E}$ from Theorem 1.1 sets up a bijection between the isomorphism classes of endofunctors of $\mathcal{O}(m, n)$ and of $\mathrm{rep}(K(m, n))$. In particular every indecomposable projective functor $G_{\Delta\Gamma}^t$ on $\mathrm{rep}(K(m, n))$ lifts to an endofunctor of $\mathcal{O}(m, n)$ and vice versa. Therefore it is enough to identify the (at first sight quite different) notions of projective functors on the two sides. This follows from Lemma 3.1, Lemma 4.7 and Theorem 8.23.

9. APPLICATIONS

In this section we give a couple of applications. First, we give a self-contained algebraic proof of a recent conjecture of Khovanov and Lauda from [KLa, §3.4] about the cyclotomic algebra R_α^Λ for level two weights in finite type A. Then we study further the graded cellular basis for R_α^Λ from Theorem 6.9, constructing a special basis for level two Specht modules. This basis has the remarkable property that it also induces a basis in the irreducible quotients of Specht modules. In particular we deduce from this a dimension formula for irreducible R_α^Λ-modules.

The Khovanov-Lauda categorification conjecture for level two. Here we briefly discuss another application of the machinery we have developed, Recall from §2 that U is the quantised enveloping algebra associated to the Lie algebra of $I^+ \times I^+$ matrices. For Λ as in (2.3), let $V(\Lambda)$ denote the irreducible U-module of highest weight Λ. The vector

$$v_+ := V_\iota = (v_{o+m} \wedge \cdots \wedge v_{o+1}) \otimes (v_{o+n} \wedge \cdots \wedge v_{o+1}) \in \textstyle\bigwedge^m V \otimes \bigwedge^n V$$

is a non-zero highest weight vector of weight Λ, and we can identify $V(\Lambda)$ with the submodule Uv_+ of $\bigwedge^m V \otimes \bigwedge^n V$. Recalling that $U_{\mathscr{A}}$ denotes Lusztig's $\mathscr{A}$-form for U, let $V(\Lambda)_{\mathscr{A}} := U_{\mathscr{A}} v_+$, which is the standard $\mathscr{A}$-form for $V(\Lambda)$.

Recall the quasi-canonical basis $\{P_\lambda \,|\, \lambda \in \Lambda(m, n; I)\}$ from (2.12). The following lemma connects this to Lusztig's canonical basis for $V(\Lambda)$.

Lemma 9.1. *The vectors $\{P_\lambda \,|\, \lambda \in \Lambda(m, n; I)^\circ\}$ give a basis for $V(\Lambda)_{\mathscr{A}}$ as a free $\mathscr{A}$-module which up to rescaling each vector by a power of q coincides with Lusztig's canonical basis. More precisely, for $\Gamma \in P(m, n; I)$, Lusztig's canonical basis for the Γ-weight space of $V(\Lambda)_{\mathscr{A}}$ is $\{q^{-\operatorname{def}(\Gamma)}P_\lambda \,|\, \lambda \in \Gamma^\circ\}$ (cf. the last statement from Lemma 3.9).*

Proof. See [BK4, Theorem 2.7], a special case of which treats the case $m \geq n$, together with [BK4, (2.48)] which explains how to deduce the case $m < n$. $\square$

Consider the following functor which is left adjoint to the graded Schur functor π from (8.17):

$$\pi^* := T_\alpha^\Lambda \otimes_{R_\alpha^\Lambda} ? : \operatorname{Rep}(R_\alpha^\Lambda) \to \operatorname{Rep}(K_\alpha^\Lambda). \tag{9.1}$$

Let $\operatorname{Proj}(R_\alpha^\Lambda)$ denote the category of finitely generated projective graded left R_α^Λ-modules, with Grothendieck group $[\operatorname{Proj}(R_\alpha^\Lambda)]$.

Theorem 9.2. *Identify the $\mathscr{A}$-modules $[\operatorname{Rep}(K(m, n; I))]$ and $\bigwedge^m V_{\mathscr{A}} \otimes \bigwedge^n V_{\mathscr{A}}$ as in Theorem 3.5. Then the functor π^* induces an $\mathscr{A}$-module isomorphism*

$$\pi^* : \bigoplus_{\alpha \in Q_+} [\operatorname{Proj}(R_\alpha^\Lambda)] \xrightarrow{\sim} V(\Lambda)_{\mathscr{A}}.$$

Moreover:

(i) *Up to a degree shift, π^* maps the isomorphism classes of projective indecomposable modules to the canonical basis of $V(\Lambda)_{\mathscr{A}}$.*

(ii) *The endomorphisms of $\bigoplus_{\alpha \in Q_+}[\operatorname{Proj}(R_\alpha^\Lambda)]$ induced by the i-restriction and i-induction functors E_i and F_i correspond to the action of the Chevalley generators E_i and F_i of $U_{\mathscr{A}}$.*

Proof. Suppose that $\Gamma := \Lambda - \alpha \in P(m, n; I)$. By Lemma 8.13(i), the free $\mathscr{A}$-module $[\operatorname{Proj}(R_\alpha^\Lambda)]$ has basis $\{[Y(\lambda)] \,|\, \lambda \in \Gamma^\circ\}$. By a standard fact about Schur functors, see e.g. [BK4, Theorem 3.7(ii)], we have that $\pi^*(Y(\lambda)) \cong P(\lambda)$ for each $\lambda \in \Gamma^\circ$. Hence, using also Theorem 3.5(i), the map π^* maps the basis $\{[Y(\lambda)] \,|\, \lambda \in \Gamma^\circ\}$ for $[\operatorname{Proj}(R_\alpha^\Lambda)]$ to $\{P_\lambda \,|\, \lambda \in \Gamma^\circ\}$. By Lemma 9.1, the latter collection of vectors is a basis for the Γ-weight space of $V(\Lambda)_{\mathscr{A}}$ that coincides with Lusztig's canonical basis up to rescaling. This establishes the first statement of the theorem and (i). For (ii), note by Lemma 8.13(iii) and Remark 8.14 that π^* intertwines the i-induction and i-restriction functors E_i and F_i with the special projective functors E_i and F_i. So we are done by Theorem 3.5(ii). $\square$

Theorem 9.2 proves the conjecture formulated by Khovanov and Lauda in [KLa, §3.4] for level two weights in finite type A.

A special basis for level two Specht modules. Fix $\alpha \in Q_+$ of height d such that $\Lambda - \alpha \in P(m, n; I)$ and set $\Gamma := \Lambda - \alpha$. Recall the graded cellular basis for R_α^Λ from Theorem 6.9. For $\lambda \in \Gamma$ we denote the corresponding cell module by $S(\lambda)$ as constructed following the general procedure of Graham and

Lehrer [GL], noting that $S(\lambda)$ is automatically a graded module because our cellular basis is graded. So as a graded vector space $S(\lambda)$ has homogeneous basis

$$\left\{ |\boldsymbol{t}^*[\boldsymbol{\gamma}^*]| \;\middle|\; \begin{array}{l} \text{for all oriented stretched cup diagrams } \boldsymbol{t}^*[\boldsymbol{\gamma}^*] \\ \text{such that } \boldsymbol{\gamma} = \gamma_0 \cdots \gamma_d \text{ with } \gamma_d = \lambda \end{array} \right\},$$

with $\mathbb{Z}$-grading defined according to (6.19). The left action of a basis vector $|\boldsymbol{s}^*[\boldsymbol{\tau}^*]\wr\boldsymbol{r}[\boldsymbol{\sigma}]| \in R_\alpha^\Lambda$ on $|\boldsymbol{t}^*[\boldsymbol{\gamma}^*]| \in S(\lambda)$ can be computed as follows. First compute the left action of $|\boldsymbol{s}^*[\boldsymbol{\tau}^*] \wr \boldsymbol{r}[\boldsymbol{\sigma}]|$ on the basis vector $|\boldsymbol{t}^*[\boldsymbol{\gamma}^*]\,\overline{\lambda}) \in \widehat{T}_\alpha^\Lambda$ using the usual procedure; in particular we get zero unless $\boldsymbol{r} = \boldsymbol{t}$ and all mirror image pairs of internal circles in $\boldsymbol{r}[\sigma]$ and $\boldsymbol{t}^*[\boldsymbol{\gamma}^*]$ are oppositely oriented. Then replace all the diagram basis vectors in the resulting expansion by zero if they do not have the weight λ decorating their top number line, and drop the cap diagram $\overline{\lambda}$ from the very top of all the remaining basis vectors to get back to an element of $S(\lambda)$.

Lemma 9.3. *For $\lambda \in \Gamma$ we have that $S(\lambda) \cong \pi(V(\lambda))$, where π is the graded Schur functor from (8.17).*

Proof. Recall from [BS1, Theorem 5.1] that $V(\lambda)$ is isomorphic to the quotient of $P(\lambda) = K_\alpha^\Lambda e_\lambda$ by the submodule $P'(\lambda)$ spanned by all basis vectors of the form $(a\mu\overline{\lambda})$ with $\mu > \lambda$ in the Bruhat order. Note that

$$\pi(P(\lambda)) = \operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, P(\lambda)) = \operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda e_\lambda) = \operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, K_\alpha^\Lambda)e_\lambda.$$

Using the isomorphism from Lemma 6.6, we deduce that $\pi(P(\lambda)) \cong \widehat{T}_\alpha^\Lambda e_\lambda$. Because $\widehat{T}_\alpha^\Lambda e_\lambda$ is realized explicitly in terms of diagrams, the same is true via this isomorphism for the Young module $Y(\lambda) = \pi(P(\lambda))$ from (8.18). In other words, we can identify $Y(\lambda)$ with the left R_α^Λ-module with basis given by all diagrams of the form (6.26) such that $b = \overline{\lambda}$. Let $Y'(\lambda)$ denote the submodule of $Y(\lambda)$ spanned by all such diagrams in with $\delta_d > \lambda$. Then it is clear from the explicit description of $S(\lambda)$ from the paragraph before the lemma that $S(\lambda) \cong Y(\lambda)/Y'(\lambda)$. Now we claim that $Y'(\lambda) \subseteq \pi(P'(\lambda))$. Given the claim, we get a surjective homorphism

$$S(\lambda) \cong Y(\lambda)/Y'(\lambda) \twoheadrightarrow \pi(P(\lambda))/\pi(P'(\lambda)) = \pi(P(\lambda)/P'(\lambda)) \cong \pi(V(\lambda))$$

and then deduce that $S(\lambda) \cong \pi(V(\lambda))$ by comparing dimensions: forgetting gradings, we have using (6.1) and adjointness that

$$\dim \pi(V(\lambda)) = \dim \operatorname{Hom}_{K_\alpha^\Lambda}(T_\alpha^\Lambda, V(\lambda)) = \sum_{\boldsymbol{i} \in I^\alpha} \dim \operatorname{Hom}_{K_0^\Lambda}(L(\iota), E_{\boldsymbol{i}^*} V(\lambda)),$$

recalling $\boldsymbol{i}^* = (i_d, \ldots, i_1)$. Using [BS2, Theorems 3.5–3.6] and then [BS2, Theorem 4.5], this is equal to the number of oriented stretched cup diagrams $\boldsymbol{t}^*[\boldsymbol{\delta}^*]$ with $\delta_d = \lambda$, i.e. it is the same as the dimension of the cell module $S(\lambda)$.

It remains to prove the claim. Take an element $y \in Y'(\lambda)$ represented under the identification $Y(\lambda) \equiv \widehat{T}_\alpha^\Lambda e_\lambda$ by a basis vector of the form $|\boldsymbol{u}^*[\boldsymbol{\delta}^*]\,\overline{\lambda})$ with $\delta_d > \lambda$. We need to show that the map $\varphi(? \otimes y) : T_\alpha^\Lambda \to P(\lambda)$ has image contained in $P'(\lambda)$. This follows from the explicit diagrammatic description of the map φ from (6.27) together with [BS1, Corollary 4.5]. $\qquad\square$

For the next corollary, recall the classification of the irreducible R_α^Λ-modules $\{D(\lambda) \mid \lambda \in \Gamma^\circ\}$ from Lemma 8.13.

Corollary 9.4. *For each $\lambda \in \Gamma$, the cell module $S(\lambda)$ is indecomposable with irreducible socle isomorphic to $D(\lambda^\circ)\langle \underline{\lambda^\circ \lambda} \rangle$, where $\lambda^\circ \in \Gamma^\circ$ is defined as in [BS2, Theorem 6.6]. Moreover if $\lambda \in \Gamma^\circ$ then $S(\lambda)$ has irreducible head isomorphic to $D(\lambda)$.*

Proof. By [BS2, Theorem 6.6], the cell module $V(\lambda)$ has irreducible socle isomorphic to $L(\lambda^\circ)\langle \deg(\underline{\lambda^\circ \lambda}) \rangle$. Given this and Lemma 8.13, a standard argument involving the Schur functor π shows that $\pi(V(\lambda))$ has irreducible socle isomorphic to $D(\lambda^\circ)\langle \deg(\underline{\lambda^\circ \lambda}) \rangle$. In view of Lemma 9.3, this proves the statement about the socle of $S(\lambda)$, hence $S(\lambda)$ is indecomposable as its socle is irreducible. Finally if $\lambda \in \Gamma^\circ$ then $V(\lambda)$ has irreducible head $L(\lambda)$ and a similar argument shows that $S(\lambda) \cong \pi(V(\lambda))$ has irreducible head $D(\lambda) = \pi(L(\lambda))$. $\square$

Corollary 9.5. *On forgetting the grading, we have that $S(\lambda) \cong \pi(\mathcal{V}(\lambda))$, where π is the ungraded Schur functor from (8.11).*

Proof. This follows from Lemma 9.3 using Corollary 8.20 and the fact that $\mathbb{E}_\alpha(\mathcal{V}(\lambda)) \cong V(\lambda)$ by Corollary 8.19. $\square$

Corollary 9.6. *For any $\lambda \in \Gamma$, the cell module $S(\lambda)$ is isomorphic to the graded Specht module from [BKW] parametrized by the bipartition obtained from λ by applying the map from Remark 2.1 (taking $e := 0$, $l := 2$ and $(k_1, k_2) := (o + m, o + n)$ in [BKW]).*

Proof. By a special case of [BK4, Theorem 3.7] if $m \geq n$, or [BK4, Theorem 4.15] if $m \leq n$, it is known that $\pi(\mathcal{V}(\lambda))$ is isomorphic to the Specht module from [BKW] as an ungraded module. Hence by Corollary 9.5 we get that $S(\lambda)$ is isomorphic to the Specht module on forgetting gradings. Since it is an indecomposable module by Corollary 9.4, it follows from this and the unicity of gradings from [BGS, Lemma 2.5.3] that $S(\lambda)$ is isomorphic to the graded Specht module up to a shift in grading. Finally to see that no shift in grading is required, we observe that $S(\lambda)$ has the same graded dimension as the graded Specht module. This follows because the two modules have homogeneous bases indexed by certain sets of oriented stretched cap diagrams and of standard bitableaux, respectively, and these two sets are in bijection in a way that respects the degrees of the two bases thanks to Remark 6.5. $\square$

By the general theory of cellular algebras, the cell module $S(\lambda)$ is equipped with a symmetric bilinear form $(.,.)$ which is associative in the sense that $(xv, w) = (v, x^* w)$ for all $x \in R_\alpha^\Lambda$ and $v, w \in S(\lambda)$, where $*$ is the antiautomorphism from (6.30); see e.g. [GL, Definition 2.3]. Using the map φ from (6.27), we can reformulate the definition of this form as follows. For basis vectors $|\boldsymbol{t}^*[\boldsymbol{\gamma}^*]|, |\boldsymbol{u}^*[\boldsymbol{\delta}^*]| \in S(\lambda)$, their inner product $(|\boldsymbol{t}^*[\boldsymbol{\gamma}^*]|, |\boldsymbol{u}^*[\boldsymbol{\delta}^*]|)$ is the coefficient of $e_\lambda = (\underline{\lambda}\lambda\overline{\lambda})$ when $\varphi((\underline{\lambda}\,\boldsymbol{u}[\boldsymbol{\delta}]| \otimes |\boldsymbol{t}^*[\boldsymbol{\gamma}^*]\,\overline{\lambda}))$ is expanded in terms of the diagram basis of K_α^Λ. The following lemma gives a more concrete description.

Lemma 9.7. *For $|\boldsymbol{t}^*[\boldsymbol{\gamma}^*]|, |\boldsymbol{u}^*[\boldsymbol{\delta}^*]| \in S(\lambda)$, the inner product $(|\boldsymbol{t}^*[\boldsymbol{\gamma}^*]|, |\boldsymbol{u}^*[\boldsymbol{\delta}^*]|)$ is equal to 1 if $\boldsymbol{t}^* = \boldsymbol{u}^*$, all matching pairs of internal circles in $\boldsymbol{t}^*[\boldsymbol{\gamma}^*]$ and*

$\boldsymbol{t}^*[\boldsymbol{\delta}^*]$ *are oppositely oriented, and all boundary circles in both diagrams are anti-clockwise; otherwise it is zero.*

Proof. This follows from the diagrammatic description of the map φ. $\qquad\square$

Note in particular that the bilinear form $(.,.)$ on $S(\lambda)$ is homogeneous of degree zero, and the $\boldsymbol{i}$-weight spaces $e(\boldsymbol{i})S(\lambda)$ for different $\boldsymbol{i} \in I^\alpha$ are orthogonal. Let $\operatorname{rad} S(\lambda)$ denote the radical of the form $(.,.)$, which is a graded R_α^Λ-submodule of $S(\lambda)$. By general theory again, the non-zero $S(\lambda)/\operatorname{rad} S(\lambda)$'s give a complete (up to grading shift) set of non-isomorphic irreducible $S(\lambda)$-modules; see [GL, Theorem 3.4].

Theorem 9.8. *For $\lambda \in \Gamma$, we have that $S(\lambda)/\operatorname{rad} S(\lambda) \neq \{0\}$ if and only if $\lambda \in \Gamma^\circ$. Moreover $S(\lambda)/\operatorname{rad} S(\lambda) \cong D(\lambda)$ for each $\lambda \in \Gamma^\circ$.*

Proof. Suppose that $S(\lambda)/\operatorname{rad} S(\lambda) \neq \{0\}$, i.e. the form $(.,.)$ on $S(\lambda)$ is non-zero. Then by Lemma 9.7 there exists at least one oriented stretched cup diagram $\boldsymbol{t}^*[\boldsymbol{\gamma}^*]$ with $\gamma_d = \lambda$ in which every generalised cup is anti-clockwise. As $\operatorname{def}(\Gamma) = \operatorname{cups}(\boldsymbol{t}^*) - \operatorname{caps}(\boldsymbol{t}^*)$, the lower reduction a of $\boldsymbol{t}^*$ has exactly $\operatorname{def}(\Gamma)$ cups. As every generalised cup in $a\lambda$ is anti-clockwise, we must have that $a = \underline{\lambda}$. Hence $\operatorname{def}(\lambda) = \operatorname{def}(\Gamma)$, so $\lambda \in \Gamma^\circ$. To complete the proof, it remains to observe that the number of isomorphism classes of irreducible R_α^Λ-modules up to grading shift is equal to the cardinality of the set Γ°. This is a consequence of Corollary 8.6, since T_α^Λ has exactly $|\Gamma^\circ|$ non-isomorphic indecomposable summands up to grading shift by Lemma 6.1. The final statement now follows using also Corollary 9.4. $\qquad\square$

Observe finally from Lemma 9.7 that the diagram basis for $S(\lambda)$ is special in the sense that it contains a basis for $\operatorname{rad} S(\lambda)$. In other words, the non-zero vectors obtained by considering the canonical images of our basis vectors in the quotient $D(\lambda) \equiv S(\lambda)/\operatorname{rad} S(\lambda)$ give a basis for the irreducible module $D(\lambda)$.

Lemma 9.9. *For $\boldsymbol{i} \in I^\alpha$, $j \in \mathbb{Z}$ and $\lambda \in \Gamma^\circ$, the dimension of the homogeneous component of $e(\boldsymbol{i})D(\lambda)$ of degree j is equal to the number of oriented stretched cap diagrams of the form $\boldsymbol{t}[\boldsymbol{\gamma}]$ with the following properties:*

   (i) *the admissible sequence underlying $\boldsymbol{t}$ is equal to $\boldsymbol{i}$;*
   (ii) *the degree of $\boldsymbol{t}[\boldsymbol{\gamma}]$ in the sense of (6.18) is equal to j.*
   (iii) $\gamma_d = \lambda$;
   (iv) *all boundary caps of $\boldsymbol{t}[\boldsymbol{\gamma}]$ are anti-clockwise.*

Proof. This follows from Lemma 9.7. $\qquad\square$

We finish with an example to illustrate Lemma 9.9.

Example 9.10. Take $o = 0$, $m = n = 2$, $\alpha = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ and $\boldsymbol{i} = (2,3,4,1,2,2,3,1)$ as in Remark 6.5. Let λ be the weight corresponding to the bipartition $((1,1),(3,3))$ under the bijection from Remark 2.1; this corresponds to the weight at the bottom of the diagram (6.22). Then the graded dimension of $e(\boldsymbol{i})D(\lambda)$ is $q + q^{-1}$, and its basis is given by the images of the diagram basis vectors of $e(\boldsymbol{i})S(\lambda)$ parametrized by the oriented stretched cap diagram from (6.22) and the one obtained from that by reversing the orientation of the

internal circle. Under the bijection from Remark 6.5 these diagrams map to the following two standard bitableaux, which are of degrees 1 and -1, respectively:

$$\left(\begin{array}{|c|}\hline 5 \\ \hline 8 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 7 \\ \hline\end{array}\right), \qquad \left(\begin{array}{|c|}\hline 6 \\ \hline 8 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline\end{array}\right).$$

Similar considerations show rad $e(\boldsymbol{i})S(\lambda)$ has graded dimension $2 + q + 2q^2 + q^3$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403, USA
E-mail address: `brundan@uoregon.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BONN, 53115 BONN, GERMANY
E-mail address: `stroppel@math.uni-bonn.de`