HILBERT SCHEMES OF SOME THREEFOLD SCROLLS OVER $\mathbb{F}_e$

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Abstract. Hilbert schemes of suitable smooth, projective threefold scrolls over the Hirzebruch surface $\mathbb{F}_e$, $e \geq 2$, are studied. An irreducible component of the Hilbert scheme parametrizing such varieties is shown to be generically smooth of the expected dimension and the general point of such a component is described.

1. Introduction

Projective varieties are distributed in *families*, obtained by suitably varying the coefficients of their defining equations. The description of such families and, in particular, of the properties of their parameter spaces is a central theme in algebraic geometry.

Milestones to approach such problems have been both the introduction of technical tools, like flatness, base change, Hilbert polynomial, etc., and the proof (due to Grothendieck with refinements by Mumford) of the existence of the so called *Hilbert scheme*, a closed, projective scheme, parametrizing families of projective varieties with suitable constant numerical/projective invariants, together with some other fundamental universal properties.

Since then, Hilbert schemes of projective varieties with given Hilbert polynomial have interested several authors over the years, especially because of the deep connections of the subject with several other important theories in algebraic geometry: zero-dimensional schemes on smooth projective varieties, Brill-Noether theory of line bundles on curves, moduli spaces of genus g curves and their stratifications in terms of suitable subvarieties, vector bundles on smooth projective varieties, just to mention a few (for an overview the reader is referred, for instance, to the bibliography in [38]).

For particular cases of projective varieties, one can find in the literature sufficiently detailed descriptions of their Hilbert schemes. For example special classes of threefolds in $\mathbb{P}^5$ were studied in [20]; results for codimension–two projective varieties are due to [17, 14, 15]; in codimension three, [32] considered the case of arithmetically Gorenstein closed subschemes in a projective space, whereas [31] dealt with determinantal schemes. For codimension greater than or equal to two, Hilbert schemes of *Palatini scrolls* in $\mathbb{P}^n$, with n odd, have been treated in [18] while in [19] Hilbert schemes of varieties defined by maximal minors were considered. We also mention results in [33] concerning Hilbert schemes of determinantal schemes.

An important class of projective varieties is that of r-*scrolls* in $\mathbb{P}^n$, namely ruled varieties over a smooth base which are embedded in $\mathbb{P}^n$ in such a way that the rulings are r-dimensional linear subspaces of $\mathbb{P}^n$. This class is important not only because it usually comes out as a fundamental special case from problems in classical adjunction theory (cf. e.g. [5, 36]), but mainly because it is strictly related to the study of vector bundles of rank $(r+1)$ over smooth projective varieties.

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For rank-two, degree d vector bundles over genus g curves (equivalently, surface scrolls of degree d and sectional genus g), apart from the classical approach of C. Segre ([37]) and of some other more recent partial results as, for instance, in [27, 3, 25, 26], a systematic study of Hilbert schemes of such surface scrolls has been developed in the series of papers [10, 11, 12, 13], where the authors bridged the Hilbert scheme approach with the vector-bundle one, showing in particular how projective geometry and degeneration techniques can be used in order to improve some known results about rank-two vector bundles on curves and also to obtain some new ones.

A similar approach has been used to study Hilbert schemes of r-scrolls, $r \geq 1$, over smooth projective surfaces S, with S either a $K3$ ([21]) or the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_1$ ([6, 7]). In the authors' opinion, it would be interesting to develop the use of projective geometry and of degeneration techniques in order to study possible limits of vector-bundles, of any rank, on classes of smooth, projective varieties.

In this paper we focus on some classes of 1–scrolls over Hirzebruch surfaces $\mathbb{F}_e$, with $e \geq 2$. Rank–two vector bundles on Hirzebruch surfaces are classified in [9]; some of their cohomological and ampleness properties are studied in [1]; moduli spaces of rank-two vector bundles on Hirzebruch surfaces are considered, for example, in [2]. On the other hand, very little is known about Hilbert schemes of 1–scrolls over $\mathbb{F}_e$.

We consider vector bundles $\mathcal{E}_e$ arising as extensions of suitable line bundles over $\mathbb{F}_e$ and with Chern classes $c_1(\mathcal{E}_e) = 3C_e + b_e f$, $c_2(\mathcal{E}_e) = k_e$, where C_e and f are respectively the section of minimal self-intersection and a fiber of $\mathbb{F}_e$, whereas b_e and k_e are integers suitably chosen (cf. Assumptions 3.1, 4.3). Such a choice of $c_1(\mathcal{E}_e) = 3C_e + b_e f$ and of the integers b_e, k_e gives the first case for which the bundle $\mathcal{E}_e$ is both *uniform* and *very-ample* (cf. § 4 and Remark 4.2).

Let therefore X_e be a threefold in $\mathbb{P}^{n_e}$ which is a scroll over $\mathbb{F}_e$, $n_e \geq 6$, $e \geq 2$, that is $X_e \cong \mathbb{P}(\mathcal{E}_e)$ is the projectivization of a rank–two vector bundle $\mathcal{E}_e$ over $\mathbb{F}_e$ as above. We assume $n_e \geq 6$ because it is known that there are no such scrolls when $n_e \leq 5$, see [36].

If one wants to parametrize varieties X_e of this type, the first tasks to be tackled are:

(i) looking at $[X_e]$ as a point of a component of $\mathcal{H}_3^{d_e,n_e}$, the Hilbert scheme parametrizing 3-dimensional subvarieties of $\mathbb{P}^{n_e}$ of degree d_e having same Hilbert polynomial $P_{X_e}(T)$ as that of X_e, and

(ii) understanding the general point of such a component in $\mathcal{H}_3^{d_e,n_e}$.

For $e = 0, 1$, the above problems have been considered in [6, 7], where the Hilbert schemes of threefold scrolls X_0 and X_1 were studied. Namely, it was proved that the irreducible component containing such scrolls is generically smooth, of the expected dimension, and its general point is actually a threefold scroll, that is the component is filled up by scrolls. The aim of this paper is to see what happens if the base of the scroll is $\mathbb{F}_e$, with $e \geq 2$.

Our main results, Theorems 5.1, and 5.7, in particular answer a question on Hilbert schemes of threefold scrolls over $\mathbb{F}_e$, $e \geq 2$, pointed out to us by C. Ciliberto and E. Sernesi and for which we thank them.

In this paper, we prove that there exists an irreducible component $\mathcal{X}_e$ of $\mathcal{H}_3^{d_e,n_e}$, containing such scrolls, which is generically smooth, of the *expected dimension* and such that $[X_e]$ belongs to the smooth locus of $\mathcal{X}_e$ (cf. Theorem 4.5). In contrast with the $e = 0, 1$ cases, we show that the family of constructed scrolls X_e's surprisingly does not fill up the component $\mathcal{X}_e$ (cf. Theorem 5.1).

We thus exhibit a smooth variety $X_\epsilon \subset \mathbb{P}^{n_e}$, which is a candidate to represent the general point of $\mathcal{X}_e$. More precisely, we show that X_ϵ corresponds to the general point of an irreducible component, of the same Hilbert scheme $\mathcal{H}_3^{d_e,n_e}$, which is generically smooth and of the expected dimension. We then show that X_ϵ flatly degenerates in $\mathbb{P}^{n_e}$ to a general threefold scroll X_e as above, in such a way that the base–scheme of the flat, embedded degeneration is entirely contained in $\mathcal{X}_e$. By the generic smoothness of $\mathcal{X}_e$, we can conclude that X_ϵ is actually the general point of $\mathcal{X}_e$ (cf. §'s 5.1, 5.2).

The paper is structured in the following way. In Section 2 notation is fixed. In Section 3, following [6, 7], we consider suitable rank-two vector bundles over $\mathbb{F}_e$, with $e \geq 2$. In Section 4 we consider Hilbert schemes parametrizing families of 3-dimensional scrolls over $\mathbb{F}_e$, $e \geq 2$. In Section 5 a description of the general point of the component $\mathcal{X}_e$ determined in Theorem 4.5 is presented. More precisely, in § 5.1 we first construct the candidate X_ϵ and analyze some of its properties, similar to those investigated for X_e in Sections 3, 4; then, in § 5.2, we show that X_ϵ actually corresponds to the general point of $\mathcal{X}_e$. Finally, Section 6 contains some concrete examples of Hilbert scheme of scrolls over some $\mathbb{F}_e$, with $e \geq 2$ and e both even and odd.

2. Notation and Preliminaries

The following notation will be used throughout this work.

X is a smooth, irreducible, projective variety of dimension 3 (or simply a threefold);
$\chi(\mathcal{F}) = \sum(-1)^i h^i(\mathcal{F})$, the Euler characteristic of $\mathcal{F}$, where $\mathcal{F}$ is any vector bundle of rank $r \geq 1$ on X;
$c_i(\mathcal{F})$, the i-th Chern class of $\mathcal{F}$;
$\mathcal{F}_{|Y}$ the restriction of $\mathcal{F}$ to a subvariety Y;
K_X the canonical bundle of X. When the context is clear, X may be dropped, so $K_X = K$;
$c_i = c_i(X)$, the i-th Chern class of X;
$d = \deg X = L^3$, the degree of X in the embedding given by a very-ample line bundle L;
$g = g(X)$, the sectional genus of (X, L) defined by $2g - 2 = (K + 2L)L^2$;
if S is a smooth surface, $\equiv$ will denote the numerical equivalence of divisors on S.

For non-reminded terminology and notation, we basically follow [29].

Definition 2.1. *A pair (X, L), where L is an ample line bundle on a threefold X, is a scroll over a normal variety Y if there exist an ample line bundle M on Y and a surjective morphism $\varphi : X \to Y$ with connected fibers such that $K_X + (4 - \dim Y)L = \varphi^*(M)$.*

In particular, if Y is smooth and (X, L) is a scroll over Y, then (see [5, Prop. 14.1.3]) $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \varphi_*(L)$ and L is the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Moreover, if $S \in |L|$ is a smooth divisor, then (see e.g. [5, Thm. 11.1.2]) S is the blow up of Y at $c_2(\mathcal{E})$ points; therefore $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S)$ and

$$d := L^3 = c_1^2(\mathcal{E}) - c_2(\mathcal{E}). \tag{2.1}$$

Throughout this work, the scroll's base Y will be the Hirzebruch surface $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, with $e \geq 0$ an integer.

Let $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$ be the natural projection onto the base. Then $\mathrm{Num}(\mathbb{F}_e) = \mathbb{Z}[C_e] \oplus \mathbb{Z}[f]$, where:
• C_e denotes the unique section corresponding to the morphism $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-e)$ on $\mathbb{P}^1$, and
• $f = \pi^*(p)$, for any $p \in \mathbb{P}^1$.
In particular

$$C_e^2 = -e, \; f^2 = 0, \; C_e f = 1.$$

Let $\mathcal{E}_e$ be a rank-two vector bundle over $\mathbb{F}_e$ and let $c_i(\mathcal{E}_e)$ be its i^{th}-Chern class. Then $c_1(\mathcal{E}_e) \equiv aC_e + bf$, for some $a, b \in \mathbb{Z}$, and $c_2(\mathcal{E}_e) \in \mathbb{Z}$.

3. Some rank-two vector bundles over $\mathbb{F}_e$, for $e \geq 2$

In [6, 7] the authors considered suitable rank-two vector bundles over $\mathbb{F}_e$, for $e = 0, 1$. In this and the following section, we will focus on the case $e \geq 2$. Therefore, unless otherwise stated, from now on we will use the following:

Assumptions 3.1. *Let $e \geq 2$, b_e, k_e be integers. Let $\mathcal{E}_e$ be a rank-two vector bundle over $\mathbb{F}_e$, with*

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f \ \text{ and } \ c_2(\mathcal{E}_e) = k_e,$$

such that

(i) $h^0(\mathcal{E}_e) \geq 7$
(ii) $b_e \geq 3e+1$
(iii) $k_e + e > b_e$

(cf. § 4 below and [1, Prop.7.2], *for motivation). Moreover, there exists an exact sequence*

$$0 \to A_e \to \mathcal{E}_e \to B_e \to 0, \tag{3.1}$$

where A_e and B_e are line bundles on $\mathbb{F}_e$ such that

$$A_e \equiv 2C_e + (2b_e - k_e - 2e)f \ \text{ and } \ B_e \equiv C_e + (k_e - b_e + 2e)f \tag{3.2}$$

(cf. [1, Prop.7.2] *and* [9]*).*

From (3.1), in particular, one has $c_1(\mathcal{E}_e) = A_e + B_e$ and $c_2(\mathcal{E}_e) = A_e B_e$.

Exact sequence (3.1) gives important preliminary information on the cohomology of $\mathcal{E}_e$, A_e and B_e. Indeed, one has

Lemma 3.2. *With Assumptions 3.1, one has*

$$h^j(\mathcal{E}_e) = h^j(A_e) = 0, \ \text{for}\ j \geq 2, \quad h^i(B_e) = 0, \ \text{for}\ i \geq 1,$$

$$h^0(A_e) = 6b_e - 3k_e - 9e + 3 + h^1(A_e), \ \ h^0(B_e) = 2k_e - 2b_e + 3e + 2$$

and

$$h^0(\mathcal{E}_e) = 4b_e - k_e - 6e + 5 + h^1(\mathcal{E}_e). \tag{3.3}$$

Proof. For dimension reasons, it is clear that $h^j(\mathcal{E}_e) = h^j(\mathbb{F}_e, A_e) = h^j(\mathbb{F}_e, B_e) = 0, \ j \geq 3$.

By Serre duality on $\mathbb{F}_e$,

$h^2(A_e) = h^0(-4C_e - (2b_e - k_e - e + 2)f) = 0$ and $h^2(B_e) = h^0(-3C_e - (k_e - b_e + 3e + 2)f) = 0$,

since $K_{\mathbb{F}_e} \equiv -2C_e - (e+2)f$. In particular, this implies that also $h^2(\mathcal{E}_e) = 0$.

We claim that, under Assumptions 3.1, we also have $h^1(B_e) = 0$. Indeed, since $B_e \equiv C_e + (k_e - b_e + 2e)f$, it follows that $R^1\pi_*(B_e) = 0$ and thus by Leray's isomorphism,

$$\begin{aligned} h^1(B_e) &= h^1(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + 2e)) \\ &= h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + 2e)) + h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + e)) = 0, \end{aligned}$$

by Assumptions 3.1-(iii).

Thus we have

$$\chi(A_e) = h^0(A_e) - h^1(A_e), \ \ \chi(B_e) = h^0(B_e), \ \ \chi(\mathcal{E}_e) = h^0(\mathcal{E}_e) - h^1(\mathcal{E}_e). \tag{3.4}$$

From the Riemann-Roch formula, we have

$$\chi(A_e) = \frac{1}{2} A_e(A_e - K_{\mathbb{F}_e}) + 1 =$$

$$\frac{1}{2}\left(2C_e + (2b_e - k_e - 2e)f\right)\left(4C_e + (2b_e - k_e - e + 2)f\right) + 1 = 6b_e - 3k_e - 9e + 3,$$

whereas

$$\chi(B_e) = h^0(B_e) = \frac{1}{2} B_e(B_e - K_{\mathbb{F}_e}) + 1 =$$

$$\frac{1}{2}\left(C_e + (k_e - b_e + 2e)f\right)\left(3C_e + (k_e - b_e + 3e + 2)f\right) + 1 = 2k_e - 2b_e + 3e + 2.$$

Since $\chi(\mathcal{E}_e) = \chi(A_e) + \chi(B_e)$, the remaining statements follow from the cohomology sequence associated with (3.1) and from (3.4). $\square$

From Lemma 3.2 we have:

$$0 \to H^0(A_e) \to H^0(\mathcal{E}_e) \to H^0(B_e) \xrightarrow{\partial} H^1(A_e) \to H^1(\mathcal{E}_e) \to 0, \tag{3.5}$$

where ∂ is the *coboundary map* determined by the extension (3.1). Thus

$$h^1(\mathcal{E}_e) \leq h^1(A_e). \tag{3.6}$$

Remark 3.3. From (3.3), Assumption 3.1(i) is equivalent to $4b_e - k_e - 6e + 5 + h^1(\mathcal{E}_e) \geq 7$, that is $k_e \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e)$.

3.1. **Vector bundles in** $\mathrm{Ext}^1(B_e, A_e)$**.** This subsection is devoted to an analysis of vector bundles fitting in the exact sequence (3.1). We need the following:

Lemma 3.4. *With Assumptions 3.1, one has*

$$\dim(\mathrm{Ext}^1(B_e, A_e)) = \begin{cases} 0 & \text{for } b_e - e < k_e < \frac{3b_e+2-5e}{2} \\ 5e + 2k_e - 3b_e - 1 & \text{for } \frac{3b_e+2-5e}{2} \leq k_e < \frac{3b_e+2-4e}{2} \\ 9e + 4k_e - 6b_e - 2 & \text{for } \frac{3b_e+2-4e}{2} \leq k_e \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e). \end{cases} \tag{3.7}$$

Proof. By standard facts, $\mathrm{Ext}^1(B_e, A_e) \cong H^1(A_e - B_e)$. From (3.2),

$$A_e - B_e \equiv C_e + (3b_e - 2k_e - 4e)f. \tag{3.8}$$

Now $R^i\pi_{e*}(C_e + (3b_e - 2k_e - 4e)f) = 0$, for $i > 0$, and $\pi_{e*}(C_e + (3b_e - 2k_e - 4e)f) \cong (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e)$, hence, from Leray's isomorphism we have

$$\begin{aligned} h^1(A_e - B_e) &= h^1(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e)) \\ &= h^1(\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e)) + h^1(\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 5e)) \end{aligned}$$

By Serre's duality on $\mathbb{P}^1$, the previous sum coincides with

$$h^0(\mathcal{O}_{\mathbb{P}^1}(2k_e + 4e - 3b_e - 2)) + h^0(\mathcal{O}_{\mathbb{P}^1}(2k_e + 5e - 3b_e - 2)).$$

Put $\alpha := 2k_e + 4e - 3b_e - 2$ and $\beta := 2k_e + 5e - 3b_e - 2$; note that $\beta = \alpha + e$.

- If $\beta < 0$ then also $\alpha < 0$ and thus $h^1(A_e - B_e) = 0$.
- If $\beta \geq 0$ and $\alpha < 0$ then $h^1(A_e - B_e) = \beta + 1$.
- Finally, if $\alpha \geq 0$ then $\beta > 0$ and thus $h^1(A_e - B_e) = \alpha + \beta + 2$.

Now observe that

$$\beta < 0 \Leftrightarrow k_e < \frac{3b_e + 2 - 5e}{2} \quad \text{and} \quad \alpha < 0 \Leftrightarrow k_e < \frac{3b_e + 2 - 4e}{2}.$$

Moreover, since $e \geq 2$, by Assumptions 3.1-(ii) one easily verifies that all such numerical conditions are compatible with Assumptions 3.1-(i) and (iii) (cf. also Rem. 3.3), in other words one has

$$b_e - e < \frac{3b_e + 2 - 5e}{2} < \frac{3b_e + 2 - 4e}{2} < 4b_e - 6e - 2 \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e).$$

Hence (3.7) follows. □

Corollary 3.5. *With Assumptions 3.1, for* $b_e - e < k_e < \frac{3b_e+2-5e}{2}$*, one has* $\mathcal{E}_e = A_e \oplus B_e$*.*

In § 5 (cf. the proof of Theorem 5.1), we shall also need to know $\dim(Aut(\mathcal{E}_e)) = h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee)$.

Lemma 3.6. *With Assumptions 3.1, take any* $\mathcal{E}_e \in \mathrm{Ext}^1(A_e, B_e)$*. Then:*

$$h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) = \begin{cases} 6b_e - 4k_e - 9e + 4 & \text{for } b_e - e < k_e < \frac{3b_e+2-5e}{2} \\ 3b_e - 2k_e - 4e + 2 & \text{for } \frac{3b_e+2-5e}{2} \leq k_e \leq \frac{3b_e-4e}{2} \text{ and } \mathcal{E}_e \text{ general} \\ 1 & \text{for } \frac{3b_e-4e}{2} < k_e \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e) \text{ and } \mathcal{E}_e \text{ general.} \end{cases} \tag{3.9}$$

Proof. (i) According to Corollary 3.5, for $b_e - e < k_e < \frac{3b_e+2-5e}{2}$, $\mathcal{E}_e = A_e \oplus B_e$. Therefore

$$\mathcal{E}_e \otimes \mathcal{E}_e^\vee \cong \mathcal{O}_{\mathbb{F}_e}^{\oplus 2} \oplus (A_e - B_e) \oplus (B_e - A_e).$$

From (3.2),

$$B_e - A_e \equiv -C_e + (2k_e - 3b_e + 4e)f, \tag{3.10}$$

so it is not effective, since it negatively intersects the irreducible, moving curve f.

From (3.8) and from the proof of Lemma 3.4, one has

$$h^0(A_e - B_e) = h^0(C_e + (3b_e - 2k_e - 4e)f) = h^0(\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e)) + h^0(\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 5e)).$$

Put $\alpha' := 3b_e - 2k_e - 4e$ and $\beta' := 3b_e - 2k_e - 5e$; note that $\beta' = \alpha' - e$

Since $k_e < \frac{3b_e-5e+2}{2}$, $\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e)$ is always effective whereas $\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 5e)$ is effective unless $3b_e - 2k_e - 5e = -1$. So $h^0(\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e)) + h^0(\mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 5e)) = 6b_e - 4k_e - 9e + 2$; taking into account also $h^0(\mathcal{O}_{\mathbb{F}_e}^{\oplus 2})$, we conclude in this case.

(ii)-(iii) We treat here the remaining cases in (3.9). Recall that the upper-bound $k_e \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e)$ comes from Assumptions 3.1-(i) (cf. Remark 3.3).

According to Lemma 3.4, when $k_e \geq \frac{3b_e+2-5e}{2}$, one has $\dim(\mathrm{Ext}^1(B_e, A_e)) > 0$. Therefore, let $\mathcal{E}_e \in \mathrm{Ext}^1(B_e, A_e)$ be general. Using the fact that $\mathcal{E}_e$ is of rank two and fits in the exact sequence (3.1), we have

$$\mathcal{E}_e^\vee \cong \mathcal{E}_e \otimes \mathcal{O}(-A_e - B_e),$$

since $c_1(\mathcal{E}_e) = A_e + B_e$. Tensoring (3.1) respectively by $\mathcal{E}_e^\vee$, $-B_e$, $-A_e$, we get the following exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & A_e - B_e & \to & \mathcal{E}_e(-B_e) & \to & \mathcal{O}_{\mathbb{F}_e} & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathcal{E}_e(-B_e) & \to & \mathcal{E}_e \otimes \mathcal{E}_e^\vee & \to & \mathcal{E}_e(-A_e) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathcal{O}_{\mathbb{F}_e} & \to & \mathcal{E}_e(-A_e) & \longrightarrow & B_e - A_e & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.11}$$

We want to compute both $h^0(\mathcal{E}_e(-B_e))$ and $h^0(\mathcal{E}_e(-A_e))$.

From the cohomology sequence associated to the first row of diagram (3.11) we get

$$0 \to H^0(A_e - B_e) \to H^0(\mathcal{E}_e(-B_e)) \to H^0(\mathcal{O}_{\mathbb{F}_e}) \xrightarrow{\widehat{\partial}} H^1(A_e - B_e).$$

Observe that the coboundary map

$$H^0(\mathcal{O}_{\mathbb{F}_e}) \xrightarrow{\widehat{\partial}} H^1(A_e - B_e),$$

has to be injective since it corresponds to the choice of the non-trivial extension class $\eta_{\mathcal{E}_e} \in \mathrm{Ext}^1(B_e, A_e)$ associated to $\mathcal{E}_e$ general. Thus

$$h^0(\mathcal{E}_e(-B_e)) = h^0(A_e - B_e)) = h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha')) + h^0(\mathcal{O}_{\mathbb{P}^1}(\beta')),$$

with α' and β' as in Case (i) above.

Since $k_e \geq \frac{3b_e+2-5e}{2}$, then $\beta' \leq -2$ hence $h^0(\mathcal{O}_{\mathbb{P}^1}(\beta')) = 0$. Thus, $h^0(\mathcal{E}_e(-B_e)) = h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha'))$. Morover, $h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha')) = 0$ if and only if $k_e > \frac{3b_e-4e}{2}$; thus

$$h^0(\mathcal{E}_e(-B_e)) = \begin{cases} 3b_e - 2k_e - 4e + 1 & \text{for } \frac{3b_e+2-5e}{2} \leq k_e \leq \frac{3b_e-4e}{2} \\ 0 & \text{for } k_e > \frac{3b_e-4e}{2} \end{cases} \tag{3.12}$$

From the third row of diagram (3.11), since $B_e - A_e$ is not effective (cf. (3.10)), it follows that $h^0(\mathcal{E}_e(-A_e)) = h^0(\mathcal{O}_{\mathbb{F}_e}) = 1$, thus $H^0(\mathcal{E}_e(-A_e)) \cong \mathbb{C}$.

From the second column of diagram (3.11), we have

$$0 \to H^0(\mathcal{E}_e(-B_e)) \to H^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) \xrightarrow{\psi} H^0(\mathcal{E}_e(-A_e)) \cong \mathbb{C} \to H^1(\mathcal{E}_e(-B_e)) \to \cdots .$$

Claim 3.7. *The map ψ is surjective.*

Proof of Claim 3.7. From the first two columns of diagram (3.11) and the fact that the coboundary map $\widehat{\partial}$ is injective, as remarked above, we have

$$\begin{array}{ccccc} & 0 & & H^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) & \\ & \downarrow & & \downarrow^{\psi} & \\ 0 \to & H^0(\mathcal{O}_{\mathbb{F}_e}) & \xrightarrow{\cong} & H^0(\mathcal{E}_e(-A_e)) & \to 0 \\ & \downarrow^{\widehat{\partial}} & & \downarrow^{\tilde{\partial}} & \\ & H^1(A_e - B_e) & \longrightarrow & H^1(\mathcal{E}_e(-B_e)) & \end{array}$$

Since $H^0(\mathcal{E}_e(-A_e))) \cong \mathbb{C}$, ψ is not surjective iff $\psi \equiv 0$, which is equivalent to $\tilde{\partial}$ injective and this is impossible since, from the first column of diagram (3.11), we have

$$H^0(\mathcal{O}_{\mathbb{F}_e}) \xrightarrow{\widehat{\partial}} H^1(A_e - B_e) \to H^1(\mathcal{E}_e(-B_e))$$

and the composition of the above two maps is $\tilde{\partial}$. This proves the claim. $\square$

From Claim 3.7, we conclude that

$$h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) = h^0(\mathcal{E}_e(-B_e)) + 1. \tag{3.13}$$

Combining (3.12) and (3.13) we determine $h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee)$ in the case $\mathcal{E}_e \in \operatorname{Ext}^1(B_e, A_e)$ is general. $\square$

Remark 3.8. (1) Note that when $\frac{3b_e+2-5e}{2} \leq k_e \leq \frac{3b_e-4e}{2}$ (which makes sense only for $e \geq 2$), any $\mathcal{E}_e \in \operatorname{Ext}^1(A_e, B_e)$ is such that $h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) > 1$, that is $\mathcal{E}_e$ is not simple. This gives a different situation with respect to cases $e = 0, 1$. Indeed, for $e = 1$, $b_1 \geq 4$, when $\dim(\operatorname{Ext}^1(B_1, A_1)) > 0$, $\mathcal{E}_1 \in \operatorname{Ext}^1(B_1, A_1)$ general is always simple (cf. [7, Lemmas 3.4, 3.6]). Similar computations hold for the case $e = 0$ (cf. (5.16) below).

(2) When $h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) = 1$ (from (3.9) this, for instance, happens when $\mathcal{E}_e \in \operatorname{Ext}^1(B_e, A_e)$ is general with $\frac{3b_e-4e}{2} < k_e \leq 4b_e - 6e + 2 + h^1(\mathcal{E}_e)$), $\mathcal{E}_e$ has to be necessarily indecomposable.

3.2. **Non-special bundles $\mathcal{E}_e$.** For our analysis in § 4, it is fundamental to deal with vector bundles $\mathcal{E}_e$ with no higher cohomology, in particular *non-special* that is with $h^1(\mathcal{E}_e) = 0$. Indeed, if $\mathcal{E}_e$ turns out to be very-ample, the fact that $\mathcal{E}_e$ has no higher cohomology not only implies that the ruled threefold $\mathbb{P}(\mathcal{E}_e)$ isomorphically embeds via the tautological linear system as a smooth, linearly normal scroll X_e in the projective space $\mathbb{P}^{n_e}$ of (the *expected*) dimension $n_e := h^0(\mathcal{E}_e) - 1$, but mainly its non-speciality ensures good behavior of the Hilbert point $[X_e]$ in its Hilbert scheme (cf. proof of Claim 4.6).

From Lemma 3.2, having $\mathcal{E}_e$ with no higher cohomology is equivalent to having $\mathcal{E}_e$ non-special. In this subsection, we therefore find sufficient conditions for the non-speciality of $\mathcal{E}_e$, coming from (3.6) and the cohomology of A_e.

Lemma 3.9. *With Assumptions 3.1, one has*

$$h^1(A_e) = \begin{cases} 0 & \text{for } b_e - e < k_e < 2b_e + 2 - 4e \\ 4e + k_e - 2b_e - 1 & \text{for } 2b_e + 2 - 4e \leq k_e < 2b_e + 2 - 3e \\ 7e + 2k_e - 4b_e - 2 & \text{for } 2b_e + 2 - 3e \leq k_e < 2b_e + 2 - 2e \\ 9e + 3k_e - 6b_e - 3 & \text{for } 2b_e + 2 - 2e \leq k_e \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e). \end{cases} \tag{3.14}$$

Proof. Fom (3.2) $\pi_{e*}(A_e) \cong Sym^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e)$ and $R^i\pi_{e*}(A_e) = 0$ for $i > 0$. Hence by Leray's isomorphism,

$$\begin{aligned} h^1(A_e) &= h^1(Sym^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e)) \\ &= h^1((\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \oplus \mathcal{O}_{\mathbb{P}^1}(-2e)) \otimes \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e)) \\ &= h^1(\mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e)) + h^1(\mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 3e)) + h^1(\mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 4e)) \end{aligned}$$

Let $\alpha' := 2e + k_e - 2b_e - 2$. By Serre Duality theorem on $\mathbb{P}^1$, from above we have

$$h^1(A_e) = h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha')) + h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha' + e)) + h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha' + 2e)).$$

- If $\alpha' + 2e < 0$, that is $k_e < 2b_e + 2 - 4e$, then $h^1(A_e) = 0$ (observe that condition $k_e < 2b_e + 2 - 4e$ is compatible with $k_e > b_e - e$, because of Assumptions 3.1-(ii)).
- If $\alpha' + e < 0 \leq \alpha' + 2e$, i.e. $2b_e + 2 - 4e \leq k_e < 2b_e + 2 - 3e$, then
$$h^1(A_e) = h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha' + 2e))) = h^0(\mathcal{O}_{\mathbb{P}^1}(4e + k_e - 2b_e - 2)) = 4e + k_e - 2b_e - 1.$$
- If $\alpha' < 0 \leq \alpha' + e$, equivalently $2b_e + 2 - 3e \leq k_e < 2b_e + 2 - 2e$, then
$$h^1(A_e) = h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha' + 2e)) + h^0(\mathcal{O}_{\mathbb{P}^1}(\alpha' + e)) = 2\alpha' + 3e + 2 = 7e + 2k_e - 4b_e - 2.$$
- Finally, if $\alpha' \geq 0$, which is $k_e \geq 2b_e + 2 - 2e$ then
$$h^1(A_e) = 3\alpha' + 3e + 3 = 9e + 3k_e - 6b_e - 3$$

(notice that condition $k_e \geq 2b_e + 2 - 2e$ is compatible with what computed in Remark 3.3; in other words one has $2b_e + 2 - 2e < 4b_e - 6e - 2 \leq 4b_e - 6e - 2 + h^1(\mathcal{E}_e)$ because of Assumptions 3.1-(ii)). Hence $h^1(A_e)$ is as in (3.14). ☐

Corollary 3.10. *Assumptions 3.1 and $k_e < 2b_e + 2 - 4e$ imply that any $\mathcal{E}_e \in \mathrm{Ext}^1(B_e, A_e)$ is such that $h^1(\mathcal{E}_e) = 0$.*

Remark 3.11. (1) Computations as in Remark 3.3 show that $k_e < 2b_e + 2 - 4e$ implies $h^0(\mathcal{E}_e) = 4b_e - k_e - 6e + 5 \geq 2b_e - 2e + 3$ which, from Assumption 3.1(iii) and $e \geq 2$, turns out to be greater than or equal to $4e + 5 \geq 13$. Therefore, conditions $b_e \geq 3e + 1$ and $b_e - e < k_e < 2b_e + 2 - 4e$ are sufficient for Assumptions 3.1 to hold.
(2) When moreover $b_e > 4e - 4$, then $\frac{3b_e - 4e}{2} < 2b_e + 2 - 4e$ holds. In this case, as observed in Remark 3.8-(2), Lemmas 3.4 and 3.6 ensure that a general $\mathcal{E}_e \in \mathrm{Ext}^1(B_e, A_e)$ is indecomposable.

Remark 3.12. As costumary, $0 \in \mathrm{Ext}^1(B_e, A_e)$ corresponds to the trivial bundle $A_e \oplus B_e$. When $k_e \geq 2b_e + 2 - 4e$ (i.e. when $h^1(A_e) > 0$), a given $\mathcal{E}_e \in \mathrm{Ext}^1(B_e, A_e) \setminus \{0\}$ is non-special if and only if the coboundary map $\partial : H^0(B_e) \to H^1(A_e)$ (corresponding to the choice of $\mathcal{E}_e$) is surjective. From (3.5), $\mathrm{Im}(\partial) \cong \mathrm{Coker}\left\{H^0(\mathcal{E}_e) \xrightarrow{\rho} H^0(B_e)\right\}$; thus the surjectivity of ∂ can be geometrically interpreted with the fact that the linear system induced by the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$ onto the section $\Sigma_e \subset \mathbb{P}(\mathcal{E}_e)$, corresponding to the quotient line bundle $\mathcal{E}_e \twoheadrightarrow B_e$, is not complete with $\mathrm{codim}_{H^0(\mathcal{O}_{\Sigma_e}(1))}(\mathrm{Im}(\rho)) = h^1(A_e)$. When $k_e \geq 2b_e + 2 - 4e$, it is a very tricky problem to find conditions granting the existence of a sublocus $\mathcal{U} \subset \mathrm{Ext}^1(B_e, A_e)$ s.t. $h^1(\mathcal{E}_e) = 0$ for any $\mathcal{E}_e \in \mathcal{U}$.

4. 3-dimensional scrolls over $\mathbb{F}_e$ and their Hilbert schemes

In this section, results from § 3 are used for the study of suitable 3-dimensional scrolls over $\mathbb{F}_e$ in projective spaces and of some components of their Hilbert schemes.

The choice of $c_1(\mathcal{E}_e) = 3C_e + b_e f$ and of the integers b_e, k_e (cf. Assumptions 3.1, 4.3), give the first case for which the bundle $\mathcal{E}_e$ is both *uniform* and *very-ample*. Indeed, if $\mathcal{E}_e$ is assumed to be ample with $c_1(\mathcal{E}_e) = 3C_e + b_e f$ then the restriction of $\mathcal{E}_{e|f}$ to any π_e-fiber f has to be ample; hence

$$\mathcal{E}_{e|f} = \mathcal{O}_f(a) \oplus \mathcal{O}_f(b), \text{ with } a, b > 0$$

and $a + b = 3$ because $c_1(\mathcal{E}_e)f = 3$. Therefore, up to reordering, the only possibility is $a = 2, b = 1$ for any π_e-fiber f, i.e. $\mathcal{E}_e$ is *uniform* (cf. e.g. [35] and [2, Def. 3]). Moreover, $c_1(\mathcal{E}_e) = 3C_e + b_e f$, together with very-ampleness hypothesis, naturally lead to Assumptions 3.1.

Indeed, one has the following necessary condition for very-ampleness:

Proposition 4.1. *(see [1, Prop. 7.2]) Let $\mathcal{E}_e$ be a very-ample, rank-two vector bundle over $\mathbb{F}_e$ such that*

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f \;\text{ and }\; c_2(\mathcal{E}_e) = k_e.$$

Then $\mathcal{E}_e$ satisfies all the hypotheses in Assumptions 3.1.

Remark 4.2. (1) By Lemma 3.4, when k_e is such that $b_e - e < k_e < \frac{3b_e+2-5e}{2}$ the only bundle in $\mathrm{Ext}^1(B_e, A_e)$ is $\mathcal{E}_e := A_e \oplus B_e$. The very-ampleness of B_e and A_e implies that of $\mathcal{E}_e := A_e \oplus B_e$, [5, Lemma 3.2.3]. On the other hand the very-ampleness of $\mathcal{E}_e := A_e \oplus B_e$ implies the ampleness of B_e and A_e, but on $\mathbb{F}_e$ ampleness of a line bundle is equivalent to very-ampleness, [29, V, Cor. 2.18], and thus $\mathcal{E}_e := A_e \oplus B_e$ very-ample implies that both B_e and A_e are very-ample. Assumption 3.1(iii) (resp., $k_e < 2b_e - 4e$) is a necessary and sufficient condition for B_e (resp., for A_e) to be very-ample. Since very-ampleness is an open condition, when $\dim(\mathrm{Ext}^1(B_e, A_e)) > 0$ and $k_e < 2b_e - 4e$ holds, then the general bundle $\mathcal{E}_e$ in $\mathrm{Ext}^1(B_e, A_e)$ is very-ample too.

(2) From the previous sections, condition $b_e - e < k_e < 2b_e - 4e$ is compatible because of Assumption 3.1(ii) and gives also that any $\mathcal{E}_e \in \mathrm{Ext}^1(B_e, A_e)$ is non-special.

(3) Comparing Lemmas 3.4 and 3.6 with this new bound on k_e, we notice that $\frac{3b_e+2-5e}{2} < 2b_e - 4e$ holds if and only if $b_e \geq 3e + 3$; similarly $\frac{3b_e+2-4e}{2} < 2b_e - 4e$ holds if and only if $b_e \geq 4e + 3$ and, finally, $\frac{3b_e-4e}{2} < 2b_e - 4e$ holds if and only if $b \geq 4e + 1$. In particular, when $b_e \geq 4e+1$ and $\frac{3b_e-4e}{2} < k_e < 2b_e - 4e$, Lemma 3.6 also ensures the existence of indecomposable bundles in $\mathrm{Ext}^1(B_e, A_e)$ (cf. Remark 3.11(2)).

From Remark (4.2), it is clear that from now on we will focus on $b_e - e < k_e < 2b_e - 4e$. In other words, Assumptions 3.1 will be replaced by:

Assumptions 4.3. *Let $e \geq 2$, k_e, b_e be integers. Let $\mathcal{E}_e$ be a rank-two vector bundle over $\mathbb{F}_e$ such that*

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f, \;\; c_2(\mathcal{E}_e) = k_e,$$

with

$$b_e \geq 3e + 1 \;\text{ and }\; b_e - e < k_e < 2b_e - 4e. \tag{4.1}$$

Let

$$(\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$$

be the 3-dimensional scroll over $\mathbb{F}_e$, and let $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$ and $\varphi : \mathbb{P}(\mathcal{E}_e) \to \mathbb{F}_e$ be the usual projections.

Proposition 4.4. *Let $\mathcal{E}_e$ be as in Assumptions 4.3. Moreover, when $\dim(\mathrm{Ext}^1(B_e, A_e)) > 0$, we further assume that $\mathcal{E}_e \in \mathrm{Ext}^1(B_e, A_e)$ is general. Then $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$ defines an embedding*

$$\Phi_e := \Phi_{|\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)|} : \mathbb{P}(\mathcal{E}_e) \hookrightarrow X_e \subset \mathbb{P}^{n_e}, \tag{4.2}$$

where $X_e = \Phi_e(\mathbb{P}(\mathcal{E}_e))$ is smooth, non-degenerate, of degree d_e, with

$$n_e = 4b_e - k_e - 6e + 4 \geq 4e + 4 \geq 12 \quad \text{and} \quad d_e = 6b_e - 9e - k_e. \tag{4.3}$$

Denoting by $(X_e, L_e) := (X_e, \mathcal{O}_{X_e}(H)) \cong (\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$, one also has

$$h^i(X_e, L_e) = 0, \;\; i \geq 1. \tag{4.4}$$

Proof. The very-ampleness of L_e is equivalent to that of $\mathcal{E}_e$, and the latter follows from Remark 4.2(1) and Assumptions 4.3. The formula on the degree d_e of X_e in (4.3) follows from (2.1). From Leray's isomorphisms, Lemma 3.2 and Corollary 3.10 we get (4.4). Finally, since $n_e+1 := h^0(X_e, L_e) = h^0(\mathbb{F}_e, \mathcal{E}_e)$, then $n_e + 1 \geq 4e + 5 \geq 13$ follows from Remark 3.11(2) and the fact that $e \geq 2$. □

4.1. **The component $\mathcal{X}_e$ of the Hilbert scheme containing $[X_e]$.** In what follows, we will be interested in studying the Hilbert scheme parametrizing subvarieties of $\mathbb{P}^{n_e}$ having the same Hilbert polynomial $P(T) := P_{X_e}(T) \in \mathbb{Q}[T]$ of X_e, which is the numerical polynomial defined by

$$(4.5)\quad P(m) = \chi(X_e, mL_e) = \frac{1}{6}m^3L_e^3 - \frac{1}{4}m^2L_e^2\cdot K + \frac{1}{12}mL_e\cdot(K^2+c_2) + \chi(\mathcal{O}_{X_e}), \text{for all } m \in \mathbb{Z},$$

as it follows from [24, Example 15.2.5, pg 291].

For basic terminology and facts on Hilbert schemes we follow, for instance, [28, 38, 39].

The scroll $X_e \subset \mathbb{P}^{n_e}$ corresponds to a point $[X_e] \in \mathcal{H}_3^{d_e,n_e}$, where $\mathcal{H}_3^{d_e,n_e}$ denotes the Hilbert scheme parametrizing 3-dimensional subvarieties of $\mathbb{P}^{n_e}$ with Hilbert polynomial $P(T)$ as above (in particular of degree d_e), where n_e and d_e are as in (4.3). When $[X_e] \in \mathcal{H}_3^{d_e,n_e}$ is a smooth point, X_e is said to be *unobstructed* in $\mathbb{P}^{n_e}$. Let

$$(4.6)\qquad N_e := N_{X_e/\mathbb{P}^{n_e}}$$

be the normal bundle of X_e in $\mathbb{P}^{n_e}$. From standard facts on Hilbert schemes (cf. e.g. [38, Corollary 3.2.7]), one has

$$(4.7)\qquad T_{[X_e]}(\mathcal{H}_3^{d_e,n_e}) \cong H^0(N_e)$$

and

$$(4.8)\qquad h^0(N_e) - h^1(N_e) \leq \dim_{[X_e]}(\mathcal{H}_3^{d_e,n_e}) \leq h^0(N_e),$$

where the left-most integer in (4.8) is the *expected dimension* of $\mathcal{H}_3^{d_e,n_e}$ at $[X_e]$ and where equality holds on the right in (4.8) iff X_e is unobstructed in $\mathbb{P}^{n_e}$.

The next result shows that X_e is unobstructed and such that $[X_e]$ sits in an irreducible component of $\mathcal{H}_3^{d_e,n_e}$ with "nice" behaviour.

THEOREM 4.5. *There exists an irreducible component $\mathcal{X}_e \subseteq \mathcal{H}_3^{d_e,n_e}$, which is generically smooth and of (the expected) dimension*

$$(4.9)\qquad \dim(\mathcal{X}_e) = n_e(n_e+1) + 3k_e - 2b_e + 3e - 5,$$

such that $[X_e]$ belongs to the smooth locus of $\mathcal{X}_e$.

Proof. By (4.7) and (4.8), the statement will follow by showing that $H^i(X_e, N_e) = 0$, for $i \geq 1$, and conducting an explicit computation of $h^0(X_e, N_e) = \chi(X_e, N_e)$.

To do this, let

$$(4.10)\qquad 0 \longrightarrow \mathcal{O}_{X_e} \longrightarrow \mathcal{O}_{X_e}(1)^{\oplus(n_e+1)} \longrightarrow T_{\mathbb{P}^{n_e}|X_e} \longrightarrow 0$$

be the Euler sequence on $\mathbb{P}^{n_e}$ restricted to X_e. Since (X_e, L_e) is a scroll over $\mathbb{F}_e$,

$$(4.11)\qquad H^i(X_e, \mathcal{O}_{X_e}) = H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}) = 0, \quad \text{for} \quad i \geq 1.$$

From (4.4), (4.11), the cohomology sequence associated to (4.10) and from the fact that X_e is non–degenerate, one has:

$$(4.12)\qquad h^0(X_e, T_{\mathbb{P}^{n_e}|X_e}) = (n_e+1)^2 - 1 \ \text{ and } \ h^i(X_e, T_{\mathbb{P}^{n_e}|X_e}) = 0, \text{ for } i \geq 1.$$

The normal sequence

$$(4.13)\qquad 0 \longrightarrow T_{X_e} \longrightarrow T_{\mathbb{P}^{n_e}|X_e} \longrightarrow N_e \longrightarrow 0$$

gives therefore

$$H^i(X_e, N_e) \cong H^{i+1}(X_e, T_{X_e}) \quad \text{for} \quad i \geq 1. \tag{4.14}$$

Claim 4.6. *$H^i(X_e, N_e) = 0$, for $i \geq 1$.*

Proof of Claim 4.6. From (4.12), (4.13) and dimension reasons, one has $h^j(X_e, N_e) = 0$, for $j \geq 3$. For the other cohomology spaces, we can use (4.14).

In order to compute $H^j(X_e, T_{X_e})$, $j = 2, 3$, we use the scroll map $\varphi : \mathbb{P}(\mathcal{E}_e) \longrightarrow \mathbb{F}_e$ and we consider the relative cotangent bundle sequence:

$$0 \to \varphi^*(\Omega^1_{\mathbb{F}_e}) \to \Omega^1_{X_e} \to \Omega^1_{X_e|\mathbb{F}_e} \longrightarrow 0. \tag{4.15}$$

From (4.15) and the Whitney sum, one obtains

$$c_1(\Omega^1_{X_e}) = c_1(\varphi^*(\Omega^1_{\mathbb{F}_e})) + c_1(\Omega^1_{X_e|\mathbb{F}_e})$$

thus

$$\Omega^1_{X_e|\mathbb{F}_e} = K_{X_e} + \varphi^*(-c_1(\Omega^1_{\mathbb{F}_e})) = K_{X_e} + \varphi^*(-K_{\mathbb{F}_e}).$$

The adjunction theoretic characterization of the scroll gives

$$K_{X_e} = -2L_e + \varphi^*(K_{\mathbb{F}_e} + c_1(\mathcal{E}_e)) = -2L_e + \varphi^*(K_{\mathbb{F}_e} + 3C_e + b_e f)$$

thus

$$\Omega^1_{X|\mathbb{F}_e} = K_{X_e} + \varphi^*(-K_{\mathbb{F}_e}) = -2L_e + \varphi^*(3C_e + b_e f)$$

which, combined with the dual of (4.15), gives

$$0 \to 2L_e - \varphi^*(3C_e + b_e f) \to T_{X_e} \to \varphi^*(T_{\mathbb{F}_e}) \to 0. \tag{4.16}$$

In what follows, we compute the cohomology of the left and right-most bundles in (4.16).

(i) First we concentrate on $\varphi^*(T_{\mathbb{F}_e})$. By Leray's isomorphism, one has

$$H^i(\varphi^*(T_{\mathbb{F}_e})) \cong H^i(T_{\mathbb{F}_e}), \text{ for any } i \geq 0.$$

Consider therefore the relative cotangent bundle sequence of $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$

$$0 \to \pi_e^* \Omega^1_{\mathbb{P}^1} \to \Omega^1_{\mathbb{F}_e} \to \Omega^1_{\mathbb{F}_e|\mathbb{P}^1} \to 0. \tag{4.17}$$

Since $\Omega^1_{\mathbb{F}_e|\mathbb{P}^1} = K_{\mathbb{F}_e} + \pi_e^* \mathcal{O}_{\mathbb{P}^1}(2) = -2C_e - ef$, dualizing (4.17) we get

$$0 \to 2C_e + ef \to T_{\mathbb{F}_e} \to \pi_e^* T_{\mathbb{P}^1} \to 0. \tag{4.18}$$

Since $\pi_e^* T_{\mathbb{P}^1} \cong \pi_e^* \mathcal{O}_{\mathbb{P}^1}(2)$, by Leray's isomorphism

$$h^0(\pi_e^* T_{\mathbb{P}^1}) = 3, \ h^i(\pi_e^* T_{\mathbb{P}^1}) = 0, \text{for } i \geq 1.$$

As in the proof of Lemma 3.9, Leray's isomorphism gives

$$h^i(2C_e + ef) = h^i(\mathbb{P}^1, [\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \oplus \mathcal{O}_{\mathbb{P}^1}(-2e)] \otimes \mathcal{O}_{\mathbb{P}^1}(e)), \text{ for any } i \geq 1.$$

Thus,

$$h^0(2C_e + ef) = e + 2, \ h^1(2C_e + ef) = e - 1, \ h^j(2C_e + ef) = 0, \text{ for } j \geq 2.$$

From [34, Lemma 10], one has

$$h^0(\mathbb{F}_e, T_{\mathbb{F}_e}) = e + 5.$$

Therefore, putting all together in the cohomology sequence associated to (4.18), we get

$$\begin{aligned} h^0(X_e, \varphi^*(T_{\mathbb{F}_e})) &= h^0(\mathbb{F}_e, T_{\mathbb{F}_e}) = e + 5, \\ h^1(X_e, \varphi^*(T_{\mathbb{F}_e})) &= h^1(\mathbb{F}_e, T_{\mathbb{F}_e}) = e - 1, \\ h^j(X_e, \varphi^*(T_{\mathbb{F}_e})) &= h^j(\mathbb{F}_e, T_{\mathbb{F}_e}) = 0, \text{ for } j \geq 2. \end{aligned} \tag{4.19}$$

(ii) We now devote our attention to the cohomology of $2L_e - \varphi^*(3C_e + b_e f)$ in (4.16). Noticing that $R^i \varphi_*(2L_e) = 0$ for $i \geq 1$ (see [29, Ex. 8.4, p. 253]), projection formula and Leray's isomorphism give

$$H^i(X_e, 2L_e - \varphi^*(3C_e + b_e f)) \cong H^i(\mathbb{F}_e, Sym^2 \mathcal{E}_e \otimes (-3C_e - b_e f)), \ \forall\, i \geq 0. \tag{4.20}$$

Therefore

$$h^j(X_e, 2L_e - \varphi^*(3C_e + b_e f)) = 0, \; j \geq 3, \tag{4.21}$$

for dimension reasons.

We now want to show that $H^2(\mathbb{F}_e, Sym^2\mathcal{E}_e \otimes (-3C_e - b_e f)) = 0$. To do this, recall that $\mathcal{E}_e$ fits in the exact sequence (3.1), with A_e and B_e as in (3.2). By [29, 5.16.(c), p. 127], there is a finite filtration of $Sym^2(\mathcal{E}_e)$,

$$Sym^2(\mathcal{E}_e) = F^0 \supseteq F^1 \supseteq F^2 \supseteq F^3 = 0$$

with quotients

$$F^p/F^{p+1} \cong Sym^p(A_e) \otimes Sym^{2-p}(B_e),$$

for each $0 \leq p \leq 2$. Hence

$$F^0/F^1 \cong Sym^0(A_e) \otimes Sym^2(B_e) = 2B_e$$

$$F^1/F^2 \cong Sym^1(A_e) \otimes Sym^1(B_e) = A_e + B_e$$

$$F^2/F^3 \cong Sym^2(A_e) \otimes Sym^0(B_e) = 2A_e, \text{ that is } F^2 = 2A_e,$$

since $F^3 = 0$. Thus, we get the following exact sequences

$$0 \to F^1 \to Sym^2(\mathcal{E}_e) \to 2B_e \to 0 \tag{4.22}$$

$$0 \to F^2 \to F^1 \to A_e + B_e \to 0 \tag{4.23}$$

$$F^2 = 2A_e \tag{4.24}$$

Twisting (4.22), (4.23) with $-c_1(\mathcal{E}_e) = -3C_e - b_e f = -A_e - B_e$ and using (4.24) we get

$$0 \to F^1(-3C_e - b_e f) \to Sym^2(\mathcal{E}_e) \otimes (-3C_e - b_e f) \to B_e - A_e \to 0 \tag{4.25}$$

$$0 \to A_e - B_e \to F^1(-3C_e - b_e f) \to \mathcal{O}_{F_e} \to 0 \tag{4.26}$$

First we focus on (4.26); from (3.8) and from the same arguments used in Lemma 3.4, one gets

$$h^i(A_e - B_e) = h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 4e) \oplus \mathcal{O}_{\mathbb{P}^1}(3b_e - 2k_e - 5e));$$

so, for dimension reasons, $h^i(A_e - B_e) = 0$, for any $i \geq 2$. Since moreover $h^i(\mathcal{O}_{\mathbb{F}_e}) = 0$ for $i \geq 1$, then (4.26) gives

$$h^2(F^1(-3C_e - b_e f)) = 0. \tag{4.27}$$

Passing to (4.25) observe that, from (3.10) and from the fact that $K_{\mathbb{F}_e} \equiv -2C_e - (e+2)f$, one gets

$$h^2(B_e - A_e) = h^0(-C_e + (3b_e - 2k_e - 5e - 2)f) = 0.$$

Thus, from (4.27), (4.25) and (4.20), one has

$$h^2(\mathbb{F}_e, Sym^2\mathcal{E}_e \otimes (-3C_e - b_e f)) = h^2(X_e, 2L_e - \varphi^*(3C_e + b_e f)) = 0. \tag{4.28}$$

Using (4.19), (4.21) and (4.28) in the cohomology sequence associated to (4.16), we get

$$h^j(X_e, T_{X_e}) = 0, \text{ for } j \geq 2. \tag{4.29}$$

Isomorphism (4.14) concludes the proof of Claim 4.6. □

Using (4.7) and (4.8), Claim 4.6 implies that there exists an irreducible component $\mathcal{X}_e$ of $\mathcal{H}_3^{d_e,n_e}$ containing $[X_e]$ as a smooth point.

Since smoothness is an open condition, $\mathcal{X}_e$ is generically smooth. Moreover, always from (4.8) and Claim 4.6, it follows that $\dim(\mathcal{X}_e) = h^0(X_e, N_e) = \chi(N_e)$ i.e. $\mathcal{X}_e$ has the expected dimension.

The Hirzebruch-Riemann-Roch theorem gives

$$\chi(N_e) = \frac{1}{6}(n_1^3 - 3n_1n_2 + 3n_3) + \frac{1}{4}c_1(n_1^2 - 2n_2) + \frac{1}{12}(c_1^2 + c_2)n_1 + (n_e - 3)\chi(\mathcal{O}_{X_e}) \tag{4.30}$$

where $n_i := c_i(N_e)$ and $c_i := c_i(X_e)$.

If $K := K_{X_e}$, the Chern classes of N_e can be obtained from (4.13):

$$\begin{aligned} n_1 &= K + (n_e+1)L_e; \\ n_2 &= \frac{1}{2}n_e(n_e+1)L_e^2 + (n_e+1)L_eK + K^2 - c_2; \\ n_3 &= \frac{1}{6}(n_e-1)n_e(n_e+1)L_e^3 + \frac{1}{2}n_e(n_e+1)KL_e^2 + (n_e+1)K^2L_e \\ &\quad -(n_e+1)c_2L_e - 2c_2K + K^3 - c_3. \end{aligned} \tag{4.31}$$

The numerical invariants of X_e can be easily computed by:

$$\begin{aligned} KL_e^2 &= -2d_e + 4b_e - 6e - 6; & K^2L_e &= 4d_e - 14b_e + 21e + 20; \\ c_2L_e &= 2b_e - 3e + 10; & K^3 &= -8d_e + 36b_e - 54e - 48; \\ -Kc_2 &= 24; & c_3 &= 8. \end{aligned}$$

Plugging these in (4.31) and then in (4.30), one gets

$$\chi(N_e) = (d_e + 3e - 2b_e + 5)n_e - 5 - 24e + 16b_e - 3d_e.$$

From (4.3), one has $d_e = 6b_e - 9e - k_e$; in particular

$$d_e + 3e - 2b_e + 5 = 4b_e - 6e - k_e + 5 = n_e + 1,$$

as it follows from (4.3). Thus

$$\chi(N_e) = (n_e+1)n_e - 5 - 3(6b_e - 9e - k_e) - 24e + 16b_e = n_e(n_e+1) + 3k_e - 2b_e + 3e - 5,$$

as in (4.9). □

Remark 4.7. The proof of Theorem 4.5 gives

$$h^0(N_e) = n_e(n_e+1) + 3k_e - 2b_e + 3e - 5, \;\; h^i(N_e) = 0, \; i \geq 1. \tag{4.32}$$

Using (4.12) and (4.32) in the exact sequence (4.13), one gets

$$\chi(T_{X_e}) = h^0(T_{\mathbb{P}^{n_e}|_{X_e}}) - h^0(N_e) = 6b_e - 4k_e + 9 - 9e. \tag{4.33}$$

Moreover, from (4.13) and (4.12), one has:

$$0 \to H^0(T_{X_e}) \to H^0(T_{\mathbb{P}^{n_e}|_{X_e}}) \xrightarrow{\alpha} H^0(N_e) \xrightarrow{\beta} H^1(T_{X_e}) \to 0, \tag{4.34}$$

In the sequel (cf. the proof of Theorem 5.1 below) we will make use of the following consequences of Theorem 4.5, interpreted via (4.34).

Corollary 4.8. *When* $\dim(\mathrm{Ext}^1(B_e, A_e)) = 0$*, one has*

$$h^0(T_{X_e}) = 6b_e - 4k_e - 8e + 8, \;\; h^1(T_{X_e}) = e - 1, \;\; h^j(T_{X_e}) = 0, \; for \; j \geq 2.$$

In particular,

$$\dim(\mathrm{Coker}(\alpha)) = e - 1, \tag{4.35}$$

where α *is the map in* (4.34).

Proof. From Lemma 3.4 and Remark 4.2(3), notice that $\dim(\mathrm{Ext}^1(B_e, A_e)) = 0$ occurs when, either $b_e = 3e+1, 3e+2$ and for any $b_e - e < k_e < 2b_e - 4e$, or for $b_e \geq 3e+3$ and $b_e - e < k_e < \frac{3b_e+2-5e}{2} < 2b_e - 4e$.

Now $h^j(T_{X_e}) = 0$, for $j \geq 2$, is (4.29) which more generally holds for any b_e, k_e as in (4.1). We thus concentrate on $h^j(T_{X_e})$, for $j = 0, 1$. Since $h^1(A_e - B_e) = \dim(\mathrm{Ext}^1(B_e, A_e)) = 0$, from (4.26) one has

$$h^0(F^1(-3C_e - b_e f)) = h^0(A_e - B_e) + 1 = 6b_e - 4k_e - 9e + 3, \quad h^1(F^1(-3C_e - b_e f)) = 0.$$

Passing to (4.25), from (3.10) and Leray's isomorphism, one has $h^i(B_e - A_e) = 0$ for any $i \geq 0$. Thus

$$h^i(Sym^2 \mathcal{E}_e \otimes (-3C_e - b_e f)) = h^i(F^1(-3C_e - b_e f)), \text{ for } 0 \leq i \leq 2,$$

and thus

$$h^0(Sym^2 \mathcal{E}_e \otimes (-3C_e - b_e f)) = 6b_e - 4k_e - 9e + 3, \ \ h^1(Sym^2 \mathcal{E}_e \otimes (-3C_e - b_e f)) = 0.$$

The cohomology sequence associated to (4.16) along with (4.20) and (4.19) gives the first part of the statement.

Finally, for (4.35), it suffices to notice that the map β in (4.34) is surjective. □

5. The general point of the component $\mathcal{X}_e$

In this section a description of the general point of $\mathcal{X}_e$, determined in Theorem 4.5, is presented. The following preliminary result shows that in general scrolls arising from Proposition 4.4 do not fill up $\mathcal{X}_e$.

THEOREM 5.1. *Let $\mathcal{Y}_e$ be the locus in $\mathcal{X}_e$ filled up by threefold scrolls X_e as in Proposition 4.4. Then*
(i) if $b_e - e < k_e < \frac{3b_e+2-5e}{2}$, one has $\mathrm{codim}_{\mathcal{X}_e}(\mathcal{Y}_e) = e - 1$,
(ii) if $\frac{3b_e+2-5e}{2} \leq k_e \leq 2b_e - 4e$, one has $\mathrm{codim}_{\mathcal{X}_e}(\mathcal{Y}_e) \leq e - 1$.

Proof. In case (i), from Lemma 3.4, $\dim(\mathrm{Ext}^1(B_e, A_e)) = 0$. Therefore $X_e \cong \mathbb{P}(A_e \oplus B_e)$ is uniquely determined, so $\dim(\mathcal{Y}_e) = \dim(\mathrm{Im}(\alpha))$, where α is the map in (4.34). Thus

$$\mathrm{codim}_{\mathcal{X}_e}(\mathcal{Y}_e) = \dim(\mathrm{Coker}(\alpha)) = e - 1$$

where the last equality comes from (4.35).

In case (ii) we have $\dim(\mathrm{Ext}^1(B_e, A_e)) > 0$; consider the following quantities.

(a) Denote by τ_e the number of parameters counting isomorphism classes of projective bundles $\mathbb{P}(\mathcal{E}_e)$ as in Proposition 4.4. In other words, τ_e takes into account *weak isomorphism classes* of extensions, which are parametrized by $\mathbb{P}(\mathrm{Ext}^1(B_e, A_e))$ (cf. [22, p. 31]), see Lemma 3.4 for the calculation of $\mathrm{Ext}^1(B_e, A_e)$. In particular, $\tau_e = \dim(\mathrm{Ext}^1(B_e, A_e)) - 1$ and, from Lemma 3.4, this number is as follows:

$$\tau_e := \begin{cases} 5e + 2k_e - 3b_e - 2 & \frac{3b_e+2-5e}{2} \leq k_e < \frac{3b_e+2-4e}{2} \\ 9e + 4k_e - 6b_e - 3 & \frac{3b_e+2-4e}{2} \leq k_e < 2b_e - 4e \end{cases} \tag{5.1}$$

(more precisely, note that if $\frac{3b_e+2-5e}{2} \leq k_e < 2b_e - 4e \leq \frac{3b_e+2-4e}{2}$, that is, when $3e+3 \leq b_e \leq 4e+2$, then (5.1) simply reads $\tau_e := 5e + 2k_e - 3b_e - 2$).

(b) $G_{X_e} \subset PGL(n_e+1, \mathbb{C})$ denotes the *projective stabilizer* of $X_e \subset \mathbb{P}^{n_e}$, i.e. the subgroup of projectivities of $\mathbb{P}^{n_e}$ fixing X_e. In particular (cf. (4.13))

$$\dim(PGL(n_e+1, \mathbb{C})) - \dim(G_{X_e}) = n_e(n_e+2) - h^0(T_{X_e}) \tag{5.2}$$

is the dimension of the full orbit of $X_e \subset \mathbb{P}^{n_e}$ under the action of all the projective transformations of $\mathbb{P}^{n_e}$. This equals $\dim(\mathrm{Im}(\alpha))$, where α is the map in (4.34).

The rest of the proof now reduces to a parameter computation to obtain a lower bound for the dimension of $\mathcal{Y}_e$. From the exact sequence (3.1), we observe that:

(*) the line bundle A_e is uniquely determined on $\mathbb{F}_e$, since $A_e \cong \mathcal{O}_{\mathbb{F}_e}(2C_e) \otimes \pi_e{}^* \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e)$;

(**) the line bundle B_e is uniquely determined on $\mathbb{F}_e$, similarly.

Let us compute how many parameters are needed to describe $\mathcal{Y}_e$. To do this, we have to add up the following quantities:

1) 0 parameters for A_e on $\mathbb{F}_e$, by (*);

2) 0 parameters for B_e, by (**);

3) τ_e as in (5.1), for isomorphism classes of $\mathbb{P}(\mathcal{E}_e)$;

4) $n_e(n_e+2) - h^0(T_{X_e})$, as in (5.2), for the dimension of the full orbit of $X_e \subset \mathbb{P}^{n_e}$ chosen.

Thus,

$$\dim(\mathcal{Y}_e) = \tau_e + n_e(n_e+2) - \dim(G_{X_e}) \tag{5.3}$$

The next step is to find an upper bound for $\dim(G_{X_e})$. It is clear that there is an obvious inclusion

$$G_{X_e} \hookrightarrow Aut(X_e), \tag{5.4}$$

where $Aut(X_e)$ denotes the algebraic group of abstract automorphisms of X_e. Since X_e, as an abstract variety, is isomorphic to $\mathbb{P}(\mathcal{E}_e)$ over $\mathbb{F}_e$, then

$$\dim(Aut(X_e)) = \dim(Aut(\mathbb{F}_e)) + \dim(Aut_{\mathbb{F}_e}(\mathbb{P}(\mathcal{E}_e))),$$

where $Aut_{\mathbb{F}_e}(\mathbb{P}(\mathcal{E}_e))$ denotes the group of automorphisms of $\mathbb{P}(\mathcal{E}_e)$ fixing the base (cf. e.g. [34]). From the fact that $Aut(\mathbb{F}_e)$ is an algebraic group, in particular smooth, it follows that

$$\dim(Aut(\mathbb{F}_e)) = h^0(\mathbb{F}_e, T_{\mathbb{F}_e}) = e + 5$$

since $e \geq 2$ (cf. [34, Lemma 10, p. 105]). On the other hand, $\dim(Aut_{\mathbb{F}_e}(\mathbb{P}(\mathcal{E}_e))) = h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) - 1$, since $Aut_{\mathbb{F}_e}(\mathbb{P}(\mathcal{E}_e))$ are given by endomorphisms of the projective bundle.

To sum up,

$$\dim(Aut(X_e)) = h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) + 4 + e.$$

From (5.4), $\dim(G_{X_e}) \leq \dim(Aut(X_e))$, then from (5.3) we deduce

$$\dim(\mathcal{Y}_e) \geq \tau_e + n_e(n_e+2) - h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) - 4 - e. \tag{5.5}$$

According to Lemma 3.6, one has

$$h^0(\mathcal{E}_e \otimes \mathcal{E}_e^\vee) = \begin{cases} 3b_e - 2k_e - 4e + 2 & \text{for } \frac{3b_e+2-5e}{2} \leq k_e < 2b_e - 4e \leq \frac{3b_e-4e}{2} \\ \\ 1 & \text{for } \frac{3b_e-4e}{2} \leq k_e < 2b_e - 4e, \end{cases}$$

As for τ_e, we use (5.1) and hence we get

(a) for $\frac{3b_e+2-5e}{2} \leq k_e < \frac{3b_e-4e}{2}$, $\tau_e = 5e + 2k_e - 3b_e - 2$ and $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = 3b_e - 2k_e - 4e + 2$,

(b) for $\frac{3b_e-4e}{2} \leq k_e < \frac{3b_e+2-4e}{2}$, $\tau_e = 5e + 2k_e - 3b_e - 2$ and $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = 1$;

(c) for $\frac{3b_e+2-4e}{2} \leq k_e < 2b_e - 4e$, $\tau_e = 9e + 4k_e - 6b_e - 3$ and $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = 1$.

In all cases, from (5.5) we get $\dim(\mathcal{Y}_e) \geq n_e(n_e+2) - 6b_e + 4k_e + 8e - 8$. From (4.9), we get

$$\begin{aligned} \mathrm{codim}_{\mathcal{X}_e}(\mathcal{Y}_e) &= \dim(\mathcal{X}_e) - \dim(\mathcal{Y}_e) \\ &\leq n_e(n_e+1) + 3k_e - 2b_e + 3e - 5 - (n_e(n_e+2) - 6b_e + 4k_e + 8e - 8) = e - 1. \end{aligned}$$

□

5.1. **A candidate for the general point of $\mathcal{X}_e$.** From Theorem 5.1, we need to exhibit a smooth variety in $\mathbb{P}^{n_e}$ which is a candidate to represent the general point of $\mathcal{X}_e$ as in Theorem 4.5. In other words, this candidate must flatly degenerate in $\mathbb{P}^{n_e}$ to the threefold scroll X_e, corresponding to $[X_e] \in \mathcal{Y}_e$ general, in such a way that the base-scheme of this flat, embedded degeneration is contained in $\mathcal{X}_e$.

In this section we first construct this candidate and analyze some of its properties similar to those investigated for X_e in §'s 3, 4. In § 5.2, we show that this candidate actually corresponds to the general point of $\mathcal{X}_e$.

For $e \geq 2$ integer, consider

$$\epsilon = 0, 1 \text{ according to } \epsilon \equiv e \pmod 2. \tag{5.6}$$

Consider the Hirzebruch surface $\mathbb{F}_\epsilon$, let $\pi_\epsilon : \mathbb{F}_\epsilon \to \mathbb{P}^1$ be the natural projection and let C_ϵ be the unique section of $\mathbb{F}_\epsilon$ corresponding to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\epsilon) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-\epsilon)$ on $\mathbb{P}^1$. Thus $C_\epsilon^2 = -\epsilon$.

With notation as in Assumptions 4.3, consider

$$b_\epsilon := b_e - \frac{3(e-\epsilon)}{2} \;\text{ and }\; k_\epsilon := k_e. \tag{5.7}$$

This choice of b_ϵ is needed in order to ensure that the Hilbert polynomial data (in particular the degree) of X_ϵ are the same as those of X_e, as it will become clear in (5.14).

Lemma 5.2. *With* (5.7) *above, conditions* (4.1) *on b_e and k_e read as*

$$b_\epsilon \geq \frac{3}{2}(e+\epsilon) + 1 \geq \frac{3\epsilon}{2} + 4 \;\text{ and }\; b_\epsilon - \epsilon < b_\epsilon + \frac{(e-3\epsilon)}{2} < k_\epsilon < 2b_\epsilon - 3\epsilon - e. \tag{5.8}$$

Proof. The proof is given by straightforward computations using (4.1) and (5.7). Indeed, by (5.7), $b_e \geq 3e+1$ in (4.1) reads as $b_\epsilon + \frac{3(e-\epsilon)}{2} \geq 3e+1$ which is $b_\epsilon \geq \frac{3}{2}e + 1 + \frac{3\epsilon}{2}$; the latter is greater than or equal to $\frac{3\epsilon}{2} + 4$ since $e \geq 2$ and from hypotheses on ϵ. Similarly, one has $b_\epsilon + \frac{(e-3\epsilon)}{2} = b_\epsilon - \epsilon + \frac{(e-\epsilon)}{2} > b_\epsilon - \epsilon$ for the same reasons.

Using $b_\epsilon = b_e - \frac{3(e-\epsilon)}{2}$, one finds

$$b_e - e = b_\epsilon + \frac{1}{2}(e - 3\epsilon). \tag{5.9}$$

Using (5.9), one gets

$$2b_e - 4e = 2(b_e - e) - 2e = 2b_\epsilon - 3\epsilon - e. \tag{5.10}$$

Since from (5.7) one has $k_\epsilon = k_e$, then one concludes by (4.1). □

Consider now the following line bundles on $\mathbb{F}_\epsilon$ (cf. (3.2)):

$$A_\epsilon \equiv 2C_\epsilon + (2b_\epsilon - k_\epsilon - 2\epsilon)f \tag{5.11}$$

and

$$B_\epsilon \equiv C_\epsilon + (k_\epsilon - b_\epsilon + 2\epsilon)f. \tag{5.12}$$

Remark 5.3. Notice that, with these choices, both A_ϵ and B_ϵ are very-ample. Indeed, from [29, V Cor. 2.18], B_ϵ is very-ample if and only if $k_\epsilon > b_\epsilon - \epsilon$, wheras A_ϵ is very-ample if and only if $k_\epsilon < 2b_\epsilon - 4\epsilon$. Both conditions are implied by (5.8), since $e \geq 2$.

As in (3.1), we consider $\mathcal{E}_\epsilon$ a rank–two vector bundle on $\mathbb{F}_\epsilon$ fitting in the exact sequence

$$0 \to A_\epsilon \to \mathcal{E}_\epsilon \to B_\epsilon \to 0. \tag{5.13}$$

Thus

$$c_1(\mathcal{E}_\epsilon) = A_\epsilon + B_\epsilon \equiv 3C_\epsilon + b_\epsilon f \;\text{ and }\; c_2(\mathcal{E}_\epsilon) = A_\epsilon B_\epsilon = k_\epsilon = k_e.$$

From (2.1) one has $\deg(\mathcal{E}_\epsilon) = (3C_\epsilon + b_\epsilon f)^2 - k_\epsilon = -9\epsilon + 6b_\epsilon - k_\epsilon$. Thus (5.7) gives

$$\deg(\mathcal{E}_\epsilon) = 6b_e - 9e - k_e = d_e, \tag{5.14}$$

where $d_e = \deg(\mathcal{E}_e)$ is as in (4.3).

Now $\mathrm{Ext}^1(B_\epsilon, A_\epsilon) \cong H^1(A_\epsilon - B_\epsilon)$, where $A_\epsilon - B_\epsilon \equiv C_\epsilon + (3b_\epsilon - 2k_\epsilon - 4\epsilon)f$ from (5.11), (5.12). In particular, $\pi_{\epsilon*}(A_\epsilon - B_\epsilon) \cong (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\epsilon)) \otimes \mathcal{O}_{\mathbb{P}^1}(3b_\epsilon - 2k_\epsilon - 4\epsilon)$. We then use similar computations as in the proofs of Lemmas 3.4 and 3.6, in the range (5.8) for k_ϵ of interest for us (recall Lemma 5.2), and we get:

$$\dim(\mathrm{Ext}^1(B_\epsilon, A_\epsilon)) = \begin{cases} 0 & \text{for } b_\epsilon + \frac{e-3\epsilon}{2} < k_\epsilon < \frac{3b_\epsilon+2-5\epsilon}{2} \\ 4k_\epsilon - 6b_\epsilon - 2 + 9\epsilon & \text{for } \frac{3b_\epsilon+2-5\epsilon}{2} \leq k_\epsilon < 2b_\epsilon - 3\epsilon - e \end{cases} \tag{5.15}$$

and

$$h^0(\mathcal{E}_\epsilon \otimes \mathcal{E}_\epsilon^\vee) = \begin{cases} 6b_\epsilon - 4k_\epsilon - 9\epsilon + 4 & \text{for } b_\epsilon + \frac{e-3\epsilon}{2} < k_\epsilon < \frac{3b_\epsilon+2-5\epsilon}{2} \\ 1 & \text{for } \frac{3b_\epsilon+2-5\epsilon}{2} \leq k_\epsilon < 2b_\epsilon - 3\epsilon - e \text{ and } \mathcal{E}_\epsilon \text{ general;} \end{cases} \tag{5.16}$$

(the reader will easily realize that the distinction of cases in (5.15) and in (5.16) occurs when $\frac{3b_\epsilon+2-5\epsilon}{2} < 2b_\epsilon - 3\epsilon - e$, that is for $b_\epsilon > 2e + \epsilon + 2$, i.e. for $b_e > \frac{7e-\epsilon}{2} + 2$ as it follows from (5.7); otherwise, only the first case in (5.15) and in (5.16) occurs, but we will not dwell on this).

Using (5.13) and same reasoning as in Lemma 3.2, under numerical assumptions (5.8) we get

$$h^j(B_\epsilon) = 0, \ \text{ for } j \geq 1. \tag{5.17}$$

Using the same strategy as in Lemma 3.2, considerations similar to (3.5), (3.6) and (3.3) can be done for $\mathcal{E}_\epsilon$ and one gets

$$h^1(\mathcal{E}_\epsilon) \leq h^1(A_\epsilon) \ \text{ and } \ h^0(\mathcal{E}_\epsilon) = 4b_\epsilon - k_\epsilon - 6\epsilon + 5 + h^1(\mathcal{E}_\epsilon) \tag{5.18}$$

In particular, from (5.7), one has:

$$h^0(\mathcal{E}_\epsilon) = 4b_e - k_e - 6e + 5 + h^1(\mathcal{E}_\epsilon) = (n_e + 1) + h^1(\mathcal{E}_\epsilon), \tag{5.19}$$

where $n_e = \chi(\mathcal{E}_e) - 1 = h^0(\mathcal{E}_e) - 1$ as in (4.3).

To compute $h^1(A_\epsilon)$ we follow the same strategy as in Lemma 3.9. Since $\pi_{\epsilon*}(A_\epsilon) \cong Sym^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\epsilon)) \otimes \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 2\epsilon)$, by Leray's isomorphism one gets that $h^1(A_\epsilon) = h^1(\pi_{\epsilon*}(A_\epsilon)) = 0$ as soon as $k_\epsilon < 2b_\epsilon + 2 - 4\epsilon$. Considering the upper–bound for k_ϵ in (5.8), we notice that $2b_\epsilon - 3\epsilon - e < 2b_\epsilon + 2 - 4\epsilon$; in other words, for k_ϵ as in (5.8), one has

$$h^1(A_\epsilon) = 0. \tag{5.20}$$

As in Corollaries 3.5, 3.10, we get therefore:

Corollary 5.4. *Assumptions* (5.8) *imply that any* $\mathcal{E}_\epsilon \in \mathrm{Ext}^1(B_\epsilon, A_\epsilon)$ *is such that* $h^1(\mathcal{E}_\epsilon) = 0$. *In particular,*

$$h^0(\mathcal{E}_\epsilon) = n_e + 1, \tag{5.21}$$

with n_e *as in* (4.3).

Proof. (5.21) follows from (5.19) and from what proved above. □

Let now $(\mathbb{P}(\mathcal{E}_\epsilon), \mathcal{O}_{\mathbb{P}(\mathcal{E}_\epsilon)}(1))$ be the 3-dimensional scroll over $\mathbb{F}_\epsilon$ associated to any $\mathcal{E}_\epsilon$ as above. From Remark 5.3, $A_\epsilon \oplus B_\epsilon$ is very-ample. Since very-ampleness is an open condition, when $\dim(\mathrm{Ext}^1(B_\epsilon, A_\epsilon)) > 0$, the general $\mathcal{E}_\epsilon \in \mathrm{Ext}^1(B_\epsilon, A_\epsilon)$ is also very-ample and thus $\mathcal{O}_{\mathbb{P}(\mathcal{E}_\epsilon)}(1)$ defines an embedding

$$\Phi_\epsilon := \Phi_{|\mathcal{O}_{\mathbb{P}(\mathcal{E}_\epsilon)}(1)|} : \mathbb{P}(\mathcal{E}_\epsilon) \hookrightarrow X_\epsilon \subset \mathbb{P}^{n_e}, \tag{5.22}$$

(see (5.21)), where $X_\epsilon := \Phi_\epsilon(\mathbb{P}(\mathcal{E}_\epsilon))$ is smooth, non-degenerate of degree d_e (cf. (5.14)). Moreover, letting $(X_\epsilon, L_\epsilon) := (X_\epsilon, \mathcal{O}_{X_\epsilon}(H)) \cong (\mathbb{P}(\mathcal{E}_\epsilon), \mathcal{O}_{\mathbb{P}(\mathcal{E}_\epsilon)}(1))$, one has $h^i(X_\epsilon, L_\epsilon) = 0, \ i \geq 1$.

One can easily see that X_ϵ and X_e have the same Hilbert polynomial $P(T)$, defined by (4.5), so $X_\epsilon \subset \mathbb{P}^{n_e}$ corresponds to a point $[X_\epsilon]$ of the Hilbert scheme $\mathcal{H}_3^{d_e, n_e}$ as in § 4.1.

Proposition 5.5. *For any ϵ, b_ϵ and k_ϵ as in* (5.6), (5.7) *and* (5.8), *there exists an irreducible component $\mathcal{X}_\epsilon \subseteq \mathcal{H}_3^{d_e,n_e}$ which is generically smooth, of (the expected) dimension*

$$\dim(\mathcal{X}_\epsilon) = n_e(n_e+1) + 3k_\epsilon - 2b_\epsilon + 3\epsilon - 5, \tag{5.23}$$

such that $[X_\epsilon]$ belongs to the smooth locus of $\mathcal{X}_\epsilon$. Moreover, the general point of $\mathcal{X}_\epsilon$ parametrizes a scroll X_ϵ as in (5.22).

Remark 5.6. Notice that, from (5.7), the right hand side of the equality in (5.23) coincides with that of (4.9), in other words $\dim(\mathcal{X}_\epsilon) = \dim(\mathcal{X}_e)$.

Proof of Proposition 5.5. Let $N_\epsilon := N_{X_\epsilon/\mathbb{P}^{n_e}}$ denote the normal bundle of X_ϵ in $\mathbb{P}^{n_e}$. As in Theorems 4.5, 5.1, the statement will follow by proving the following intermediate steps:
(a) show that $H^i(X_\epsilon, N_\epsilon) = (0)$, for $i \geq 1$,
(b) conduct an explicit computation of $h^0(X_\epsilon, N_\epsilon) = \chi(X_\epsilon, N_\epsilon)$,
(c) perform a parameter computation to estimate the dimension of the locus $\mathcal{Y}_\epsilon$ filled up by scrolls X_ϵ as in (5.22). Therefore $\dim(\mathcal{Y}_\epsilon)$ gives a lower bound for $\dim(\mathcal{X}_\epsilon)$. Finally,
(d) show that $\dim(\mathcal{Y}_\epsilon)$ equals the number in (5.23).

<u>Case $\epsilon = 1$</u>. From (5.8), we have $5 \leq b_1 \leq b_1 + \frac{e-3}{2} < k_1 < 2b_1 - 3 - e$, indeed by (5.6) the case e odd gives $e \geq 3$. Notice that the upper and lower bound are compatible since, by (5.8), $b_1 \geq \frac{3}{2}(e+1) + 1$. Using (5.11), (5.12), we get

$$A_1 \equiv 2C_1 + (2b_1 - k_1 - 2)f \quad \text{and} \quad B_1 \equiv C_1 + (k_1 - b_1 + 2)f.$$

All steps (a)-(d) are already proved in [7, Prop. 5.5, Thm. 5.7] (cases considered here all come from cases therein coming from the first line of [7, (16) in Lemma 3.7]).

<u>Case $\epsilon = 0$</u>. In this case, we have $b_0 + \frac{e}{2} < k_0 < 2b_0 - e$ where, from (4.1), $b_0 > 3$ for $e \geq 2$ even and the upper and lower bound on k_0 are compatible. By (5.11), (5.12), we have

$$A_0 \equiv 2C_0 + (2b_0 - k_0 - 2)f \quad \text{and} \quad B_0 \equiv C_0 + (k_0 - b_0 + 2)f,$$

where C_0 and f are generators of the two different rulings on $\mathbb{F}_0$.

For Steps (a) and (b), we will use the same strategy of Theorem 4.5. By Corollary 5.4, $H^i(X_0, L_0) = 0$, for $i \geq 1$.

Thus, using the Euler sequence restricted to X_0 as in (4.10), the fact that (X_0, L_0) is a scroll over $\mathbb{F}_0$, non–degenerate in $\mathbb{P}^{n_e}$ (cf. (4.11) and (4.12)) and the normal sequence of $X_0 \subset \mathbb{P}^{n_e}$ as in (4.13), we get

$$H^i(X_0, N_0) \cong H^{i+1}(X_0, T_{X_0}) \quad \text{for} \quad i \geq 1. \tag{5.24}$$

Consequently $h^3(X_0, N_0) = 0$ for dimension reasons; for $h^1(X_0, N_0)$, $h^2(X_0, N_0)$, we can use (5.24).

In order to compute $H^j(X_0, T_{X_0})$, $j = 2, 3$, let $\varphi : \mathbb{P}(\mathcal{E}_0) \longrightarrow \mathbb{F}_0$ be the scroll map. We use the relative cotangent bundle sequence as in (4.15) and adjunction on X_0 to get, as in (4.16), the exact sequence

$$0 \to 2L_0 - \varphi^*(3C_0 + b_0 f) \to T_{X_0} \to \varphi^*(T_{\mathbb{F}_0}) \to 0. \tag{5.25}$$

By Leray's isomorphism, one has $H^j(\varphi^*(T_{\mathbb{F}_0})) \cong H^j(T_{\mathbb{F}_0})$, for any $j \geq 0$. Since $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $h^j(T_{\mathbb{F}_0}) = 2h^j(\mathcal{O}_{\mathbb{P}^1}(2))$, for any $j \geq 0$. Thus,

$$h^0(X_0, \varphi^*(T_{\mathbb{F}_0})) = h^0(\mathbb{F}_0, T_{\mathbb{F}_0}) = 6 \ \text{ and } \ h^j(X_0, \varphi^*(T_{\mathbb{F}_0})) = h^j(\mathbb{F}_0, T_{\mathbb{F}_0}) = 0, \text{ for } j \geq 1. \tag{5.26}$$

For the cohomology of $2L_0 - \varphi^*(3C_0 + b_0 f)$, since $R^i\varphi_*(2L_0) = 0$ for $i \geq 1$ (see [29, Ex. 8.4, p. 253]), projection formula and Leray's isomorphism give

$$H^i(X_0, 2L_0 - \varphi^*(3C_0 + b_0 f)) \cong H^i(\mathbb{F}_0, Sym^2\mathcal{E}_0 \otimes (-3C_0 - b_0 f)), \ \forall\, i \geq 0. \tag{5.27}$$

Therefore

$$h^j(X_0, 2L_0 - \varphi^*(3C_0 + b_0 f)) = 0, \ j \geq 3, \tag{5.28}$$

for dimension reasons. Finally, we use filtrations as in (4.22), (4.23), (4.24) and argue as in the proof of Claim 4.6-(ii), to get also

$$h^2(\mathbb{F}_0, Sym^2\mathcal{E}_0 \otimes (-3C_0 - b_0 f)) = h^2(X_0, 2L_0 - \varphi^*(3C_0 + b_0 f)) = 0. \tag{5.29}$$

From (5.25), using (5.26), (5.27) and (5.28), we deduce that $h^j(X_0, T_{X_0}) = 0$, for any $j \geq 2$, so from (5.24) we get $h^i(N_0) = 0$, for $i \geq 1$.

In particular, generic smoothness of $\mathcal{X}_0$ and the fact that it has the expected dimension follow from (4.7), (4.8).

To compute the expected dimension (i.e. Step (*b*)), we use the Hirzebruch-Riemann-Roch theorem as in (4.30), with values as in (4.31). This gives

$$h^0(N_0) = \chi(N_0) = (d_0 - 2b_0 + 5)n_0 - 5 + 16b_0 - 3d_0.$$

Using (4.3) and (5.7), one gets

$$h^0(N_0) = (n_0 + 1)n_0 + 3k_0 - 2b_0 - 5.$$

As for Step (*c*), consider the exact sequence (5.13). A_0 and B_0 are uniquely determined on $\mathbb{F}_0$. As in the proof of Theorem 5.1, to compute $\dim(\mathcal{Y}_0)$ we have to add up the quantities τ_0, that is the number of parameters counting isomorphism classes of projective bundles $\mathbb{P}(\mathcal{E}_0)$, and the dimension of the full orbit of $X_0 \subset \mathbb{P}^{n_0}$ under the action of $PGL(n_0 + 1, \mathbb{C})$.

From (5.15) we get

$$\tau_0 = \begin{cases} 0 & \text{for } b_0 + \frac{e}{2} < k_0 < \frac{3b_0+2}{2} \\ 4k_0 - 6b_0 - 3 & \text{for } \frac{3b_0+2}{2} \leq k_0 < 2b_0 - e, \end{cases}$$

(cf. the proof of Theorem 5.1).

The dimension of the orbit of X_0 is given by

$$\dim(PGL(n_0 + 1, \mathbb{C})) - \dim(G_{X_0}) = n_0(n_0 + 2) - h^0(T_{X_0}),$$

where $G_{X_0} \subset PGL(n_0 + 1, \mathbb{C})$ is the projective stabilizer. In particular,

$$\dim(\mathcal{Y}_0) = \tau_0 + n_0(n_0 + 2) - \dim(G_{X_0}).$$

As in the proof of Theorem 5.1, one obviously has

$$\dim(G_{X_0}) \leq \dim(Aut(X_0)) = \dim(Aut(\mathbb{F}_0)) + \dim(Aut_{\mathbb{F}_0}(\mathbb{P}(\mathcal{E}_0)),$$

where $Aut(X_0)$ denotes the algebraic group of abstract automorphisms of X_0 whereas $Aut_{\mathbb{F}_0}(\mathbb{P}(\mathcal{E}_0))$ the group of automorphisms of $\mathbb{P}(\mathcal{E}_0)$ fixing the base (cf. e.g. [34]).

From (5.26), we have $\dim(Aut(\mathbb{F}_0)) = 6$ (cf. also [34, Lemma 10]).

For $\dim(Aut_{\mathbb{F}_0}(\mathbb{P}(\mathcal{E}_0)))$, from (5.16) one gets

$$\dim(Aut_{\mathbb{F}_0}(\mathbb{P}(\mathcal{E}_0))) = \begin{cases} 6b_0 - 4k_0 + 3 & \text{for } b_0 + \frac{e}{2} < k_0 < \frac{3b_0+2}{2} \\ 0 & \text{for } \frac{3b_0+2}{2} < k_0 < 2b_0 - e \text{ and } \mathcal{E}_0 \text{ general} \end{cases}$$

In all cases, one gets

$$\dim(\mathcal{Y}_0) \geq n_0(n_0 + 2) + 4k_0 - 6b_0 - 9.$$

For Step (*d*), we recall (5.23). So we have

$$n_0(n_0 + 1) + 3k_0 - 2b_0 - 5 = \dim(\mathcal{X}_0) \geq \dim(\mathcal{Y}_0) \geq n_0(n_0 + 2) + 4k_0 - 6b_0 - 9.$$

Observe that the left and right most sides of the previous inequalities are equal: indeed $n_0(n_0+1) + 3k_0 - 2b_0 - 5 - (n_0(n_0+2) + 4k_0 - 6b_0 - 9) = 4b_0 + 4 - k_0 - n_0 = 0$ as it follows from (5.21). Thus $\dim(\mathcal{X}_0) = \dim(\mathcal{Y}_0)$ which concludes the proof. ☐

5.2. **The components $\mathcal{X}_e$ and $\mathcal{X}_\epsilon$ coincide.**

THEOREM 5.7. *With Assumptions 4.3, one has $\mathcal{X}_e = \mathcal{X}_\epsilon$.*

Proof. Notice that, from the proof of Lemma 5.2, Assumptions 4.3 are equivalent to conditions in (5.8) which are exactly the values for which $\mathcal{X}_\epsilon$ has been constructed.

Recall that $\mathcal{X}_e$ and $\mathcal{X}_\epsilon$ have the same dimension (cf. Remark 5.6) and are both components of the same Hilbert scheme $\mathcal{H}_3^{d_e,n_e}$ as in § 4.1, since X_e and X_ϵ have the same Hilbert polinomial (cf. § 5.1). From Theorems 4.5 and 5.1, we furthermore have that $[X_e] \in \mathcal{Y}_e$ general is a smooth point for $\mathcal{X}_e$ and similarly, Proposition 5.5 states that $[X_\epsilon] \in \mathcal{X}_\epsilon$ general is a smooth point too. Thus, by smoothness and the fact that $\dim(\mathcal{X}_\epsilon) = \dim(\mathcal{X}_e)$, to prove the theorem it will be enough to exhibit a flat, embedded (in $\mathbb{P}^{n_e}$) degeneration of X_ϵ to X_e which is entirely contained in the smooth locus of $\mathcal{X}_\epsilon$; in other words, we need to show that there exist a flat family

$$\begin{array}{ccc} \mathfrak{F} & \subset & \mathbb{P}^{n_e} \times \Delta \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_2} & \\ \Delta & & \end{array}$$

where Δ is a smooth, irreducible affine curve, pr_2 is the projection onto the second factor, $\mathfrak{F} \subset \mathbb{P}^{n_e} \times \Delta$ is a closed subscheme of relative dimension three, π is the restriction to it of pr_2, which is proper, flat and such that $\pi^{-1}(t) := \mathfrak{F}_t \cong X_\epsilon$, for $t \neq 0$, and $\pi^{-1}(0) = \mathfrak{F}_0 \cong X_e$, and Δ maps to an (affine) irreducible curve in $\mathcal{H}_3^{d_e,n_e}$ (which, by abuse of notation, we will always denote by Δ) connecting $[X_\epsilon]$ with $[X_e]$ and such that $\Delta \subset (\mathcal{X}_\epsilon)_{sm}$, the smooth locus of $\mathcal{X}_\epsilon$.

To exhibit this degeneration, recall that X_e and X_ϵ are respectively determined by the pairs $(\mathbb{F}_e, \mathcal{E}_e)$ and $(\mathbb{F}_\epsilon, \mathcal{E}_\epsilon)$ (cf. Prop. 4.4 and (5.22)). According to what was proved in the previous sections, when $\dim(\mathrm{Ext}^1(B_e, A_e)) > 0$ it is clear that the bundle $\mathcal{E}_e$ flatly degenerates (or specializes, in the sense of [4, p. 126]) inside the vector space $\mathrm{Ext}^1(B_e, A_e)$ to the decomposable bundle $A_e \oplus B_e$; when otherwise $\dim(\mathrm{Ext}^1(B_e, A_e)) = 0$ one simply has $\mathcal{E}_e = A_e \oplus B_e$. The same occurs for bundles in $\mathrm{Ext}^1(B_\epsilon, A_\epsilon)$ on $\mathbb{F}_\epsilon$.

Denote by D_e (respectively D_ϵ) the *decomposable* scroll determined by the pair $(\mathbb{P}(A_e \oplus B_e), \mathcal{O}_{\mathbb{P}(A_e \oplus B_e)}(1))$ (respectively $(\mathbb{P}(A_\epsilon \oplus B_\epsilon), \mathcal{O}_{\mathbb{P}(A_\epsilon \oplus B_\epsilon)}(1))$).

From the proofs of Theorem 4.5 and Proposition 5.5, $[X_e]$, $[D_e]$, $[X_\epsilon]$ and $[D_\epsilon]$ are all smooth points of the Hilbert scheme $\mathcal{H}_3^{d_e,n_e}$ and the flat (abstract) degenerations of general bundles in $\mathrm{Ext}^1(B_e, A_e)$ and in $\mathrm{Ext}^1(B_\epsilon, A_\epsilon)$ to the decomposable ones $A_e \oplus B_e$ and $A_\epsilon \oplus B_\epsilon$, respectively, clearly give rise to flat degenerations, embedded in $\mathbb{P}^{n_e}$, of X_e to D_e and of X_ϵ to D_ϵ, which are contained in the smooth locus of $\mathcal{X}_e$ and $\mathcal{X}_\epsilon$, respectively. The assertions follow from the fact that, since all the bundles involved are very-ample and with no higher cohomology (cf. previous sections), the corresponding threefold scrolls are smooth with non-special normal bundles in $\mathbb{P}^{n_e}$.

It is therefore enough to show that there exists a flat, embedded degeneration of D_ϵ to D_e which is entirely contained in the smooth locus of $\mathcal{X}_\epsilon$; if this is the case, by smoothness at each step and by $\dim(\mathcal{X}_e) = \dim(\mathcal{X}_\epsilon)$, we must have $\mathcal{X}_e = \mathcal{X}_\epsilon$ as desired.

Now, the decomposable scroll D_e has two disjoint sections, say S^{α_e} and S^{β_e}, where $\alpha_e := \deg(S^{\alpha_e}) = \deg(A_e) = 8b_e - 4k_e - 12e$ and $\beta_e := \deg(S^{\beta_e}) = \deg(B_e) = 2k_e - 2b_e + 3e$ (cf. (3.2)), which correspond to the two quotients $A_e \oplus B_e \twoheadrightarrow A_e$ and $A_e \oplus B_e \twoheadrightarrow B_e$ respectively. Precisely, S^{α_e} (respectively S^{β_e}) is given by the embedding of $\mathbb{F}_e$ via the very-ample linear system $|A_e|$ (respectively $|B_e|$); from Lemma 3.2 and the non-speciality of both A_e and B_e, the projective linear spans of such surfaces $\langle S^{\alpha_e} \rangle \cong \mathbb{P}^{\ell_e}$ and $\langle S^{\beta_e} \rangle \cong \mathbb{P}^{r_e}$, where $\ell_e := h^0(A_e) - 1 = 6b_e - 3k_e - 9e + 2$ and $r_e := h^0(B_e) - 1 = 2k_e - 2b_e + 3e + 1 = \beta_e + 1$, are skew, spanning the whole $\mathbb{P}^{n_e}$, and D_e turns out to be the join of these two surfaces.

Similarly D_ϵ is the joint in $\mathbb{P}^{n_e}$ of two smooth, rational surfaces S^{α_ϵ} and S^{β_ϵ}, with S^{α_ϵ} and S^{β_ϵ} respectively given by the embedding of $\mathbb{F}_\epsilon$ via $|A_\epsilon|$ and $|B_\epsilon|$, where $\alpha_\epsilon = \deg(S^{\alpha_\epsilon}) = \deg(A_\epsilon) = \alpha_e$ and $\beta_\epsilon = \deg(S^{\beta_\epsilon}) = \deg(B_\epsilon) = \beta_e$, the last equalities following from (5.7), (5.11), (5.12). As above, these two surfaces are (disjoint) sections of D_ϵ, whose linear spans

$\langle S^{\alpha_\epsilon}\rangle \cong \mathbb{P}^{\ell_e}$ and $\langle S^{\beta_\epsilon}\rangle \cong \mathbb{P}^{r_e}$ are skew, spanning the whole $\mathbb{P}^{n_e}$ (all the assertions follow from (5.7), (5.11), (5.12), the non–speciality of A_ϵ and of B_ϵ and the fact that the bundle is decomposable).

Since $|B_\epsilon|$ (respectively $|B_e|$) is very-ample and unisecant to the fibers of $\mathbb{F}_\epsilon$ (respectively of $\mathbb{F}_e$), the image surface S^{β_ϵ} (respectively S^{β_e}) is a smooth, rational normal scroll inside $\mathbb{P}^{r_e}$. If we denote by $\mathcal{H}_2^{\beta_e,r_e}$ the Hilbert scheme of rational normal scrolls of degree β_e in $\mathbb{P}^{r_e}$, it is well-known that it is irreducible, smooth at points corresponding to smooth scrolls and that its general point is given by *balanced* scrolls, i.e. those arising from $\mathbb{F}_\epsilon$. In particular, there is a flat degeneration of S^{β_ϵ} to S^{β_e}, embedded in $\mathbb{P}^{r_e}$, represented by an affine curve denoted by Δ, connecting the Hilbert point $[S^{\beta_\epsilon}]$ to $[S^{\beta_e}]$ and which is entirely contained in the smooth locus of $\mathcal{H}_2^{\beta_e,r_e}$ (cf. e.g. [10, Def. 2.15, Rem. 3.9] and [16, Lemma 3] or read details below for the case with $|A_\epsilon|$ and $|A_e|$).

Similarly, denoting by $\mathcal{H}_2^{\alpha_e,\ell_e}$ the Hilbert scheme of closed subschemes of $\mathbb{P}^{\ell_e}$ having the same Hilbert polynomial as S^{α_ϵ} (equivalently S^{α_e}), one can easily show that there exists a flat embedded (in $\mathbb{P}^{\ell_e}$) degeneration of S^{α_ϵ} to S^{α_e}, represented by the same Δ as above, connecting $[S^{\alpha_\epsilon}]$ to $[S^{\alpha_e}]$ and which is entirely contained in the smooth locus of a component of $\mathcal{H}_2^{\alpha_e,\ell_e}$.

To do this, for simplicity we focus on the case e even, i.e. $\epsilon = 0$, since for e odd the arguments hold almost verbatim. Take therefore for a moment $e = 2k \geq 2$; the non-trivial extension $0 \to \mathcal{O}_{\mathbb{P}^1}(-k) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(k) \to 0$ over $\mathbb{P}^1$ gives rise to a line of the vector space $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(-k)) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-e))$, which can be identified with a 1-dimensional, affine base scheme Δ of a flat degeneration (or specialization, in the sense of [4, p. 126]) of the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ to $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k)$ over $\mathbb{P}^1$, and so of $\mathbb{F}_0$ to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k)) \cong \mathbb{F}_e$.

Since $\mathbb{F}_0$ and $\mathbb{F}_{2k}$ are endowed with very-ample line bundles A_0 and A_{2k}, respectively, of same degree and same projective dimension, it is a standard procedure to identify Δ as above with also the base scheme of a flat, embedded (in $\mathbb{P}^{\ell_0}$) degeneration of smooth, rational surfaces S^{α_0} to $S^{\alpha_{2k}}$ (cf. e.g. [23] and [10, Constr. 3.6, 3.7] for procedures in even more degenerate situations). Briefly, one takes the trivial family $\mathcal{T} := \mathbb{F}_0 \times \Delta \xrightarrow{\mathrm{pr}_2} \Delta$, which is also endowed with a relative line bundle $\mathcal{A}$ resticting to A_0 on any pr_2-fiber. One then performs standard operations involving: (1) blowing-ups and blowing-downs in the central fiber of $\mathcal{T}$, and (2) twisting $\mathcal{A}$ by components of the central fiber. Doing this, one gets a birational modification of the (original) central fiber $(\mathcal{T}_0, \mathcal{A}_{|_{\mathcal{T}_0}}) = (\mathbb{F}_0, A_0)$ and a (no more trivial) proper, flat family $\mathcal{T}' \xrightarrow{\pi'} \Delta$, together with a relative line bundle $\mathcal{A}' \to \mathcal{T}'$ s.t.: the total space $\mathcal{T}'$ is smooth, if $\mathcal{T}'_t := \pi'^{-1}(t)$ for $t \in \Delta$, then $h^0(\mathcal{A}'_{|_{\mathcal{T}'_t}}) = \alpha_0 + 1$, for any $t \in \Delta$, $(\mathcal{T}'_t, \mathcal{A}'_{|_{\mathcal{T}'_t}}) = (\mathcal{T}_t, \mathcal{A}_{|_{\mathcal{T}_t}}) = (\mathbb{F}_0, A_0) \cong S^{\alpha_0} \subset \mathbb{P}^{\ell_0}$, for $t \neq 0$, whereas $(\mathcal{T}'_0, \mathcal{A}'_{|_{\mathcal{T}'_0}}) \cong (\mathbb{F}_{2k}, A_{2k}) \cong S^{\alpha_{2k}} \subset \mathbb{P}^{\ell_0}$ (cf. e.g. [23] and [10] for full details). This means that Δ can be identified as an affine curve, always denoted by Δ, in $\mathcal{H}_2^{\alpha_e,\ell_e}$ with the desired properties (the fact that Δ is entirely contained in the smooth locus of a component of $\mathcal{H}_2^{\alpha_e,\ell_e}$ follows from the fact that the normal bundles in $\mathbb{P}^{\ell_0}$ of both S^{α_0} and $S^{\alpha_{2k}}$ are non-special, as it follows from the Euler sequence restricted to them).

Turning back to the general case with any e and $\epsilon = 0, 1$, it is then clear that for $t \in \Delta \setminus \{0\}$ approaching to 0 we have "simultaneous" specializations of S^{α_ϵ} to S^{α_e} in $\mathbb{P}^{\ell_e}$ and of S^{β_ϵ} to S^{β_e} in $\mathbb{P}^{r_e}$ and so of their respective join in $\mathbb{P}^{n_e}$. Formally one applies the same procedures explained above to both pairs $(\mathbb{F}_\epsilon, A_\epsilon)$ and $(\mathbb{F}_\epsilon, B_\epsilon)$ and so also to $(\mathbb{F}_\epsilon, A_\epsilon \oplus B_\epsilon)$; in this way Δ can be identified with the base scheme of the desired flat family $\mathfrak{F} \xrightarrow{\pi} \Delta$ as in the beginning of the proof, whose general fiber is given by $(\mathbb{F}_\epsilon, A_\epsilon \oplus B_\epsilon) \cong D_\epsilon$ and whose central fiber is $(\mathbb{F}_e, A_e \oplus B_e) = D_e$ (notice that flatness of $\mathfrak{F}$ over Δ follows from the facts that Δ is integral and that all the fibers have the same Hilbert polynomial as in (4.5), cf.[38, Prop. 4.2.1 (ii)]). Very-ampleness and non-speciality of $A_e \oplus B_e$ imply that D_e and D_ϵ are smooth, non-special threefold scrolls in $\mathbb{P}^{n_e}$ with $h^1(N_{D_\epsilon/\mathbb{P}^{n_e}}) = h^1(N_{D_e/\mathbb{P}^{n_e}}) = 0$ (cf. proofs of Claim 4.6 and of Prop. 5.5), i.e. the curve Δ is entirely contained in the smooth locus of $\mathcal{H}_3^{d_e,n_e}$ and so of $\mathcal{X}_\epsilon$, being one irreducible component of the Hilbert scheme. This forces $\mathcal{X}_\epsilon = \mathcal{X}_e$ as desired. □

Remark 5.8. The proof of Theorem 5.7 can be interpreted as a projective-geometry counterpart of (abstract) specializations of rank-five vector bundles over $\mathbb{P}^1$ as in [4, Prop. 2.3]. Applying the direct image functors $R^j\pi_{e*}$ to the exact sequence (3.1) gives the following exact sequence of bundles on $\mathbb{P}^1$

$$\begin{aligned}0 \to &\pi_{e*}(A_e) \cong Sym^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e) \to \pi_{e*}(\mathcal{E}_e)\\ \to &\pi_{e*}(B_e) \cong (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + 2e) \to 0,\end{aligned}$$

that is

$$\begin{aligned}0 \to &\mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 3e) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 4e) \to \pi_{e*}(\mathcal{E}_e)\\ \to &\mathcal{O}_{\mathbb{P}^1}(k_e - b_e + 2e) \oplus \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + e) \to 0.\end{aligned}$$

Thus the push-forward via π_{e*} defines a natural map

$$\mathrm{Ext}^1(B_e, A_e) \xrightarrow{\Pi_e} \mathrm{Ext}^1(\pi_{e*}(B_e), \pi_{e*}(A_e)), \ \text{ s.t. } \Pi_e(\mathcal{E}_e) := \pi_{e*}(\mathcal{E}_e).$$

Now $\pi_{e*}(\mathcal{E}_e)$ is a rank-five vector bundle on $\mathbb{P}^1$ with $\delta_e := \deg(\pi_{e*}(\mathcal{E}_e)) = 4b_e - k_e - 6e$ so $\pi_{e*}(\mathcal{E}_e) = \bigoplus_{i=1}^5 \mathcal{O}_{\mathbb{P}^1}(\alpha_i)$, for some $\alpha_i \in \mathbb{Z}$ with $\Sigma_{i=1}^5 \alpha_i = 4b_e - k_e - 6e$.

Similarly, from (5.13) one gets

$$\begin{aligned}0 \to &\pi_{\epsilon*}(A_\epsilon) \cong Sym^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\epsilon)) \otimes \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 2e) \to \pi_{\epsilon*}(\mathcal{E}_\epsilon)\\ \to &\pi_{\epsilon*}(B_\epsilon) \cong (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\epsilon)) \otimes \mathcal{O}_{\mathbb{P}^1}(k_\epsilon - b_\epsilon + 2e) \to 0\end{aligned}$$

which reads also

$$\begin{aligned}0 \to &\mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 2\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 3\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 4\epsilon) \to \pi_{\epsilon*}(\mathcal{E}_\epsilon)\\ \to &\mathcal{O}_{\mathbb{P}^1}(k_\epsilon - b_\epsilon + 2\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(k_\epsilon - b_\epsilon + \epsilon) \to 0.\end{aligned}$$

As above $\pi_{\epsilon*}(\mathcal{E}_\epsilon)$ is decomposable, of rank five on $\mathbb{P}^1$, with $\deg(\pi_{\epsilon*}(\mathcal{E}_\epsilon)) = 4b_\epsilon - k_\epsilon - 6\epsilon$. From (5.7) one has $4b_\epsilon - k_\epsilon - 6\epsilon = 4b_e - k_e - 6e$, i.e. $\deg(\pi_{\epsilon*}(\mathcal{E}_\epsilon)) = \deg(\pi_{e*}(\mathcal{E}_e)) = \delta_e$.

It is clear that, inside $\mathrm{Ext}^1(\pi_{e*}(B_e), \pi_{e*}(A_e))$, the bundle $\pi_{e*}(\mathcal{E}_e)$ flatly degenerates (or is equal) to the bundle

$$\mathcal{T}_e := \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 2e) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 3e) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 4e) \oplus \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + 2e) \oplus \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + e).$$

For simplicitly, put

$$\xi_1' := 2b_e - k_e - 2e, \ \xi_2' := 2b_e - k_e - 3e, \ \xi_3' := 2b_e - k_e - 4e$$

and

$$\eta_1' := k_e - b_e + 2e, \ \eta_2' := k_e - b_e + e.$$

Similarly, inside $\mathrm{Ext}^1(\pi_{\epsilon*}(B_\epsilon), \pi_{\epsilon*}(A_\epsilon))$, the vector bundle $\pi_{\epsilon*}(\mathcal{E}_\epsilon)$ flatly degenerates (or is equal) to

$$\mathcal{T}_\epsilon := \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 2\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 3\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_\epsilon - k_\epsilon - 4\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(k_\epsilon - b_\epsilon + 2\epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(k_\epsilon - b_\epsilon + \epsilon).$$

Using (5.7), the latter reads

$$\begin{aligned}\mathcal{T}_\epsilon \ = \ &\mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 3e + \epsilon) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 3e) \oplus \mathcal{O}_{\mathbb{P}^1}(2b_e - k_e - 3e - \epsilon)\\ &\oplus \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + \tfrac{(3e+\epsilon)}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(k_e - b_e + \tfrac{(3e-\epsilon)}{2}).\end{aligned}$$

As above, for simplicity, put

$$\xi_1 := 2b_e - k_e - 3e + \epsilon, \ \xi_2 := 2b_e - k_e - 3e, \ \xi_3 := 2b_e - k_e - 3e - \epsilon$$

and

$$\eta_1 := k_e - b_e + \frac{(3e+\epsilon)}{2}, \ \eta_2 := k_e - b_e + \frac{(3e-\epsilon)}{2}.$$

By [4, Prop 2.3], one deduces that $\mathcal{T}_e$ is a flat specialization of $\mathcal{T}_\epsilon$; indeed, they have same rank and same degree but the latter is more balanced since, for any $1 \leq i \leq 2$:

$$\{0, 1\} \ni \epsilon = \eta_2 - \eta_1 = \xi_{i+1} - \xi_i \ \text{ whereas } \ 2 \leq e = \eta_2' - \eta_1' = \xi_{i+1}' - \xi_i'.$$

6. Examples

We give some examples of Hilbert schemes of threefold scrolls over $\mathbb{F}_e$, with e both even and odd. We use notation and assumptions as in the previous sections.

(1) Take $e = 2$, $b_2 = 11$, $k_2 = 11$, which are compatible with (4.1). Consider vector bundles $\mathcal{E}_2$ over $\mathbb{F}_2$ fitting in

$$0 \to A_2 = 2C_2 + 7f \to \mathcal{E}_2 \to B_2 = C_2 + 4f \to 0.$$

More precisely, since $\mathrm{Ext}^1(B_2, A_2) \cong H^1(C_2 + 3f) = (0)$, then $\mathcal{E}_2 = (2C_2 + 7f) \oplus (C_2 + 4f)$. One has $h^0(\mathcal{E}_2) = 26$, $h^i(\mathcal{E}_2) = 0$, for $i \geq 1$, and $d_2 = \deg(\mathcal{E}_2) = 37$.

For $X_2 \subset \mathbb{P}^{25}$ we know that $h^1(N_2) = h^1(N_{X_2/\mathbb{P}^{25}}) = 0$ (cf. the proof of Claim 4.6). Then $[X_2] \in \mathcal{X}_2$ is a smooth point, where $\mathcal{X}_2$ is generically smooth of dimension 662.

From (5.7), on $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ we take vector bundles $\mathcal{E}_0$ fitting in

$$0 \to A_0 = 2C_0 + 5f \to \mathcal{E}_0 \to B_0 = C_0 + 3f \to 0,$$

compatible with (5.8). As above, since $\mathrm{Ext}^1(B_0, A_0) \cong H^1(C_0 + 2f) = (0)$, then $\mathcal{E}_0 = (2C_0 + 5f) \oplus (C_0 + 3f)$. $\mathcal{E}_0$ has the same degree and the same cohomology as that of $\mathcal{E}_2$. Let $X_0 \subset \mathbb{P}^{25}$ be the associated threefold scroll. From Proposition 5.5 and Theorem 5.7, $[X_0] \in \mathcal{X}_2$ is the general point.

In terms of vector bundles as in Remark 5.8, notice that up to a descending reorder of the summands we have

$$\pi_{2*}(\mathcal{E}_2) = \mathcal{T}_2 = \mathcal{O}_{\mathbb{P}^1}(7) \oplus \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

and

$$\pi_{0*}(\mathcal{E}_0) = \mathcal{T}_0 = \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 2}$$

so $\pi_{2*}(\mathcal{E}_2) = \mathcal{T}_2$ is a flat specialization of $\pi_{0*}(\mathcal{E}_0) = \mathcal{T}_0$ ([4, Prop. 2.3]).

(2) From (4.1), take $e = 3$, $b_3 = 15$, $k_3 = 15$. Consider vector bundles $\mathcal{E}_3$ over $\mathbb{F}_3$ fitting in

$$0 \to A_3 = 2C_3 + 8f \to \mathcal{E}_3 \to B_3 = C_3 + 7f \to 0.$$

Since $15 = k_3 < 2b_3 - 4e = 18$, from the first line of (3.14), $h^1(A_3) = 0$ so the same holds for any $\mathcal{E}_3 \in \mathrm{Ext}^1(B_3, A_3) \cong H^1(C_3 + f) \cong \mathbb{C}$ (cf. Corollary 3.10). All $\mathcal{E}_3$'s have degree $d_3 = 47$, $h^0(\mathcal{E}_3) = 32$ and no higher cohomology. Moreover, any $\mathcal{E}_3$ corresponding to a non-zero vector in $\mathrm{Ext}^1(B_3, A_3)$ flatly degenerates inside this vector space to the trivial bundle $\mathcal{T}_3 := A_3 \oplus B_3$.

From (5.7), on $\mathbb{F}_1$ we correspondingly take

$$0 \to A_1 = 2C_1 + 6f \to \mathcal{E}_1 \to B_1 = C_1 + 6f \to 0.$$

Now $\mathrm{Ext}^1(B_1, A_1) \cong H^1(C_1) \cong (0)$ and thus $\mathcal{E}_1 = A_1 \oplus B_1$ is the unique bundle. From the proof of Theorem 5.7, these all correspond to smooth points of the Hilbert scheme $\mathcal{H}_3^{27,31}$, in particular contained in the same irreducible component $\mathcal{X}_3$, which is generically smooth.

In terms of vector bundles on $\mathbb{P}^1$, we have that

$$\pi_{3*}(\mathcal{T}_3) := \mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(7) \oplus \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4),$$

which corresponds to the zero-vector of $\mathrm{Ext}^1(\pi_{3*}(B_3), \pi_{3*}(A_3))$. Similarly,

$$\pi_{1*}(\mathcal{T}_1) = \mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(5)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(4).$$

The bundle $\pi_{1*}(\mathcal{T}_1)$ degenerates to $\pi_{3*}(\mathcal{T}_3)$ since it is more balanced than $\pi_{3*}(\mathcal{T}_3)$ (apply [4, Prop 2.3]).

(3) Take $e = 4$, $b_4 = 18$, $k_4 = 18$. Consider vector bundles $\mathcal{E}_4$ over $\mathbb{F}_4$ fitting in

$$0 \to A_4 = 2C_4 + 10f \to \mathcal{E}_4 \to B_4 = C_4 + 8f \to 0.$$

As above, $\mathrm{Ext}^1(B_4, A_4) \cong \mathbb{C}$, all bundles have degree $d_4 = 58$ and are such that $h^i(\mathcal{E}_4) = 0$, for $i \geq 1$, and $h^0(\mathcal{E}_4) = 35$. The general element in $\mathrm{Ext}^1(B_4, A_4)$ flatly degenerates to the trivial one $\mathcal{T}_4 = A_4 \oplus B_4$.

On $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ consider bundles fitting in

$$0 \to A_0 = 2C_0 + 6f \to \mathcal{E}_0 \to B_0 = C_0 + 6f \to 0.$$

Now $\mathrm{Ext}^1(B_0, A_0) \cong H^1(C_0) = (0)$. Similarly as in (2),

$$\pi_{4*}(\mathcal{T}_4) = \mathcal{O}_{\mathbb{P}^1}(10) \oplus \mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

corresponds to the zero-vector of $\mathrm{Ext}^1(\pi_{4*}(B_4), \pi_{4*}(A_4))$ wheras

$$\pi_{0*}(\mathcal{E}_0) = \mathcal{O}_{\mathbb{P}^1}^{\oplus 5}(6)$$

flatly degenerates to $\pi_{4*}(\mathcal{T}_4)$, since it is more balanced (apply e.g. [4, Prop 2.3]). As in example (2), we can conclude.

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