S. SASAO AND H. TAKAHASHI
KODAI MATH. J.
2 (1979), 139—147

HIGHLY CONNECTED POINCARÉ COMPLEXES

Dedicated to Professor A. Komatu on his 70th birthday

BY SEIYA SASAO AND HIDEO TAKAHASHI

Introduction.

We are interested in the following problem proposed by Wall in [2] "Classify up to homotopy $(n-1)$-connected Poincaré complexes of dimension $2n+1$ and $2n+2$."

In this paper we shall discuss the case of dimension $2n+2$ under some additional conditions. Let K be a Poincaré complex which is $(n-1)$-connected and of dimension $2n+2$. If K has the same rational homology as the sphere, then the homology $H_*(K;Z)$ is as follows

$$H_0(K;Z)=Z=H_{2n+2}(K;Z)$$

$$H_n(K;Z)=G=H_{n+1}(K;Z)$$

$$H_i(K;Z)=0 \qquad \text{for other dimensions,}$$

where G denotes a finite abelian group. We denote by $P(n, n+1;G)$ the complex K such as above and call it a Poincaré complex of type $(n, n+1;G)$. Then our main results are

THEOREM A. *Let $n\geqq 3$ and $G\otimes Z_2=0$. Then $P(n, n+1;G)$ has the same homotopy type as the connected sum of $P(n, n+1;G_1)$ and $P(n, n+1;G_2)$ if G is a direct sum of G_1 and G_2.*

THEOREM B. *Under the same conditions as Theorem* A, *if $P(n, n+1;G)$ is S-reducible it's homotopy type is unique with respect to n and G.*

By applying these theorems to the case of manifolds we shall prove

THEOREM C. *Let M be a $(n-1)$-connected rational homology sphere which is a smooth manifold of dimension $2n+2$ with no 2-torsion. Then M is uniquely determined up to homotopy by homology for $n\equiv 0, 1 \bmod 4$.*

The case of $G\otimes Z_2\neq 0$ (essencially, G is a 2-group) is more complicated, therefore we shall discuss it in the subsequent paper.

Received October 14, 1977

The plan of this paper is as follows. First, in § 1, we study the homotopy of Moore spaces and in § 2 characterize Poincaré complexes of type $(n, n+1; G)$. In § 3 we shall prove Theorem A and B, and in § 4 the proof of Theorem C shall be given. Throughout this paper we assume that groups $G, H, \cdots$ are finite abelian with no 2-torsion and $n \geqq 3$.

§ **1.** Homotopy of Moore spaces.

We denote by M_G^n the Moore space of type (n, G) and by $\sharp$ the integer $2n+1$. We first note the following easy

LEMMA 1.1. *$\pi_i(M_G^n)$ is trivial for $i=n+1$, $n+2$.*

Now we define a homomorphism

$$\mu_H^G : \pi_\sharp(M_G^n \vee M_H^{n+1}) \longrightarrow \mathrm{Hom}(G, H)$$

by $$\mu_H^G(f)=\mu_f\frown : G=H^{n+1}(M_G^n ; Z)=H^{n+1}(c(f) ; Z) \longrightarrow H_{n+1}(c(f) ; Z)=H_{n+1}(M_H^{n+1} ; Z)=H,$$

where $c(f)$ denotes the mapping cone for a map $f: S^\sharp \to M_G^n \vee M_H^{n+1}$ and μ_f is the oriented generator of $H_{2n+2}(c(f) ; Z)$. Let h be a map $M_G^n \vee M_H^{n+1} \to M_{G'}^n \vee M_{H'}^{n+1}$. Clearly h is decomposed into the sum of four maps;

$$h_1 : M_G^n \longrightarrow M_{G'}^n,\ h_2 : M_G^n \longrightarrow M_{H'}^{n+1},\ h_3 : M_H^{n+1} \longrightarrow M_{G'}^n \text{ and } h_4 : M_H^{n+1} \to M_{H'}^{n+1}.$$

Then, from the commutative diagram

$$\begin{array}{ccc} H^{n+1}(c(f) ; Z) & \xrightarrow{\mu_f \frown} & H_{n+1}(c(f) ; Z) \\ h_1^* \uparrow & & \downarrow h_{4*} \\ H^{n+1}(c(hf) ; Z) & \xrightarrow[\mu_{hf}\frown]{} & H_{n+1}(c(hf) ; Z), \end{array}$$

we obtain

LEMMA 1.2. *(The naturality of μ_H^G) $\mu_{H'}^{G'}(hf)=h_{4*}\mu_H^G(f)h_1^*$.*

Now we prove

PROPOSITION 1.3. $\pi_\sharp(M_G^n \vee M_H^{n+1})=\pi_\sharp(M_G^n) \oplus \pi_\sharp(M_H^{n+1}) \oplus \mathrm{Hom}(G, H)$

Proof. The proof follows from the standard isomorphism

$$\pi_\sharp(M_G^n \vee M_H^{n+1})=\pi_\sharp(M_G^n) \oplus \pi_\sharp(M_H^{n+1}) \oplus \partial\pi_{\sharp+1}(M_G^n \times M_H^{n+1}, M_G^n \vee M_H^{n+1})$$

if we can show that the restriction μ_H^G on the third factor is an isomorphism. Thus, by using isomorphisms

$$\pi_{\sharp+1}(M_G^n \times M_H^{n+1}, M_G^n \vee M_H^{n+1})=\pi_{\sharp+1}(M_G^n \wedge M_H^{n+1})=\pi_{\sharp+1}(M_{G\otimes H}^{2n+1} \vee M_{G*H}^{2n+2}),$$

where $\wedge$ denotes the smash product, the proof can be reduced to the case of $G=Z_{p^i}$ and Z_{p^j}. Let α be the generator of

$$\pi_{\#+1}(M_G^n\times M_H^{n+1},\ M_G^n\vee M_H^{n+1})\cong H_{\#+1}(M_G^n\times M_H^{n+1},\ M_G^n\vee M_H^{n+1};Z)\,.$$

Then there exists a map $\varphi: c(f)\to M_G^n\times M_H^{n+1}$ ($f=\partial\alpha$) such that $\varphi_*: H_{2n+2}(c(f),Z)\to H_{2n+2}(M_G^n\times M_H^{n+1};Z)$ is surjective and $\varphi|M_G^n\vee M_H^{n+1}=$identity. Consider the commutative diagram

$$\begin{array}{ccc}
Z_{p^i}=H^{n+1}(c(f);Z) & \xrightarrow{\quad\mu_f\cap\quad} & H_{n+1}(c(f);Z)=Z_{p^j} \\
\uparrow & & \downarrow \\
Z_{p^i}=H^{n+1}(M_G^n\times M_H^{n+1};Z) & \xrightarrow[\varphi_*(\mu_f)\cap]{} & H_{n+1}(M_G^n\times M_H^{n+1};Z)=Z_{p^j}\,.
\end{array}$$

Then the proof is obtained from $\varphi_*(\mu_f)\cap 1=p^{j-k}(1)$ ($k=\min(i,\ j)$).

Now we investigate the N-fold suspension

$$E^N:\ \pi_\#(M_i^n\vee M_i^{n+1})\longrightarrow\pi_{\#+N}(M_i^{n+N}\vee M_i^{N+1+n})\qquad(N\longrightarrow\infty)\,,$$

where M_i^n denotes M_G^n for $G=Z_{p^i}$. First, in the decomposition given by Proposition 1.3, we can easily obtain

$$E(\mathrm{Hom}\,(Z_{p^i},\ Z_{p^i}))=0\quad\text{and}\quad E^{-N}(0)\cap\pi_\#(M_i^{n+1})=[\pi_{n+1}(M_i^{n+1}),\ \pi_{\#+1}(M_i^{n+1})]\,,$$

where $[\ ,\ ]$ denotes the Whitehead product. Next, let $M_{i,\infty}^n$ be the reduced product for M_i^n. Using $\pi_{\#+1}(M_{i,\infty}^n,\ M_i^n)=0$ and the homotopy exact sequence of the pair $(M_{i,\infty}^n,\ M_i^n)$, we have

LEMMA 1.4. *$E:\ \pi_\#(M_i^n)\longrightarrow\pi_{\#+1}(M_i^{n+1})$ is injective.*

For the investigation of $E:\ \pi_{\#+1}(M_i^{n+1})\to\pi_{\#+2}(M_i^{n+2})$ we define a homomorphism $h_n:\pi_{2n}(M_i^n)\to Z_{p^i}$ as follows. Let $c(f)=M_i^n\cup e^\#$ be the mapping cone for a map $f: S^{2n}\to M_i^n$ and let α, β, γ be generators of $H^n(c(f);Z_{p^i})$, $H^{n+1}(c(f);Z_{p^i})$ and $H^{2n+1}(c(f);Z_{p^i})$ respectively. Then put $\mu f\cap(\alpha\cup\beta)=h_n(f)$.

LEMMA 1.5. (1) $h_n(E\pi_{2n-1}(M_i^{n-1}))=0$
(2) *if n is even, h_n is trivial*
(3) *if n is odd, h_n is surjective*

Proof. (1) follows from the definition of h_n and (2) is deduced from applying the Bockstein operator. For (3), consider the boundary homomorphism ∂: $\pi_{2n+1}(M_{i,\infty}^n,\ M_i^n)=Z_{p^i}\to\pi_{2n}(M_i^n)$. We assert

$$h_n(\partial(1))=\text{a generator of } Z_{p^i}\,.$$

Clearly there exists a map $\psi:\ c(f)\to M_{i,\infty}^n$ such that $\psi|M_i^n=$identity and $\psi_*:\ H_{2n+1}(c(f);Z)=Z\to H_{2n+1}(M_{i,\infty}^n;Z)$ is surjective. Then our assertion follows from the cohomologyring structure of $M_{i,\infty}^n$.

LEMMA 1.6. *$E^2:\ \pi_\#(M_i^n)\to\pi_{\#+2}(M_i^{n+2})$ is injective.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & & & \pi_{\#}(M_{\iota}^{n}) & & \\ & & & & \downarrow & & \\ \pi_{\#+2}(M_{\iota,\infty}^{n+1}) & \xrightarrow[j]{} & \pi_{\#+2}(M_{\iota,\infty}^{n+1}, M_{\iota}^{n+1}) & \longrightarrow & \pi_{\#+1}(M_{\iota}^{n+1}) & \xrightarrow[i]{} & \pi_{\#+1}(M_{\iota,\infty}^{n+1}) \\ & & \| & & \downarrow h_{n+1} & & \\ & & Z_{p^{\iota}} & & Z_{p^{\iota}} & & \end{array}$$

If n is even, the proof follows from lemma 1.4 and (1) of lemma 1.5.
If n is odd, j is surjective by (1) and (3) of lemma 1.5, and hence i is injective. Thus the proof is completed.

Thus, from combining lemmas, we have

PROPOSITION 1.7. *The kernel of E^N is the subgroup*

$$[\pi_{n+1}(M_{\iota}^{n+1}), \pi_{n+1}(M_{\iota}^{n+1})] \oplus \mathrm{Hom}(Z_{p^{\iota}}, Z_{p^{\iota}}).$$

Now let ι_{n+1} be the generator of $\pi_{n+1}(M_{\iota}^{n+1})$ and define the map $\nu_r: M_{\iota}^{n} \vee M_{\iota}^{n+1} \to M_{\iota}^{n} \vee M_{\iota}^{n+1}$ by $\nu_r | M_{\iota}^{n+1} = \text{identity}$ and $\nu_r | M_{\iota}^{n} = \text{identity} + r\iota_{n+1} \circ id/S^n$. For the later, we note

LEMMA 1.8. *For $id \in \mathrm{Hom}(Z_{p^{\iota}}, Z_{p^{\iota}}) \subset \pi_{\#}(M_{\iota}^{n} \vee M_{\iota}^{n+1})$ we have $\nu_{r*}(id) = r[\iota_{n+1}, \iota_{n+1}] + id$.*

Proof. Since $E^N(id)=0$, by Proposition 1.8, $\nu_{r*}(id)$ has a representation

$$\nu_{r*}(id) = x[\iota_{n+1}, \iota_{n+1}] + y(id)$$

for some integers x and y. Then $y=1$ follows from the naturality of cup-product and $x=r$ is easily deduced from the cohomology ring structure of the mapping cone for id.

§2. Poincaré complexes of type $(n, n+1; G)$.

First we note

LEMMA 2.1. *$P(n, n+1; G)$ has the same homotopy type as the mapping cone for a map $f: S^{\#} \to M_G^n \vee M_G^{n+1}$.*

Remark: This is not true in the case of $G \otimes Z_2 \neq 0$.

Proof. Let X be a Poincaré complex of type $(n, n+1; G)$. Since $\pi_i(X)=0$ $(0 \leqq i \leqq n-1)$ and $\pi_n(X)=G$, we may regard M_G^n as a subcomplex of X. Then we have

$$\pi_{n+1}(X) \cong \pi_{n+1}(X, M_G^n) \cong H_{n+1}(X, M_G^n) \cong H_{n+1}(X) \cong G,$$

using lemma 1.1 and the homotopy-homology exact sequence of the pair (X, M_G^n). Hence there is a map $g: M_G^{n+1} \to X$ such that

$$g_*: H_{n+1}(M_G^{n+1}; Z) \longrightarrow H_{n+1}(X; Z)$$

is an isomorphism. Then, since the map $id \vee g: M_G^n \vee M_G^{n+1} \to X$ induces an isomorphism of homology up to dimension $2n+1$ the proof is completed by the standard argument.

Thus, from the point of view of homotopy, we can replace a complex of type $(n, n+1; G)$ with $c(f)$.

LEMMA 2.2 *$c(f)$ is a Poincaré complex if and only if $f(\in \pi_{\#}(M_G^n \vee M_G^{n+1}))$ is contained in the subgroup*

$$\pi_{\#}(M_G^n) \oplus \pi_{\#}(M_G^{n+1}) \oplus \mathrm{Aut}\, G .$$

Proof. The part "only if" follows from the definition of decomposition in Proposition 1.3. For the part "if" we must show that two homomorphisms

(1) $\mu_f \frown : H^{n+1}(c(f); Z) \longrightarrow H_{n+1}(c(f); Z)$

(2) $\mu_f \frown : H^{n+2}(c(f); Z) \longrightarrow H_n(c(f); Z)$

are both isomorphisms, where μ_f denotes the generator of $H_{2n+2}(c(f); Z)$.

Clearly (1) holds by the definition. Let Z_{pi}, Z_{pj} be two direct summands of G and let p_i (p_j) be the projection $G \to Z_{pi}$ (Z_{pj}). Since p_i, p_j naturally induce the maps

$$\hat{p}_i : M_G^n \longrightarrow M_i^n \quad \text{and} \quad \hat{p}_j : M_G^{n+1} \longrightarrow M_j^{n+1} \qquad (M_i^n = M_{Z_{pi}}^n),$$

we have the map

$$\hat{p}_i \vee \hat{p}_j = p : M_G^n \vee M_G^{n+1} \longrightarrow M_i^n \vee M_j^{n+1} .$$

On the other hand, by lemma 1.2, we may suppose that f has a representation $f = \alpha \oplus \beta \oplus id$ (Proposition 1.3). Then we have

$$p_*(f) = \hat{p}_{i*}(\alpha) \oplus \hat{p}_{j*}(\beta) \oplus id \qquad \text{if} \quad Z_{pi} = Z_{pj} \tag{2.3}$$

$$= \hat{p}_{i*}(\alpha) \oplus \hat{p}_{j*}(\beta) \qquad \text{if} \quad Z_{pi} \neq Z_{pj} , \tag{2.4}$$

using lemma 1.2. Let $\hat{p}$ be the map: $c(f) \to c(pf)$ which is the natural extension of p and consider the commutative diagram

$$\begin{array}{ccc} G = H^{n+2}(c(f); Z) & \xrightarrow{\mu_f \frown} & H_n(c(f); Z) = G \\ \uparrow \hat{p}_* & & \downarrow \hat{p}_* = p_i \\ Z_{pj} = H^{n+2}(c(pf); Z) & \xrightarrow{\mu_{pf} \frown} & H_n(c(pf); Z) = Z_{pi} . \end{array}$$

We assert that

$$\mu_{pf} \frown Z_{pi} = 0 \qquad \text{if} \quad Z_{pi} \neq Z_{pj}$$

$$= Z_{pi} \qquad \text{if} \quad Z_{pi} = Z_{pj} .$$

The case of $Z_{pi} \neq Z_{pj}$. By (2.4) there exists a map

$$q: c(pf) \longrightarrow c(\hat{p}_i\alpha)\vee c(\hat{p}_j\beta)$$

such that $q|M_i^n\vee M_j^{n+1}=\iota d$ and $q_*(\mu_{pf})=\mu_{\hat{p}_i\alpha}+\mu_{\hat{p}_j\beta}$. Since $\mu_{\hat{p}_i\alpha}$ and $\mu_{\hat{p}_j\beta}$ are both trivial we have that $\mu_{pf}\frown$ is also trivial.

The case of $Z_{p^\iota}=Z_{p^j}$. For our purpose it is sufficient to consider Z_p-coefficient instead of Z-coefficient. Then we can take generators $x(\in H^n(c(pf); Z_p))$ and $y(\in H^{n+1}(c(pf); Z_p))$ such that $\beta_\iota x$ and $\beta_j y$ both generators, where β_ι denotes the Bockstein operator. Thus, using Kronecker product and (2.3), we have

$$\langle x, \mu_{pf}\frown\beta_\iota y\rangle=\pm\langle x\cup\beta_\iota y, \mu_{pf}\rangle=\pm\langle\beta_\iota x\cup y, \mu_{pf}\rangle$$
$$=\pm\langle y, \mu_{pf}\frown\beta_\iota x\rangle=\pm 1\,.$$

These show our assertion, and therefore the proof of (2) is completed.

3. The proof of Theorem A and B.

First we replace a space of type $(n, n+1; G)$ with $c(f)$ by lemma 2.1. Let $G=G_1\oplus G_2$ and let $Z_{p^i}(x)$, $Z_{p^j}(y)$ be direct summands of G_1 and G_2 respectively. By the decomposition

$$\pi_\#(M_G^n\vee M_G^{n+1})=\pi_\#(M_G^n)\oplus\pi_\#(M_G^{n+1})\oplus \mathrm{Hom}\,(G, G)$$
$$=\pi_\#(M_{G_1}^n)\oplus\pi_\#(M_{G_2}^n)\oplus\pi_\#(M_{G_1}^{n+1})\oplus\pi_\#(M_{G_2}^{n+1})\oplus[G_1, G_2]\oplus \mathrm{Hom}\,(G, G)\,,$$

where we identify G_ι with $\pi_{n+1}(M_{G_i}^{n+1})$, we may suppose that f has the representation

$$f=\alpha_1+\alpha_2+\beta_1+\beta_2+\sum_{x,y} s[x, y]+id\,.$$

For fixed $Z_{p^i}(x_0)$ and $Z_{p^j}(y_0)$, let p_0 be the map $M_G^n\to M_\iota^n$ induced by the projection $G_1\to Z_{p^\iota}(x_0)$ and let p_0^r be the composite map

$$M_{G_1}^n \underset{p_0}{\longrightarrow} M_\iota^n \longrightarrow M_\iota^n/S^n=S^{n+1} \underset{ry_0}{\longrightarrow} M_{G_2}^{n+1}\,.$$

Consider the map $F_r: M_G^n\vee M_G^{n+1}\to M_G^n\vee M_G^{n+1}$ defined by $F_r|M_{G_1}^{n+1}$=identity, $F_r|M_{G_2}^{n+1}$=identity, $F_r|M_{G_2}^n$=identity and $F_r|M_{G_1}^n$=identity$+p_0^r$. F_r is clearly a homotopy equivalence and we prove

(1) $F_{r*}(\alpha_2)=\alpha_2\,,\qquad F_{r*}(\beta_\iota)=\beta_\iota\qquad(\iota=1, 2)$

(2) $F_{r*}([x, y])=[x, y]$

(3) $F_{r*}(\alpha_1)=\alpha_1+p_{0*}^r(\alpha_1)$

(4) $F_{r*}(\iota d)=\iota d+r[x_0, y_0]\,.$

For, (1) and (2) are obvious by the definition of F_r and (3) follows from $E\pi_{2n}(M_{G_1}^{n-1})=\pi_{2n+1}(M_{G_1}^n)$. Since it is easy to obtain

$$F_{r*}(\iota d)=\iota d+\sum_{x,y} a[x, y]$$

we must determine a for each x, y. Now consider the commutative diagram

$$\begin{array}{ccc} M_{G_1}^n \vee M_{G_1}^{n+1} \vee M_{G_2}^n \vee M_{G_2}^{n+1} & \xrightarrow{F_r} & M_{G_1}^n \vee M_{G_1}^{n+1} \vee M_{G_2}^n \vee M_{G_2}^{n+1} \\ {\scriptstyle p_x^n \vee p_y^{n+1} \vee p_x^n \vee p_y^{n+1}} \downarrow & & \downarrow {\scriptstyle p_x^n \vee p_x^{n+1} \vee p_y^n \vee p_y^{n+1}} \\ M_i^n \vee M_i^{n+1} \vee M_j^n \vee M_j^{n+1} & \xrightarrow[G_r]{} & M_i^n \vee M_i^{n+1} \vee M_j^n \vee M_j^{n+1} = X_{x,y}\,, \end{array}$$

where $G_r = \iota d \vee \iota d \vee \iota d \vee \iota d$ $((x, y) \neq (x_0, y_0))$, p_x^n is the map $M_G^n \to M_i^n$ induced by the projection $G \to Z_{pi}(x)$, and

$$G_r = (\iota d + r y_0 \circ M_i^n / S^n) \vee \iota d \vee \iota d \vee \iota d \qquad ((x, y) = (x_0, y_0))\,.$$

Then we have

$$G_{r*}(\iota d) = \iota d + a[x, y]\,.$$

Let α_x, β_x be generators for $H^{n+1}(M_i^n; Z_{pk})$ and $H^{n+1}(M_i^{n+1}; Z_{pk})$ (k=mim (ι, j)) respectively and we denote by $\hat{X}_{x,y}$ the mapping cone for $\iota d \in \pi_\#(X_{x,y})$. In the cohomology ring $H^*(\hat{X}_{x,y}; Z_{pk})$, we have

$$\alpha_x \cup \beta_x = a \text{ generator} \quad \text{and} \quad \beta_x \cup \beta_y = 0\,.$$

On the other hand, in the cohomology ring $H^*(c(G_r(\iota d))$, we have $\beta_x \cup \beta_y = a(1)$. Hence the proof of (4) follows from

$$a(1) = G_{r*}(\beta_x) \cup G_{r*}(\beta_y) = \beta_x \cup \beta_y = 0 \qquad ((x, y) \neq (x_0, y_0))$$

$$= \beta_x \cup (r\alpha_x + \beta_y) = r(1) \qquad ((x, y) = (x_0, y_0))\,.$$

Thus the proof of Theorem A is completed by using iteratedly F_r for various r.

Especially we have

COROLLARY 3.1. *Let* $G = \sum_p \sum_\iota \sum Z_{p\iota}$ *be the direct-sum decomposition of* G. *Then* $P(n, n+1; G)$ *has the same homotopy type as the connected sum of* $P(n, n+1; Z_{p\iota})$*s.*

Next we consider the proof of Theorem B. Let $G = \sum_p \sum_\iota \sum Z_{p\iota}$ and let x be the generator of a $Z_{p\iota}$-component. We denote by $M_i^n(x)$ the Moore space corresponding to the $Z_{p\iota}$-component generated by x. By Corollary 3.1 we may assume that $P(n, n+1; G)$ has a decomposition

$$P(n, n+1; G) = (\bigvee_x M(x)) \bigcup_f e^{2n+2}\,, \qquad f = \bigoplus_x \sigma_x \qquad (\sigma_x = f_x + f_x' + \iota d)\,,$$

where $M(x)$ is the space $M_i^n(x) \vee M_i^{n+1}(x)$ and $\sigma_x \in \pi_\#(M(x))$. If $P(n, n+1; G)$ is S-reducible we can know from Proposition 1.7 that

$$f_x = 0 \quad \text{and} \quad f_x' \in [\pi_{n+1}(M_i^{n+1}(x)),\ \pi_{n+1}(M_i^{n+1}(x))]$$

Then, by applying the map F_r, the proof is completed.

§ **4.** π-manifolds.

We describe a closed smooth manifold as a manifold of type $(n, n+1; G)$ if it's underlying Poincaré complex is of type $(n, n+1; G)$.

If M is a π-manifold of type $(n, n+1; G)$, M is S-reducible and hence it's homotopy type is unique with respect to n and G by Theorem B. Conversely we prove

PROPOSITION 4.1. *If K is a S-reducible Poincaré complex of type $(n, n+1; G)$, then K has the homotopy type of a π-manifold.*

Proof. Consider the product manifold $S^n \times S^{n+2}$ and let ι be the generator of $\pi_n(S^n \times S^{n+2})$. Since $S^n \times S^{n+2}$ is a π-manifold, a new π-manifold K_m is obtained from killing the class $m\iota$ (Theorem 2 of [1]). Clearly K_m is a π-manifold of type $(n, n+1; Z_m)$ and hence it's homotopy type is unique. Then the proof is completed by Theorem B and Corollary 3.1.

Next, for the proof of Theorem C, we prove

PROPOSITION 4.2. *Let $n \equiv 0,1$ mod 4. Then manifolds of type $(n, n+1; G)$ are all π-manifolds.*

Proof. Let M be a manifold of type $(n, n+1; G)$ and let ν_M be the stable normal bundle for M. By lemma 2.1 we may suppose

$$M = (M_G^n \vee M_G^{n+1}) \cup e^{2n+2} \qquad \text{(up to homotopy)}$$

Let P be the natural map $M \to S^{2n+2} = M / M_G^n \vee M_G^{n+1}$. Then, from Puppe's sequence, we obtain two isomorphisms

$$P^*: Z = [S^{2n+2}, BO]_0 \longrightarrow [M, BO]_0 \qquad (n \equiv 1 \bmod 4)$$

$$P^*: Z_2 = [S^{2n+2}, BO]_0 \longrightarrow [M, BO]_0 \qquad (n \equiv 0 \bmod 4).$$

Thus, there exists a bundle ξ over S^{2n+2} with $P^*(\xi) = \nu_M$. Since the Thom space $T(\nu_M)$ is S-reducible and P is of degree 1, $T(\xi)$ is also reducible, hence we have $J(\xi)=0$. If $n \equiv 1$ mod 4, $J(\xi)=0$ is equivalent to $\xi=0$. Therefore we have $\nu_M = p^*(\xi) = 0$. If $n \equiv 1$ mod 4, ξ is determined by it's Pontrijagin class. Using Hirzeburch formula for ν_M and Index $(M)=0$, we can know that the top Pontrijagin class of ν_M is zero. Thus we get $\xi=0$, i.e. $\nu_M=0$.

Now Theorem C is clear from Proposition 4.2. Finally we note

PROPOSITION 4.3. *Let M be an almost parallerizable manifold of type $(n, n+1; G)$. Then M is a π-manifold and hence it's homotopy type is unique with respect to n and G.*

Proof. Let ν_M be the stable normal bundle for M. Since the restriction $\nu_M | M_G^n \vee M_G^{n+1}$ is trivial, the proof follows from the same argument as the proof of Proposition 4.2.

REFERENCES

[1] J. W. MILNOR, A procedure for killing homotopy groups of differentiable manifolds, Differential Geometry: Symposia in Pure Math, A. M. S. 1961, 39-55.
[2] C. T. C. WALL, Manifolds-Tokyo 1973, 429-430.

DEPT. OF MATH.
TOKYO INST. OF TECHNOLOGY.
OH-OKAYAMA MEGURO-KU,
TOKYO, JAPAN.