Hilbert C^*-bimodules over commutative C^*-algebras and an isomorphism condition for quantum Heisenberg manifolds.

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Abstract

Abstract: A study of Hilbert C^*-bimodules over commutative C^*-algebras is carried out and used to establish a sufficient condition for two quantum Heisenberg manifolds to be isomorphic.

Introduction. In [AEE], a theory of crossed products of C^*-algebras by Hilbert C^*-bimodules was introduced and used to describe certain deformations of Heisenberg manifolds constructed by Rieffel (see [Rf4] and [AEE, 3.3]). This deformation consists of a family of C^*-algebras, denoted $D^c_{\mu\nu}$, depending on two real parameters μ and ν, and a positive integer c. In case $\mu = \nu = 0$, $D^c_{\mu\nu}$ turns out to be isomorphic to the algebra of continuous functions on the Heisenberg manifold M^c.

For K-theoretical reasons [Ab2], $D^c_{\mu\nu}$ and $D^{c'}_{\mu'\nu'}$ cannot be isomorphic unless $c = c'$. It is the main purpose of this work to show that the C^*-algebras $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic when (μ, ν) and (μ', ν') are in the same orbit under the usual action of $GL_2(Z)$ on the torus T^2 (here the parameters are

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viewed as running in T^2, since $D^c_{\mu\nu}$ and $D^c_{\mu+n,\nu+m}$ are isomorphic for any integers m and n).

As indicated above, the quantum Heisenberg manifold $D^c_{\mu\nu}$ may be described as a crossed product of the commutative C^*-algebra $C(\mathbf{T}^2)$ by a Hilbert C^*-bimodule. Motivated by this, we are led to study some special features of Hilbert C^*-bimodules over commutative C^*-algebras, which are relevant to our purposes.

In Section 1 we consider, for a commutative C^*-algebra A, two subgroups of its Picard group Pic(A): the group of automorphisms of A (embedded in Pic(A) as in [BGR]), and the classical Picard group CPic(A) (see, for instance, [DG]) consisting of Hilbert line bundles over the spectrum of A. Namely, we prove that Pic(A) is the semidirect product of CPic(A) by Aut(A). This result carries over a slightly more general setting, and a similar statement (see Proposition 1.1) holds for Hilbert C^*-bimodules that are not full, partial automorphisms playing then the role of Aut(A). These results provide a tool that enables us to deal with Pic($C(\mathbf{T}^2)$) in order to prove our isomorphism theorem for quantum Heisenberg manifolds, which is done in Section 2.

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1 The Picard group and the classical Picard group.

Notation. Let A be C^*-algebra. If M is a Hilbert C^*-bimodule over A (in the sense of [BMS, 1.8]) we denote by $\langle\ ,\ \rangle^M_L$, and $\langle\ ,\ \rangle^M_R$, respectively, the left and right A-valued inner products, and drop the superscript whenever the context is clear enough. If M is a left (resp. right) Hilbert C^*-module over A, we denote by $K({}_AM)$ (resp. $K(M_A)$) the C^*-algebra of compact operators on M. When M is a Hilbert C^*-bimodule over A we will view the elements of $\langle M, M\rangle_R$ (resp. $\langle M, M\rangle_L$) as compact operators on the left (resp. right) module M, as well as elements of A, via the well-known identity:

$$\langle m, n\rangle_{L} p = m\langle n, p\rangle_{R},$$

for $m, n, p \in M$.

The bimodule denoted by $\tilde{M}$ is the dual bimodule of M, as defined in [Rf1, 6.17].

By an isomorphism of left (resp. right) Hilbert C^*-modules we mean an isomorphism of left (resp. right) modules that preserves the left (resp. right) inner product. An isomorphism of Hilbert C^*-bimodules is an isomorphism of both left and right Hilbert C^*-modules. We recall from [BGR, 3] that $\mathrm{Pic}(A)$, the Picard group of A, consists of isomorphism classes of full Hilbert C^*-bimodules over A (that is, Hilbert C^*-bimodules M such that $\langle M, M\rangle_{L} = \langle M, M\rangle_{R} = A$), equipped with the tensor product, as defined in [Rf1, 5.9].

It was shown in [BGR, 3.1] that there is an anti-homomorphism from $\mathrm{Aut}(A)$ to $\mathrm{Pic}(A)$ such that the sequence

$$1 \longrightarrow \mathrm{Gin}(A) \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathrm{Pic}(A)$$

is exact, where $\mathrm{Gin}(A)$ is the group of generalized inner automorphisms of A. By this correspondence, an automorphism α is mapped to a bimodule that corresponds to the one we denote by $A_{\alpha^{-1}}$ (see below), so that $\alpha \mapsto A_\alpha$ is a group homomorphism having $\mathrm{Gin}(A)$ as its kernel.

Given a partial automorphism (I, J, θ) of a C^*-algebra A, we denote by J_θ the corresponding ([AEE, 3.2]) Hilbert C^*-bimodule over A. That is, J_θ consists of the vector space J endowed with the A-actions:

$$a \cdot x = ax, \quad x \cdot a = \theta[\theta^{-1}(x)a],$$

and the inner products

$$\langle x, y\rangle_{L} = xy^*,$$

and

$$\langle x, y\rangle_{R} = \theta^{-1}(x^*y),$$

for $x, y \in J$, and $a \in A$. If M is a Hilbert C^*-bimodule over A, we denote by M_θ the Hilbert C^*-bimodule obtained by taking the tensor product $M \otimes_A J_\theta$.

The map $m \otimes j \mapsto mj$, for $m \in M$, $j \in J$, identifies M_θ with the vector space MJ equipped with the A-actions:

$$a \cdot mj = amj, \quad mj \cdot a = m\theta[\theta^{-1}(j)a],$$

and the inner products

$$\langle x, y\rangle_L^{M_\theta} = \langle x, y\rangle_L^M,$$

and

$$\langle x, y\rangle_R^{M_\theta} = \theta^{-1}(\langle x, y\rangle_R^M),$$

where $m \in M$, $j \in J$, $x, y \in MJ$, and $a \in A$.

As mentioned above, when M is a C^*-algebra A, equipped with its usual structure of Hilbert C^*-bimodule over A, and $\theta \in \mathrm{Aut}(A)$ the bimodule A_θ corresponds to the element of $\mathrm{Pic}(A)$ denoted by $X_{\theta^{-1}}$ in [BGR, 3], so we have $A_\theta \otimes A_\sigma \cong A_{\theta\sigma}$ and $\widetilde{A_\theta} \cong A_{\theta^{-1}}$ for all $\theta, \sigma \in \mathrm{Aut}(A)$.

In this section we discuss the interdependence between the left and the right structure of a Hilbert C^*-bimodule. Proposition 1.1 shows that the right structure is determined, up to a partial isomorphism, by the left one. By specializing this result to the case of full Hilbert C^*-bimodules over a commutative C^*-algebra, we are able to describe $\mathrm{Pic}(A)$ as the semidirect product of the classical Picard group of A by the group of automorphisms of A.

Proposition 1.1 *Let M and N be Hilbert C^*-bimodules over a C^*-algebra A. If $\Phi : M \longrightarrow N$ is an isomorphism of left A-Hilbert C^*-modules, then there is a partial automorphism (I, J, θ) of A such that $\Phi : M_\theta \longrightarrow N$ is an isomorphism of $A - A$ Hilbert C^*-bimodules. Namely, $I = \langle N, N\rangle_R$, $J = \langle M, M\rangle_R$ and $\theta(\langle \Phi(m_0), \Phi(m_1)\rangle_R) = \langle m_0, m_1\rangle_R$.*

Proof: Let $\Phi : M \longrightarrow N$ be a left A-Hilbert C^*-module isomorphism. Notice that, if $m \in M$, and $\|m\| = 1$, then, for all $m_i, m_i' \in M$, and $i = 1, ..., n$:

$$
\begin{aligned}
\| \sum m\langle m_i, m_i'\rangle_R\| &= \| \sum\langle m, m_i\rangle_L m_i'\| \\
&= \|\Phi(\sum\langle m, m_i\rangle_L m_i')\| \\
&= \| \sum\langle m, m_i\rangle_L \Phi(m_i')\| \\
&= \| \sum\langle \Phi(m), \Phi(m_i)\rangle_L \Phi(m_i')\| \\
&= \| \sum \Phi(m)\langle \Phi(m_i), \Phi(m_i')\rangle_R\|.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\| \sum\langle m_i, m_i'\rangle_R\| &= \sup_{\{m:\|m\|=1\}} \| \sum m\langle m_i, m_i'\rangle_R\| \\
&= \sup_{\{m:\|m\|=1\}} \| \sum \Phi(m)\langle \Phi(m_i), \Phi(m_i')\rangle_R\| \\
&= \| \sum\langle \Phi(m_i), \Phi(m_i')\rangle_R\|,
\end{aligned}
$$

Set $I = \langle N, N\rangle_R$, and $J = \langle M, M\rangle_R$, and let $\theta : I \longrightarrow J$ be the isometry defined by

$$
\theta(\langle \Phi(m_1), \Phi(m_2)\rangle_R) = \langle m_1, m_2\rangle_R,
$$

for $m_1, m_2 \in M$. Then,

$$
\begin{aligned}
\theta(\langle \Phi(m_1), \Phi(m_2)\rangle_R^*) &= \theta(\langle \Phi(m_2), \Phi(m_1)\rangle_R) \\
&= \langle m_2, m_1\rangle_R \\
&= \langle m_1, m_2\rangle_R^* \\
&= \theta(\langle \Phi(m_1), \Phi(m_2)\rangle_R)^*,
\end{aligned}
$$

and

$$
\begin{aligned}
\theta(\langle \Phi(m_1), \Phi(m_2)\rangle_R\langle \Phi(m_1'), \Phi(m_2')\rangle_R) &= \theta(\langle \Phi(m_1), \Phi(m_2)\langle \Phi(m_1'), \Phi(m_2')\rangle_R\rangle_R) \\
&= \theta(\langle \Phi(m_1), \langle \Phi(m_2), \Phi(m_1')\rangle_L \Phi(m_2')\rangle_R) \\
&= \langle m_1, \langle \Phi(m_2), \Phi(m_1')\rangle_L m_2'\rangle_R \\
&= \langle m_1, \langle m_2, m_1'\rangle_L m_2'\rangle_R \\
&= \langle m_1, m_2\langle m_1', m_2'\rangle_R\rangle_R \\
&= \langle m_1, m_2\rangle_R\langle m_1', m_2'\rangle_R \\
&= \theta(\langle m_1, m_2\rangle_R)\theta(\langle m_1', m_2'\rangle_R),
\end{aligned}
$$

which shows that θ is an isomorphism.

Besides, $\Phi : M_\theta \longrightarrow N$ is a Hilbert C^*-bimodule isomorphism:

$$\begin{aligned}
\Phi(m\langle m_1, m_2\rangle_R \cdot a) &= \Phi(m\theta[\theta^{-1}(\langle m_1, m_2\rangle_R)a] \\
&= \Phi(m\theta(\langle \Phi(m_1), \Phi(m_2)a\rangle_R)) \\
&= \Phi(m\langle m_1, \Phi^{-1}(\Phi(m_2)a)\rangle_R) \\
&= \Phi(\langle m, m_1\rangle_L \Phi^{-1}(\Phi(m_2)a)) \\
&= \langle m, m_1\rangle_L \Phi(m_2)a \\
&= \Phi(\langle m, m_1\rangle_L m_2)a \\
&= \Phi(m\langle m_1, m_2\rangle_R)a,
\end{aligned}$$

and

$$\langle \Phi(m_1), \Phi(m_2)\rangle_R = \theta^{-1}(\langle m_1, m_2\rangle_R^M) = \langle m_1, m_2\rangle_R^{M_\theta}.$$

Finally, Φ is onto because

$$\Phi(M_\theta) = \Phi(M\langle M, M\rangle_R) = \Phi(M) = N.$$

Q.E.D.

Corollary 1.2 *Let M and N be Hilbert C^*-bimodules over a C^*-algebra A, and let $\Phi : M \longrightarrow N$ be a an isomorphism of left Hilbert C^*-modules. Then Φ is an isomorphism of Hilbert C^*-bimodules if and only if Φ preserves either the right inner product or the right A-action.*

Proof: Let θ be as in Proposition 1.1, so that $\Phi : M_\theta \longrightarrow N$ is a Hilbert C^*-bimodule isomorphism. If Φ preserves the right inner product, then θ is the identity map on $\langle M, M\rangle_R$ and $M_\theta = M$.

If Φ preserves the right action of A, then, for $m_0, m_1, m_2 \in M$ we have:

$$\begin{aligned}
\Phi(m_0)\langle \Phi(m_1), \Phi(m_2)\rangle_R &= \langle \Phi(m_0), \Phi(m_1)\rangle_L \Phi(m_2) \\
&= \langle m_0, m_1\rangle_L \Phi(m_2) \\
&= \Phi(m_0\langle m_1, m_2\rangle_R) \\
&= \Phi(m_0)\langle m_1, m_2\rangle_R,
\end{aligned}$$

so Φ preserves the right inner product as well. **Q.E.D.**

Proposition 1.3 *Let M and N be left Hilbert C^*-modules over a C^*-algebra A. If M and N are isomorphic as left A-modules, and $K({}_AM)$ is unital, then M and N are isomorphic as left Hilbert C^*-modules.*

Proof: First recall that any A-linear map $T : M \longrightarrow N$ is adjointable. For if $m_i, m_i' \in M$, $i = 1, ..., n$ are such that $\sum \langle m_i, m_i' \rangle_R = 1_{K({}_AM)}$, then for any $m \in M$:

$$T(m) = T(\sum \langle m, m_i \rangle_L m_i') = \sum \langle m, m_i \rangle_L T(m_i') = (\sum \xi_{m_i, Tm_i'})(m),$$

wher $\xi_{m,n} : M \longrightarrow N$ is the compact operator (see, for instance, [La, 1]) defined by $\xi_{m,n}(m_0) = \langle m_0, m \rangle_L n$, for $m \in M$, and $n \in N$, which is adjointable. Let $T : M \longrightarrow N$ be an isomorphism of left modules, and set $S : M \longrightarrow N$, $S = T(T^*T)^{-1/2}$. Then S is an A-linear map, therefore adjointable. Furthermore, S is a left Hilbert C^*-module isomorphism: if $m_0, m_1 \in M$, then

$$\begin{aligned} \langle S(m_0), S(m_1) \rangle_L &= \langle T(T^*T)^{-1/2} m_0, T(T^*T)^{-1/2} m_1 \rangle_L \\ &= \langle m_0, (T^*T)^{-1/2} T^*T (T^*T)^{-1/2} m_1 \rangle_L \\ &= \langle m_0, m_1 \rangle_L. \end{aligned}$$

Q.E.D.

We next discuss the Picard group of a C^*-algebra A. Proposition 1.1 shows that the left structure of a full Hilbert C^*-bimodule over A is determined, up to an isomorphism of A, by its left structure.

This suggests describing Pic(A) in terms of the subgroup Aut(A) together with a cross-section of the equivalence classes under left Hilbert C^*-modules isomorphisms. When A is commutative there is a natural choice for this cross-section: the family of symmetric Hilbert C^*-bimodules (see Definition 1.5). That is the reason why we now concentrate on commutative C^*-algebras and their symmetric Hilbert C^*-bimodules.

Proposition 1.4 *Let A be a commutative C^* algebra and M a Hilbert C^*-bimodule over A. Then $\langle m, n \rangle_L p = \langle p, n \rangle_L m$ for all $m, n, p \in M$.*

Proof: We first prove the proposition for $m = n$, the statement will then follow from polarization identities.

Let $m, p \in M$, then:

$$\langle \langle m, m\rangle_L p - \langle p, m\rangle_L m,\ \langle m, m\rangle_L p - \langle p, m\rangle_L m\rangle_L$$

$$\begin{aligned}
&= \langle \langle m, m\rangle_L p, \langle m, m\rangle_L p\rangle_L - \langle \langle m, m\rangle_L p, \langle p, m\rangle_L m\rangle_L \\
&- \langle \langle p, m\rangle_L m, \langle m, m\rangle_L p\rangle_L + \langle \langle p, m\rangle_L m, \langle p, m\rangle_L m\rangle_L
\end{aligned}$$

$$\begin{aligned}
&= \langle m\langle m, p\rangle_R \langle p, m\rangle_R, m\rangle_L - \langle m, m\rangle_L \langle p, m\rangle_L \langle m, p\rangle_L \\
&- \langle p, m\rangle_L \langle m, p\rangle_L \langle m, m\rangle_L + \langle p, m\rangle_L \langle m, m\rangle_L \langle m, p\rangle_L
\end{aligned}$$

$$= \langle m\langle p, m\rangle_R \langle m, p\rangle_R, m\rangle_L - \langle m, m\rangle_L \langle p, m\rangle_L \langle m, p\rangle_L$$

$$= \langle m\langle p, m\rangle_R, m\langle p, m\rangle_R\rangle_L - \langle m, m\rangle_L \langle p, m\rangle_L \langle m, p\rangle_L$$

$$= \langle \langle m, p\rangle_L m, \langle m, p\rangle_L m\rangle_L - \langle m, m\rangle_L \langle p, m\rangle_L \langle m, p\rangle_L$$

$$= 0.$$

Now, for $m, n, p \in M$, we have:

$$\begin{aligned}
\langle m, n\rangle_L p &= \tfrac{1}{4} \textstyle\sum_{k=0}^{3} i^k \langle m + i^k n, m + i^k n\rangle_L p \\
&= \tfrac{1}{4} \textstyle\sum_{k=0}^{3} 1^k \langle p, m + i^k n\rangle_L (m + i^k n) \\
&= \tfrac{1}{4} \textstyle\sum_{k=0}^{3} i^k p\langle m + i^k n, m + i^k n\rangle_R \\
&= p\langle n, m\rangle_R \\
&= \langle p, n\rangle_L m.
\end{aligned}$$

Definition 1.5 *Let A be a commutative C^*-algebra. A Hilbert C^*-bimodule M over A is said to be* symmetric *if $am = ma$ for all $m \in M$, and $a \in A$.*

If M is a Hilbert C^*-bimodule over A, the symmetrization *of M is the symmetric Hilbert C^*-bimodule M^s, whose underlying vector space is M with its given left Hilbert-module structure, and right structure defined by:*

$$m \cdot a = am, \quad \langle m_0, m_1 \rangle_R^{M^s} = \langle m_1, m_0 \rangle_L^M,$$

for $a \in A$, $m, m_0, m_1 \in M^s$. The commutativity of A guarantees the compatibility of the left and right actions. As for the inner products, we have, in view of Proposition 1.4:

$$\begin{aligned} \langle m_0, m_1 \rangle_L^{M^s} \cdot m_2 &= \langle m_0, m_1 \rangle_L^M m_2 \\ &= \langle m_2, m_1 \rangle_L^M m_0 \\ &= m_0 \cdot \langle m_2, m_1 \rangle_L^M \\ &= m_0 \cdot \langle m_1, m_2 \rangle_R^{M^s}, \end{aligned}$$

for all $m_0, m_1, m_2 \in M^s$.

Remark 1.6 *By Corollary 1.2 the bimodule M^s is, up to isomorphism, the only symmetric Hilbert C^*-bimodule that is isomorphic to M as a left Hilbert module.*

Remark 1.7 *Let M be a symmetric Hilbert C^*-bimodule over a commutative C^*-algebra A such that $K({}_AM)$ is unital. By Remark 1.6 and Proposition 1.3, a symmetric Hilbert C^*-bimodule over A is isomorphic to M if and only if it is isomorphic to M as a left module.*

Example 1.8 *Let $A = C(X)$ be a commutative unital C^*-algebra, and let M be a Hilbert C^*-bimodule over A that is, as a left Hilbert C^*-module, isomorphic to $A^n p$, for some $p \in Proj(M_n(A))$. This implies that $pM_n(A)p \cong K({}_AM)$ is isomorphic to a C^*-subalgebra of A and is, in particular, commutative. By viewing $M_n(A)$ as $C(X, M_n(\mathbb{C}))$ one gets that $p(x)M_n(\mathbb{C})p(x)$ is a commutative C^*-algebra, hence rank $p(x) \leq 1$ for all $x \in X$.*

Conversely, let $A = C(X)$ *be as above, and let* $p : X \longrightarrow Proj(M_n(\mathbb{C}))$ *be a continuous map, such that rank* $p(x) \leq 1$ *for all* $x \in X$*. Then* $A^n p$ *is a Hilbert* C^**-bimodule over* A *for its usual left structure, the right action of* A *by pointwise multiplication, and right inner product given by:*

$$\langle m, r\rangle_L = \tau(m^* r),$$

for $m, r \in A^n p$, $a \in A$, *and where* τ *is the usual* A*-valued trace on* $M_n(A)$ *(that is,* $\tau[(a_{ij})] = \sum a_{ii}$*).*

To show the compatibility of the inner products, notice that for any $T \in M_n(A)$, *and* $x \in X$ *we have:*

$$(pTp)(x) = p(x)T(x)p(x) = [trace(p(x)T(x)p(x))]p(x),$$

which implies that $pTp = \tau(pTp)p$*. Then, for* $m, r, s \in M$*:*

$$\langle m, r\rangle_L s = mpr^* sp = m\tau(pr^* sp)p = m\tau(r^* s) = m \cdot \langle r, s\rangle_R.$$

Besides, $A^n p$ *is symmetric:*

$$\langle m, r\rangle_R = \tau(m^* r) = \sum_{i=1}^{n} m_i^* r_i = \langle r, m\rangle_L,$$

for $m = (m_1, m_2, ..., m_n)$, $r = (r_1, r_2, ...r_n) \in M$*.*

Therefore, by Remark 1.7, if $p, q \in Proj(M_n(A))$*, the Hilbert* C^**-bimodules* $A^n p$ *and* $A^n q$ *described above are isomorphic if and only if* p *and* q *are Murray-von Neumann equivalent. Notice that the identity of* $K({}_A A^n p)$ *is* $\tau(p)$*, that is, the characteristic function of the set* $\{x \in X : rank\ p(x) = 1\}$*. Therefore* $A^n p$ *is full as a right module if and only if rank* $p(x) = 1$ *for all* $x \in X$*, which happens in particular when* X *is connected, and* $p \neq 0$*.*

Proposition 1.9 *Let* A *be a commutative* C^**-algebra. For any Hilbert* C^**-bimodule* M *over* A *there is a partial automorphism* $(\langle M, M\rangle_R, \langle M, M\rangle_L, \theta)$ *of* A *such that the map* $i : (M^s)_\theta \longrightarrow M$ *defined by* $i(m) = m$ *is an isomorphism of Hilbert* C^**-bimodules .*

Proof: The map $i : M^s \longrightarrow M$ is a left Hilbert C^*-modules isomorphism. The existence of θ, with $I = \langle M, M\rangle_R$ and $J = \langle M^s, M^s\rangle_R = \langle M, M\rangle_L$, follows from Proposition 1.1.

Q.E.D.

We now turn to the discussion of the group Pic(A) for a commutative C^*-algebra A. For a full Hilbert C^*-bimodule M over A, we denote by $[M]$ its equivalence class in Pic(A). For a commutative C^*-algebra A, the group Gin(A) is trivial, so the map $\alpha \mapsto A_\alpha$ is one-to-one. In what follows we identify, via that map, $Aut(A)$ with a subgroup of Pic(A).

Symmetric full Hilbert C^*-bimodules over a commutative C^*-algebra $A = C(X)$ are known to correspond to line bundles over X. The subgroup of Pic(A) consisting of isomorphism classes of symmetric Hilbert C^*-bimodules is usually called the classical Picard group of A, and will be denoted by CPic(A). We next specialize the result above to the case of full bimodules.

Notation 1.10 *For $\alpha \in Aut(A)$, and M a Hilbert C^*-bimodule over A, we denote by $\alpha(M)$ the Hilbert C^*-bimodule $\alpha(M) = A_\alpha \otimes M \otimes A_{\alpha^{-1}}$.*

Remark 1.11 *The map $a \otimes m \otimes b \mapsto amb$ identifies $A_\alpha \otimes M \otimes A_{\alpha^{-1}}$ with M equipped with the actions:*

$$a \cdot m = \alpha^{-1}(a)m, \quad m \cdot a = m\alpha^{-1}(a),$$

and inner products

$$\langle m_0, m_1\rangle_L = \alpha(\langle m_0, m_1\rangle_L^M),$$

and

$$\langle m_0, m_1\rangle_R = \alpha(\langle m_0, m_1\rangle_R^M),$$

for $a \in A$, and $m, m_0, m_1 \in M$.

Theorem 1.12 *Let A be a commutative C^*-algebra. Then $CPic(A)$ is a normal subgroup of $Pic(A)$ and*

$$Pic(A) = CPic(A) \rtimes Aut(A),$$

where the action of $Aut(A)$ is given by conjugation, that is $\alpha \cdot M = \alpha(M)$.

Proof: Given $[M] \in \mathrm{Pic}(A)$ write, as in Proposition 1.9, $M \cong M^s_\theta$, θ being an isomorphism from $\langle M, M\rangle_R = A$ onto $\langle M, M\rangle_L = A$.

Therefore $M \cong M^s \otimes A_\theta$, where $[M^s] \in \mathrm{CPic}(A)$ and $\theta \in \mathrm{Aut}(A)$. If $[S] \in \mathrm{CPic}(A)$ and $\alpha \in \mathrm{Aut}(A)$ are such that $M \cong S \otimes A_\alpha$, then S and M^s are symmetric bimodules, and they are both isomorphic to M as left Hilbert C^*-modules. This implies, by Remark 1.6, that they are isomorphic. Thus we have:

$$M^s \otimes A_\theta \cong M^s \otimes A_\alpha \Rightarrow A_\theta \cong \widetilde{M^s} \otimes M^s \otimes A_\theta \cong \widetilde{M^s} \otimes M^s \otimes A_\alpha \cong A_\alpha,$$

which implies ([BGR, 3.1]) that $\theta\alpha^{-1} \in \mathrm{Gin}(A) = \{id\}$, so $\alpha = \theta$, and the decomposition above is unique.

It only remains to show that $\mathrm{CPic}(A)$ is normal in $\mathrm{Pic}(A)$, and it suffices to prove that $[A_\alpha \otimes S \otimes A_{\alpha^{-1}}] \in \mathrm{CPic}(A)$ for all $[S] \in \mathrm{CPic}(A)$, and $\alpha \in \mathrm{Aut}(A)$, which follows from Remark 1.11.

Q.E.D.

Notation 1.13 *If $\alpha \in Aut(A)$, then for any positive integers k, l, we still denote by α the automorphism of $M_{k\times l}(A)$ defined by $\alpha[(a_{ij})] = (\alpha(a_{ij}))$.*

Lemma 1.14 *Let A be a commutative unital C^*-algebra, and $p \in Proj(M_n(A))$ be such that $A^n p$ is a symmetric Hilbert C^*-bimodule over A, for the structure described in Example 1.8. If $\alpha \in Aut(A)$, then $\alpha(A^n p) \cong A^n \alpha(p)$.*

Proof: Set $J : \alpha(A^n p) \longrightarrow A^n\alpha(p)$, $\quad J(m \otimes x \otimes r) = m\alpha(xr)$, for $m \in A_\alpha$, $r \in A_{\alpha^{-1}}$, and $x \in A^n p$. Notice that

$$m\alpha(xr) = m\alpha(xpr) = m\alpha(xr)\alpha(p) \in A^n\alpha(p).$$

Besides, if $a \in A$

$$\begin{aligned} J(m \cdot a \otimes x \otimes r) &= J(m\alpha(a) \otimes x \otimes r) \\ &= m\alpha(axr) \\ &= J(m \otimes a \cdot x \otimes r), \end{aligned}$$

and

$$\begin{aligned} J(m \otimes x \cdot a \otimes r) &= m\alpha(xar) \\ &= J(m \otimes x \otimes a \cdot r), \end{aligned}$$

so the definition above makes sense. We now show that J is a Hilbert C^*-bimodule isomorphism. For $m \in A_\alpha$, $n \in A_{\alpha^{-1}}$, $x \in A^n p$, and $a \in A$, we have:

$$\begin{aligned} J(a \cdot (m \otimes x \otimes r)) &= J(am \otimes x \otimes r) \\ &= am\alpha(xr) \\ &= a \cdot J(m \otimes x \otimes r), \end{aligned}$$

and

$$\begin{aligned} J(m \otimes x \otimes r \cdot a) &= m\alpha(x(r\alpha^{-1}(a)) \\ &= m\alpha(xr)a \\ &= J((m \otimes x \otimes r) \cdot a) \end{aligned}$$

Finally,

$$\begin{aligned} \langle J(m \otimes x \otimes r), J(m' \otimes x' \otimes r')\rangle_L &= \langle m\alpha(xr), m'\alpha(x'r')\rangle_L \\ &= \langle m \cdot [(xr)(x'r')^*], m'\rangle_L \\ &= \langle m \cdot \langle x \cdot \langle r, r''\rangle_L^A, x'\rangle_L^{A^n p}, m'\rangle_L \\ &= \langle m \cdot \langle x \otimes r, x' \otimes r'\rangle_L^{A^n p \otimes A_{\alpha^{-1}}}, m'\rangle_L \\ &= \langle m \otimes x \otimes r, m' \otimes x' \otimes r'\rangle_L, \end{aligned}$$

which shows, by Corollary 1.2, that J is a Hilbert C^*-bimodule isomorphism.

Q.E.D.

Proposition 1.15 *Let A be a commutative unital C^*-algebra and M a Hilbert C^*-bimodule over A. If $\alpha \in Aut(A)$ is homotopic to the identity, then*

$$A_\alpha \otimes M \cong M \otimes A_{\gamma^{-1}\alpha\gamma},$$

where $\gamma \in Aut(A)$ is such that $M \cong (M^s)_\gamma$.

Proof: We then have that $K({}_AM)$ is unital so, in view of Proposition 1.3 we can assume that $M^s = A^n p$ with the Hilbert C^*-bimodule structure described in Example 1.8, for some positive integer n, and $p \in \mathrm{Proj}(M_n(A))$. Since p and $\alpha(p)$ are homotopic, they are Murray-von Neumann equivalent ([Bl, 4]). Then, by Lemma 1.14 and Example 1.8, we have

$$A_\alpha \otimes M \cong A_\alpha \otimes M^s \otimes A_\gamma \cong M^s \otimes A_{\alpha\gamma} \cong M \otimes A_{\gamma^{-1}\alpha\gamma}.$$

Q.E.D.

We turn now to the discussion of crossed products by Hilbert C^*-bimodules, as defined in [AEE]. For a Hilbert C^*-bimodule M over a C^*-algebra A, we denote by $A \rtimes_M \mathbb{Z}$ the crossed product C^*-algebra. We next establish some general results that will be used later.

Notation 1.16 *In what follows, for $A - A$ Hilbert C^*-bimodules M and N we write $M \overset{cp}{\cong} N$ to denote $A \rtimes_M \mathbb{Z} \cong A \rtimes_N \mathbb{Z}$.*

Proposition 1.17 *Let A be a C^*-algebra, M an $A - A$ Hilbert C^*-bimodule and $\alpha \in Aut(A)$. Then*

i) $M \overset{cp}{\cong} \tilde{M}$.

ii) $M \overset{cp}{\cong} \alpha(M)$.

Proof: Let i_A and i_M denote the standard embeddings of A and M in $A \rtimes_M \mathbb{Z}$, respectively.

i) Set

$$i_{\tilde{M}} : \tilde{M} \longrightarrow A \rtimes_M \mathbb{Z}, \quad i_{\tilde{M}}(\tilde{m}) = i_M(m)^*.$$

Then $(i_A, i_{\tilde{M}})$ is covariant for $(A, \tilde{M})$:

$$i_{\tilde{M}}(a \cdot \tilde{m}) = i_{\tilde{M}}(\widetilde{ma^*}) = [i_M(ma^*)]^* = i_A(a) i_M(m)^* = i_A(a) i_{\tilde{M}}(\tilde{m}),$$

$$i_{\tilde{M}}(\tilde{m_1})i_{\tilde{M}}(\tilde{m_2})^* = i_M(m_1)^* i_M(m_2) = i_A(\langle m_0, m_1\rangle_R^M) = i_A(\langle m_0, m_1\rangle_L^{\tilde{M}}),$$

for $a \in A$ and $m, m_0, m_1 \in M$. Analogous computations prove covariance on the right. By the universal property of the crossed products there is a homomorphism from $A \rtimes_{\tilde{M}} \mathbb{Z}$ onto $A \rtimes_M \mathbb{Z}$. Since $\tilde{\tilde{M}} = M$, by reversing the construction above one gets the inverse of J.

ii) Set

$$j_A : A \longrightarrow A \rtimes_M \mathbb{Z}, \quad j_{\alpha(M)} : M \longrightarrow A \rtimes_M \mathbb{Z},$$

defined by $j_A = i_A \circ \alpha^{-1}$, $j_{\alpha(M)}(m) = i_M(m)$, where the sets M and $\alpha(M)$ are identified as in Remark 1.11. Then $(j_A, j_{\alpha(M)})$ is covariant for $(A, \alpha(M))$:

$$j_{\alpha(M)}(a \cdot m) = j_{\alpha(M)}(\alpha^{-1}(a)m) = i_A(\alpha^{-1}(a))i_M(m) = j_A(a)i_{\alpha(M)}(m),$$

$$j_{\alpha(M)}(m_0)j_{\alpha(M)}(m_1)^* = i_M(m_0)i_M(m_1)^* = i_A(\langle m_0, m_1\rangle_L^M) =$$
$$= j_A(\alpha\langle m_0, m_1\rangle_L^M) = j_A(\langle m_0, m_1\rangle_L^{\alpha(M)}),$$

for $a \in A$, $m, m_0, m_1 \in M$, and analogously on the right. Therefore there is a homomorphism

$$J : A \rtimes_{\alpha(M)} \mathbb{Z} \longrightarrow A \rtimes_M \mathbb{Z},$$

whose inverse is obtained by applying the construction above to α^{-1}.

Q.E.D.

2 An application: isomorphism classes for quantum Heisenberg manifolds.

For $\mu, \nu \in \mathbb{R}$ and a positive integer c, the quantum Heisenberg manifold $D^c_{\mu\nu}$ ([Rf4]) is isomorphic ([AEE, Ex.3.3]) to the crossed product $C(\mathbf{T}^2) \rtimes_{(X^c_\nu)_{\alpha_{\mu\nu}}} \mathbb{Z}$, where X^c_ν is the vector space of continuous functions on $\mathbb{R} \times \mathbf{T}$ satisfying $f(x+1, y) = e(-c(y-\nu))f(x,y)$. The left and right actions of $C(\mathbf{T}^2)$ are defined by pointwise multiplication, the inner products by $\langle f, g\rangle_L = f\overline{g}$, and $\langle f, g\rangle_R = \overline{f}g$, and $\alpha_{\mu\nu} \in \mathrm{Aut}(C(\mathbf{T}^2))$ is given by $\alpha_{\mu\nu}(x,y) = (x+2\mu, y+2\nu)$, and, for $t \in \mathbb{R}$, $e(t) = exp(2\pi i t)$.

Our purpose is to find isomorphisms in the family $\{D^c_{\mu\nu} : \mu, \nu \in I\!R, c \in Z\!\!Z, c > 0\}$. We concentrate in fixed values of c, because $K_0(D^c_{\mu\nu}) \cong Z\!\!Z^3 \oplus Z\!\!Z_c$([Ab2]). Besides, since $\alpha_{\mu\nu} = \alpha_{\mu+m,\nu+n}$ for all $m, n \in Z\!\!Z$, we view from now on the parameters μ and ν as running in $\mathbf{T}$.

Let M^c denote the set of continuous functions on $I\!R \times \mathbf{T}$ satisfying $f(x+1,y) = e(-cy)f(x,y)$. Then M^c is a Hilbert C^*-bimodule over $C(\mathbf{T}^2)$, for pointwise action and inner products given by the same formulas as in X^c.

The map $f \mapsto \tilde{f}$, where $\tilde{f}(x,y) = f(x, y+\nu)$, is a Hilbert C^*-bimodule isomorphism between $(X^c_\nu)_{\alpha_{\mu\nu}}$ and $C(\mathbf{T}^2)_\sigma \otimes M^c \otimes C(\mathbf{T}^2)_\rho$, where $\sigma(x,y) = (x, y+\nu)$, and $\rho(x,y) = (x+2\mu, y+\nu)$. In view of Proposition 1.17 we have:

$$D^c_{\mu\nu} \cong C(\mathbf{T}^2) \rtimes_{C(\mathbf{T}^2)_\sigma \otimes M^c \otimes C(\mathbf{T}^2)_\rho} Z\!\!Z \cong$$

$$\cong C(\mathbf{T}^2) \rtimes_{(M^c)_{\rho\sigma}} Z\!\!Z \cong C(\mathbf{T}^2) \rtimes_{M^c_{\alpha_{\mu\nu}}} Z\!\!Z.$$

As a left module over $C(\mathbf{T}^2)$, M^c corresponds to the module denoted by $X(1,c)$ in [Rf3, 3.7]. It is shown there that M^c represents the element $(1,c)$ of $K_0(C(\mathbf{T}^2)) \cong Z\!\!Z^2$, where the last correspondence is given by $[X] \mapsto (a,b)$, a being the dimension of the vector bundle corresponding to X and $-b$ its twist. It is also proven in [Rf3] that any line bundle over $C(\mathbf{T}^2)$ corresponds to the left module M^c, for exactly one value of the integer c, and that $M^c \otimes M^d$ and M^{c+d} are isomorphic as left modules. It follows now, by putting these results together, that the map $c \mapsto [M^c]$ is a group isomorphism from $Z\!\!Z$ to $\mathrm{CPic}(C(\mathbf{T}^2))$.

Lemma 2.1

$$Pic(C(\mathbf{T}^2)) \cong Z\!\!Z \rtimes_\delta Aut(C(\mathbf{T}^2)),$$

where $\delta_\alpha(c) = \det\alpha_* \cdot c$, *for* $\alpha \in Aut(C(\mathbf{T}^2))$, *and* $c \in Z\!\!Z$; α_* *being the usual automorphism of* $K_0(C(\mathbf{T}^2)) \cong Z\!\!Z^2$, *viewed as an element of* $GL_2(Z\!\!Z)$.

Proof: By Theorem 1.12 we have:

$$\mathrm{Pic}(C(\mathbf{T}^2)) \cong \mathrm{CPic}(C(\mathbf{T}^2)) \rtimes_\delta \mathrm{Aut}(C(\mathbf{T}^2)).$$

If we identify CPic($C(\mathbf{T}^2)$) with $\mathbb{Z}$ as above, it only remains to show that $\alpha(M^c) \cong M^{\det\alpha_* \cdot c}$. Let us view $\alpha_* \in GL_2(\mathbb{Z})$ as above. Since α_* preserves the dimension of a bundle, and takes $C(\mathbf{T}^2)$ (that is, the element $(1,0) \in \mathbb{Z}^2$) to itself, we have

$$\alpha_* = \begin{pmatrix} 1 & 0 \\ 0 & \det\alpha_* \end{pmatrix}$$

Now,

$$\alpha_*(M^c) = \alpha_*(1,c) = (1, \det\alpha_* \cdot c) = M^{\det\alpha_* \cdot c}.$$

Since there is cancellation in the positive semigroup of finitely generated projective modules over $C(\mathbf{T}^2)$ ([Rf3]), the result above implies that $\alpha_*(M^c)$ and $M^{\det\alpha_* \cdot c}$ are isomorphic as left modules. Therefore, by Remark 1.7, they are isomorphic as Hilbert C^*-bimodules .

Q.E.D.

Theorem 2.2 *If (μ, ν) and (μ', ν') belong to the same orbit under the usual action of $GL(2, \mathbb{Z})$ on $\mathbf{T}^2$, then the quantum Heisenberg manifolds $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic.*

Proof: If (μ, ν) and (μ', ν') belong to the same orbit under the action of $GL(2, \mathbb{Z})$, then $\alpha_{\mu'\nu'} = \sigma\alpha_{\mu\nu}\sigma^{-1}$, for some $\sigma \in GL(2, \mathbb{Z})$. Therefore, by Lemma 2.1 and Proposition 1.17:

$$M^c_{\alpha_{\mu'\nu'}} \cong M^c_{\sigma\alpha_{\mu\nu}\sigma^{-1}} \cong M^c \otimes C(\mathbf{T}^2)_{\sigma\alpha\sigma^{-1}} \cong$$

$$\cong C(\mathbf{T}^2)_\sigma \otimes M^{\det\sigma_*^{-1} \cdot c} \otimes C(\mathbf{T}^2)_{\alpha_{\mu\nu}\sigma^{-1}} \cong \sigma(M^{\det\sigma_* \cdot c}_{\alpha_{\mu\nu}}) \overset{cp}{\cong} M^{\det\sigma_* \cdot c}_{\alpha_{\mu\nu}}.$$

In case $\det\sigma_* = -1$ we have

$$M^{\det\sigma_* \cdot c}_{\alpha_{\mu\nu}} \cong M^{-c}_{\alpha_{\mu\nu}} \overset{cp}{\cong} \widetilde{M^{-c}_{\alpha_{\mu\nu}}} \cong C(\mathbf{T}^2)_{\alpha^{-1}_{\mu\nu}} \otimes M^c \cong (M^c)_{\alpha^{-1}_{\mu\nu}},$$

since $\det\alpha_* = 1$, because $\alpha_{\mu\nu}$ is homotopic to the identity.

On the other hand, it was shown in [Ab1, 0.3] that $M^c_{\alpha^{-1}_{\mu,\nu}} \overset{cp}{\cong} M^c_{\alpha_{\mu\nu}}$.

Thus, in any case, $M^c_{\alpha_{\mu'\nu'}} \overset{cp}{\cong} M^c_{\alpha_{\mu\nu}}$. Therefore

$$D^c_{\mu'\nu'} \cong C(\mathbf{T}^2) \rtimes_{M^c_{\alpha_{\mu'\nu'}}} \mathbb{Z} \cong C(\mathbf{T}^2) \rtimes_{M^c_{\alpha_{\mu\nu}}} \mathbb{Z} \cong D^c_{\mu\nu}.$$

Q.E.D.

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