Tôhoku Math. J.
48 (1996), 149–163

HILBERT SPACES RELATED TO HARMONIC FUNCTIONS

KEIKO FUJITA

(Received December 26, 1994, revised May 31, 1995)

Abstract. We construct a Hilbert space with a reproducing kernel by using a measure which is not positive. The space is unitarily isomorphic to a Hilbert space on the spherical sphere under the Fourier transformation. Then we study Poisson transform of Sobolev space on the n-dimensional unit sphere.

Introduction. In the study of harmonic functions on the Euclidean space $\boldsymbol{R}^{n+1}$, the complex light cone $\tilde{M}=\{z\in\boldsymbol{C}^{n+1};\, z^2\equiv z_1^2+z_2^2+\cdots+z_{n+1}^2=0\}$ plays an important role. Let

$$M=\{z=x+iy\in\tilde{M};\, \|x\|=1/2\}$$

be the spherical sphere, where $\|x\|$ is the Euclidean norm.

We define the Fourier transformation $\mathscr{F}$ on $L^2(M)$ by

$$\mathscr{F}: f \mapsto \mathscr{F}f(x)=\int_M f(z)\exp(\bar{z}\cdot x)dM(z)\,,$$

where dM is the normalized $\mathrm{O}(n+1)$-invariant measure on M.

We denote by $\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ the space of harmonic functions on $\boldsymbol{R}^{n+1}$. We define a sesquilinear form $(\ ,\ )_{\boldsymbol{R}^{n+1}}$ by

$$(f, g)_{\boldsymbol{R}^{n+1}}=\int_{\boldsymbol{R}^{n+1}} f(x)\overline{g(x)}d\mu(x)\,,$$

where the measure $d\mu$ is constructed by means of the function ρ_n which is introduced in Ii [2] and Wada [7]. Note that $d\mu$ is not a positive measure.

In this paper, we assume $n\geq 2$ and we shall show that the sesquilinear form $(\ ,\ )_{\boldsymbol{R}^{n+1}}$ is a non-degenerate inner product on

$$L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})=\{f\in\mathscr{A}_\Delta(\boldsymbol{R}^{n+1});\, \|f\|^2_{\boldsymbol{R}^{n+1}}\equiv(f, f)_{\boldsymbol{R}^{n+1}}<\infty\}\,,$$

although the measure $d\mu$ is not positive and that $(L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}), (\ ,\ )_{\boldsymbol{R}^{n+1}})$ is a Hilbert space with a reproducing kernel. Then we construct the reproducing kernel concretely.

We denote by $\mathscr{O}(\tilde{M}[1])$ the space of holomorphic functions in a neighborhood of $\tilde{M}[1]=\{z=x+iy\in\tilde{M};\, \|x\|\leq 1/2\}$ and by $L^2\mathscr{O}(M)$ the closure of $\mathscr{O}(\tilde{M}[1])$ in $L^2(M)$. The second aim of this paper is to show that $L^2\mathscr{O}(M)$ is unitarily isomorphic to $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$

1991 *Mathematics Subject Classification*. Primary 46E22; Secondary 42B10, 42B30.

under the Fourier transformation $\mathscr{F}$. The outline of the above results was announced in [1].

Let $S=S^n$ be the n-dimensional unit sphere. We know that the Poisson transformation $\mathscr{P}_M$ maps $L^2(S)$ into $L^2\mathscr{O}(M)$. In the last section, we shall determine the image of $L^2(S)$ under $\mathscr{P}_M$ as a "Hardy-Sobolev" space. This result describes a result of Lebeau [3] more precisely.

The author would like to thank Professor Mitsuo Morimoto of Sophia University for useful advice.

1. A Hilbert space of harmonic functions. We denote by $\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$ the space of k-homogeneous harmonic polynomials on $\boldsymbol{R}^{n+1}$ and by $N(k,n)$ the dimension of $\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$. We know

$$N(k,n)=(2k+n-1)(k+n-2)!/(k!(n-1)!)=O(k^{n-1}).$$

The following lemma is known:

LEMMA 1.1. *Let $f_k\in\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$ and $g_l\in\mathscr{P}^l_\Delta(\boldsymbol{R}^{n+1})$. If $k\neq l$, then*

$$\int_S f_k(\omega)g_l(\omega)dS(\omega)=0.$$

We denote by $\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ the space of harmonic functions on $\boldsymbol{R}^{n+1}$ equipped with the topology of uniform convergence on compact sets. Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$. Define the k-homogeneous harmonic component f_k of $f\in\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ by

$$f_k(x)=N(k,n)(\sqrt{x^2})^k\int_S f(\tau)P_{k,n}\left(\frac{x}{\sqrt{x^2}}\cdot\tau\right)dS(\tau), \qquad x\in\boldsymbol{R}^{n+1}, \tag{1}$$

where $x\cdot y=x_1y_1+x_2y_2+\cdots+x_{n+1}y_{n+1}$ and dS is the normalized $\mathrm{O}(n+1)$-invariant measure on S. Then, the following lemma is also known:

LEMMA 1.2. *Let $f\in\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ and f_k the k-homogeneous harmonic component of f defined by* (1). *Then the expansion $\sum_{k=0}^\infty f_k$ converges to f in the topology of $\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$.*

We denote the modified Bessel function by

$$K_\nu(r)=\int_0^\infty \exp(-r\cosh t)\cosh\nu t\,dt, \qquad \nu\in\boldsymbol{R}, \quad 0<r<\infty.$$

Ii [2] and Wada [7] introduced the function

$$\rho_n(r)=\begin{cases}\displaystyle\sum_{l=0}^{(n-1)/2} a_{nl}r^{l+1}K_l(r), & \text{if } n \text{ is odd},\\ \displaystyle\sum_{l=0}^{n/2} a_{nl}r^{l+1/2}K_{l-1/2}(r), & \text{if } n \text{ is even},\end{cases}$$

where the constants a_{nl}, $l=0, 1, 2, \dots, [n/2]$, are defined uniquely by

$$(2)\quad \int_0^\infty r^{2k+n-1}\rho_n(r)dr = \frac{N(k, n)k!\Gamma(k+(n+1)/2)2^{2k}}{\Gamma((n+1)/2)} \equiv C(k, n), \quad k=0, 1, 2, \dots$$

(see [7, Lemma 2.2]). Note that $\rho_n(r)$ is not positive but there is $R_n>0$ such that $\rho_n(r)>0$ for $r\geq R_n$. The function ρ_n is estimated as follows:

$$(3)\quad \begin{cases} |\rho_n(r)| \leq \sqrt{r}\,P_{(n-1)/2}(r)\exp(-r), & \text{if } n \text{ is odd}, \\ \rho_n(r) = P_{n/2}(r)\exp(-r), & \text{if } n \text{ is even}, \end{cases}$$

where $P_{(n-1)/2}(r)$ and $P_{n/2}(r)$ are polynomials of degree $[n/2]$ (see [7, p. 429]).

We define a measure $d\mu$ on $\boldsymbol{R}^{n+1}$ by

$$\int_{\boldsymbol{R}^{n+1}} f(x)d\mu(x) = \int_0^\infty \int_S f(r\omega)dS(\omega)r^{n-1}\rho_n(r)dr$$

and a sesquilinear form $(\ ,\ )_{\boldsymbol{R}^{n+1}}$ by

$$(f, g)_{\boldsymbol{R}^{n+1}} = \int_{\boldsymbol{R}^{n+1}} f(x)\overline{g(x)}d\mu(x) .$$

Although $\rho_n(r)$ is not positive, the sesquilinear form $(\ ,\ )_{\boldsymbol{R}^{n+1}}$ is an inner product on

$$L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}) = \{f \in \mathscr{A}_\Delta(\boldsymbol{R}^{n+1});\ \|f\|^2_{\boldsymbol{R}^{n+1}} \equiv (f, f)_{\boldsymbol{R}^{n+1}} < \infty\}$$

by the following proposition:

PROPOSITION 1.3. *Let* $f=\sum f_k \in \mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$\begin{aligned}(f, f)_{\boldsymbol{R}^{n+1}} &= \sum_{k=0}^\infty (f_k, f_k)_{\boldsymbol{R}^{n+1}} \\ &= \sum_{k=0}^\infty C(k, n)\int_S f_k(\omega)\overline{f_k(\omega)}dS(\omega) \geq 0 ,\end{aligned}$$

i.e., either both sides are infinite or both sides are finite and equal.

PROOF. For $R>0$ we put $C_R(k, n)=\int_0^R r^{2k+n-1}\rho_n(r)dr$ and

$$I(R) = \int_{B(R)} |f(x)|^2 d\mu(x) ,$$

where $B(R)=\{x\in \boldsymbol{R}^{n+1};\ \|x\|<R\}$. Since $\rho_n(r)>0$ for $r\geq R_n$, $I(R)$ is monotone increasing for $R\geq R_n$ and $(f, f)_{\boldsymbol{R}^{n+1}} = \lim_{R\to\infty} I(R)$. By Lemmas 1.2 and 1.1,

$$I(R) = \sum_{k=0}^\infty C_R(k, n)/C(k, n)(f_k, f_k)_{\boldsymbol{R}^{n+1}} .$$

Choose sufficiently large $R\geq R_n$ so that $C_R(k, n)>0$, $k=0, 1, 2, \dots$, and take the limit.

Then by Fatou's lemma we have

$$\lim_{R\to\infty}\sum_{k=0}^{\infty} C_R(k, n)/C(k, n)(f_k, f_k)_{\boldsymbol{R}^{n+1}}=\sum_{k=0}^{\infty}(f_k, f_k)_{\boldsymbol{R}^{n+1}}\,.$$

q.e.d.

LEMMA 1.4. *Let $f\in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Then we have*

$$|f(x)|\le\sqrt{\Gamma((n+1)/2)}\exp(\|x\|/2)\|f\|_{\boldsymbol{R}^{n+1}}\,.$$

PROOF. By Lemma 1.2, $f\in\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ can be expanded as follows:

$$f(x)=\sum_{k=0}^{\infty} N(k, n)\|x\|^k\int_S f_k(\omega)P_{k,n}\left(\frac{x}{\|x\|}\cdot\omega\right)dS(\omega)\,.$$

Since $N(k, n)\int_S(P_{k,n}(\omega\cdot x/\|x\|))^2dS(\omega)=1$, we have

$$\begin{aligned}|f(x)|&\le\sum_{k=0}^{\infty} N(k, n)\|x\|^k\int_S\left|f_k(\omega)P_{k,n}\left(\frac{x}{\|x\|}\cdot\omega\right)\right|dS(\omega)\\ &\le\sum_{k=0}^{\infty}\|x\|^k\sqrt{N(k, n)/C(k, n)}\left(C(k, n)\int_S|f_k(\omega)|^2dS(\omega)\right)^{1/2}\\ &\le\|f\|_{\boldsymbol{R}^{n+1}}\sum_{k=0}^{\infty}\|x\|^k\sqrt{N(k, n)/C(k, n)}\\ &\le\sqrt{\Gamma((n+1)/2)}\|f\|_{\boldsymbol{R}^{n+1}}\sum_{k=0}^{\infty}\|x\|^k/(k!2^k)\\ &=\sqrt{\Gamma((n+1)/2)}\exp(\|x\|/2)\|f\|_{\boldsymbol{R}^{n+1}}\,.\end{aligned}$$

q.e.d.

THEOREM 1.5. *$(L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}), (\ ,\ )_{\boldsymbol{R}^{n+1}})$ is a Hilbert space.*

PROOF. We have only to prove the completeness of the pre-Hilbert space $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Let $\{f_N\}$ be a Cauchy sequence in $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Then by Lemma 1.4 and the Poisson integral formula, $\{f_N\}$ converges uniformly on every compact set to a function $f\in\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Choose sufficiently large $R\ge R_n$ so that

$$C_R(k, n)>0\,,\qquad k=0, 1, 2, \dots\,. \tag{4}$$

Then divide the integral of $\|f_N-f\|^2_{\boldsymbol{R}^{n+1}}$ into

$$I_1(N)=\int_{B(R)}|f_N(x)-f(x)|^2d\mu(x)$$

and

$$I_2(N)=\int_{\boldsymbol{R}^{n+1}\setminus B(R)} |f_N(x)-f(x)|^2 d\mu(x).$$

Since the integral domain of $I_1(N)$ is compact, $|I_1(N)|<\infty$. Since $\rho_n(r)>0$ for $r\geq R\geq R_n$, by Fatou's lemma and (4),

$$\begin{aligned}I_2(N)&=\int_{\boldsymbol{R}^{n+1}\setminus B(R)} \liminf_{M\to\infty}|f_N(x)-f_M(x)|^2 d\mu(x)\\&\leq\liminf_{M\to\infty}\int_{\boldsymbol{R}^{n+1}\setminus B(R)} |f_N(x)-f_M(x)|^2 d\mu(x)\\&\leq\liminf_{M\to\infty}\|f_N(x)-f_M(x)\|^2_{\boldsymbol{R}^{n+1}}.\end{aligned}$$

Since $\{f_N\}$ is a Cauchy sequence, $I_2(N)$ tends to 0 as $N\to\infty$. Therefore, $\|f\|_{\boldsymbol{R}^{n+1}}\leq \|f-f_N\|_{\boldsymbol{R}^{n+1}}+\|f_N\|_{\boldsymbol{R}^{n+1}}<\infty$ and $\|f-f_N\|_{\boldsymbol{R}^{n+1}}=I_1(N)+I_2(N)$ tends to 0 as $N\to\infty$.

q.e.d.

COROLLARY 1.6. *Let $f\in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ and f_k the k-homogeneous harmonic component of f defined by* (1). *Then the expansion $\sum_{k=0}^{\infty} f_k$ converges to f in the topology of $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$.*

From this corollary, Proposition 1.3 and Lemma 1.1, we get the following theorem:

THEOREM 1.7. *The Hilbert space $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ is the direct sum of the spaces $\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$:*

$$L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})=\bigoplus_{k=0}^{\infty}\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1}).$$

The mapping $f\mapsto f_k$ defined by (1) *is the orthogonal projection of $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ onto $\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$.*

By Lemma 1.4, there is a reproducing kernel on the Hilbert space $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Now, we construct the reproducing kernel on $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Put

$$E_1(x, y)=\int_M \exp(\zeta\cdot x)\exp(y\cdot\bar{\zeta})dM(\zeta).$$

Then $E_1(x, y)$ is real-valued, symmetric and satisfies

$$\Delta_x E_1(x, y)\equiv(\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_{n+1}^2)E_1(x, y)=0.$$

Put

$$\tilde{P}_{k,n}(z, w)=(\sqrt{z^2})^k(\sqrt{w^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right).$$

LEMMA 1.8 ([7, Lemma 1.3]).

$$\int_M (\zeta\cdot z)^k(w\cdot\bar{\zeta})^l dM(\zeta)=\frac{\delta_{kl}}{2^k N(k,n)\gamma_{k,n}}\tilde{P}_{k,n}(z,w)\,,\qquad z, w\in \boldsymbol{C}^{n+1}\,,$$

where $\gamma_{k,n}$ is the coefficient of the highest power of the Legendre polynomial $P_{k,n}(t)$:

$$\gamma_{k,n}=2^k\Gamma(k+(n+1)/2)/(N(k,n)\Gamma((n+1)/2)k!)\,.$$

By this lemma, $E_1(x, y)$ is expanded as follows:

$$(5)\qquad E_1(x,y)=\sum_{k=0}^{\infty} N(k,n)/C(k,n)\tilde{P}_{k,n}(x,y)$$

$$=\sum_{k=0}^{\infty}\Gamma((n+1)/2)/(k!\,\Gamma(k+(n+1)/2)2^{2k})\tilde{P}_{k,n}(x,y)\,.$$

Therefore, there is a constant C such that

$$|E_1(x,y)|\le C\exp(\|x\|/(2A))\exp(A\|y\|/2)\,,\qquad x, y\in \boldsymbol{R}^{n+1}$$

for any $A>0$. Moreover, we have

$$\|E_1(\cdot, y)\|^2_{\boldsymbol{R}^{n+1}}\le\Gamma((n+1)/2)J_0(i\|y\|)<\Gamma((n+1)/2)\exp(\|y\|)\,,$$

where $J_0(t)$ is the Bessel function of degree 0.

In particular, $E_1(x, \cdot)$ and $E_1(\cdot, y)$ belong to $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$.

THEOREM 1.9. *E_1 is the reproducing kernel on the Hilbert space $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$; that is, for $f\in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ we have*

$$f(y)=(f_x, E_1(y,x))_{\boldsymbol{R}^{n+1}}=\int_{\boldsymbol{R}^{n+1}} f(x)\overline{E_1(y,x)}\,d\mu(x)\,,\qquad y\in\boldsymbol{R}^{n+1}\,.$$

PROOF. Since $E_1(y, \cdot)\in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$, Corollary 1.6, (5), Lemma 1.1 and (2) imply

$$\int_{\boldsymbol{R}^{n+1}} f(x)\overline{E_1(y,x)}\,d\mu(x)$$

$$=\int_0^\infty\int_S\sum_{l=0}^{\infty} f_l(r\omega)\sum_{k=0}^{\infty}N(k,n)/C(k,n)r^k\tilde{P}_{k,n}(\omega,y)dS(\omega)r^{n-1}\rho_n(r)dr$$

$$=\int_0^\infty\sum_{k=0}^{\infty} f_k(y)/C(k,n)r^{2k+n-1}\rho_n(r)dr$$

$$=\sum_{k=0}^{\infty} f_k(y)=f(y)\,.$$

q.e.d.

2. Complex harmonic functions. We denote by $\mathscr{P}_\Delta^k(C^{n+1})$ the space of the k-homogeneous complex harmonic polynomials; that is, if $f \in \mathscr{P}_\Delta^k(C^{n+1})$, $\Delta_z f(z) \equiv (\partial^2/\partial z_1^2 + \cdots + \partial^2/\partial z_{n+1}^2) f(z) = 0$.

By definition, $\mathscr{P}_\Delta^k(C^{n+1})|_{R^{n+1}} = \mathscr{P}_\Delta^k(R^{n+1})$. For $f_k \in \mathscr{P}_\Delta^k(R^{n+1})$, the harmonic extension $\tilde{f}_k$ of f_k is given by

$$\tilde{f}_k(z) = N(k, n) \int_S f_k(\tau) \tilde{P}_{k,n}(z, \tau) dS(\tau), \qquad z \in C^{n+1}. \tag{6}$$

The cross norm $L(z)$ on C^{n+1} corresponding to the Euclidean norm $\|x\|$ is the Lie norm given by

$$L(z) = L(x+iy) = [\|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - (x \cdot y)^2}]^{1/2},$$

and the dual Lie norm $L^*(z)$ is given by

$$L^*(z) = \sup\{|z \cdot \zeta|;\ L(\zeta) \le 1\}$$
$$= \frac{1}{\sqrt{2}} [\|x\|^2 + \|y\|^2 + \sqrt{(\|x\|^2 - \|y\|^2)^2 + 4(x \cdot y)^2}]^{1/2}.$$

The open and the closed Lie balls of radius R with center at 0 are defined by

$$\tilde{B}(R) = \{z \in C^{n+1};\ L(z) < R\}, \qquad 0 < R \le \infty,$$

and by

$$\tilde{B}[R] = \{z \in C^{n+1};\ L(z) \le R\}, \qquad 0 \le R < \infty,$$

respectively. Put

$$\tilde{M}(R) = \tilde{B}(R) \cap \tilde{M}, \qquad \tilde{M}[R] = \tilde{B}[R] \cap \tilde{M}.$$

We denote by $\mathcal{O}(\tilde{B}(R))$ (resp. $\mathcal{O}(\tilde{M}(R))$) the space of holomorphic functions on $\tilde{B}(R)$ (resp. $\tilde{M}(R)$) equipped with the topology of uniform convergence on compact sets. We call

$$\mathcal{O}_\Delta(\tilde{B}(R)) = \{f \in \mathcal{O}(\tilde{B}(R));\ \Delta_z f(z) = 0\}, \qquad 0 < R \le \infty$$

the space of complex harmonic functions on $\tilde{B}(R)$.

The following lemmas are known:

LEMMA 2.1. *The restriction mapping α_B establishes the following linear topological isomorphism;*

$$\alpha_B : \mathcal{O}_\Delta(\tilde{B}(R)) \xrightarrow{\sim} \mathscr{A}_\Delta(B(R)),$$

where $\mathscr{A}_\Delta(B(R))$ is the space of harmonic functions on $B(R)$ equipped with the topology of uniform convergence on compact sets.

Moreover, the inverse mapping α_B^{-1} is given by the Poisson integral $\mathscr{P}$:

$$\mathscr{P}: f \mapsto \mathscr{P}f(z) = \int_S f(\rho\omega) K_1(z, \omega/\rho) dS(\omega) ,$$

where $0 < \rho < R$ *and*

$$K_1(z, w) = \frac{1 - z^2 w^2}{(1 + z^2 w^2 - 2z \cdot w)^{(n+1)/2}} , \qquad L(z)L(w) < 1 \tag{7}$$

is the Poisson kernel.

LEMMA 2.2 (cf. [4]). *The restriction mapping* α_M *establishes the following linear topological isomorphism*:

$$\alpha_M : \mathscr{O}_\Delta(\tilde{B}(R)) \xrightarrow{\sim} \mathscr{O}(\tilde{M}(R)) .$$

Moreover, the inverse mapping α_M^{-1} *is given by the Cauchy integral* $\mathscr{C}$:

$$\mathscr{C}: f \mapsto \mathscr{C}f(z) = \int_M f(\rho w) K_0(z, \bar{w}/\rho) dM(w) ,$$

where $0 < \rho < R$ *and*

$$K_0(z, w) = \frac{1 + 2z \cdot w}{(1 - 2z \cdot w)^n} , \qquad w \in \tilde{M} , \quad L(z)L(w) = 2L(z)L^*(w) < 1 \tag{8}$$

is the Cauchy kernel.

LEMMA 2.3 (cf. [2, Lemma 1.7] and [7, Lemma 1.4]). *Let* $f_k \in \mathscr{P}^k_\Delta(\boldsymbol{C}^{n+1})$ *and* $g_l \in \mathscr{P}^l_\Delta(\boldsymbol{C}^{n+1})$. *Then*

$$\int_M f_k(w) \overline{g_k(w)} dM(w) = \frac{\gamma_{k,n}}{2^k} \int_S f_k(\omega) \overline{g_k(\omega)} dS(\omega) ,$$

$$\int_M f_k(w) \overline{g_l(w)} dM(w) = 0 , \qquad k \neq l .$$

3. A Hilbert space on the spherical sphere. We denote by $L^2(M)$ the space of square integrable functions on M with the inner product

$$(f, g)_M = \int_M f(w) \overline{g(w)} dM(w) ,$$

and by $\mathscr{P}^k(M)$ the space of the k-homogeneous polynomials on M. Define the k-homogeneous component f_k of $f \in L^2(M)$ by

$$f_k(z) = 2^k N(k, n) \int_M f(w)(z \cdot \bar{w})^k dM(w) , \qquad z \in M . \tag{9}$$

The harmonic extension of f_k is given by

$$\tilde{f}_k(z) = 2^k N(k, n) \int_M f_k(w)(z \cdot \bar{w})^k dM(w) , \qquad z \in \boldsymbol{C}^{n+1} .$$

By Lemma 2.3, $\mathscr{P}^k(M)$ and $\mathscr{P}^l(M)$ are mutually orthogonal for $k \neq l$.

Let $L^2\mathcal{O}(M)$ be the closed subspace of $L^2(M)$ generated by $\mathscr{P}^k(M)$, $k = 0, 1, 2, \ldots$. Then by definition we have the following lemma:

LEMMA 3.1. *Let $f \in L^2\mathcal{O}(M)$ and f_k the k-homogeneous component of f defined by* (9). *Then the expansion $\sum_{k=0}^{\infty} f_k$ converges to f in the topology of $L^2\mathcal{O}(M)$; that is, we have the Hilbert direct sum decomposition*:

$$L^2\mathcal{O}(M) = \bigoplus_{k=0}^{\infty} \mathscr{P}^k(M) .$$

For any function f on $\tilde{M}(1)$, define the function f^t on M by $f^t(z) = f(tz)$ for $0 < t < 1$. If $f \in \mathcal{O}(\tilde{M}(1))$, then $f^t \in L^2\mathcal{O}(M)$.

LEMMA 3.2. *If $f \in L^2\mathcal{O}(M)$, then there is $\tilde{f} \in \mathcal{O}(\tilde{M}(1))$ such that* $\lim_{t\uparrow 1} \|f - \tilde{f}^t\|_M = 0$.
Conversely, if $\tilde{f} \in \mathcal{O}(\tilde{M}(1))$ satisfies $\sup_{0<t<1} \|\tilde{f}^t\|_M < \infty$, *then* $f = \lim_{t\uparrow 1} \tilde{f}^t$ *belongs to $L^2\mathcal{O}(M)$.*

PROOF. Let $f = \sum_{k=0}^{\infty} f_k \in L^2\mathcal{O}(M)$. Define

$$\tilde{f}(w) = \int_M f(\zeta) K_0(\bar{\zeta}, w) dM(\zeta) , \qquad L(w) < 1 ,$$

then $\tilde{f} \in \mathcal{O}(\tilde{M}(1))$. Put $w = tz$ for $z \in M$ and $0 < t < 1$, then we have $\tilde{f}(w) = \tilde{f}^t(z) = \sum_{k=0}^{\infty} f_k(tz)$. Since $\|f - \tilde{f}^t\|_M^2 = \sum_{k=0}^{\infty} (1 - t^k)^2 \|f_k\|_M^2$, we have $\lim_{t\uparrow 1} \|f - \tilde{f}^t\|_M = 0$ by Fatou's lemma.

Conversely, assume that $\tilde{f} \in \mathcal{O}(\tilde{M}(1))$ satisfies $\sup_{0<t<1} \|\tilde{f}^t\|_M < \infty$. Expand $\tilde{f}^t$ by $\sum_{k=0}^{\infty} \tilde{f}_k^t$. Then by Fatou's lemma, we have

$$\infty > \lim_{t\uparrow 1} \|\tilde{f}^t\|_M^2 = \lim_{t\uparrow 1} \sum_{k=0}^{\infty} \|\tilde{f}_k^t\|_M^2 \geq \sum_{k=0}^{\infty} \lim_{t\uparrow 1} \|\tilde{f}_k^t\|_M^2 = \|f\|_M^2 .$$

q.e.d.

From this lemma, we have

$$L^2\mathcal{O}(M) = \{f \in \mathcal{O}(\tilde{M}(1));\ \sup_{0<t<1} \|f^t\|_M < \infty\} .$$

COROLLARY 3.3. *Let $f \in L^2\mathcal{O}(M)$. Then we have*

$$f(z) = \lim_{t\uparrow 1} \mathscr{C} f(tz) = \lim_{t\uparrow 1} \int_M f(w) K_0(\bar{w}, tz) dM(w) , \qquad z \in M ,$$

where the limit is taken in $L^2(M)$.

4. The Fourier transformation. We define the Fourier transform $\mathscr{F}f$ of $f \in L^2(M)$ by

$$\mathscr{F}f(x) = \int_M f(w)\overline{\exp(x \cdot w)}dM(w), \qquad x \in \boldsymbol{R}^{n+1}.$$

Then by Lemmas 3.1 and 2.3 and (9), for $f = \sum_{k=0}^{\infty} f_k \in L^2\mathcal{O}(M)$ we have

$$\mathscr{F}f(x) = \sum_{k=0}^{\infty} \frac{1}{N(k, n)k!2^k} f_k(x), \qquad x \in \boldsymbol{R}^{n+1}. \tag{10}$$

We call the mapping $\mathscr{F} : f \mapsto \mathscr{F}f$ the Fourier transformation.

THEOREM 4.1. *The Fourier transformation $\mathscr{F}$ is a unitary isomorphism of $L^2\mathcal{O}(M)$ onto $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$.*

PROOF. Let $F \in L^2\mathcal{O}(M)$. By Lemmas 3.1 and 2.3 and (10),

$$\begin{aligned} \infty > (F, F)_M &= \sum_{k=0}^{\infty} \int_M F_k(w)\overline{F_k(w)}dM(w) \\ &= \sum_{k=0}^{\infty} C(k, n) \int_S \frac{F_k(\omega)}{N(k, n)k!2^k} \frac{\overline{F_k(\omega)}}{N(k, n)k!2^k} dS(\omega) \\ &= (\mathscr{F}F, \mathscr{F}F)_{\boldsymbol{R}^{n+1}}. \end{aligned}$$

Thus $\mathscr{F}$ is an isometric mapping of $L^2\mathcal{O}(M)$ into $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$.

Now, we prove that $\mathscr{F}$ is surjective. Let $f = \sum_{k=0}^{\infty} f_k \in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. Then Proposition 1.3 and Lemmas 1.1 and 2.3 imply

$$\begin{aligned} \infty > \int_{\boldsymbol{R}^{n+1}} f(x)\overline{f(x)}d\mu(x) &= \sum_{k=0}^{\infty} C(k, n) \int_S f_k(\omega)\overline{f_k(\omega)}dS(\omega) \\ &= \sum_{k=0}^{\infty} N(k, n)^2 k!^2 2^{2k} \int_M f_k(w)\overline{f_k(w)}dM(w) \\ &= \int_M \left(\sum_{k=0}^{\infty} N(k, n)k!2^k f_k(w) \right) \left(\overline{\sum_{l=0}^{\infty} N(l, n)l!2^l f_l(w)} \right) dM(w). \end{aligned}$$

Therefore, $F(w) = \sum_{k=0}^{\infty} N(k, n)k!2^k f_k(w)$ belongs to $L^2\mathcal{O}(M)$. By (10), $\mathscr{F}F(z) = f(z)$.

q.e.d.

Especially for $f \in \mathcal{O}(\tilde{M}[1])|_M \subset L^2\mathcal{O}(M)$, we have

$$\mathscr{F} : \mathcal{O}(\tilde{M}[1]) \xrightarrow{\sim} \mathrm{Exp}_\Delta(\boldsymbol{R}^{n+1}; [1/2]),$$

where

$$\mathrm{Exp}_\Delta(\boldsymbol{R}^{n+1}; [1/2]) = \{f \in \mathscr{A}_\Delta(\boldsymbol{R}^{n+1});\ \exists B < 1/2,\ \exists C > 0 \text{ s.t. } |f(x)| \leq C\exp(B\|x\|)\} \tag{11}$$

(cf. [4]).

THEOREM 4.2. *If* $f \in \mathrm{Exp}_\Delta(\boldsymbol{R}^{n+1}; [1/2])$, *then*

$$\mathscr{F}^{-1} f(z) = \int_{\boldsymbol{R}^{n+1}} \exp(x \cdot z) f(x) d\mu(x), \qquad z \in M. \tag{12}$$

PROOF. Because of $|\exp(x \cdot z)| \leq \exp(\|x\|/2)$ for $z \in M$, the integral on the right-hand side in (12) converges absolutely by (3) and (11), which we denote by $F(z)$. Then by the Fubini theorem and Theorem 1.9, $\mathscr{F} F(x) = f(x)$. q.e.d.

For $f(x) \in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ and $0 < t < 1$, $\tilde{f}^t(x) = f(tx) \in \mathrm{Exp}_\Delta(\boldsymbol{R}^{n+1}, [1/2])$ and $\lim_{t \uparrow 1} \|f - \tilde{f}^t\|_{\boldsymbol{R}^{n+1}} = 0$. Therefore, we have the following corollary:

COROLLARY 4.3. *Let* $f \in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$\mathscr{F}^{-1} f(z) = \lim_{t \uparrow 1} \int_{\boldsymbol{R}^{n+1}} \exp(x \cdot z) f(tx) d\mu(x), \qquad z \in M,$$

where the limit is taken in $L^2(M)$.

THEOREM 4.4. *Let* $f \in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$\mathscr{F}^{-1} f(z) = \lim_{R \to \infty} \int_{B(R)} \exp(x \cdot z) f(x) d\mu(x), \qquad z \in M,$$

where the limit is taken in $L^2(M)$.

PROOF. Let $f \in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$ and f_k the k-homogeneous harmonic component of f. Put

$$f^R(z) = \int_{B(R)} \exp(x \cdot z) f(x) d\mu(x), \qquad z \in M,$$

$$f_k^R(z) = \int_{B(R)} \exp(x \cdot z) f_k(x) d\mu(x), \qquad z \in M.$$

Then by using the Fubini theorem and Lemmas 1.8 and 2.3, we have

$$\begin{aligned}
\mathscr{F} f_k^R(w) &= \int_M \int_{B(R)} \exp(x \cdot z) f_k(x) d\mu(x) \exp(w \cdot \bar{z}) dM(z), \qquad x = r\omega, \\
&= C_R(k, n) \int_M \int_S \frac{(\omega \cdot z)^k}{k!} f_k(\omega) dS(\omega) \exp(w \cdot \bar{z}) dM(z) \\
&= \frac{C_R(k, n)}{C(k, n)} f_k(\omega).
\end{aligned}$$

By the uniform convergence of $\sum_{k=0}^{\infty} f_k$ on $B[R] = \{x \in \boldsymbol{R}^{n+1}; \|x\| \leq R\}$, we have

$$\mathscr{F}f^R(w)=\sum_{k=0}^{\infty} C_R(k, n)/C(k, n)f_k(w) .$$

By Proposition 1.3,

$$\lim_{R\to\infty} \|f-\mathscr{F}f^R\|^2_{\boldsymbol{R}^{n+1}}=\lim_{R\to\infty}\sum_{k=0}^{\infty}(1-C_R(k, n)/C(k, n))^2\|f_k\|^2_{\boldsymbol{R}^{n+1}}=0 .$$

Since $\mathscr{F}$ is a unitary isomorphism, $\mathscr{F}^{-1}f=\lim_{R\to\infty} f^R$ in $L^2(M)$. q.e.d.

5. The Poisson transformation. Let $L^2(S)$ be the space of square integrable functions on S with respect to the inner product

$$(f, g)_S=\int_S f(\omega)\overline{g(\omega)}dS(\omega) .$$

We call $\mathscr{H}^k(S)=\{P|_S;\ P\in\mathscr{P}^k_\Delta(\boldsymbol{C}^{n+1})\}$ the space of k-spherical harmonics. For $f\in L^2(S)$, the k-spherical harmonic component f_k of f is defined by

(13) $$f_k(\omega)=N(k, n)\int_S f(\tau)P_{k,n}(\omega\cdot\tau)dS(\tau) .$$

Note that (13) is the restriction of (1) on S and the harmonic extension of $f_k\in\mathscr{H}^k(S)$ is given by (6). The following lemmas are known:

LEMMA 5.1. *Let $f\in L^2(S)$ and f_k be the k-spherical harmonic component of f defined by* (13). *Then the expansion $\sum_{k=0}^{\infty} f_k$ converges to f in the topology of $L^2(S)$; that is, we have the Hilbert direct sum decomposition*:

$$L^2(S)=\bigoplus_{k=0}^{\infty}\mathscr{H}^k(S) .$$

LEMMA 5.2. *Let $f\in L^2(S)$. Then we have*

$$f(\omega)=\lim_{t\uparrow 1}\mathscr{P}f(t\omega)=\lim_{t\uparrow 1}\int_S f(\eta)K_1(\eta, t\omega)dS(\eta) ,\qquad \omega\in S ,$$

where the limit is taken in $L^2(S)$.

Put $\|f\|^2_S=(f, f)_S$. By Lemma 2.3, for $f_k\in\mathscr{P}^k_\Delta(\boldsymbol{C}^{n+1})$ we have

(14) $$\|f_k\|^2_S=2^k/\gamma_{k,n}\|f_k\|^2_M .$$

Thus for $f=\sum f_k$, $f_k\in\mathscr{H}^k(S)$ we have

$$\lim_{t\uparrow 1}\sum t^{2k}\|\tilde{f}_k\|^2_M=\lim_{t\uparrow 1}\sum\gamma_{k,n}/2^k t^{2k}\|\tilde{f}_k\|^2_S ,$$

where $\tilde{f}_k$ is the harmonic extension of f_k.

Since

$$2^k/\gamma_{k,n}=N(k,n)\Gamma((n+1)/2)k!/\Gamma(k+(n+1)/2)=O(k^{(n-1)/2}), \tag{15}$$

$\mathscr{P}f(tz)$ converges in $L^2\mathcal{O}(M)$ as $t\uparrow 1$. Therefore, we can define the Poisson transform $\mathscr{P}_M f$ of $f\in L^2(S)$ by

$$\mathscr{P}_M f(z)=\lim_{t\uparrow 1}\int_S f(\omega)K_1(tz,\omega)dS(\omega), \qquad z\in M,$$

where the limit is taken in $L^2(M)$. We call the mapping $\mathscr{P}_M: f\mapsto \mathscr{P}_M f$ the Poisson transformation.

To determine the image of $\mathscr{P}_M$ more exactly, we introduce the following spaces. Let $l\geq 0$ and let Δ_S be the Laplace-Beltrami operator on S. Considering $\Delta_S^l f_k=\{-k(k+n-1)\}^l f_k$ for $f_k\in\mathscr{H}^k(S)$, we define the Sobolev space on S by

$$H^l(S)=\left\{f\in L^2(S);\ \sum_{k=0}^{\infty}(1+k^2)^l\|f_k\|_S^2<\infty\right\},$$

where f_k is the k-spherical harmonic component of f defined by (13). We denote the norm on $H^l(S)$ by $\|\cdot\|_{H^l(S)}$.

Similarly, we define the "Hardy-Sobolev" space on M by

$$H^l\mathcal{O}(M)=\left\{f\in L^2\mathcal{O}(M);\ \sum_{k=0}^{\infty}(1+k^2)^l\|f_k\|_M^2<\infty\right\},$$

where f_k is the k-homogeneous component of f defined by (9). We denote the norm on $H^l\mathcal{O}(M)$ by $\|\cdot\|_{H^l\mathcal{O}(M)}$. Note that $H^0(S)=L^2(S)$ and $H^0\mathcal{O}(M)=L^2\mathcal{O}(M)$.

Because of (14) and (15), for $f\in L^2(S)$ we have

$$\begin{aligned}\|\mathscr{P}_M f(z)\|^2_{H^{(n-1)/4}\mathcal{O}(M)}&=\lim_{t\uparrow 1}\sum_{k=0}^{\infty}t^{2k}(1+k^2)^{(n-1)/4}\|\tilde{f}_k\|_M^2\\&=\lim_{t\uparrow 1}\sum_{k=0}^{\infty}t^{2k}\gamma_{k,n}/2^k(1+k^2)^{(n-1)/4}\|\tilde{f}_k\|_S^2<\infty.\end{aligned} \tag{16}$$

Thus

$$\mathscr{P}_M: L^2(S)\to H^{(n-1)/4}\mathcal{O}(M).$$

Since $\|\tilde{f}_k\|_S^2=\|\alpha_B\circ\mathscr{C}\tilde{f}_k\|_S^2$ in (16), we can define the Cauchy transform $\mathscr{C}_S g$ of $g\in H^{(n-1)/4}\mathcal{O}(M)$ by

$$\mathscr{C}_S g(\omega)=\lim_{t\uparrow 1}\int_M g(z)K_0(t\omega,\bar{z})dM(z), \qquad \omega\in S,$$

where the limit is taken in $L^2(S)$. We call the mapping $\mathscr{C}_S: g\mapsto\mathscr{C}_S g$ the Cauchy transformation.

PROPOSITION 5.3. *Let $l \geq 0$. Then the Poisson transformation $\mathscr{P}_M$ establishes the following linear topological isomorphism*:

$$\mathscr{P}_M : H^l(S) \xrightarrow{\sim} H^{l+(n-1)/4}\mathcal{O}(M) .$$

Moreover, the inverse mapping of $\mathscr{P}_M$ is given by $\mathscr{C}_S$; that is, $\mathscr{P}_M^{-1} = \mathscr{C}_S$.

PROOF. Let $f \in H^l(S)$. By the same argument as above, $\mathscr{P}_M f$ belongs to $H^{l+(n-1)/4}\mathcal{O}(M) \subset H^{(n-1)/4}\mathcal{O}(M)$. Thus we can consider $\mathscr{C}_S \circ \mathscr{P}_M f$. By (7), (8), Lemma 5.2 and the Fubini theorem, we have

$$\mathscr{C}_S \circ \mathscr{P}_M f(\omega) = f(\omega) , \qquad f \in H^l(S) .$$

Therefore $\mathscr{C}_S \circ \mathscr{P}_M = \mathrm{id}$ and $\mathscr{P}_M$ is injective.

Let $g \in H^{l+(n-1)/4}\mathcal{O}(M)$. By the same argument as above, $\mathscr{C}_S g$ belongs to $H^l(S) \subset L^2(S)$. Thus we can consider $\mathscr{P}_M \circ \mathscr{C}_S g$. By (7), (8), Corollary 3.3 and the Fubini theorem, we have

$$\mathscr{P}_M \circ \mathscr{C}_S g(z) = g(z) , \qquad g \in H^{l+(n-1)/4}\mathcal{O}(M) .$$

Therefore $\mathscr{P}_M$ is surjective.

The continuities of $\mathscr{P}_M$ and $\mathscr{P}_M^{-1}$ are clear. q.e.d.

For $f = \sum_{k=0}^{\infty} f_k \in \mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$, we have

$$\Delta_x f(x) = \left(\frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S \right) f(\omega) = 0 , \qquad x = r\omega , \quad \omega \in S .$$

Thus $\Delta_S^l f_k = \sum_{k=0}^{\infty} \{-k(k+n-1)\}^l f_k$ for $f_k \in \mathscr{P}_\Delta^k(\boldsymbol{R}^{n+1})$. Put

$$H^l\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}) = \{ f \in \mathscr{A}_\Delta(\boldsymbol{R}^{n+1}) ; ((1+\Delta_S)^l f, (1+\Delta_S)^l f)_{\boldsymbol{R}^{n+1}} < \infty \} ,$$

then we have the following linear topological isomorphism:

$$\mathscr{F} : H^l\mathcal{O}(M) \xrightarrow{\sim} H^l\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}) .$$

REFERENCES

[1] K. FUJITA, A Hilbert space of harmonic functions, Proc. Japan Acad. 70, Ser. A (1994), 286–289.

[2] K. II, On a Bargmann-type transform and a Hilbert space of holomorphic functions, Tôhoku Math. J. 38 (1986), 57–69.

[3] G. LEBEAU, Fonctions harmoniques et spectre singulier, Ann. Scient. Ecole Norm. Sup. 4e série, 13 (1980), 269–291.

[4] M. MORIMOTO AND K. FUJITA, Analytic functionals and entire functionals on the complex light cone, Hiroshima Math. J. 25 (1995), 493–512 .

[5] M. MORIMOTO, Analytic functionals on the sphere and their Fourier-Borel transformations, Complex Analysis, Banach Center Publications 11 PWN-Polish Scientific Publishers, Warsaw, 1983, 223–250.

[6] C. MÜLLER, Spherical Harmonics, Lecture Notes in Math. 17 (1966), Springer.

[7] R. WADA, On the Fourier-Borel transformations of analytic functionals on the complex sphere, Tôhoku Math. J. 38 (1986), 417–432.

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
SAGA UNIVERSITY
SAGA 840
JAPAN