Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 977 – 984
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doi:10.3906/mat-1401-58

Hilbert series of the braid monoid MB_4 in band generators

Zaffar IQBAL*, **Shamaila YOUSAF**
Department of Mathematics, University of Gujrat, Pakistan

Received: 23.01.2014 • **Accepted:** 17.05.2014 • **Published Online:** 24.10.2014 • **Printed:** 21.11.2014

Abstract: L. A. Bokut gave a Gröbner–Shirshov basis of the braid group B_n in band generators. Using this presentation and solving all the ambiguities we construct a linear system for irreducible words and compute the Hilbert series of the braid monoid MB_4.

Key words: Braid monoids, Gröbner–Shirshov basis, reducible and irreducible words, Hilbert series

1. Introduction

The braid group B_n admits the following classical presentation given by E. Artin [3]:

$$B_n = \left\langle x_1, \dots, x_{n-1} \;\middle|\; \begin{array}{l} x_i x_j = x_j x_i \text{ if } \mid i-j \mid \geq 2 \\ x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i \text{ if } 1 \leq i \leq n-2 \end{array} \right\rangle.$$

Elements of B_n are *words* expressed in the generators $x_1, \dots, x_{n-1}$. The braid group B_n admits another presentation called the band presentation given by J. Birman, K. H. Ko, and S. J. Lee [5]. This presentation consists of the generators a_{ts}, $n \geq t > s \geq 1$, where a_{ts} represents the braid in which the t^{th} string crosses over the s^{th} string while the s^{th} and t^{th} strings cross in front of all intermediate strings. Therefore, the band presentation of the braid group B_n is given by

$$B_n = \left\langle a_{ts}, n \geq t > s \geq 1 \;\middle|\; \begin{array}{l} a_{ts} a_{rq} = a_{rq} a_{ts},\, (t-r)(s-r)(s-q)(t-q) > 0 \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr},\, n \geq t > s \geq 1 \end{array} \right\rangle.$$

The *braid monoid* MB_n consisting of only positive crossings admits the same presentation of the braid group B_n:

$$MB_n = \left\langle a_{ts}, n \geq t > s \geq 1 \;\middle|\; \begin{array}{l} a_{ts} a_{rq} = a_{rq} a_{ts},\, (t-r)(s-r)(s-q)(t-q) > 0 \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr},\, n \geq t > s \geq 1 \end{array} \right\rangle.$$

In [10] we constructed a linear system for the braid monoid MB_n in Artin generators and computed the Hilbert series for the braid monoids MB_3 and MB_4. In this paper we construct a similar kind of linear system to compute the Hilbert series of MB_4 in band generators. This linear system is the key behind all the computations to compute the Hilbert series of MB_4.

*Correspondence: zaffar.iqbal@uog.edu.pk

2010 *AMS Mathematics Subject Classification:* 20F36, 20F05, 13D40.

In the Hilbert series $\frac{1}{q_n(t)}$ of MB_n (for $n = 3, 4, 5, 6$) for Artin generators, the degrees of the polynomials $q_n(t)$ are 3, 6, 10, 15 respectively (for details see [11]), whereas, in the case of band generators, the degrees of $q_n(t)$ are 2 and 3 (for $n = 3$ and 4). The advantage of the Hilbert series (for band generators) is that the growth of $q_n(t)$ for band generators is much slower than the growth for Artin generators.

2. Preliminaries

In a presentation of a monoid we fix a total order of the generators (between the generators we choose the natural order $a_{21} < a_{31} < a_{32} < a_{41} < \cdots < a_{n(n-1)}$). In the monoid the relation $\alpha = \beta$ will be written as $\alpha > \beta$ in the length-lexicographic order. Let $\alpha_1 = uw$ and $\alpha_2 = wv$; then the word of the form uwv is said to be an *ambiguity* (for details see [4]). If $\alpha_1 v = u\alpha_2$ as a relation as well as in the length-lexicographic order then we say that the ambiguity uwv is solvable (or solved). Such a presentation is *complete* if and only if all the ambiguities are solvable (for details see [4], [8]). Corresponding to the relations $\alpha = \beta$, the changes $\gamma\alpha\delta \to \gamma\beta\delta$ give a rewriting system. A complete presentation is equivalent to a confluent rewriting system.

In a complete presentation (or in the general presentation) of MB_n a word containing α will be called a *reducible word* and a word that does not contain α will be called an *irreducible word* (also called normal form of the word). We will denote $B_*^{(m)}$ as the set of reducible words and $A_*^{(m)}$ as the set of irreducible words in MB_n.

Let U and V be nonempty words; then the word $Ua_{ij}V$ will be denoted as $Ua_{ij} \times_{ij} a_{ij}V$.

Definition 2.1 *[9] Let G be a finitely generated group and S be a finite set of generators of G. The word lenth $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exists $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.*

Definition 2.2 *[9] Let G be a finitely generated group and S be a finite set of generators of G. The growth function of the pair (G, S) associates to an integer $k \geq 0$ the number $a(k)$ of elements $g \in G$ such that $l_S(g) = k$ and the corresponding spherical growth series or the Hilbert series is given by $P_G(t) = \sum_{k=0}^{\infty} a(k)t^k$.*

For a sequence $\{s_k\}_{k\geq 1}$ of positive numbers, we define the growth rate:

Definition 2.3 *Let r be a positive real number; then the growth rate r of the sequence $\{s_k\}_{k\geq 1}$ of positive numbers is defined as*

$$\overline{\lim_k} \exp\left(\frac{\log s_k}{k}\right).$$

In 2008, L. A. Bokut [6] gave the Gröbner–Shirshov basis (GSB) of B_n in band generators. The notion of this basis is in [2, 4, 7, 8, 12] under different names: complete presentation, presentation with solvable ambiguities, Gröbner–Shirshov basis, rewriting system, and so on. In [1] we proved that a subset of the GSB of B_n given by Bokut [6] is a GSB of MB_n. Using the notations (used in [1]) (t, s) for generator a_{ts} and $V_{[t,s]}$ or $W_{[t,s]}$ for words in (k, l) such that $t \geq k > l \geq s$, we proved in [1] that:

Theorem 2.4 *[1] A* GSB *of braid monoid MB_n consists of the following relations:*

$$(k, l)(i, j) = (i, j)(k, l), \quad k > l > i > j,$$

$$(k,l)V_{[j-1,1]}(i,j) = (i,j)(k,l)V_{[j-1,1]}, \quad k > i > j > l,$$

$$(t_3,t_2)(t_2,t_1) = (t_2,t_1)(t_3,t_1),$$

$$(t_3,t_1)V_{[t_2-1,1]}(t_3,t_2) = (t_2,t_1)(t_3,t_1)V_{[t_2-1,1]},$$

$$(t,s)V_{[t_2-1,1]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_3,t_2)(t,s)V_{[t_2-1,1]}(t_2,t_1)W'_{[t_3-1,t_1]},$$

$$(t_3,s)V_{[t_2-1,1]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_2,s)(t_3,s)V_{[t_2-1,1]}(t_2,t_1)W'_{[t_3-1,t_1]},$$

for $t_3 > t_2 > t_1,\ t > t_3, t_2 > s,$ and $W_{[t_3-1,t_1]}(t_3,t_1) = (t_3,t_1)W'_{[t_3-1,t_1]}$ where

$$W'_{[t_3-1,t_1]} = W_{[t_3-1,t_1]}|_{(p,q)\mapsto(p,q)} \text{ if } q \neq t_1;\ (p,t_1) \mapsto (t_3,p).$$

3. Hilbert series of MB_4 in Birman–Ko–Lee generators

For the band presentation of the braid monoid MB_3 we gave its Hilbert series in [1] as

$$P_M^{(3)}(t) \quad = \quad \frac{1}{(1-t)(1-2t)}.$$

In this paper we compute the Hilbert series of MB_4 (in band presentation). From 1 we have the following band-presentation of MB_4:

$$\left\langle a_{43}, a_{42}, a_{41}, a_{32}, a_{31}, a_{21} \middle| R_1^{(3)}, R_2^{(3)}, R_i^{(4)}, i = 3, \ldots, 10 \right\rangle,$$

where $R_1^{(3)} : a_{31}a_{32} = a_{21}a_{31}\,,$ $R_2^{(3)} : a_{32}a_{21} = a_{21}a_{31}\,,$ $R_3^{(4)} : a_{41}a_{32} = a_{32}a_{41}\,,$ $R_4^{(4)} : a_{41}a_{42} = a_{21}a_{41}\,,$ $R_5^{(4)} : a_{41}a_{43} = a_{31}a_{41}\,,$ $R_6^{(4)} : a_{42}a_{21} = a_{21}a_{41}\,,$ $R_7^{(4)} : a_{42}a_{43} = a_{32}a_{42}\,,$ $R_8^{(4)} : a_{43}a_{21} = a_{21}a_{43}\,,$ $R_9^{(4)} : a_{43}a_{31} = a_{31}a_{41}$ and $R_{10}^{(4)} : a_{43}a_{32} = a_{32}a_{42}$ are the given basic relations.

For the braid monoid MB_4 we give another form of Theorem 2.4 that is directly used to compute the Hilbert series of MB_4. This form is obtained by solving all the ambiguities in the presentation of MB_4.

Proposition 3.1 *A complete presentation of MB_4 for band generators is given by*

$$\left\langle a_{43}, a_{42}, a_{41}, a_{32}, a_{31}, a_{21} \middle| R_1^{(3)}, R_2^{(3)}, R_{19}^{(3)}, R_i^{(4)}, i = 3, \ldots, 18 \right\rangle,$$

where $R_1^{(3)}, R_2^{(3)}, R_i^{(4)}, i = 3, \ldots, 10$ are the basic relations and the new relations $R_{11}^{(4)}, \ldots, R_{18}^{(4)}$ are given as follows:

$R_{11}^{(4)} : a_{41}a_{21}^{s+1}a_{31} = a_{32}a_{41}a_{21}a_{32}^{s}\,,$

$R_{12}^{(4)} : a_{41}a_{21}^{s+1}a_{41} = a_{21}a_{41}a_{21}a_{42}^{s}\,,$

$R_{13}^{(4)} : a_{41}a_{21}^{r}a_{32}^{s+1}a_{42} = a_{31}a_{41}a_{21}^{r}a_{32}a_{43}^{s}\,,$

$R_{14}^{(4)} : a_{41}a_{21}^{r}a_{43} = a_{31}a_{41}a_{21}^{r}\,,$

$R_{15}^{(4)} : a_{42}a_{32}^{s+1}a_{42} = a_{32}a_{42}a_{32}a_{43}^{s}\,,$

$R_{16}^{(4)} : a_{41}a_{21}^{s+1}W(32)a_{41} = a_{21}a_{41}a_{21}W'(32)\,,$

$R_{17}^{(4)}: a_{41}a_{31}W(31)a_{41} = a_{31}a_{41}a_{31}W'(31)\,,$

$R_{18}^{(4)}: a_{42}a_{31}W(31)a_{41} = a_{32}a_{42}a_{31}W'(31)\,,$

$R_{19}^{(4)}: a_{31}a_{21}^{s+1}a_{31} = a_{21}a_{31}a_{21}a_{32}^{s}\,,$

where r is a positive and s is a nonnegative integer, $W(3k)$ an irreducible word in MB_3 starting with $a_{3k}, (k=1,2)$ and $W'(3k) = W(3k): a_{32} \to a_{32}\,,\ a_{21} \to a_{41}\,,\ a_{31} \to a_{43}$ (as mentioned in Theorem 2.4).

Proof It is obvious from solving all the ambiguities and from the presentation given in [1]. Hence the proof is omitted. □

As defined above, the set $A_*^{(m)}$ denotes the set of irreducible words and $B_*^{(m)}$ the set of reducible words in MB_m. In particular, $B_{ij\cdot kl;uv}^{(m)}$ denotes the set of reducible words starting with $a_{ij}a_{kl}$ and ending with a_{uv} and $B_{ij\cdot kl\cdot pq;uv}^{(m)}$ denotes the set of reducible words starting with $a_{ij}a_{kl}^{s+1}a_{pq}$ and ending with a_{uv}. Hence we have the following sets of reducible words in MB_4:

$B_{31\cdot 21;31}^{(3)} = \{a_{31}a_{21}^{s+1}a_{31}\},\ B_{42\cdot 32;42}^{(4)} = \{a_{42}a_{32}^{s+1}a_{42}\},\ B_{41\cdot 21;41}^{(4)} = \{a_{41}a_{21}^{s+1}a_{41}\},$

$B_{41\cdot 21;43}^{(4)} = \{a_{41}a_{21}^{r}a_{43}\},\ B_{41\cdot 21;31}^{(4)} = \{a_{41}a_{21}^{s+1}a_{31}\},$

$B_{41\cdot 21;42}^{(4)} = \{a_{41}a_{21}^{r}a_{32}^{s+1}a_{42}\},\ B_{41\cdot 21\cdot 32;41}^{(4)} = \{a_{41}a_{21}^{s+1}W(32)a_{41}\},$

$B_{41\cdot 31;41}^{(4)} = \{a_{41}a_{31}W(31)a_{41}\},\ B_{42\cdot 31;41}^{(4)} = \{a_{42}a_{31}W(31)a_{41}\}.$

We are using the other notions as follows:

- We denote the set $\{a_{21}, a_{21}^2, a_{21}^3, \ldots\}$ by $A_{21}^{(2)}$.
- $A_{ij}^{(n)}$ denotes the set of irreducible words starting with a_{ij} and $A_{\overline{ij}}^{(n)}$ denotes $\{a_{ij}, a_{ij}^2, a_{ij}^3, \ldots\}$.
- $A_{nj\cdot kl}^{(n)}$ denotes the set of irreducible words starting with $a_{nj}a_{kl}$, where $j=2,3,\ k=2,3$ and $l=1,2$.
- The Hilbert series of $B_*^{(m)}$, $A_*^{(m)}$, and MB_4 are denoted by $Q_*^{(m)}$, $P_*^{(m)}$, and $P_M^{(4)}(t)$ respectively. It is obvious that $P_{\overline{k(k-1)}}^{(k)} = P_{21}^{(2)}$, where $P_{21}^{(2)} = \frac{t}{1-t}$.

Note that as $B_{31\cdot 21;31}^{(3)} = \{a_{31}\} \times A_{21}^{(2)} \times \{a_{31}\}$ and $P_{21}^{(2)} = \frac{t}{1-t}$, hence $Q_{31\cdot 21;31}^{(3)} = \frac{t^3}{1-t}$.

Now we construct a linear system for reducible words in MB_4.

Proposition 3.2 *The following equalities hold for reducible words in MB_4.*

1) $Q_{41\cdot 21;31}^{(4)} = \frac{t^3}{1-t}\,,$

2) $Q_{41\cdot 21;41}^{(4)} = \frac{t^3}{1-t}\,,$

3) $Q_{41\cdot 21\cdot 32;41}^{(4)} = \frac{t^4}{(1-t)(1-2t)}\,,$

4) $Q_{41\cdot 21;43}^{(4)} = \frac{t^3}{1-t}\,,$

5) $Q_{41\cdot 21;42}^{(4)} = \frac{t^4}{(1-t)^2}\,,$

6) $Q_{41\cdot 31;41}^{(4)} = \frac{t^3}{1-2t}\,,$

7) $Q_{42\cdot 32;42}^{(4)} = \frac{t^3}{1-t}\,,$

8) $Q_{42\cdot 31;41}^{(4)} = \frac{t^3}{1-2t}\,,$

Proof Using simply the decomposition of words we have:

1) $B^{(4)}_{41\cdot 21;31} = \{a_{41}a_{21}^{r+1}a_{31}\} = \{a_{41}\} \times A^{(2)}_{21} \times \{a_{31}\}$ implies $Q^{(4)}_{41\cdot 21;31} = \frac{t^3}{1-t}$.

2) $B^{(4)}_{41\cdot 21;41} = \{a_{41}a_{21}^{r+1}a_{41}\} = \{a_{41}\} \times A^{(2)}_{21} \times \{a_{41}\}$ gives us $Q^{(4)}_{41\cdot 21;41} = \frac{t^3}{1-t}$.

3) The decomposition $B^{(4)}_{41\cdot 21\cdot 32;41} = \{a_{41}a_{21}^{r+1}W(32)a_{41}\} = \{a_{41}\} \times A^{(2)}_{21} \times A^{(3)}_{32} \times \{a_{41}\}$ gives the Hilbert series $Q^{(4)}_{41\cdot 21\cdot 31;41} = \frac{t^4}{(1-t)(1-2t)}$.

4) $B^{(4)}_{41\cdot 21;43} = \{a_{41}a_{21}^{r}a_{43}\} = \{a_{41}\} \times A^{(2)}_{21} \times \{a_{43}\}$ implies $Q^{(4)}_{41\cdot 21;43} = \frac{t^3}{1-t}$.

5) The decomposition $B^{(4)}_{41\cdot 21;42} = \{a_{41}a_{21}^{r}a_{32}^{s+1}a_{42}\} = \{a_{41}\} \times A^{(2)}_{21} \times A^{(3)}_{32} \times \{a_{41}\}$ gives the Hilbert series $Q^{(4)}_{41\cdot 21;42} = \frac{t^4}{(1-t)^2}$.

6) $B^{(4)}_{41\cdot 31;41} = \{a_{41}a_{31}W(31)a_{41}\} = \{a_{41}\} \times A^{(3)}_{31} \times \{a_{41}\}$ implies $Q^{(4)}_{41\cdot 31;41} = \frac{t^3}{1-2t}$.

7) $B^{(4)}_{42\cdot 32;42} = \{a_{42}a_{32}^{r+1}a_{42}\} = \{a_{42}\} \times A^{(3)}_{32} \times \{a_{42}\}$ implies $Q^{(4)}_{42\cdot 32;42} = \frac{t^3}{1-t}$.

8) $B^{(4)}_{42\cdot 31;41} = \{a_{42}a_{31}W(31)a_{41}\} = \{a_{42}\} \times A^{(3)}_{31} \times \{a_{41}\}$ implies $Q^{(4)}_{42\cdot 31;41} = \frac{t^3}{1-2t}$. □

Next we construct a linear system for canonical forms in MB_4.

Proposition 3.3 *The following equalities hold for irreducible words in MB_4.*

1) $P^{(4)}_{31} = \frac{t}{1-2t}\big(1 + \sum_{i=1}^{3} P^{(4)}_{4i}\big)$,

2) $P^{(4)}_{32} = \frac{t}{1-2t}\big(1 + \sum_{i=1}^{3} P^{(4)}_{4i}\big)$,

3) $P^{(4)}_{21} = \frac{t(1-2t)}{1-t}\big(1 + \sum_{i=1}^{3} P^{(4)}_{4i}\big)$,

4) $P^{(4)}_{41} = t + tP^{(4)}_{41} + \sum_{i=2}^{3} P^{(4)}_{41\cdot i1}$,

5) $P^{(4)}_{42} = t + t\sum_{i=1}^{2} P^{(4)}_{4i} + \sum_{i=1}^{2} P^{(4)}_{42\cdot 3i}$,

6) $P^{(4)}_{43} = t + t\sum_{i=1}^{3} P^{(4)}_{4i}$,

7) $P^{(4)}_{41\cdot 31} = tP^{(4)}_{31} - \frac{t^2}{1-2t}P^{(4)}_{41}$,

8) $P^{(4)}_{42\cdot 31} = tP^{(4)}_{31} - \frac{t^2}{1-2t}P^{(4)}_{41}$,

9) $P^{(4)}_{42\cdot 32} = tP^{(4)}_{32} - \frac{t^2}{1-t}P^{(4)}_{42}$,

10) $P^{(4)}_{41\cdot 21} = tP^{(4)}_{21} - \frac{t^2}{1-t}P^{(4)}_{31} - \frac{t^2}{1-2t}P^{(4)}_{41} - \frac{t^3}{(1-t)^2}P^{(4)}_{42} - \frac{t^2}{1-t}P^{(4)}_{43}$.

Proof We compute the Hilbert series inductively. Here we use the series of the irreducible words of MB_3, which we have computed in [1]. The series are: $P^{(3)}_{31} = \frac{t}{1-2t}$, $P^{(3)}_{32} = \frac{t}{1-2t}$, $P^{(3)}_{21} = \frac{t(1-2t)}{1-t}$. If $\sqcup$ denotes the disjoint union of sets, then using the GSB of MB_4 and the decomposition of words we have:

1) $A_{31}^{(4)} = A_{31}^{(3)} \sqcup \Big(A_{31}^{(3)} \times A_{41}^{(4)}\Big) \sqcup \Big(A_{31}^{(3)} \times A_{42}^{(4)}\Big) \sqcup \Big(A_{31}^{(3)} \times A_{43}^{(4)}\Big)$. This gives us

$$\begin{aligned} P_{31}^{(4)} &= \Big(1 + \sum_{i=1}^{3} P_{4i}^{(4)}\Big) P_{31}^{(3)} \\ &= \frac{t}{1-2t}\Big(1 + \sum_{i=1}^{3} P_{4i}^{(4)}\Big). \end{aligned}$$

2) $A_{32}^{(4)} = A_{32}^{(3)} \sqcup \Big(A_{32}^{(3)} \times A_{41}^{(4)}\Big) \sqcup \Big(A_{32}^{(3)} \times A_{42}^{(4)}\Big) \sqcup \Big(A_{32}^{(3)} \times A_{43}^{(4)}\Big)$ implies

$$\begin{aligned} P_{32}^{(4)} &= \Big(1 + \sum_{i=1}^{3} P_{4i}^{(4)}\Big) P_{32}^{(3)} \\ &= \frac{t}{1-2t}\Big(1 + \sum_{i=1}^{3} P_{4i}^{(4)}\Big). \end{aligned}$$

3) $A_{21}^{(4)} = A_{21}^{(3)} \sqcup \Big(A_{21}^{(3)} \times A_{41}^{(4)}\Big) \sqcup \Big(A_{21}^{(3)} \times A_{42}^{(4)}\Big) \sqcup \Big(A_{21}^{(3)} \times A_{43}^{(4)}\Big)$ implies

$$\begin{aligned} P_{21}^{(4)} &= \Big(1 + \sum_{i=1}^{3} P_{4i}^{(4)}\Big) P_{21}^{(3)} \\ &= \frac{t(1-2t)}{1-t}\Big(1 + \sum_{i=1}^{3} P_{4i}^{(4)}\Big). \end{aligned}$$

The set $A_{4i}^{(4)}$ consists of all the words starting with the generator a_{4i}. Therefore, the set $\{a_{4i}\} \times A_{4i}^{(4)}$ is a subset of $A_{4i}^{(4)}$ consisting of all the words starting with a_{4i}^2. We apply this concept in the proofs of (4), (5), and (6).

4) The set $A_{41}^{(4)}$ is a disjoint union of the sets $\{a_{41}\}$, $\{a_{41}\} \times A_{41}^{(4)}$, $A_{41\cdot 21}^{(4)}$, and $A_{41\cdot 31}^{(4)}$, i.e. $A_{41}^{(4)} = \{a_{41}\} \sqcup \Big(\{a_{41}\} \times A_{41}^{(4)}\Big) \sqcup A_{41\cdot 21}^{(4)} \sqcup A_{41\cdot 31}^{(4)}$. Therefore, we have

$$P_{41}^{(4)} = t + tP_{41}^{(4)} + \sum_{i=2}^{3} P_{41\cdot i1}^{(4)}.$$

Similarly, we have

5) $A_{42}^{(4)} = \{a_{42}\} \sqcup \Big(\{a_{42}\} \times A_{42}^{(4)}\Big) \sqcup \Big(\{a_{42}\} \times A_{41}^{(4)}\Big) \sqcup A_{42\cdot 31}^{(4)} \sqcup A_{42\cdot 32}^{(4)}$ implies

$$P_{42}^{(4)} = t + t\sum_{i=1}^{2} P_{4i}^{(4)} + \sum_{i=1}^{2} P_{42\cdot 3i}^{(4)}.$$

6) $A_{43}^{(4)} = \{a_{43}\} \sqcup \Big(\{a_{43}\} \times A_{41}^{(4)}\Big) \sqcup \Big(\{a_{43}\} \times A_{42}^{(4)}\Big) \sqcup \Big(\{a_{43}\} \times A_{43}^{(4)}\Big)$ implies

$$P_{43}^{[4]} = t + t\sum_{i=1}^{3} P_{4i}^{(4)}.$$

7) $A^{(4)}_{41\cdot 31} = \{a_{41}\} \times A^{(4)}_{31} \diagdown \left(B_{41\cdot 31;41} \times_{41} A^{(4)}_{41}\right)$ implies

$$P^{(4)}_{41\cdot 31} = tP^{(4)}_{31} - \frac{t^2}{1-2t}P^{(4)}_{41}.$$

8) $A^{(4)}_{42\cdot 31} = \{a_{42}\} \times A^{(4)}_{31} \diagdown \left(B_{42\cdot 31;41} \times_{41} A^{(4)}_{41}\right)$ implies

$$P^{(4)}_{42\cdot 31} = tP^{(4)}_{31} - \frac{t^2}{1-2t}P^{(4)}_{41}.$$

9) $A^{(4)}_{42\cdot 32} = \{a_{42}\} \times A^{(4)}_{32} \diagdown \left(B_{42\cdot 32;42} \times_{42} A^{(4)}_{42}\right)$ implies

$$P^{(4)}_{42\cdot 32} = tP^{(4)}_{32} - \frac{t^2}{1-2t}P^{(4)}_{42}.$$

10) $A^{(4)}_{41\cdot 21} = \{a_{41}\} \times A^{(4)}_{21} \diagdown \Big[\left(B_{41\cdot 21;41} \times_{41} A^{(4)}_{41}\right) \sqcup \left(B_{41\cdot 21;31} \times_{31} A^{(4)}_{31}\right) \sqcup \left(B_{41\cdot 21;43} \times_{43} A^{(4)}_{43}\right) \sqcup \left(B_{41.21;42} \times_{42} A^{(4)}_{42}\right)\Big]$ implies

$$P^{(4)}_{41\cdot 21} = tP^{(4)}_{21} - \frac{t^2}{1-t}P^{(4)}_{31} - \frac{t^2}{1-2t}P^{(4)}_{41} - \frac{t^3}{(1-t)^2}P^{(4)}_{42} - \frac{t^2}{1-t}P^{(4)}_{43}.$$

□

Theorem 3.4 *The Hilbert series of the braid monoid MB_4 in band generators is given by*

$$P^{(4)}_M(t) = \frac{1}{(1-t)(1-5t+5t^2)}.$$

Proof Solving the system of linear equations constructed in Proposition 3.3 we get $P^{(4)}_{21} = \frac{t}{(1-t)(1-5t+5t^2)}$, $P^{(4)}_{31} = \frac{t}{1-5t+5t^2}$, $P^{(4)}_{32} = \frac{t}{1-5t+5t^2}$, $P^{(4)}_{41} = \frac{t-2t^2}{1-5t+5t^2}$, $P^{(4)}_{42} = \frac{t-t^2}{1-5t+5t^2}$, $P^{(4)}_{43} = \frac{t-2t^2}{1-5t+5t^2}$. Therefore, we have the Hilbert series of the braid monoid MB_4 as

$$\begin{aligned} P^{(4)}_M(t) &= 1 + P^{(4)}_{21} + \sum_{i=1}^{2} P^{(4)}_{3i} + \sum_{j=1}^{3} P^{(4)}_{4j}; \\ &= \frac{1}{(1-t)(1-5t+5t^2)}. \end{aligned}$$

□

Corollary 3.5 *The growth rate of braid monoid MB_4 (in Birman–Ko–Lee generators) is* 3.618*.*

Proof By partial fractions we have $\frac{1}{(1-t)(1-5t+5t^2)} = \frac{1}{1-t} - \frac{\sqrt{5}}{1-\frac{5-\sqrt{5}}{2}t} + \frac{\sqrt{5}}{1-\frac{5+\sqrt{5}}{2}t}$. The only term that contributes in approximation of the series is $\frac{\sqrt{5}}{1-\frac{5+\sqrt{5}}{2}t}$ and $\frac{\sqrt{5}}{1-\frac{5+\sqrt{5}}{2}t} = \sqrt{5}\big(1 + \frac{5+\sqrt{5}}{2}t + (\frac{5+\sqrt{5}}{2})^2 t^2 + \cdots\big)$. Therefore, the growth function is $a^{(4)}_k = \sqrt{5}(\frac{5+\sqrt{5}}{2})^k$ and hence the growth rate of MB_4 is $\frac{5+\sqrt{5}}{2}$ (approximately equal to 3.618). The growth rate of MB_3 is 2. □

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