Hinged and supported plates with corners

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Abstract. We consider the Kirchhoff-Love model for the supported plate, that is, the fourth order differential equation $\Delta^2 u = f$ with appropriate boundary conditions. Due to the expectation that a downwardly directed force f will imply that the plate, which is supported at its boundary, touches that support everywhere, one commonly identifies those boundary conditions with the ones for the so-called hinged plate: $u = 0 = \Delta u - (1 - \sigma)\,\kappa u_n$. Engineers however are usually aware that rectangular roofs tend to bend upwards near the corners and this would mean that $u = 0$ is not appropriate. We will confirm this behaviour and show the difference of the supported and the hinged plates in case of domains with corners.

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1. Introduction

1.1. Description and motivation

Consider a thin plate subjected to a negative (downward) vertical load that lies freely at its sides on a supporting structure. Whenever the plate touches its support, the corresponding boundary condition fixes the height. A second boundary condition, which is necessary for the Kirchhoff-Love model of the plate in order to find a well-posed boundary value problem, comes naturally from the variational model describing the energy. This set of boundary conditions are known as hinged. However, if the plate does not touch its supporting structure, one finds a different set of boundary conditions. So it means that a supported plate may satisfy different sets of boundary conditions at different parts at the boundary. For a downward force one expects the plate to touch at least at some boundary parts.

In the mathematical and engineering literature the (simply) supported and hinged boundary conditions are often confused; see also the comments by Blaauwendraad on [4, Chapter 13.4]. So before deriving a mathematical formulation let us clearly describe these two types of boundary conditions:

- hinged: the deflection of the plate is zero on the boundary;
- supported: the deflection of the plate cannot become negative on the boundary.

So the main question is:

Does a plate which is supported at its boundary by walls of constant height and is pushed downwards, touch this supporting structure everywhere?

In engineering literature that considers supported plates, such as [4, 5], one finds that a rectangular supported plate will lift at the corners when pushed downwards. A rule of thumb is described by Figure 1. One approximates a thin plate by a configuration of 9 rigid tiles, elastically connected to each other, and supposes that the force is distributed over 12 points at the boundary. Pushed downwards by a uniformly distributed weight of size 1, the forces working on these 12 points act as depicted in Figure 1. That is, upward forces appear at the corners which, if the roof is not fixed to its supporting walls, will tend to move the plate upwards.

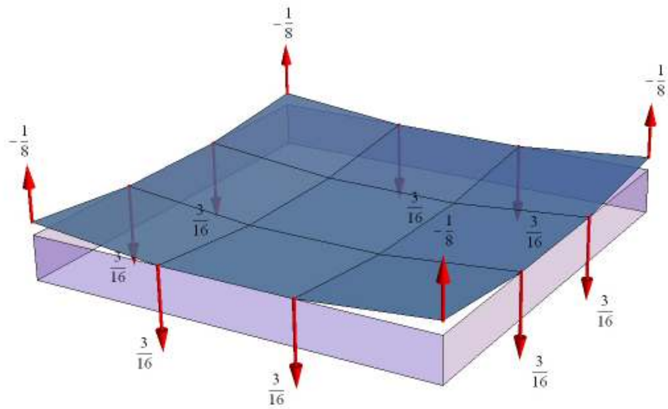

Figure 1. A discretized square roof with homogeneous weight distribution lifts at the corners (see [4, Chapter 9.3] or [4, p. 18]).

Here we will show that a similar result comes out of the analysis of a model that uses the continuous formulation. Within the framework of this Kirchhoff-Love model, a negatively loaded, simply supported plate will exhibit bending moments, concentrated at the corners, which will force the plate to lift there. The unilateral boundary condition that is involved makes the present model of the supported plate nonlinear.

Friedrichs in [9] is one of the first to give a modern variational formulation for the plate. See also [8]. Kirchhoff plates under several linear boundary conditions with angular corners have been studied in the seminal paper by Blum and Rannacher [6] from 1980. For numerical approaches to this biharmonic plate model we refer to a paper by Babuška and Li [3]. Corner singularities for clamped and so-called 'supported' Kirchhoff plates have been numerically dealt with in [23], [22]. In [4, Chapter 13.4] one finds numerical evidence that a square, supported plate under a uniform load will move upwards near the corners.

1.2. The mathematical setting

The Kirchhoff-Love model for thin elastic plates can be considered as the Euler-Lagrange equation that arises in the following minimization problem:

Find $u_0 : \Omega \longmapsto \mathbb{R}$ in an appropriate family of functions $\mathcal{V}$ such that:

$$u_0 = \operatorname{argmin}\{u \in \mathcal{V};\ J_\sigma(u)\} \tag{1.1}$$

with

$$J_\sigma(u) = \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - fu\right) d\lambda \tag{1.2}$$

Here Ω represents the shape of the plate and $u(x)$ the deflection at $x \in \Omega$ under the vertical load density $f(x)$. The mechanical problem that we are interested in concerns a downward force $f \leq 0$ resulting in a largely negative deflection u. The parameter σ denotes the Poisson ratio of the plate, constant for the homogeneous situation considere here, and, depending on the material, varying from -1 up to 0.5. The minimal value of the functional corresponds to the elastic energy of the deformed plate. The minimizer u, if it exists, gives the deflection of the plate. Introducing the boundary conditions through an appropriate set of functions $\mathcal{V}$, one models the different cases; see [31] for a detailed discussion.

Assuming that the minimizer is smooth enough, one can perform the usual integration by parts of the weak Euler-Lagrange equation $J'_\sigma(u;\varphi)=0$, that is

$$\int_\Omega (\Delta u \Delta \varphi + (1-\sigma)(2u_{xy}\varphi_{xy} - u_{xx}\varphi_{yy} - u_{yy}\varphi_{xx}) - f\varphi)\, d\lambda = 0 \text{ for all } \varphi \in \mathcal{V},$$

to obtain, for a sufficiently smooth domain, the differential equation and the corresponding natural boundary conditions.

- For the hinged plate the appropriate function space for the weak setting is

$$\mathcal{V} = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

  for the notation see e.g. [2]. A strong solution should satisfy the boundary value problem:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \sigma\Delta u + (1-\sigma)u_{nn} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

- For the simply supported plate the weak setting uses

$$\mathcal{V} = \left\{ u \in W^{2,2}(\Omega) \text{ and } \min(u,0) \in W_0^{1,2}(\Omega) \right\}.$$

  In this case we are minimizing in a closed subset of $W^{2,2}(\Omega)$ and the solution will satisfy a variational inequality. Applying local arguments one sees that a strong solution u should satisfy:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \\ \sigma\Delta u + (1-\sigma)u_{nn} = 0 & \text{on } \partial\Omega, \\ u(x) = 0 \text{ or } \partial_n\left(\Delta u(x)\right) + (1-\sigma)u_{n\tau\tau}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{1.4}$$

  Here n and τ represent the exterior normal and the counter-clockwise tangent vectors respectively. The third equation in (1.3) and (1.4) can be rewritten as

$$\sigma\Delta u + (1-\sigma)u_{nn} = \Delta u - (1-\sigma)u_{\tau\tau} - (1-\sigma)\kappa u_n \text{ on } \partial\Omega$$

and $u_{\tau\tau}|_{\partial\Omega} = 0$ when $u|_{\partial\Omega} = 0$. The function κ is the signed curvature of the boundary taken positive on strictly convex boundary parts.

The Poisson ratio gives a measure of the tendency of materials to expand or contract in the other directions when they are forced to expand or contract in one direction. Most materials tend to expand when forcefully contracted in one direction and thus possess a positive Poisson ratio; cork has almost zero and metals are close to 0.3, whereas for some exotic foam polymers $\sigma < 0$, i.e. they can contract in all directions when they are forced to do so only in one. See [31] and references therein.

1.3. The set-up of this paper

We deal with hinged and supported plates that have at most finitely many convex or concave corners with smooth remaining boundary parts. The existence results in the weak setting do not depend to strongly on the presence of these corners and will be considered in Section 2.

After having existence for the hinged and the supported case, we want to compare both solutions. In order to do so we will need some more regularity and this depends strongly on the presence of corners. This is dealt with at Section 3. We need to distinguish between convex and concave corners. Moreover, for the hinged problem on convex corners there is a difference between acute ($< \frac{1}{2}\pi$), square ($= \frac{1}{2}\pi$) and obtuse ($\in \left(\frac{1}{2}\pi, \pi\right)$). We will measure angles from inside and assume C^4-smoothness outside those corners.

- By a concave corner we mean that after a possible rotation and a translation to put the corner in O, we may write locally, for some open neighborhood U of O, that

$$\Omega \cap U = \{(x_1,x_2)\,; x_2 > \min(\varphi_1(x_1), \varphi_2(x_1))\} \cap U$$

  with $\varphi_1, \varphi_2 \in C^4(-1,1)$ and $\varphi_1(0) = \varphi_2(0) = 0$, $\varphi_1'(0) > \varphi_2'(0)$.

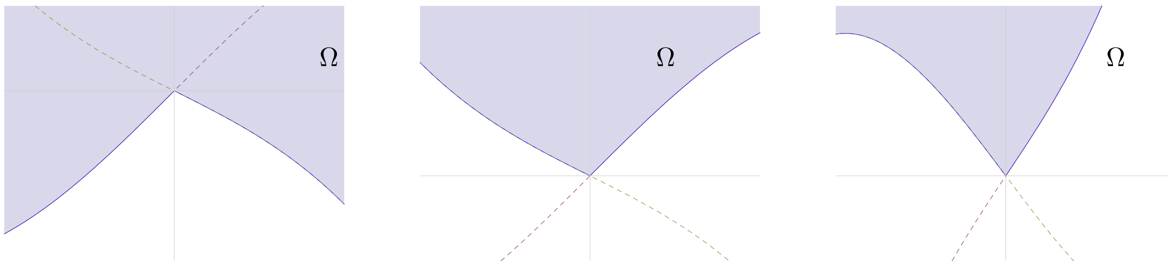

Figure 2. Respectively a concave, an obtuse-convex and acute-convex corner.

- By a convex corner we mean that after a possible rotation and a translation we may write locally, that
$$\Omega \cap U = \{(x_1, x_2)\,; x_2 > \max(\varphi_1(x_1), \varphi_2(x_1))\} \cap U$$
with $\varphi_1, \varphi_2 \in C^4(-1,1)$ and $\varphi_1(0) = \varphi_2(0) = 0$, $\varphi_1'(0) > \varphi_2'(0)$. In this setting acute means $\varphi_1'(0)\varphi_2'(0) < -1$ and obtuse $\varphi_1'(0)\varphi_2'(0) > -1$. For a square angle it holds that $\varphi_1'(0)\varphi_2'(0) = -1$.

In two dimensions such a curvilinear corner can be mapped through a two-side-bounded biconformal mapping on a domain which is locally polygonal with a corner of the same angle. For a conformal mapping h one finds $\Delta(u \circ h) = |h'|^2 (\Delta u) \circ h$. Two-side-bounded biconformal mapping h is means $0 < c_1 \leq |h'|^2 \leq c_2 < \infty$. Since a polygonal angle will simplify the mathematics involved, we will sometimes just consider this setting.

The case of square angles presents a special case for the hinged plate. If the force f equals zero in a neighborhood of the corner, the presence of this corner no longer puts a restriction on the regularity. An outline of this case was presented in [30]. The details can be found in Section 3.1.

For plates with general angles we need to use the theory due to Kondratiev, see [18], which is based on the asymptotic formula due to [32].

In Section 4 we will show our main result, namely that the supported plate generically is not supported everywhere on the boundary but comes loose at certain subsets of the boundary. In other words, the hinged and the supported plate behave differently.

Finally, in the Appendix, we show some numerical results for a supported L-shaped plate with different loads.

2. Existence results

Let us start with some notation. For the appropriate function classes $\mathcal{V}$ we write

- hinged case: $H_0(\Omega) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,
- supported case: $H_+(\Omega) := \left\{ u \in W^{2,2}(\Omega) \,\middle|\, u^- := -\min(u,0) \in W_0^{1,2}(\Omega) \right\}$.

We let $d\lambda$ and ds denote respectively the 2D and 1D boundary Lebesgue measures. Moreover, differentiation will be denoted with subscripts (e.g. u_x, u_y, u_{xy}), n and τ will denote the exterior normal and counter-clockwise oriented tangent vector respectively and the L^p and $W^{m,p}$ norms will be written as $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, whenever there is no confusion on the domain of integration. We will also use the following semi-norm

$$|u|_{2,2} = \left\| \left| \nabla^2 u \right| \right\|_2 .$$

Note that $|\cdot|_{2,2}$ is a norm in $H_0(\Omega)$ which is equivalent to $\|\cdot\|_{2,2}$. See Appendix 5.

Remark 2.1. One should notice that $H_+(\Omega)$ is closed in $W^{2,2}(\Omega)$. Indeed, let $u_n \in H_+(\Omega)$ with $u_n \to u \in W^{2,2}(\Omega)$ in the $W^{2,2}$ norm topology. Then $u_n \to u$ in $W^{1,2}(\Omega)$, which implies that $W_0^{1,2}(\Omega) \ni u_n^- \to u^-$. Since $W_0^{1,2}(\Omega)$ is closed in $W^{1,2}(\Omega)$ it holds that $u^- \in W_0^{1,2}(\Omega)$.

2.1. The hinged case

We establish the existence of a minimizer for J_σ in the aforementioned cases. When the plate is hinged, existence is a straightforward and a classical result but we include it here for the sake of completeness.

Theorem 2.2 (Existence and uniqueness for the hinged case). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with Lipschitz boundary. Let J_σ be as in (1.2) with $-1 < \sigma < 1$ and $f \in L^2(\Omega)$. Then J_σ possesses a unique minimizer in $H_0(\Omega)$.*

Proof. Defining the bilinear form

$$\alpha_\sigma(u,v) = \int_\Omega \Big(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\Big)\, d\lambda, \tag{2.1}$$

we can write

$$J_\sigma(u) = \frac{1}{2}\alpha_\sigma(u,u) - \int_\Omega fu\, d\lambda \quad \text{and} \quad J'_\sigma(u;v) = \alpha_\sigma(u,v) - \int_\Omega fv\, d\lambda.$$

One has the following estimate

$$\begin{aligned}
\alpha_\sigma(u,u) &= \int_\Omega \big((\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\big)\, d\lambda \\
&= 2\int_\Omega \big(\tfrac{1}{2}u_{xx}^2 + \tfrac{1}{2}u_{yy}^2 + (1-\sigma)u_{xy}^2 + \sigma\, u_{xx}u_{yy}\big)\, d\lambda \\
&\geq 2(1-|\sigma|)\int_\Omega \big(\tfrac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) + u_{xy}^2\big)\, d\lambda = (1-|\sigma|)\, |u|_{2,2}^2
\end{aligned} \tag{2.2}$$

and coercivity is implied by Corollary 5.4. Since $\alpha_\sigma(u,v)$ is a continuous bilinear form and satisfies (2.2), a direct application of the Lax-Milgram Lemma (e.g. [21, page 57]) completes the proof. □

2.2. The supported case and the variational inequality

Proving the existence of minimizers in $H_+(\Omega)$ is not so straightforward. Two problems appear that make it more appropriate to consider an alternative approach: the nature of the boundary conditions and the fact that the corresponding bilinear form is not obviously coercive. Thus, we will prove existence by studying the corresponding variational inequality and its regularization.

The connection between variational inequalities and minimization problems is well known (see [17]) and illustrated by the following Lemma which we include for the sake of completeness.

Lemma 2.3. *Let X be a Banach space, $F \in C^1(X;\mathbb{R})$ a convex functional, i.e.*

$$F\Big(u + t\,(v-u)\Big) \leq F(u) + t\Big(F(v) - F(u)\Big) \text{ for } u, v \in X \text{ and } t \in [0,1],$$

and let $K \subset X$ be closed and convex. For $u \in K$ the following statements are equivalent.

(i) $F'(u; v-u) \geq 0$ *for all* $v \in K$,

(ii) $F(u) = \min_{v\in K} F(v)$.

Here $F'(u;h)$ denotes the Gâteaux derivative of F at u in the direction of h.

Proof. $(i) \Rightarrow (ii)$. Since F is convex, one has that

$$F'(u; v-u) = \lim_{t\downarrow 0} \frac{F\Big(u + t(v-u)\Big) - F(u)}{t} \leq F(v) - F(u), \text{ for all } v \in K,$$

i.e. $F(u) \leq F(v)$ for all $v \in K$.

$(ii) \Rightarrow (i)$. Assume that u minimizes F in K and let $v \in K$. Since K is convex it holds that $u + t(v-u) \in K$, for all $t \in [0,1]$. This implies that the C^1 function

$$g(t) := F\Big(u + t(v-u)\Big) \text{ with } t \in [0,1],$$

attains its minimum at $t = 0$, i.e. $g'(0) \geq 0$ or

$$F'(u; v - u) \geq 0.$$

$\square$

We will now move on to the theorem.

Theorem 2.4 (Existence for the supported case). *Let $\Omega \in \mathbb{R}^2$ with a Lipschitz boundary and $-1 < \sigma < 1$. Moreover assume that $0 \not\equiv f \in L^2(\Omega)$ such that*

$$\int_\Omega f\zeta \, d\lambda < 0, \text{ for all nontrivial } \zeta \in H_+(\Omega) \text{ with } \alpha_\sigma(\zeta, \zeta) = 0. \tag{2.3}$$

Then, there exists a minimizer $u_\sigma \in H_+(\Omega)$ of J_σ.

Remark 2.5. *(i)* The functions ζ which satisfy (2.3) are nothing more than the affine functions with nonnegative boundary values. They represent the rigid motions of an unloaded plate. Assumption (2.3) implies that the force density is such that all nontrivial nonnegative rigid motions will increase the energy: $J_\sigma(u + r) > J_\sigma(u)$ for all $r = a \cdot x + b \gneqq 0$ on Ω. For the condition (2.3) to be satisfied it is not necessary that f is nonpositive everywhere.

(ii) The condition also implies that there exists $x_0 \in \partial\Omega$ such that $u_\sigma(x_0) = 0$: assume that the plate does not touch $\partial\Omega$ and so $h = \min_{x \in \partial\Omega} u_\sigma(x) > 0$. Then $u_\sigma - h \in H_+(\Omega)$ and

$$J_\sigma(u_\sigma - h) = J_\sigma(u_\sigma) + \int_\Omega fh \, d\lambda < J_\sigma(u_\sigma),$$

which is a contradiction, since u_σ is supposed to be a minimizer. In fact, the same argument shows that $h = 0$ is the only affine function such that $u_\sigma - h \in H_+(\Omega)$.

(iii) A failure to fulfill (2.3) will result in the existence of multiple minimizers or even the non-existence of such. To see this, assume that u_σ is a minimizer and that ζ_0 is a nontrivial affine function in $H_+(\Omega)$ for which $\int_\Omega f\zeta_0 \, d\lambda \geq 0$. Then

$$J_\sigma(u_\sigma + \zeta_0) = J_\sigma(u_\sigma) - \int_\Omega f\zeta_0 \, d\lambda \leq J_\sigma(u_\sigma).$$

Hence u_σ, if it exists, is not unique. If $\int_\Omega f\zeta_0 \, d\lambda > 0$ then no minimizer exists, since $u_\sigma + t\,\zeta_0 \in H_+(\Omega)$ for all $t \geq 0$ and

$$\lim_{t \to \infty} J_\sigma(u_\sigma + t\zeta_0) = -\infty.$$

Proof of Theorem 2.4. Following Lemma 2.3 a minimizer is a function $u_\sigma \in H_+(\Omega)$ such that

$$J'_\sigma(u_\sigma; v - u_\sigma) \geq 0 \text{ for all } v \in H_+(\Omega). \tag{2.4}$$

Since the functional J_σ is not coercive on $W^{2,2}(\Omega)$ or $H_+(\Omega)$ we are going to consider an elliptic regularization of α_σ.

Define the inner product $(\!(\cdot,\cdot)\!)$ on $W^{2,2}(\Omega)$ by

$$(\!(u, v)\!) = \int_\Omega \left(uv + \nabla u \cdot \nabla v + \nabla^2 u \cdot \nabla^2 v\right) d\lambda$$

and consider for $\varepsilon > 0$:

$$\alpha_{\sigma,\varepsilon}(u, v) = \alpha_\sigma(u, v) + \varepsilon\,(\!(u, v)\!) \text{ for } u, v \in W^{2,2}(\Omega). \tag{2.5}$$

Let $J_{\sigma,\varepsilon}$ be the corresponding regularized functional, i.e.

$$J_{\sigma,\varepsilon}(u) = \frac{1}{2}\alpha_{\sigma,\varepsilon}(u, u) - \int_\Omega fu \, d\lambda,$$

and let $J'_{\sigma,\varepsilon} : H_+(\Omega) \to \left(W^{2,2}(\Omega)\right)'$ denote the Gâteaux derivative of $J_{\sigma,\varepsilon}$ given by

$$J'_{\sigma,\varepsilon}(u; v) = \alpha_{\sigma,\varepsilon}(u, v) - \int_\Omega fv \, d\lambda.$$

We stay within the setting of [17, Chapter III]. Since $u \mapsto \sqrt{\alpha_{\sigma,\varepsilon}(u,u)}$ is a norm on $W^{2,2}(\Omega)$, the mapping $u \mapsto J'_{\sigma,\varepsilon}(u;\cdot)$ is continuous and strictly monotone:

$$J'_{\sigma,\varepsilon}(u;u-v) - J'_{\sigma,\varepsilon}(v;u-v) = \alpha_{\sigma,\varepsilon}(u-v,u-v) \geq 0, \quad \text{for } \ u,v \in H_+(\Omega) \tag{2.6}$$

with a strict inequality for $u \neq v$, and coercive in the sense that:

$$\lim_{\substack{u\in H_+(\Omega)\\ \|u\|_{2,2}\to\infty}} \frac{J'_{\sigma,\varepsilon}(u;u)}{\|u\|_{2,2}} = +\infty$$

(take $\varphi \equiv 0$ in [17, Definition 1.3, p. 84]). We also have that $H_+(\Omega)$ is closed (see Remark 2.1) and convex in $W^{2,2}(\Omega)$. Then [17, Corollary 1.8, p. 87] implies the existence of $u_\varepsilon \in H_+(\Omega)$ satisfying

$$J'_{\sigma,\varepsilon}(u_\varepsilon; v-u_\varepsilon) \geq 0 \text{ for all } v \in H_+(\Omega). \tag{2.7}$$

By the strict monotonicity u_ε is unique: if $u_\varepsilon, \tilde{u}_\varepsilon$ would both satisfy (2.7), then

$$0 \geq J'_{\sigma,\varepsilon}(u_\varepsilon; u_\varepsilon - v_\varepsilon) - J'_{\sigma,\varepsilon}(v_\varepsilon; u_\varepsilon - v_\varepsilon) = \alpha_{\sigma,\varepsilon}(u-v,u-v) > 0.$$

Rephrased (2.7) means that

$$\alpha_{\sigma,\varepsilon}(u_\varepsilon, u_\varepsilon - v) \leq \int_\Omega f(u_\varepsilon - v)\, d\lambda \text{ for all } v \in H_+(\Omega), \tag{2.8}$$

which implies

$$\begin{aligned} \alpha_{\sigma,\varepsilon}(v, u_\varepsilon - v) &= \alpha_{\sigma,\varepsilon}(u_\varepsilon, u_\varepsilon - v) - \alpha_{\sigma,\varepsilon}(u_\varepsilon - v, u_\varepsilon - v) \\ &\leq \int_\Omega f(u_\varepsilon - v)\, d\lambda \text{ for all } v \in H_+(\Omega). \end{aligned} \tag{2.9}$$

In fact (2.8) and (2.9) are equivalent. This equivalence is known as Minty's lemma (see [17, Lemma 1.5, p. 84]).

So we have a unique minimizer u_ε of $J_\sigma(u) + \frac{1}{2}\varepsilon\, (\!(u,u)\!)$ in $H_+(\Omega)$.

What happens if we let $\varepsilon \downarrow 0$? If $\|u_\varepsilon\|_{2,2}$ is uniformly bounded, then, since bounded sets in $W^{2,2}(\Omega)$ are weakly precompact, there exists $u_\varepsilon \in W^{2,2}(\Omega)$ and a weakly convergent sequence $u_{\varepsilon_n} \rightharpoonup u_\sigma$. The weak lower semicontinuity of $u \mapsto J_\sigma(u)$ implies

$$\begin{aligned} J_\sigma(u_\sigma) &\leq \liminf_{n\to\infty} J_\sigma(u_{\varepsilon_n}) = \liminf_{n\to\infty} \left(J_{\sigma,\varepsilon_n}(u_{\varepsilon_n}) - \tfrac{1}{2}\varepsilon_n\, (\!(u_{\varepsilon_n}, u_{\varepsilon_n})\!)\right) \\ &= \liminf_{n\to\infty} J_{\sigma,\varepsilon_n}(u_{\varepsilon_n}) \leq \liminf_{n\to\infty} J_{\sigma,\varepsilon_n}(v) = J_\sigma(v)\,. \end{aligned}$$

for any $v \in H_+(\Omega)$. So, J_σ has a minimizer u_σ and we are done. See also [17, Theorem 2.1, p. 88].

Now suppose that $\|u_\varepsilon\|_{2,2}$ is not uniformly bounded, that is, there exists a subsequence $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and $\|u_{\varepsilon_n}\|_{2,2} \to \infty$. Setting $w_n = \|u_{\varepsilon_n}\|_{2,2}^{-1} u_{\varepsilon_n} \in H_+(\Omega)$, there exists a subsequence, again denoted w_n, that weakly converges in $W^{2,2}(\Omega)$, say $w_n \rightharpoonup w$. Since $H_+(\Omega)$ is closed and convex it is also weakly closed by Mazur's Lemma (see [21, Theorem 6 page 103]). Hence $w \in H_+(\Omega)$. Use (2.8) and estimate (2.2) with $v = 0$, $\varepsilon = \varepsilon_n$ and divide by $\|u_{\varepsilon_n}\|_{2,2}^2$ to get

$$\begin{aligned} 0 \leq (1-|\sigma|)\, |w_n|_{2,2}^2 &\leq \alpha_\sigma(w_n, w_n) = \frac{a_{\sigma,\varepsilon}(u_{\varepsilon_n}, u_{\varepsilon_n}) - \varepsilon_n\, |u_{\varepsilon_n}|_{2,2}^2}{\|u_{\varepsilon_n}\|_{2,2}^2} \\ &\leq \frac{1}{\|u_{\varepsilon_n}\|_{2,2}} \int_\Omega f w_n\, d\lambda - \varepsilon_n \leq \frac{\|f\|_2\, \|w_{\varepsilon_n}\|_2}{\|u_{\varepsilon_n}\|_{2,2}} - \varepsilon_n \\ &\leq \frac{\|f\|_2\, \|w_{\varepsilon_n}\|_{2,2}}{\|u_{\varepsilon_n}\|_{2,2}} - \varepsilon_n = \frac{\|f\|_2}{\|u_{\varepsilon_n}\|_{2,2}} - \varepsilon_n. \end{aligned} \tag{2.10}$$

Thus it follows that $|w_n|_{2,2} \to 0$ for $n \to \infty$. Moreover the functional $|\cdot|_{2,2} : W^{2,2}(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous. Indeed

$$\begin{aligned}\int_\Omega |\partial_\alpha w_n|^2 \, d\lambda - \int_\Omega |\partial_\alpha w|^2 \, d\lambda &= \int_\Omega (\partial_\alpha w_n - \partial_\alpha w)(\partial_\alpha w_n + \partial_\alpha w) \, d\lambda \\ &= \int_\Omega (\partial_\alpha w_n - \partial_\alpha w)(\partial_\alpha w_n - \partial_\alpha w + 2\partial_\alpha w) \, d\lambda \\ &= \int_\Omega (\partial_\alpha w_n - \partial_\alpha w)^2 \, d\lambda + 2 \int_\Omega (\partial_\alpha w_n - \partial_\alpha w) \, \partial_\alpha w \, d\lambda \\ &\geq 2 \int_\Omega (\partial_\alpha w_n - \partial_\alpha w) \, \partial_\alpha w \, d\lambda\end{aligned}$$

for any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = 2$. Since $\int_\Omega (\partial_\alpha w_n - \partial_\alpha w) \, \partial_\alpha w \, d\lambda \to 0$ as $n \to \infty$ we obtain the claim and thus $|w|_{2,2} = 0$. Hence w is affine.

Dividing (2.9) by $\|u_{\varepsilon_n}\|_{2,2}$ we find

$$\alpha_{\sigma,\varepsilon_n} \left(v, w_n - \|u_{\varepsilon_n}\|_{2,2}^{-1} v \right) \leq \int_\Omega f \, \left(w_n - \|u_{\varepsilon_n}\|_{2,2}^{-1} v \right) \, d\lambda \text{ for all } v \in H_+(\Omega). \tag{2.11}$$

Since $w_n \rightharpoonup w$ in $W^{2,2}(\Omega)$ and since $\|w_n\|_{2,2} = 1$ we find that

$$\alpha_{\sigma,\varepsilon_n} (v, w_n) \to a_\sigma (v, w) \text{ for } n \to \infty.$$

Since moreover $\|u_{\varepsilon_n}\|_{2,2}^{-1} \to 0$ and $\int_\Omega f \; w_n \; d\lambda \to \int_\Omega f \; w \; d\lambda$, one finds from (2.11) that

$$\alpha_\sigma(v, w) \leq \int_\Omega f \; w \; d\lambda \text{ for all } v \in H_+(\Omega). \tag{2.12}$$

Since w is affine, it follows that

$$0 = \alpha_\sigma(v, w) \leq \int_\Omega f \; w \; d\lambda \leq 0 \text{ for all } v \in H_+(\Omega),$$

with a strict inequality and hence a contradiction unless $w \equiv 0$. So $w \equiv 0$.

By (2.10) we have $|w_n|_{2,2} \to 0$ for $n \to \infty$. The compact embedding of $W^{2,2}(\Omega)$ into $W^{1,2}(\Omega)$ implies that $w_n \to w$ strongly in $W^{1,2}(\Omega)$ and it follows that $\|w_n\|_{1,2} \to 0$. Since $\|.\|_{1,2} + |.|_{2,2}$ and $\|.\|_{2,2}$ are equivalent norms one finds that $\|w_n\|_{2,2} \to 0$ for $n \to \infty$ which contradicts $\|w_n\|_{2,2} = 1$. □

Next we show the uniqueness of the minimizer.

Proposition 2.6. *Having the same assumptions as in Theorem 2.4, the minimizer u_σ of J_σ is unique in $H_+(\Omega)$.*

Proof. Let $u, v \in H_+(\Omega)$. Then one has that

$$\begin{aligned}&J'_\sigma(u; u - v) - J'_\sigma(v; u - v) = \\ &= \int_\Omega \left((u-v)_{xx}^2 + (u-v)_{yy}^2 + 2\sigma(u-v)_{xx}(u-v)_{yy} + 2(1-\sigma)(u-v)_{xy}^2 \right) d\lambda \\ &\geq (1 - |\sigma|) \int_\Omega \left((u-v)_{xx}^2 + (u-v)_{yy}^2 + 2(u-v)_{xy}^2 \right) d\lambda \\ &= (1 - |\sigma|)|u - v|_{2,2} \geq 0\end{aligned} \tag{2.13}$$

with equality if and only if $u - v$ is affine. Now let $v_\sigma \in H_+(\Omega)$ with $v_\sigma \not\equiv u_\sigma$ such that

$$J'_\sigma(v_\sigma; v - v_\sigma) \geq 0, \text{ for all } v \in H_+(\Omega). \tag{2.14}$$

Assume that $u_\sigma - v_\sigma$ is not affine. Then, using (2.13) we obtain

$$J'_\sigma(v_\sigma; u_\sigma - v_\sigma) < J'_\sigma(u_\sigma; u_\sigma - v_\sigma) = -J'_\sigma(u_\sigma; v_\sigma - u_\sigma) \leq 0$$

which is contradicting (2.14), since $J'_\sigma(u_\sigma; v - u_\sigma) \geq 0$ for all $v \in H_+(\Omega)$. That means that $w := u_\sigma - v_\sigma$ is affine and one has

$$J_\sigma(u_\sigma) = J_\sigma(v_\sigma + w) = J_\sigma(v_\sigma) - \int_\Omega f w \, d\lambda > J_\sigma(u_\sigma)$$

if and only if $w \not\equiv 0$. □

3. The regularity of hinged plates

A corner with angle $\frac{1}{2}\pi$ forms an exceptional case, in the sense that the solution has more regularity then for corners near $\frac{1}{2}\pi$. We will first consider rectangular plate and would like to remark that similar results hold locally whenever the plate has a corner with angle $\frac{1}{2}\pi$ without being rectangular.

3.1. The hinged rectangular plate

3.1.1. An extension and a density lemma on a rectangle. When one considers functions on $\overline{\Omega}$ that are 0 on $\partial\Omega$, corners in the boundary $\partial\Omega$ may imply loss of regularity or demand extra conditions for the behaviour of the function near such corners. Usually regularity near the boundary is obtained by defining an extension operator on functions on $\overline{\Omega}$ to those that live on a neighbourhood of Ω. For domains with corners such an extension operator may be, if it exists, rather technical. The straight angles of a rectangle

$$\mathcal{R} = (0, a) \times (0, b) \tag{3.1}$$

with $a, b > 0$ however allow the following straightforward extension:

Lemma 3.1. *Let $\mathcal{R}$ be as in (3.1). For $u : \overline{\mathcal{R}} \to \mathbb{R}$ let us define*

$$Eu\,(x,y) = \begin{cases} u\,(x,y) & \text{for } (x,y) \in \mathcal{R}, \\ -u\,(-x,y) & \text{for } (-x,y) \in \mathcal{R}, \\ -u\,(x,-y) & \text{for } (x,-y) \in \mathcal{R}, \\ u\,(-x,-y) & \text{for } (-x,-y) \in \mathcal{R}, \\ 0 & \text{elsewhere.} \end{cases}$$

Hence Eu defines a function from $[-a, a] \times [-b, b]$ to $\mathbb{R}$. Set $\mathcal{R}_0 = (-a, a) \times (-b, b)$. Then

1. *Let $\gamma \in [0, 1]$. The operator $E : C^{1,\gamma}\left(\overline{\mathcal{R}}\right) \cap C_0\left(\overline{\mathcal{R}}\right) \to C^{1,\gamma}\left(\overline{\mathcal{R}_0}\right) \cap C_0\left(\overline{\mathcal{R}_0}\right)$ is continuous.*
2. *The operator $E : W^{2,2}(\mathcal{R}) \cap W^{1,2}_0(\mathcal{R}) \to W^{2,2}(\mathcal{R}_0) \cap W^{1,2}_0(\mathcal{R}_0)$ is continuous.*

Figure 3. The construction of this extension can be viewed as unfolding a bulging doubly folded piece of paper.

Proof. We will prove that the range of E is well defined and contained in the appropriate spaces. The continuity is then immediate. Let us consider

$$E_1 u\,(x,y) = \begin{cases} u\,(x,y) & \text{for } (x,y) \in \mathcal{R}, \\ -u\,(-x,y) & \text{for } (-x,y) \in \mathcal{R}, \\ 0 & \text{elsewhere,} \end{cases}$$

which defines a first antisymmetric reflection to $[-a, a] \times [0, b]$. With a similarly defined E_2 in the y-direction one finds $E = E_2 \circ E_1$. It is thus enough to give the proof for E_1.

For the first item it is sufficient to notice that due to $u(0,y)=0$ the function $E_1 u$ and its first derivatives are continuous over $\{0\}\times[0,b]$.

A short proof of the second item uses elliptic regularity. Set

$$f(x,y)=\begin{cases} -(\Delta u)(x,y) & \text{for } (x,y)\in\mathcal{R},\\ (\Delta u)(-x,y) & \text{for } (-x,y)\in\mathcal{R},\\ 0 & \text{elsewhere.}\end{cases}$$

Then $f\in L^2(\Omega)$. Let $\mathcal{R}_1=(-a,a)\times(0,b)$ and consider the following

$$\begin{cases} -\Delta\tilde{u}=f & \text{on } \mathcal{R}_1,\\ \tilde{u}=0 & \text{on } \partial\mathcal{R}_1,\end{cases} \tag{3.2}$$

Problem (3.2) has a unique weak solution $\tilde{u}\in W_0^{1,2}(\mathcal{R}_1)$ and since $\mathcal{R}_1$ is convex one even finds $\tilde{u}\in W^{2,2}(\mathcal{R}_1)$ (see [16]). Now define

$$\hat{u}(x,y)=-\tilde{u}(-x,y).$$

Then $\hat{u}\in W^{2,2}(\mathcal{R}_1)$ and it satisfies (3.2). Since strong solutions of (3.2) are unique (see [11]) we find $\tilde{u}\equiv\hat{u}$ and thus

$$\tilde{u}(0,y)=\hat{u}(0,y)=-\tilde{u}(0,y),\ \text{i.e. } \tilde{u}(0,y)=0.$$

Thus $\tilde{u}-u\in W^{2,2}(\mathcal{R})$ and since $-\Delta(\tilde{u}-u)=0$ in $\mathcal{R}$ and $\tilde{u}=u=0$ on $\partial\mathcal{R}$ we find by uniqueness that $\tilde{u}\equiv u$ on $\overline{\mathcal{R}}$, that is $\tilde{u}\equiv E_1 u\in W^{2,2}(\mathcal{R}_1)$. □

Remark 3.2. Let $G_{\mathcal{R}_1}$ denote the solution operator for (3.2). Then the last steps of the proof say

$$EG_{\mathcal{R}}=G_{\mathcal{R}_0}E. \tag{3.3}$$

We recall the following definitions

$$\begin{aligned} C^k(\overline{\mathcal{R}}) &= \{u\in C^k(\mathcal{R});\ \partial_\alpha u \text{ bounded, uniformly continuous in } \mathcal{R},\\ &\qquad \text{for all } \alpha\in\mathbb{N}\times\mathbb{N} \text{ with } |\alpha|\le k\},\\ C^\infty(\overline{\mathcal{R}}) &= \bigcap_{k=0}^{\infty} C^k(\overline{\mathcal{R}}),\\ C_0(\overline{\mathcal{R}}) &= \{u\in C(\overline{\mathcal{R}});\ u=0 \text{ on } \partial\mathcal{R}\}.\end{aligned}$$

Since $\mathcal{R}$ is bounded and has a Lipschitz boundary there exists a total extension operator for $\mathcal{R}$ (see [2, Theorem 5.24, p. 154]) and thus $C^\infty(\overline{\mathcal{R}})$ coincides with the space of functions in $C^\infty(\mathbb{R}^2)$, restricted to $\overline{\mathcal{R}}$.

The result that follows is closely related to [15, Theorem 1.6.2].

Corollary 3.3. $\overline{C^\infty(\overline{\mathcal{R}})\cap C_0(\overline{\mathcal{R}})}^{\|\cdot\|_{2,2}}=H_0(\mathcal{R})$.

Proof. Since $C^\infty(\overline{\mathcal{R}})\cap C_0(\overline{\mathcal{R}})\subset H_0(\mathcal{R})$ and $H_0(\mathcal{R})$ is closed the $\subset$-direction is immediate. Let us write $\mathcal{R}_{00}=(-2a,2a)\times(-2b,2b)$. For the other inclusion we may use Lemma 3.1 twice to define a extension operator

$$\tilde{E}:\mathbb{R}^{\mathcal{R}}\to\mathbb{R}^{\mathcal{R}_{00}}, \tag{3.4}$$

that is, from functions on $\mathcal{R}$ to functions on $\mathcal{R}_{00}$. First we extend in an odd way as for E from $[0,a]\times[0,b]$ to $[0,2a]\times[0,2b]$ and next again in an odd way, which could also be called a periodic extension, from $[0,2a]\times[0,2b]$ to $[-2a,2a]\times[-2b,2b]$. See Figure 4. By Lemma 3.1 $\tilde{E}$ is continuous as operator from $W^{2,2}(\mathcal{R})\cap W_0^{1,2}(\mathcal{R})$ to $W^{2,2}(\mathcal{R}_{00})\cap W_0^{1,2}(\mathcal{R}_{00})$.

Next we define a function $\chi\in C^\infty(\mathcal{R}_{00})$ with $0\le\chi\le 1$ which satisfies

$$\chi=\begin{cases} 1 & \text{on } \left(-\frac{4}{3}a,\frac{4}{3}a\right)\times\left(-\frac{4}{3}b,\frac{4}{3}b\right),\\ 0 & \text{on } \mathcal{R}_{00}\setminus\left(-\frac{5}{3}a,\frac{5}{3}a\right)\times\left(-\frac{5}{3}b,\frac{5}{3}b\right).\end{cases}$$

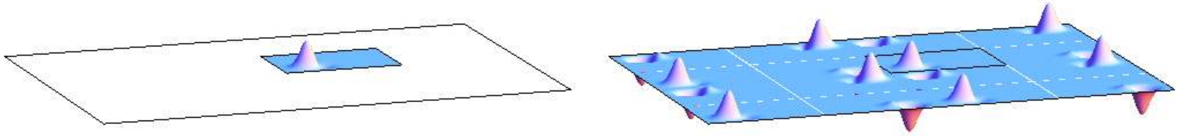

Figure 4. $\tilde{E}$ extends a function on $\mathcal{R}$ to a function on $\mathcal{R}_{00}$ by respectively 'unfolding' to east, north, west and south.

If $u \in H_0(\mathcal{R})$ then $\tilde{E}u \in H_0(\mathcal{R}_{00})$ and $\chi\tilde{E}u \in W_0^{2,2}(\mathcal{R}_{00})$. Using the standard mollifier with $z = (x,y) \in \mathbb{R}^2$

$$\varphi_1(z) := \begin{cases} ce^{-\frac{1}{1-|z|^2}}, & \text{for } |z| < 1, \\ 0, & \text{for } |z| \geq 1 \end{cases}$$

with $c^{-1} = \int_{\mathbb{R}^2} e^{-\frac{1}{1-|z|^2}}\, d\lambda$ and $\varphi_\varepsilon(z) = \varepsilon^{-2}\varphi_1(z/\varepsilon)$, we find for the convolution

$$\varphi_\varepsilon * \chi\tilde{E}u \in C_0^\infty(\mathcal{R}_{00}) \text{ for } \varepsilon < \operatorname{dist}\left(\operatorname{supp}\chi\tilde{E}u, \partial\mathcal{R}_{00}\right) = \min\left(\frac{a}{3}, \frac{b}{3}\right)$$

and

$$\left\|\varphi_\varepsilon * \chi\tilde{E}u - \tilde{E}u\right\|_{W^{2,2}(\mathcal{R}_{00})} \to 0 \text{ for } \varepsilon \downarrow 0.$$

It follows that

$$\left\|\left(\varphi_\varepsilon * \chi\tilde{E}u\right)_{|\mathcal{R}} - u\right\|_{W^{2,2}(\mathcal{R})} \to 0 \text{ for } \varepsilon \downarrow 0.$$

By the symmetry of $\tilde{E}$ and φ_ε and the fact that $\chi = 1$ near $\partial\mathcal{R}$ it follows that $\varphi_\varepsilon * \chi\tilde{E}u = 0$ on $\partial\mathcal{R}$ for ε small enough. Hence $\left(\varphi_\varepsilon * \chi\tilde{E}u\right)_{|\overline{\mathcal{R}}} \in C_0\left(\overline{\mathcal{R}}\right)$ for those small ε. □

3.1.2. The Dirichlet Laplace problem on a rectangle. We will show that a hinged rectangular plate solves the Navier Bilaplace problem. To that end we recall some results for the Dirichlet Laplace problem on a rectangle.

In the proof of Lemma 3.1 we have used properties of the solution of the Dirichlet-Laplace problem. Indeed, if $f \in L^2(\mathcal{R})$ then the solution of

$$\begin{cases} -\Delta u = f & \text{on } \mathcal{R}, \\ u = 0 & \text{on } \partial\mathcal{R}, \end{cases} \tag{3.5}$$

satisfies $u \in W^{2,2}(\mathcal{R}) \cap W_0^{1,2}(\mathcal{R})$. If $f \in C^\gamma(\mathcal{R})$ with $\gamma \in (0,1)$, then $\tilde{E}f \in L^\infty(\mathcal{R}_{00})$ and in general $\tilde{E}f \notin C^\gamma(\mathcal{R}_{00})$, such that for the solution we may conclude using regularity on $\mathcal{R}_{00}$ that $u \in W^{2,p}(\mathcal{R})$ for any $p \in (1,\infty)$ and through a Sobolev imbedding that $u \in C^{1,\theta}\left(\overline{\mathcal{R}}\right)$ for any $\theta \in (0,1)$. This is the optimal regularity if we refrain from putting additional restrictions on f.

3.1.3. Iterated Dirichlet Laplace and Navier Bilaplace. Concerning the Navier boundary conditions for the Bilaplace operator, i.e. the problem

$$\begin{cases} \Delta^2 u = f & \text{on } \mathcal{R}, \\ \Delta u = u = 0 & \text{on } \partial\mathcal{R}, \end{cases} \tag{3.6}$$

an iterated use of the regularity for (3.5) yields a function $u \in W^{2,2}(\mathcal{R})$ with $\Delta u \in W^{2,2}(\mathcal{R})$ satisfying

$$\begin{cases} -\Delta u = w & \text{in } \mathcal{R}, \\ u = 0 & \text{on } \partial\mathcal{R}, \end{cases} \qquad \begin{cases} -\Delta w = f & \text{in } \mathcal{R}, \\ w = 0 & \text{on } \partial\mathcal{R}. \end{cases} \tag{3.7}$$

But this does not give a priori the optimal result: For any bounded domain Ω and $f \in L^2(\Omega)$ one obtains a unique weak solution $\hat{u} \in H_0(\Omega)$ of (3.6) by minimizing the functional

$$J_1(u) = \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) d\lambda.$$

On smooth boundary parts one finds that this solution $\hat{u}$ satisfies $\Delta \hat{u} = 0$. However, this function is not necessarily the same as the system solution: $\Delta u \in W^{2,2}(\Omega)$ does not imply $u \in W^{4,2}(\Omega)$ in general (see [27]).

However, if $f = 0$ on $\partial\mathcal{R}$ and $f \in C^\gamma(\mathcal{R})$, then $\tilde{E}f \in C^\gamma(\mathcal{R}_{00})$. This implies that when we consider (3.6) as an iterated Dirichlet Laplacian a better regularity result is available for the second step.

Lemma 3.4. *If $f \in L^2(\mathcal{R})$ then the weak solution $(u, -\Delta u) \in W_0^{1,2}(\mathcal{R}) \times W_0^{1,2}(\mathcal{R})$ of (3.7) satisfies $u \in W^{4,2}(\mathcal{R})$.*

Proof. Assuming that $\tilde{E}$ and $\mathcal{R}_{00}$ are as in (3.4), we have that $\tilde{E}f \in L^2(\mathcal{R}_{00})$. Solving

$$\begin{cases} \Delta^2 \tilde{u} = \tilde{E}f & \text{on } \mathcal{R}_{00}, \\ \Delta \tilde{u} = \tilde{u} = 0 & \text{on } \partial\mathcal{R}_{00}, \end{cases} \tag{3.8}$$

one finds by standard regularity theory (see [1]) for the weak solution that $\tilde{u} \in W^{4,2}_{loc}(\Omega)$ for any domain Ω with $\overline{\Omega} \subset \mathcal{R}_{00}$. This implies that $\tilde{u}_{|\mathcal{R}} \in W^{4,2}(\mathcal{R})$. Since $\tilde{E}f$ is antisymmetric and the weak solution of (3.8) is unique, applying an argument similar to that in the proof of Lemma 3.1(2) one finds that $\tilde{u}$ satisfies

$$\tilde{u}(x,y) = -\tilde{u}(-x,y) = -\tilde{u}(x,-y) = \tilde{u}(-x,-y)$$

for $x \in \mathcal{R}_0 = (0,2a) \times (0,2b)$. Thus

$$\Delta\tilde{u}(x,y) = -\Delta\tilde{u}(-x,y) = -\Delta\tilde{u}(x,-y) = \Delta\tilde{u}(-x,-y)$$

which implies that $\tilde{u} = \Delta\tilde{u} = 0$ on $\partial\mathcal{R}$. So we have found that $\left(\tilde{u}_{|\mathcal{R}}, -\Delta\tilde{u}_{|\mathcal{R}}\right)$ is a solution to (3.6). Hence we may conclude that $u \equiv \tilde{u}_{|\mathcal{R}} \in W^{4,2}(\mathcal{R})$. □

3.1.4. A regularity result for the rectangular plate. Let us use the following notation

$$\mathcal{K}(u) := \int_\mathcal{R} \det(\nabla^2 u) d\lambda, \text{ for } u \in W^{2,2}(\mathcal{R}),$$

where $\nabla^2 u$ is the Hessian matrix of u and $\det(\nabla^2 u) = u_{xx}u_{yy} - u_{xy}^2$. Defining

$$J_1(u) := \int_\mathcal{R} \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) d\lambda$$

we obtain the following decomposition of the energy functional:

$$J_\sigma(u) = J_1(u) - (1-\sigma)\mathcal{K}(u). \tag{3.9}$$

In fact $\mathcal{K}(u)$ will turn out to be a boundary term and its behaviour is going to yield the corresponding natural boundary conditions for a Kirchhoff plate (see also [15, Lemma 2.2.2]). When the domain is smooth, one can apply the argumentation found in [28] or [31]. For our rectangular domain we use Corollary 3.3.

Lemma 3.5. *Let $u \in C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$. Then for all $v \in C^\infty(\overline{\mathcal{R}})$*

$$\mathcal{K}'(u;v) = -\int_{\partial\mathcal{R}} u_{\tau n} v_\tau d\tau = 2\left[u_{xy}v\right]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} + \int_{\partial\mathcal{R}} u_{\tau\tau n} v d\tau.$$

Here the entries above have a plus sign, those below a minus sign.

Proof. Standard use of Fubini and integrating by parts gives

$$\begin{aligned}\int_{\mathcal{R}} u_{xy}v_{xy}\,d\lambda &= \int_0^a \left[u_{xy}(x,y)v_x(x,y)\right]_{y=0}^b dx - \int_0^a\int_0^b u_{xyy}(x,y)v_x(x,y)\,dy\,dx\\ &= \int_0^b \left[u_{xy}(x,y)v_y(x,y)\right]_{x=0}^a dy - \int_0^b\int_0^a u_{xxy}(x,y)v_y(x,y)\,dx\,dy.\end{aligned}$$

Since $\mathcal{R}$ has boundary parts parallel to the axes,

$$u(x,0) = u(x,b) = u(0,y) = u(a,y) = 0$$

implies that

$$u_{xx}(x,0) = u_{xx}(x,b) = u_{yy}(0,y) = u_{yy}(a,y) = 0,$$

and one obtains

$$\begin{aligned}\int_{\mathcal{R}} u_{xx}v_{yy}\,d\lambda &= \int_0^a \left[u_{xx}(x,y)v_y(x,y)\right]_{y=0}^b dx - \int_0^a\int_0^b u_{xxy}(x,y)v_y(x,y)\,dy\,dx\\ &= -\int_0^a\int_0^b u_{xxy}(x,y)v_y(x,y)\,dy\,dx\end{aligned}$$

and similarly

$$\begin{aligned}\int_{\mathcal{R}} u_{yy}v_{xx}\,d\lambda &= \int_0^b \left[u_{yy}(x,y)v_x(x,y)\right]_{x=0}^a dy - \int_0^b\int_0^a u_{xyy}(x,y)v_x(x,y)\,dx\,dy\\ &= -\int_0^b\int_0^a u_{xyy}(x,y)v_x(x,y)\,dx\,dy.\end{aligned}$$

Thus, a direct calculation yields that

$$\begin{aligned}\mathcal{K}'(u;v) &= \int_{\mathcal{R}} \left(u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}\right)\,d\lambda\\ &= -\int_0^a \left[u_{xy}(x,y)v_x(x,y)\right]_{y=0}^b dx - \int_0^b \left[u_{xy}(x,y)v_y(x,y)\right]_{x=0}^a dy\\ &= -\int_{\partial\mathcal{R}} u_{\tau n}v_\tau d\tau.\end{aligned}$$

Moreover, by

$$\begin{aligned}-\int_0^a u_{xy}(x,b)v_x(x,b)dx &= -\left[u_{xy}v\right]_{(0,b)}^{(a,b)} + \int_0^a u_{xxy}(x,b)v(x,b)dx,\\ \int_0^a u_{xy}(x,0)v_x(x,0)dx &= \left[u_{xy}v\right]_{(0,0)}^{(a,0)} - \int_0^a u_{xxy}(x,b)v(x,b)dx,\\ -\int_0^b u_{xy}(a,y)v_y(a,y)dy &= -\left[u_{xy}v\right]_{(a,0)}^{(a,b)} + \int_0^b u_{xyy}(a,y)v(a,y)dy,\\ \int_0^b u_{xy}(0,y)v_y(0,y)dy &= \left[u_{xy}v\right]_{(0,0)}^{(0,b)} - \int_0^b u_{xyy}(0,y)v(0,y)dy,\end{aligned}$$

we find

$$\mathcal{K}'(u;v) = 2\left[u_{xy}v\right]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} + \int_{\partial\mathcal{R}} u_{\tau\tau n}v d\tau$$

and the lemma is proved. □

Corollary 3.6. $\mathcal{K}(u) = 0$ *for all* $u \in H_0(\mathcal{R})$.

Proof. Let $u \in H_0(\mathcal{R})$. Using Corollary 3.3 we can find a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$ such that $u_k \to u$ in $H_0(\overline{\mathcal{R}})$ for $k \to \infty$. For $u_k \in C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$ one finds

$$\mathcal{K}(u_k) = \frac{1}{2}\mathcal{K}'\left(u_k; u_k\right)$$

and by Lemma 3.5

$$\mathcal{K}'\left(u_k; u_k\right) = -\int_{\partial\mathcal{R}} \left(u_k\right)_{\tau n}\left(u_k\right)_\tau \, d\tau = 0.$$

Since $(u, v) \mapsto \mathcal{K}'(u; v)$ is continuous on $H_0(\mathcal{R}) \times H_0(\mathcal{R})$, one has

$$\mathcal{K}(u) = \frac{1}{2}\mathcal{K}'\left(u; u\right) = \frac{1}{2}\lim_{k\to\infty} \mathcal{K}'\left(u_k; u_k\right) = 0.$$

□

Remark 3.7. Thus, in the case of a rectangular plate with fixed boundary the total energy functional becomes

$$J_\sigma(u) = J_1(u) = \int_{\mathcal{R}} \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) d\lambda.$$

Corollary 3.8. *If $\tilde{u}$ is a minimizer of J_σ in $H_0(\mathcal{R})$ with $f \in L^2(\mathcal{R})$ then $u \in W^{4,2}(\mathcal{R})$.*

Proof. This is a direct result of Remark 3.7 and Lemma 3.4. □

3.2. A plate with corners of arbitrary opening angle

Here we consider the general case where the corners of the plate have an arbitrary opening angle $\omega \in (0, 2\pi)$, measured from the inside. Note that one does not expect that the solution will have the regularity that a rectangular plate exhibits, unless some orthogonality conditions are fulfilled. Following the theory developed by Kondrat'ev and Williams [18, 32], the solutions will have an expansion near the corner consisting of a regular part and a singular one. The coefficients of these "singular eigenfunctions" depend on the domain and f; they are going to be zero only when f is orthogonal to a set of "adjoint eigenfunctions". Before we move on we will need the following definitions:

Definition 3.9 (Weighted Sobolev spaces on cones.). Let $m \in \mathbb{N}^2$ and (r, θ) be a polar coordinate system centered at $x_0 \in \mathbb{R}^n$. For $\omega \in (0, 2\pi)$ let

$$\mathcal{K}_{x_0,\omega} := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ r > 0 \text{ and } 0 < \theta < \omega\}.$$

Then we define

$$\|u\|_{V_\alpha^k(\mathcal{K}_{x_0,\omega})} := \left(\sum_{|m|\le k} \int_{\mathcal{K}_{x_0,\omega}} r^{2(\alpha-k-|m|)} |\partial_m u|^2 \, dx\right)^{\frac{1}{2}} \tag{3.10}$$

and the following weighted space on $\mathcal{K}_{x_0}$:

$$V_\alpha^k(\mathcal{K}_{x_0,\omega}) := \overline{C_0^\infty(\overline{\mathcal{K}}_{x_0,\omega} \setminus \{x_0\})}^{\|\cdot\|_{V_\alpha^k}}. \tag{3.11}$$

Remark 3.10. For $\mathcal{S} \subset \overline{\Omega}$ we take

$$C_0^\infty\left(\overline{\Omega} \setminus \mathcal{S}\right) := \left\{u \in C(\overline{\Omega});\ \exists v \in C^\infty(\mathbb{R}^n),\ \text{such that } v_{|\overline{\Omega}} = u \text{ and } \operatorname{supp} u \subset \overline{\Omega} \setminus \mathcal{S}\right\}.$$

Next we are going to define a weighted Sobolev space over a bounded domain $\Omega \subset \mathbb{R}^2$. For that, we are assuming that $\partial\Omega$ is C^2-smooth with the exception of a finite number of points, where it locally coincides with a cone, that is, if $x \in \partial\Omega$ is a singular point, there exists $\varepsilon > 0$ and $\omega \in (0, 2\pi)$ such that

$$\Omega \cap B_\varepsilon(x) = \mathcal{K}_{x,\omega} \cap B_\varepsilon(x), \tag{3.12}$$

where $\mathcal{K}_{x,\omega}$ is as in the above Definition.

Definition 3.11 (Weighted Sobolev spaces on domains.). Let $\mathcal{S}$ be the set of conical points of $\partial\Omega$. Let $\varepsilon > 0$ be such that (3.12) holds for every $x \in \mathcal{S}$ and for these $x \in \mathcal{S}$ let $\zeta_x \in C^\infty(\overline{\Omega})$ be a cut-off function with

$$\zeta_x = \begin{cases} 1 & \text{in } \overline{\Omega} \cap B_{\varepsilon/2}(x), \\ 0 & \text{in } \overline{\Omega} \setminus B_\varepsilon(x). \end{cases}$$

Moreover, set $\zeta = 1 - \sum_{x\in\mathcal{S}} \zeta_x$ and

$$\widehat{\zeta_x u} = \begin{cases} \zeta_x u & \text{in } \overline{\Omega} \cap B_\varepsilon(x) \subset \overline{\mathcal{K}}_x, \\ 0 & \text{in } \overline{\mathcal{K}}_x \setminus B_\varepsilon(x). \end{cases}$$

Then we define

$$V_\alpha^k(\Omega) := \overline{C_0^\infty(\overline{\Omega} \setminus \mathcal{S})}^{\|\cdot\|_{V_\alpha^k(\Omega)}}, \tag{3.13}$$

where

$$\|u\|_{V_\alpha^k(\Omega)} := \|\zeta u\|_{W^{k,2}(\Omega)} + \sum_{x\in\mathcal{S}} \left\|\widehat{\zeta_x u}\right\|_{V_\alpha^k(\mathcal{K}_x)}. \tag{3.14}$$

Definition 3.12. We define $V_\alpha^{k-1/2}(\partial\Omega)$ to be the space of traces on $\partial\Omega$ of functions in $V_\alpha^k(\Omega)$ with the norm

$$\|u\|_{V_\alpha^{k-1/2}(\partial\Omega)} := \inf_{v\in V_\alpha^k(\Omega)} \|v\|_{V_\alpha^k(\Omega)}.$$

For boundary value problems on polygonal domains we refer to the monographs [14, 15]. For more general elliptic operators with a wider class of singularities see [19, 20, 25, 26]. The biharmonic problem for a so-called freely supported plate is studied in [24].

3.2.1. The boundary value problem. We start with two useful observations that are going to enable us to compare minimizers and solutions for the hinged plate boundary value problem in weighted spaces. The goal is to show that a minimizer of J_σ in $H_0(\Omega)$ satisfies a boundary value problem away from the cornerpoints in a weighted Sobolev space and vice versa. Thus we are going to be able to prove existence of a strong solution in a weighted space.

Lemma 3.13. *Let u_σ be a minimizer of J_σ in $H_+(\Omega)$. Then*

$$J'(u_\sigma; u_\sigma) = 0 \quad \text{and } J'(u_\sigma; v) \geq 0 \text{ for all } v \in H_+(\Omega). \tag{3.15}$$

Proof. Since u_σ is a minimizer, it satisfies the variational inequality

$$J'(u_\sigma; v - u_\sigma) \geq 0 \text{ for all } v \in H_+(\Omega).$$

Taking $v = 2u_\sigma \in H_+(\Omega)$ and $v \equiv 0 \in H_+(\Omega)$ completes the proof. □

In the next lemma we consider for $\omega \in (0, 2\pi)$ the bounded cone

$$\Omega_\omega := \left\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ 0 < r < 1 \text{ and } 0 < \theta < \omega\right\}.$$

Lemma 3.14. *Let $\partial^{\mathbf{i}}$, for $\mathbf{i} = \mathbf{1}, ..., \mathbf{4}$ a multiindex with $i = |\mathbf{i}|$, denote any partial derivative of order i. Then the bilinear forms*

(i) $b_1(u,v) := \displaystyle\int_{\partial\Omega_\omega} \partial^{\mathbf{2}} u\, \partial^{\mathbf{1}} v\, ds$,

(ii) $b_2(u,v) := \displaystyle\int_{\Omega_\omega} \partial^{\mathbf{2}} u\, \partial^{\mathbf{2}} v\, dxdy$ *and*

(iii) $b_3(u,v) := \displaystyle\int_{\Omega_\omega} \partial^{\mathbf{4}} u\, v\, dxdy$

are continuous in $V_2^4(\Omega_\omega) \times H_0(\Omega_\omega)$.

Proof. *(i)* Since $u \in V_2^4(\Omega_\omega)$, one gets that $\partial^{\mathbf{2}} u \in V_4^2(\Omega_\omega)$, that is

$$\partial^{\mathbf{2}} u_{|\partial\Omega_\omega} \in V_4^{3/2}(\partial\Omega_\omega) \subset V_{1/2}^0(\partial\Omega_\omega),$$

with the last imbedding taken from [19, Lemma 6.1.2]. On the other hand, we have that

$$H_0(\Omega_\omega) \subset V_0^2(\Omega_\omega) \cap V_{-1}^1(\Omega_\omega)$$

(see [27, Lemma 3.4]) and thus $\partial^{\mathbf{1}} v_{|\partial\Omega_\omega} \in V_0^{1/2}(\partial\Omega_\omega) \subset V_{-1/2}^0(\partial\Omega_\omega)$. Then, by Cauchy-Schwarz, one gets

$$\begin{aligned} |b_1(u,v)| &= \int_{\partial\Omega_\omega} (r^{\frac{1}{2}} \partial_{\mathbf{2}} u)\ (r^{-\frac{1}{2}} \partial_{\mathbf{1}} v)\ ds \\ &\leq \left(\int_{\partial\Omega_\omega} r\ (\partial^{\mathbf{2}} u)^2\ ds \right)^{\frac{1}{2}} \left(\int_{\partial\Omega_\omega} r^{-1} (\partial^{\mathbf{1}} v)^2\ ds \right)^{\frac{1}{2}} \\ &\leq c \left\| \partial^{\mathbf{2}} u \right\|_{V_2^2(\Omega_\omega)} \|v\|_{V_0^2(\Omega_\omega)}. \end{aligned}$$

(ii) The result is immediate since $\partial^{\mathbf{2}} u \in V_2^2(\Omega_\omega) \subset V_0^0(\Omega_\omega) = L^2(\Omega_\omega)$.

(iii) One has the following estimate

$$\begin{aligned} |b_3(u,v)| &= \int_{\Omega_\omega} (r^2 \partial^{\mathbf{4}} u)(r^{-2} v)\ dxdy \\ &\leq \underbrace{\left(\int_{\Omega_\omega} r^4 |\partial^{\mathbf{4}} u|^2\ dxdy \right)^{\frac{1}{2}}}_{A} \underbrace{\left(\int_{\Omega_\omega} r^{-4} |v|^2\ dxdy \right)^{\frac{1}{2}}}_{B}. \end{aligned} \tag{3.16}$$

Moreover, since $u \in V_2^4(\Omega_\omega)$, we get that

$$A^2 \leq \sum_{|m| \leq 4} \int_{\Omega_\omega} r^{2|m|-4} |\partial^m u|^2\ dxdy = \|u\|^2_{V_2^4(\Omega_\omega)} \tag{3.17}$$

and since $v \in H_0(\Omega_\omega) \subset V_0^2(\Omega_\omega)$ (see [27, Lemma 3.4]) it holds with Corollary 5.4 that

$$B^2 \leq \sum_{|m| \leq 2} \int_{\Omega_\omega} r^{2|m|-4} |\partial^m v|^2\ dxdy = \|v\|^2_{V_0^2(\Omega_\omega)} \leq c \|v\|^2_{2,2}. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18) completes the proof. □

The above Lemma enables us to integrate by parts functions which belong to a weighted space.

Corollary 3.15. *Let $\Omega \subset \mathbb{R}^2$ be bounded, piecewise smooth with corner boundary singularities, $u \in V_2^4(\Omega)$ and $v \in H_0(\Omega)$. Then the following Green's identity holds:*

$$\int_\Omega \Delta u \Delta v\ dxdy = \int_\Omega \Delta^2 u\ v\ dxdy + \int_{\partial\Omega} \Delta u\ \partial_n v\ ds. \tag{3.19}$$

Proof. Let $\mathcal{S}$ be the set of cornerpoints of $\partial\Omega$ and $\{u_k\}_{k\in\mathbb{N}} \subset C_0^\infty\left(\overline{\Omega} \setminus \mathcal{S}\right)$, such that

$$\lim_{k\to\infty} \|u_k - u\|_{V_2^4(\Omega)} = 0.$$

Then, (3.19) holds true for $u \equiv u_k$ and Lemma 3.14 allows us to take the limit as $k \to \infty$ to complete the proof. □

As in previous sections, we define

$$\mathcal{K}(u) := \int_\Omega \det(\nabla^2 u)\ dxdy$$

for $u \in W^{2,2}(\Omega)$, where $\nabla^2 u$ denotes the Hessian matrix of u.

Corollary 3.16. *Let $\Omega \subset \mathbb{R}^2$ be bounded and piecewise smooth with corner boundary singularities and let $\mathcal{S}$ be the set containing the corners of $\partial\Omega$. Then the following holds true:*

(i) *For all $u \in H_0(\Omega)$ and $v \in W^{3,2}(\Omega)$ it holds that*

$$\mathcal{K}'(u;v) = \int_{\partial\Omega} (\kappa(s)\, \partial_n u\, \partial_n v + \partial_n u\, \partial_{\tau\tau} v)\ ds. \tag{3.20}$$

(ii) *For all $u \in H_0(\Omega)$ we have*

$$\mathcal{K}(u) = \frac{1}{2} \int_{\partial\Omega} \kappa(s)(\partial_n u)^2\ ds. \tag{3.21}$$

(iii) *$(u,v) \longmapsto \mathcal{K}'(u;v)$ is a continuous bilinear form on $V_2^4(\Omega) \times H_0(\Omega)$.*

Proof. For smooth u, v one obtains *(i)* and *(ii)* by a direct computation; see for example [10, 28]. The proof follows then by a density argument and by using [15, Theorem 1.6.2].

Concerning *(iii)*, one gets

$$\mathcal{K}'(u;v) = \int_{\partial\Omega} (\kappa(s)\, \partial_n u\, \partial_n v + \partial_n v\, \partial_{\tau\tau} u)\ ds$$

for $u \in C_0^\infty\big(\overline{\Omega}\backslash\mathcal{S}\big)$ and $v \in H_0(\Omega)$. The result follows then by density with the help of Lemma 3.14. $\square$

Now we are able to give the relationship between the minimization and the boundary value problem for a hinged plate.

Corollary 3.17. *Let $f \in L^2(\Omega)$ and $-1 < \sigma < 1$.*

- *The solution of the hinged plate, i.e. the unique minimizer of J_σ in $H_0(\Omega)$ lies in $W^{4,2}(\Omega_1)$, for any open Ω_1 with $\bar\Omega_1 \subset \bar\Omega \setminus \mathcal{S}$, and satisfies*

$$\begin{cases} \Delta^2 u = f & a.e.\ in\ \Omega, \\ u = 0 & on\ \partial\Omega, \\ \Delta u - (1-\sigma)\kappa\partial_n u = 0 & on\ \partial\Omega \setminus \mathcal{S}, \end{cases} \tag{3.22}$$

  where $\mathcal{S}$ is the set of corners of $\partial\Omega$.
- *If $u \in V_2^4(\Omega)$ satisfies* (3.22) *then it is a minimizer of J_σ in $H_0(\Omega)$.*

Proof. Let $\varepsilon > 0$ and define

$$B(\mathcal{S}) := \bigcup_{x\in\mathcal{S}} B_\varepsilon(x).$$

Since problem (3.22) is regular on $\partial\Omega \setminus \mathcal{S}$ and the boundary of the domain is smooth away from the corners, one can show, using standard regularity techiniques, that the minimizer $u \in H_0(\Omega)$ of J_σ lies in $W^{4,2}\big(\Omega \setminus \overline{B(\mathcal{S})}\big)$. Then one has that $J'(u;\varphi) = 0$ for all $\varphi \in C^\infty(\overline{\Omega}) \cap C_0(\overline{\Omega})$ compactly supported with $\operatorname{supp}\varphi \subset \overline{\Omega} \setminus \overline{B(\mathcal{S})}$ and an integration by parts is allowed:

$$\begin{aligned} 0 &= \int_{\partial\Omega\setminus\overline{B(\mathcal{S})}} (\Delta u \Delta\varphi - f\varphi)\, dxdy - (1-\sigma)\mathcal{K}'(u;\varphi) \\ &= \int_{\Omega\setminus\overline{B(\mathcal{S})}} \big(\Delta^2 u - f\big)\, \varphi\ dx + \int_{\partial\Omega\setminus\overline{B(\mathcal{S})}} \Big(\Delta u - (1-\sigma)\,\kappa\partial_n u\Big)\, \partial_n\varphi\ ds \\ &\quad - \int_{\partial\Omega\setminus\overline{B(\mathcal{S})}} \Big((1-\sigma)\,\partial_{\tau\tau n} u + \partial_n \Delta u\Big)\, \varphi\ ds. \end{aligned} \tag{3.23}$$

Thus, one obtains the differential equation in $\Omega \setminus \overline{B(\mathcal{S})}$ and the natural boundary condition on $\partial\Omega \setminus \overline{B(\mathcal{S})}$. Letting $\varepsilon \to 0$ we get that

$$\Delta^2 u = f \text{ in } \Omega \ \text{ and } \ \Delta u - (1-\sigma)\kappa\partial_n u = 0 \text{ on } \partial\Omega \setminus \mathcal{S}.$$

On the other hand, (3.20) and Green's identity (3.19) imply that if for the solution u to (3.22) holds that $u \in V_2^4(\Omega)$, then u will satisfy the weak Euler-Lagrange equation $J_\sigma(u;v) = 0$ for all $v \in H_0(\Omega)$. $\square$

3.2.2. Kondrat'ev's expansion near a corner. Let $\omega \in (0, 2\pi)$ and define

$$\mathcal{K}_\omega := \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ r > 0 \text{ and } 0 < \theta < \omega \right\}, \tag{3.24}$$

an infinite circular sector of $\mathbb{R}^2$, centered at the origin with an opening angle ω. Consider the following problem

$$\begin{cases} \Delta^2 u = f & \text{in } \mathcal{K}_\omega, \\ u = \Delta u = 0 & \text{on } \partial\mathcal{K}_\omega. \end{cases} \tag{3.25}$$

According to [18], for solving (3.25) one needs to find the nonzero solutions for the following two-point boundary value problem

$$\begin{cases} v''''^{2})v''^{2}(\lambda - 2)^2 v(\theta) = 0, & \text{when } \theta \in (0, \omega), \\ \quad v(0) = v''(0) = 0, \\ \quad v(\omega) = v''(\omega) = 0. \end{cases} \tag{3.26}$$

We assume that Ω is smooth with the exception of N corners with interior opening angles $\omega_i \in (0, 2\pi)$ for $i = 1, .., N$. Using the results of the previous section we are able to prove a regularity assertion for a hinged plate.

Proposition 3.18. *Let $f \in L^2(\Omega)$ and assume for all $i = 1, .., N$ that $\omega_i \in (0, 2\pi)$. Then the weak solution u for* (3.22)*, i.e. the minimizer of J_σ in $H_0(\Omega)$, belongs to $V_2^4(\Omega)$.*

Proof. We first show the existence of a solution for (3.22) in $V_2^4(\Omega)$, when f belongs in a larger space than $L^2(\Omega)$. Assume, for the time being, that $f \in V_2^0(\Omega)$ and define the operator

$$L : V_2^4(\Omega) \to V_2^0(\Omega) \ \text{ with } \ Lu := \Delta^2 u \ \text{ and}$$
$$\mathcal{D}(L) := \{u \in V_2^4(\Omega);\ u = 0 \text{ on } \partial\Omega \ \text{ and } \ \Delta u - (1-\sigma)\kappa\partial_n u = 0 \text{ on } \partial\Omega \setminus \mathcal{S}\}.$$

Note that functions in $V_2^4(\Omega)$ lie in $W^{2,2}(\Omega)$ and are hence continuous since $\Omega \subset \mathbb{R}^2$; so u is defined pointwise. Away from corners the function u is C^2 up to the boundary and thus the second boundary condition is well defined. So $\mathcal{D}(L)$ and hence the operator L is well defined. Now we apply [19, Theorem 6.3.3] to find that L is Fredholm when $\lambda \neq 1$, where λ is any eigenvalue of problem (3.26). We can directly solve (3.26) by assuming exponential type solutions, to obtain for each j a pair of eigenvalues λ_j, μ_j corresponding to the same type of eigenfunctions Φ_j:

$$\lambda_j = \frac{j\pi}{\omega} \text{ and } \mu_j = \frac{j\pi}{\omega} + 2 \text{ with } \Phi_j = \sin\left(\frac{j\pi}{\omega}\theta\right).$$

Thus, L is Fredholm when $\omega_i \neq 0, \pi, 2\pi$ and its range coincides with the set of all functions $f \in V_2^0(\Omega)$ such that

$$\int_\Omega f v \, dxdy = 0 \ \text{ for all } v \in \ker L^\dagger,$$

where the operator $L^\dagger : V_2^4(\Omega) \to V_2^0(\Omega)$ is defined similarly to L but by considering the formally adjoint problem to (3.22): Let $u, v \in C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$ and calculate

$$\begin{aligned} &\int_\Omega \left(\Delta^2 u\right) v \, dxdy + \int_{\partial\Omega} \left(\Delta u - (1-\sigma)\kappa\partial_n u\right) \partial_n v \, ds \\ = &\int_\Omega \Delta u \Delta v \, dxdy - (1-\sigma) \int_{\partial\Omega} \kappa \partial_n u \, \partial_n v \, ds \\ = &\int_\Omega \left(\Delta^2 v\right) u \, dxdy + \int_{\partial\Omega} \left(\Delta v - (1-\sigma)\kappa\partial_n v\right) \partial_n u \, ds. \end{aligned}$$

Thus, in view of [19, Section 6.2.3], we obtain $L = L^\dagger$, that is, the problem (3.22) is formally self-adjoint. To complete the proof of this step we need to show that $\ker L = \{0\}$. Let $u \in V_2^4(\Omega)$ be such that

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa\partial_n u = 0 & \text{on } \partial\Omega \setminus \mathcal{S}. \end{cases} \tag{3.27}$$

Since $V_2^4(\Omega) \subset V_0^2(\Omega) \subset W^{2,2}(\Omega)$, Corollary 3.17 implies that u is the unique minimizer of J_σ in $H_0(\Omega)$ with $f \equiv 0$, that is $u \equiv 0$. □

4. The comparison argument

4.1. A rectangular plate

Finally we can state our main result. The right angles of our rectangle will allow us to deploy an argument based on Serrin's corner point Lemma ([29]).

Theorem 4.1. *Let $f \in L^2(\mathcal{R})$ with $0 \not\equiv f \le 0$. Then, the minimizer $\tilde{u}$ of J_1 in $H_0(\mathcal{R})$ cannot be a minimizer of J_σ in $H_+(\mathcal{R})$.*

Proof. We proceed by contradiction and assume that $\tilde{u} \in H_0(\mathcal{R})$ minimizes J_σ also in $H_+(\mathcal{R})$. By Corollary 3.8 we find that $\tilde{u} \in W^{4,2}(\mathcal{R})$. Consequently, Sobolev's embedding Theorem implies that $\tilde{u} \in C^{2,\theta}(\overline{\mathcal{R}})$ for $0 < \theta < 1$ and that the traces of 3rd order derivatives of $\tilde{u}$ are well defined in $L^2(\partial\mathcal{R})$.

Letting $v \in C^\infty(\overline{\mathcal{R}})$ and integrating by parts the corresponding variational inequality we find

$$\begin{aligned} J'_\sigma(\tilde{u}; v - \tilde{u}) &= J'_\sigma(\tilde{u}; v) - J'_\sigma(\tilde{u}; \tilde{u}) = J'_\sigma(\tilde{u}; v) \\ &= J'_1(\tilde{u}; v) - (1-\sigma)\mathcal{K}'(\tilde{u}; v) \\ &= \int_{\mathcal{R}} (\Delta\tilde{u}\Delta v - f v)\, d\lambda - (1-\sigma)\mathcal{K}'(\tilde{u}; v). \end{aligned} \tag{4.1}$$

Moreover, since $\tilde{u}$ also minimizes J_σ in $H_0(\mathcal{R})$, Remark 3.7 yields that $\Delta\tilde{u} = 0$ on $\partial\mathcal{R}$. Hence we have

$$\begin{aligned} \int_{\mathcal{R}} \Delta\tilde{u}\Delta v\, d\lambda &= \int_0^a [\Delta\tilde{u}\ v_y - \Delta\tilde{u}_y\ v]_{y=0}^b\, dx + \int_0^b [\Delta\tilde{u}\ v_x - \Delta\tilde{u}_x\ v]_{x=0}^a\, dy \\ &\quad + \int_{\mathcal{R}} \Delta^2\tilde{u}\ v\ d\lambda \\ &= \int_{\mathcal{R}} \Delta^2\tilde{u}\ v\ d\lambda - \int_0^a [\Delta\tilde{u}_y\ v]_{y=0}^b\, dx - \int_0^b [\Delta\tilde{u}_x\ v]_{x=0}^a\, dy \\ &= \int_{\mathcal{R}} \Delta^2\tilde{u}\ v\ d\lambda - \int_{\partial\mathcal{R}} \partial_n(\Delta\tilde{u})\, v\, d\tau. \end{aligned}$$

Using Lemma 3.5, Corollary 3.3 and the density of smooth functions into $L^2(\partial\mathcal{R})$ we find

$$\begin{aligned} J'_\sigma(\tilde{u}; v - \tilde{u}) &= \int_{\mathcal{R}} \left(\Delta^2\tilde{u} - f\right) v\, d\lambda - \int_{\partial\mathcal{R}} \partial_n\left(\Delta\tilde{u} + (1-\sigma)\,\tilde{u}_{\tau\tau}\right) v\, d\tau \\ &\quad -2\,(1-\sigma)\,[\tilde{u}_{xy}v]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)}. \end{aligned}$$

Since we assumed that $\tilde{u}$ is a minimizer in $H_+(\mathcal{R})$ one has $J'_\sigma(\tilde{u}; v - \tilde{u}) \ge 0$ for all $v \ge 0$ on $\partial\mathcal{R}$ and thus $\Delta^2\tilde{u} = f$ and

$$\int_{\partial\mathcal{R}} \partial_n\left(\Delta\tilde{u} + (1-\sigma)\,\tilde{u}_{\tau\tau}\right) v\, d\tau + 2\,(1-\sigma)\,[\tilde{u}_{xy}v]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} \le 0. \tag{4.2}$$

Since $\tilde{u} = 0$ on $\partial\mathcal{R}$ and $f \le 0$ is nontrivial, then $\tilde{u}_{xy}$ will have a sign at these corners. We claim that $\tilde{u}_{xy}(0,0) < 0$. Since $\tilde{u} \in W^{4,2}(\mathcal{R})$ the function $\tilde{u}$ solves

$$\begin{cases} -\Delta\tilde{u} = w & \text{in } \mathcal{R}, \\ \tilde{u} = 0 & \text{on } \partial\mathcal{R}, \end{cases}$$

where w solves

$$\begin{cases} -\Delta w = f & \text{in } \mathcal{R}, \\ w = 0 & \text{on } \partial\mathcal{R}. \end{cases}$$

By the maximum principle it follows for $f \le 0$ and nontrivial, that $w < 0$ in $\mathcal{R}$. A application of the maximum principle to $\tilde{u}$ implies $\tilde{u} < 0$ and by Hopf's boundary point Lemma even that, on a boundary point which is not a corner, that $\tilde{u}_n > 0$. At corners, $\tilde{u} = 0$ on $\partial\mathcal{R}$ and the C^2 smoothness imply

$\nabla\tilde{u}=0$. Hence we may use Serrin's corner point lemma (see [29]) which implies that $\tilde{u}_{\gamma\gamma}(0,0)<0$ for all directions γ, entering $\mathcal{R}$ non-tangentially. Taking $\gamma=\left(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}\right)$ we have

$$\tilde{u}_{\gamma\gamma}(0,0)=\tfrac{1}{2}\tilde{u}_{xx}(0,0)+\tilde{u}_{xy}(0,0)+\tfrac{1}{2}\tilde{u}_{yy}(0,0)=\tilde{u}_{xy}(0,0).$$

So we find $\tilde{u}_{xy}(0,0)<0$. Let $\varepsilon>0$ and consider the test function $v_\varepsilon\left(x,y\right)=e^{-\left(x^2+y^2\right)/\varepsilon}$. We then find that

$$\int_{\partial\mathcal{R}}\partial_n\left(\Delta\tilde{u}+\left(1-\sigma\right)\tilde{u}_{\tau\tau}\right)v_\varepsilon\,d\tau=\mathcal{O}\left(\varepsilon\right)\ \text{ for }\varepsilon\downarrow 0$$

and

$$2\left(1-\sigma\right)\left[\tilde{u}_{xy}v_\varepsilon\right]_{(a,b)}^{(a,0)\,\&\,(0,b)}=\mathcal{O}\left(e^{-\min(a,b)^2/\varepsilon}\right)\leq\mathcal{O}\left(\varepsilon\right)\ \text{ for }\varepsilon\downarrow 0.$$

However

$$2\left(1-\sigma\right)\left[\tilde{u}_{xy}v_\varepsilon\right]_{(0,0)}=-2\left(1-\sigma\right)\tilde{u}_{xy}\left(0,0\right)>0,$$

and we find

$$\begin{aligned}\int_{\partial\mathcal{R}}\partial_n\left(\Delta\tilde{u}+\left(1-\sigma\right)\tilde{u}_{\tau\tau}\right)v_\varepsilon\,d\tau+2\left(1-\sigma\right)\left[\tilde{u}_{xy}v_\varepsilon\right]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)}&=\\=\mathcal{O}\left(\varepsilon\right)-2\left(1-\sigma\right)\tilde{u}_{xy}\left(0,0\right)>0,&\end{aligned}$$

for ε sufficiently small, a contradiction to (4.2). $\square$

4.2. Comparison using Kondrat'ev's "singular eigenfunctions"

We need to have a more accurate picture of the behaviour of the solution to the hinged plate problem in the neighbourhood of a corner. To that end, we compute the asymptotic expansion given by Kondrat'ev's theory, developed in the previous section. From now on and in order to avoid technicalities, we will assume that the function f is smooth.

Corollary 4.2. *Assume that $u\in V_2^4(\Omega)$ solves (3.22) for $f\in C_0^\infty\left(\overline{\Omega}\setminus\mathcal{S}\right)$. Then, for each corner of opening angle $\omega_i\in(0,2\pi)\setminus\{\pi\}$, there exist constants m_{ω_i},c_j,d_j, such that, in a neighbourhood of this corner, u has the following expansion:*

$$u=\sum_{0<\frac{j\pi}{\omega}<m+3}c_jr^{\frac{j\pi}{\omega}}\sin\left(\frac{j\pi}{\omega}\theta\right)+\sum_{0<2+\frac{j\pi}{\omega}<m+3}d_jr^{2+\frac{j\pi}{\omega}}\sin\left(\frac{|j|\pi}{\omega}\theta\right)+w \tag{4.3}$$

with $m>m_{\omega_i}$ and $w\in V_0^{m+4}(\Omega)$. Moreover, if u_l denotes the lowest order term in the above expansion (i.e. the smallest power of r), then for

- $\omega\in(0,\pi):u_l=c_1\ r^{\frac{\pi}{\omega}}\sin\left(\frac{\pi}{\omega}\theta\right)$,
- $\omega\in\left(\pi,\frac{3\pi}{2}\right):u_l=d_{-1}\ r^{2-\frac{\pi}{\omega}}\sin\left(\frac{\pi}{\omega}\theta\right)$,
- $\omega=\frac{3\pi}{2}:u_l=r^{\frac{4}{3}}\left(d_{-1}\sin\left(\frac{2}{3}\theta\right)+c_2\sin\left(\frac{4}{3}\theta\right)\right)$,
- $\omega\in\left(\frac{3\pi}{2},2\pi\right):u_l=c_2\ r^{\frac{2\pi}{\omega}}\sin\left(\frac{2\pi}{\omega}\theta\right)$.

Note that in the last case the lowest order term is sign-changing.

Proof. Since $f\in C_0^\infty\left(\overline{\Omega}\setminus\mathcal{S}\right)$, we have that $f\in V_0^m(\Omega)$ for all $m\in\mathbb{N}$. A solution of the hinged plate problem satisfies $u\in V_2^4(\Omega)\subset V_3^4(\Omega)$ and thus one can apply [18, Theorem 3.3] with $k_1=m$, $k=\alpha_1=0$, and $\alpha=3$ to obtain that the solution will have the expansion

$$u=\sum_{0<\frac{j\pi}{\omega}<m+3}c_jr^{\frac{j\pi}{\omega}}\sin\left(\frac{j\pi}{\omega}\theta\right)+\sum_{0<2+\frac{j\pi}{\omega}<m+3}d_jr^{2+\frac{j\pi}{\omega}}\sin\left(\frac{|j|\pi}{\omega}\theta\right)+w \tag{4.4}$$

whenever $\frac{j\pi}{\omega}\neq m+3$ with $w\in V_0^{m+4}(\Omega)$. Since $\omega\neq\pi,2\pi$, we can always choose m as large as needed, such that

$$j=\frac{\omega}{\pi}(m+3) \tag{4.5}$$

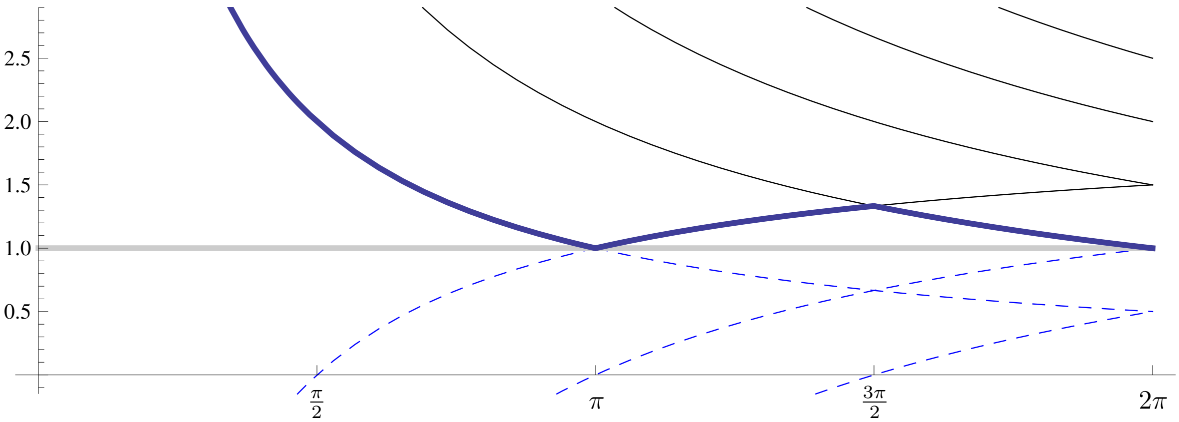

Figure 5. The exponents of the radial part of the expansion (4.3). The lowest order terms are given by the thick blue line; everything below the horizontal grey line is "too singular", i.e. does not belong in $W^{2,2}(\Omega)$.

is not a positive integer: If $\frac{\omega}{\pi} \in \mathbb{Q}$ and $\frac{\omega}{\pi}(m+3) \in \mathbb{N}$, then $\frac{\omega}{\pi}(m+1+3) \notin \mathbb{N}$. Moreover, there will exist at least one term in the above sum when

$$\omega > \frac{\pi}{m+3}. \tag{4.6}$$

Summing up, for a given opening angle ω we choose m such that (4.6) holds and (4.5) gives that j is not a positive integer. □

Before we move on with the comparison of the hinged and supported plate, it is important to have a certain estimate on the coefficients of the lowest order terms in the expansion (4.3). We would wish to have a general answer to the sign of the coefficients of the lowest order terms. This depends highly on Ω and f and thus a general answer is not to be expected. However, when the boundary of the domain has only convex corners, then one can give the following estimate based on the maximum principle.

Lemma 4.3. *Assume that Ω is a convex polygon and let $f \in C_0^\infty\left(\overline{\Omega} \setminus \mathcal{S}\right)$ with $f \leq 0$ and $f \not\equiv 0$. Then $c_1 < 0$.*

Remark 4.4. The proof of this lemma is based on the existence of a comparison principle for convex polygonal plates, since, in that case, the hinged plate problem coincides with the Navier bilaplace problem. When the boundary of the domain has curved parts, then these two problems are no longer the same. However, for the hinged plate problem on convex domains with $C^{2,1}$-smooth boundary, a sign preserving property holds true; see [28]. Judging from the behaviour of hinged plates in polygonal and smooth convex domains, we suppose that a similar result should hold on general convex domains with nonsmooth boundary.

Proof of Lemma 4.3. If all corners of the boundary are convex, one obtains that u has the expansion

$$u = c_1 r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) + d_1 r^{2+\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) + \text{higher order terms}$$

in an ε-neighbourhood of a corner, where the first term is harmonic and the higher order terms smoother than the first two. Then we have

$$\int_0^\varepsilon \left(r^{\frac{\pi}{\omega}-2}\right)^2 r \, dr < \infty,$$

which implies that $\Delta u \in W^{2,2}(\Omega)$. Thus, a hinged plate will satisfy

$$\left\{\begin{array}{rl} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rl} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{array}\right. \tag{4.7}$$

and an iterated application of the maximum principle yields that $u < 0$ in Ω. Now u as a solution of the Dirichlet Laplacian with right hand side v has the expansion

$$u = \sum_{0<j<(m+2)\frac{\omega}{\pi}} c_j r^{\frac{\pi}{\omega}} \sin\left(\frac{j\pi}{\omega}\theta\right) + w$$

with $w \in V_0^{m+2}(\Omega)$ (see [19, Section 6.6.1]). Note that $V_0^{m+2}(\Omega) \subset V_{-m}^2(\Omega)$ and one can apply [27, Lemma 6.7] with $\gamma = -m$ to find that there exists a positive constant C, such that

$$|w| \leq C r^{1+m}$$

sufficiently close to the corner. Since m can be taken arbitrarily large and the functions $\sin\left(\frac{j\pi}{\omega}\theta\right)$ are sign-changing for $j > 1$, we get that $u = o\left(r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)\right)$ and thus $c_1 < 0$. $\square$

We will make the comparison of the hinged and supported plates via a contradiction and we thus need a way to check for solutions to the supported case. A criterion for checking whether a hinged plate is also a solution to the supported problem is given by the following

Lemma 4.5. *Let $f \in C_0^\infty\left(\overline{\Omega} \setminus \mathcal{S}\right)$ and assume that the minimizer $u \in H_0(\Omega)$ of J_σ is also a minimizer in $H_+(\Omega)$. Then*

$$\partial_n \Delta u + (1-\sigma)\partial_{n\tau\tau} u \leq 0 \text{ on } \partial\Omega \setminus \mathcal{S}, \tag{4.8}$$

where $\mathcal{S}$ is the set containing the corners of $\partial\Omega$.

Proof. Similarly as in the proof of Corollary 3.17 one can show that for $f \in C_0^\infty\left(\overline{\Omega} \setminus \mathcal{S}\right) \subset W^{1,2}(\Omega)$ we get $u \in W^{5,2}\left(\Omega \setminus \overline{B(\mathcal{S})}\right)$ and thus all third order derivatives of u are continuous on the boundary. Testing the variational inequality $J'(u; v) \geq 0$ with functions v nonnegative on the boundary and supported away from the corners proves the Lemma. $\square$

Now we move on to compare the hinged and supported plates. For simplicity we assume that $\partial\Omega$ contains only one corner at the origin, of opening angle ω.

Theorem 4.6. *Let $\omega \in (0, 2\pi) \setminus \{\pi\}$, $f \in C_0^\infty(\overline{\Omega} \setminus \{0\})$ and suppose:*

(i) *for $\omega \in (0, \pi)$ that $c_1 < 0$,*
(ii) *for $\omega \in \left(\pi, \frac{3\pi}{2}\right)$ that $d_{-1} < 0$,*
(iii) *for $\omega = \frac{3\pi}{2}$ that $d_{-1} < \frac{2(1-\sigma)}{4-\sigma} |c_2|$,*
(iv) *for $\omega \in \left(\frac{3\pi}{2}, 2\pi\right)$ that $c_2 \neq 0$ or $d_{-1} < 0$.*

In each of these cases the minimizer $u \in H_0(\Omega)$ of J_σ cannot be a minimizer in $H_+(\Omega)$.

Remark 4.7. Note that f should also satisfy assumption (2.3) on the existence of a minimizer of J_σ in $H_+(\Omega)$. Otherwise the result of the theorem is trivial: There would exist no minimizer in $H_+(\Omega)$.

Proof of Theorem 4.6. Here we are going to use Lemma 4.5 in the following way: we assume that the solution u to the hinged plate problem minimizes J_σ in $H_+(\Omega)$ and thus the "supported" boundary condition

$$N(u) := \partial_n \Delta u + (1-\sigma)\partial_{n\tau\tau} u \leq 0 \text{ on } \partial\Omega \setminus \{0\} \tag{4.9}$$

holds true by Lemma 4.5. Then, if u_l denotes the lowest order term in all cases of Corollary 4.2, we will see that $N(u_l)$ is also the leading term of $N(u)$. Thus, calculating $N(u_l)$, we will see that it does not satisfy the supported condition near the origin, that is, $N(u_l) > 0$ sufficiently close to 0, which yields a contradiction. We thus conclude that the solution to the hinged plate problem is not a solution in the supported case.

A hinged plate, i.e. the minimizer of J_σ in $H_0(\Omega)$ has the following expansion in a neighbourhood of the origin:

$$u = u_l + \text{higher order terms} + w,$$

where $w \in V_0^{m+4}(\Omega)$ with m arbitrarily large. Thus one has for its third order derivatives that $\partial^{\mathbf{3}} w \in V_0^{m+1}(\Omega) \subset V_{1-m}^2$ and therefore

$$|(\partial^{\mathbf{3}} w)(r,\theta)| \leq c\, \|w\|_{V_0^5(\Omega)}\, r^m \tag{4.10}$$

(see [27, Lemma 6.7]). Hence, there exists a sufficiently large m such that $N(u) \sim N(u_l)$. For $\theta = \omega$ we get $\partial_n = \partial_\theta$ and $\partial_{\tau\tau} = \partial_{rr}$. Consider the following cases:

(i) $\omega \in (0,\pi)$: We get

$$\begin{aligned} N\left(c_1 r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)\right)\Big|_{\theta=\omega} &= (1-\sigma)\,\frac{c_1\pi^2\left(\frac{\pi}{\omega}-1\right)}{\omega^2} r^{\frac{\pi}{\omega}-2}\cos\left(\frac{\pi\theta}{\omega}\right)\Bigg|_{\theta=\omega} \\ &= (1-\sigma)\frac{c_1\pi^2\left(1-\frac{\pi}{\omega}\right)}{\omega^2} r^{\frac{\pi}{\omega}-2} > 0. \end{aligned}$$

(ii) $\omega \in \left(\pi, \frac{3\pi}{2}\right)$: We compute

$$N\left(d_{-1} r^{2-\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)\right)\Big|_{\theta=\omega} = \frac{d_{-1}\pi(\omega-\pi)\big(\sigma(2\omega-\pi)+\pi-6\omega\big)}{\omega^3} r^{-\frac{\pi}{\omega}} \longrightarrow +\infty,$$

since for $\sigma < 1 \leq \frac{6\omega-\pi}{2\omega-\pi}$ we find that $\sigma(2\omega-\pi)+\pi-6\omega < 0$.

(iii) $\omega = \frac{3\pi}{2}$: The calculation here is a bit more complex. We have

$$\begin{aligned} & N\left(r^{\frac{4}{3}}\left(d_{-1}\sin\left(\tfrac{2}{3}\theta\right) + c_2\sin\left(\tfrac{4}{3}\theta\right)\right)\right)\Big|_{\theta=\frac{3\pi}{2}} \\ = \quad & -\frac{8}{27} r^{-2/3}\left(d_{-1}(-4+\sigma)\cos\left(\tfrac{2\theta}{3}\right) + 2c_2(-1+\sigma)\cos\left(\tfrac{4\theta}{3}\right)\right)\Big|_{\theta=\frac{3\pi}{2}} \\ = \quad & \frac{8}{27} r^{-2/3}(d_{-1}(-4+\sigma) + 2c_2(1-\sigma)), \end{aligned}$$

whereas for $\theta = 0$ we have that $\partial_n = -\partial_\theta$ and thus, similarly,

$$N\left(r^{\frac{4}{3}}\left(d_{-1}\sin\left(\tfrac{2}{3}\theta\right) + c_2\sin\left(\tfrac{4}{3}\theta\right)\right)\right)\Big|_{\theta=0} = \frac{8}{27} r^{-2/3}(d_{-1}(-4+\sigma) - 2c_2(1-\sigma)).$$

Using our assumptions for this case, we obtain again that $N(u_l) \to +\infty$ as $r \to 0$, either for $\theta = 0$ or $\theta = \omega$.

(iv) $\omega \in \left(\frac{3\pi}{2}, 2\pi\right)$: Similar to the previous cases we compute for $\theta = \omega$ that

$$\begin{aligned} N\left(c_2 r^{\frac{2\pi}{\omega}} \sin\left(\frac{2\pi}{\omega}\theta\right)\right)\Bigg|_{\theta=\omega} &= (1-\sigma)\,\frac{4c_2\pi^2\left(\frac{2\pi}{\omega}-1\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2}\cos\left(\frac{2\pi\theta}{\omega}\right)\Bigg|_{\theta=\omega} \\ &= (1-\sigma)\frac{4c_2\pi^2\left(\frac{2\pi}{\omega}-1\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2} \end{aligned}$$

and for $\theta = 0$ we have that

$$N\left(c_2 r^{\frac{2\pi}{\omega}} \sin\left(\frac{2\pi}{\omega}\theta\right)\right)\Bigg|_{\theta=0} = -(1-\sigma)\frac{4c_2\pi^2\left(\frac{2\pi}{\omega}-1\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2}.$$

In the last case the result follows for $c_2 \neq 0$ since the corresponding function is sign-changing. If $c_2 = 0$, then we have to consider the next leading term, which is the one with coefficient d_{-1} and proceed as in case *(ii)*. □

5. Appendix

5.1. An inequality of Poincaré type

It is well known that for bounded domains Ω it holds that $\|u\|_{L^2(\Omega)} \le c_\Omega \, \||\nabla u|\|_{L^2(\Omega)}$ for $u \in W_0^{1,2}(\Omega)$. It is less known that $\||\nabla u|\|_{L^2(\Omega)} \le c_\Omega \left\||\nabla^2 u|\right\|_{L^2(\Omega)}$ for $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and we will include a proof. Consider $x = (x_1, ..., x_n) \in \Omega$ and write dx and ds for the n-D Lebesgue and $(n-1)$-D surface measure.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a simply connected bounded domain. Then there exists a constant $c > 0$, such that*

$$\int_\Omega |\nabla u(x)|^2 dx \le c \ |u|_{2,2}^2 \quad \text{for all } u \in W^{2,2}(\Omega) \cap C^2(\Omega) \cap C_0(\overline{\Omega})$$

Proof. First we consider the 1D case and assume that $u \in W^{2,2}(0,l) \cap C^2(0,l) \cap C_0[0,l]$. Since $u(0) = u(l) = 0$, the mean value theorem shows that there exists $x_0 \in (0,l)$ such that $u'(x_0) = 0$. Using twice Hölder's inequality one has that

$$\begin{aligned}
\int_{x_0}^l |u'(x)|^2 \, dx &= \int_{x_0}^l \left| \int_{x_0}^x u''(t) \, dt \right|^2 dx \ \le \ \int_{x_0}^l \left(\int_{x_0}^x |u''(t)| \, dt \right)^2 dx \\
&\le \int_{x_0}^l \left[\left(\int_{x_0}^x |u''(t)|^2 \, dt \right) \left(\int_{x_0}^x dt \right) \right] dx \\
&\le \left(\int_{x_0}^l |u''(t)|^2 \, dt \right) \int_{x_0}^l (x - x_0) \, dx = \frac{1}{2} (l - x_0)^2 \int_{x_0}^l |u''(t)|^2 \, dt.
\end{aligned}$$

Since $\int_0^l |u'(x)|^2 \, dx = \int_0^{x_0} |u'(x)|^2 \, dx + \int_{x_0}^l |u'(x)|^2 \, dx$, one obtains the desired result.
In the $n-$dimensional case we will proceed using Fubini's theorem and the above result for one variable at a time:

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx &= \int_{x_1} \int_{x_2} ... \int_{x_n} |\nabla u(x_1, x_2, ..., x_n)|^2 \, dx_n ... dx_2 dx_1 \\
&= \int_{x_1} \int_{x_2} ... \int_{x_n} \left(|u_{x_1}|^2 + |u_{x_2}|^2 + ... + |u_{x_n}|^2 \right) dx_n ... dx_2 dx_1 \\
&\le c \int_{x_1} \int_{x_2} ... \int_{x_n} \left(|u_{x_1 x_1}|^2 + |u_{x_2 x_2}|^2 + ... + |u_{x_n x_n}|^2 \right) dx_n ... dx_2 dx_1 \\
&\le c |u|_{2,2}^2
\end{aligned}$$

and the assertion is proved. □

Remark 5.2. A related result will hold for functions with nonzero boundary conditions. Let Ω be a simply connected bounded domain of $\mathbb{R}^n$. Then there exists $c_1 > 0$ such that

$$\int_\Omega |\nabla u|^2 \, dx \le c_1 \left(|u|_{2,2}^2 + \int_{\partial\Omega} |\nabla u|^2 \, ds \right) \text{ for all } u \in W^{2,2}(\Omega) \cap C^2(\Omega) \cap C^1(\overline{\Omega}).$$

Based on a remark on the Theorem of Meyers and Serrin (see [2]), stated in [12], one can prove the following.

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^n$ bounded, with Lipschitz boundary and define $C_0^k(\overline{\Omega})$ as the space of k-differentiable functions whose derivatives up to order k are zero on $\partial\Omega$. Then, for $m \ge k+1$*

$$\overline{W^{m,p}(\Omega) \cap C^\infty(\Omega) \cap C_0^k(\overline{\Omega})}^{\|\cdot\|_{m,p}} = W^{m,p}(\Omega) \cap W_0^{k+1,p}(\Omega).$$

Proof. For consistency reasons we will first outline the proof given by Meyers and Serrin. Let $k \in \mathbb{N}$ and define

$$\Omega_k := \left\{ x \in \Omega \,\middle|\, |x| < k \text{ and } d(x, \partial\Omega) > \frac{1}{k} \right\},$$

$$\Omega_{-1} = \Omega_0 := \emptyset.$$

Setting $U_k := \Omega_{k+1} \cap \left(\overline{\Omega_{k-1}}\right)^c$ one sees that $\{U_k\}_{k\in\mathbb{N}}$ is an open covering of Ω, so there exists a partition of unity Ψ subordinate to $\{U_k\}_{k\in\mathbb{N}}$. Moreover, let ψ_k be the sum of the finitely many functions $\psi \in \Psi$ with support in U_k and η_ε the standard mollifier. Fixing

$$0 < \varepsilon < \frac{1}{(k+1)(k+2)}$$

one can see that $\eta_\varepsilon * (\psi_k u)$ has support in $V_k := \Omega_{k+2} \cap (\Omega_{k-2})^c \subset\subset \Omega$. Choose ε_k small enough, such that

$$\|\eta_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{W^{m,p}(\Omega)} = \|\eta_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{W^{m,p}(V_k)} < \frac{\varepsilon}{2^k} \tag{5.1}$$

and set

$$\phi_\varepsilon := \sum_{k=1}^{\infty} \eta_{\varepsilon_k} * (\psi_k u).$$

Then $\phi_\varepsilon \in C^\infty(\Omega)$ and, since ε is independent of V_k one obtains that $\|\phi_\varepsilon - u\|_{m,p,\Omega} < \varepsilon$.

Following [12, Remark 1.18, p. 16] we consider $\delta > 0$, $\rho > 0$, $x_0 \in \partial\Omega$ and $k_0 = \left\lceil \frac{1}{\rho} \right\rceil - 2$. Then one has that for all $x \in B_\rho(x_0) \cap \Omega$

$$\phi_\delta(x) - u(x) = \sum_{k=k_0}^{\infty} \left(\eta_{\delta_k} * \Big(\psi_k(x) u(x) \Big) - \psi_k(x) u(x) \right).$$

Now, estimate (5.1) yields

$$\|u - \phi_\delta\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} \le \sum_{k=k_0}^{\infty} \|J_{\delta_k} * (\psi_k u) - \psi_k u\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} \le \delta \sum_{k=k_0}^{\infty} \frac{1}{2^k} = 2^{-\frac{1}{\rho}} 8\delta.$$

Assuming that the norm on the left is not identically zero, there exist constants $c(\delta), \rho(\delta) > 0$ such that for all positive $\rho < \rho(\delta)$:

$$c(\delta)\rho^2 \le \|u - \phi_\delta\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} \le 8\delta\, 2^{-\frac{1}{\rho}}$$

which cannot hold. Thus for ρ small enough

$$\|u - \phi_\delta\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} = 0$$

which together with the Lebesgue's differentation Theorem implies that the traces of the $m-1$ order derivatives of the approximating sequence φ_δ agree in an L^p-sense with the ones of u (which are well defined since $\partial\Omega$ is Lipschitz) and the claim is proved. □

Combining Lemma 5.1 and Proposition 5.3 we obtain the desired result.

Corollary 5.4. *Let $\Omega \subset \mathbb{R}^n$ bounded with a Lipschitz boundary. Then $|\cdot|_{2,2}$ and $\|\cdot\|_{2,2}$ are equivalent norms on $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

5.2. Numerical computations for the supported plate

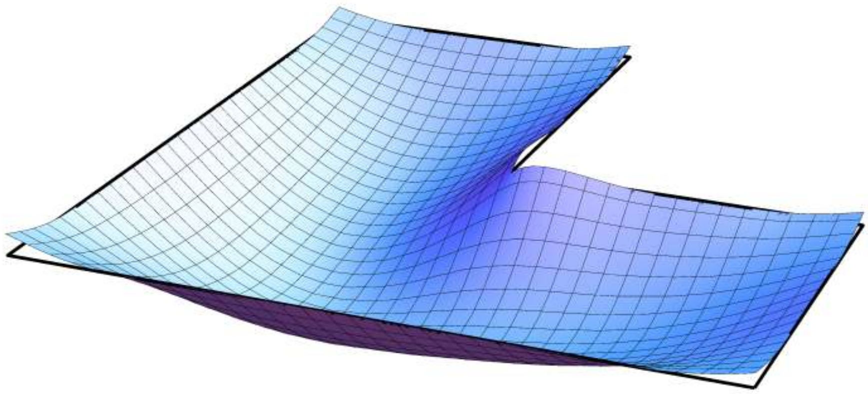

Figure 6. An L-shaped plate with a uniform load ($f = -1$) leads to a solution that moves upwards near all corner points.

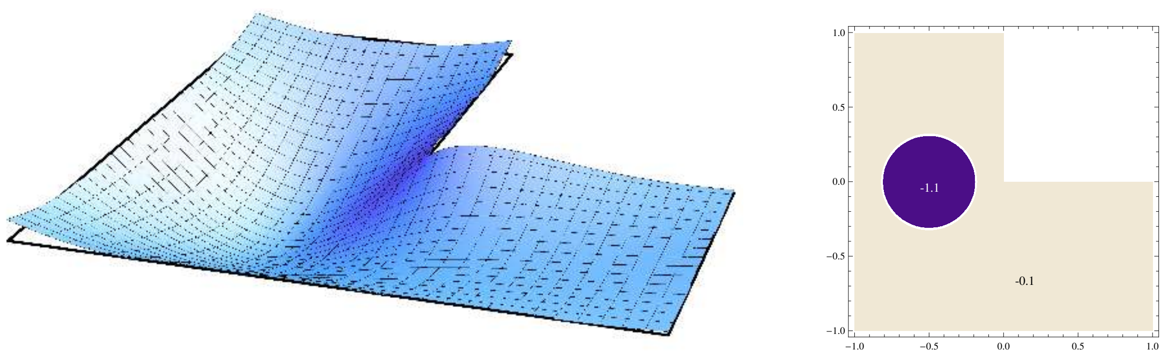

Figure 7. An L-shaped plate with small load ($f = -.1$) everywhere except for a local heavier load ($f = -1.1$) on the dark circular area. On the right this force density.

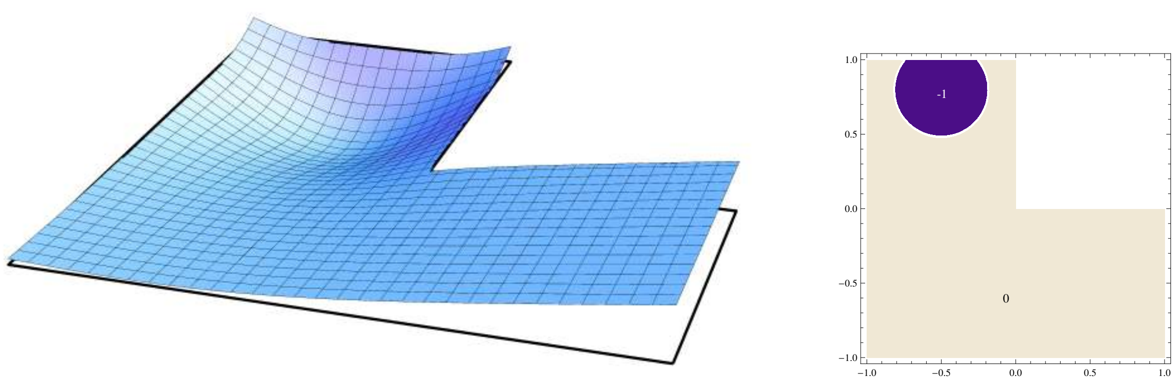

Figure 8. An L-shaped plate loaded locally on one side results to a large free boundary on the other side.

In order to illustrate these analytical results we display some numerical evidence. Note that it is not our intention to prove convergence of the numerical scheme we used. For the sake of completeness and, since numerics for fourth order variational inequalities on domains with corners are far from trivial, a short sketch follows.

We have used a damped Newton method to solve the penalized problem corresponding to the variational inequality (2.4) (see [13]): Find $u \in W^{2,2}(\Omega)$ such that

$$J'_\sigma(u;v) - \int_{\partial\Omega} \beta(u) v \, ds = 0 \quad \text{for all} \quad v \in W^{2,2}(\Omega),$$

where

$$\beta(s) := \begin{cases} -10^{10} s & \text{for } s < 0, \\ 0 & \text{for } s \geq 0. \end{cases} \tag{5.2}$$

The calculations were done using a negative right hand side f (see Figures 6-8) and $C^1 \cap W^{2,2}$-Argyris elements (see [7, Theorem 2.2.13]).

Define by $V_N := \operatorname{span}\left(\{e_i\}_{i=1}^N\right)$, the linear span of the Argyris basis element functions $\{e_i\}_{i=1}^N$, defined on a triangulation of Ω consisting of N triangles (see [7]). A finite element approximation of (5.2) would consist in solving the system of nonlinear equations

$$F(u) = 0, \tag{5.3}$$

for $u = \sum_{i=1}^N u_i e_i$, where $F : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$F(u) = \begin{pmatrix} F_1(u) \\ \vdots \\ F_N(u) \end{pmatrix} := \begin{pmatrix} J'_\sigma(u;e_1) - \int_{\partial\Omega} \beta(u) e_1 \, ds \\ \vdots \\ J'_\sigma(u;e_N) - \int_{\partial\Omega} \beta(u) e_N \, ds \end{pmatrix}.$$

To solve (5.3), we start with an initial guess $u^{(0)} = \left(u_1^{(0)}, ..., u_N^{(0)}\right)^\top$ and calculate the iterations via the formula

$$\begin{pmatrix} u_1^{(k+1)} \\ \vdots \\ u_N^{(k+1)} \end{pmatrix} = \begin{pmatrix} u_1^{(k)} \\ \vdots \\ u_N^{(k)} \end{pmatrix} - \vartheta \cdot \left(F'(u^{(k)})\right)^{-1} \cdot F(u^{(k)}).$$

The standard Newton method uses $\vartheta = 1$ but is primarily equipped for differentiable functionals. Our functional is not differentiable near 0 and for convergence we had to use a rather small damping factor $\vartheta \in (0,1)$. The matrix $F'(u)$ is the Jacobian of F, given by

$$F'(u) := \left(\frac{\partial F_j}{\partial u_i}\right)_{i,j=1,\dots,N},$$

where

$$\begin{aligned} \frac{\partial F_j}{\partial u_i} &= \int_\Omega \Big(\Delta e_i \Delta e_j + (1-\sigma)(2\partial_{xy} e_i \partial_{xy} e_j - \partial_{xx} e_i \partial_{yy} e_j - \partial_{yy} e_i \partial_{xx} e_j)\Big)\, dx \\ &\quad - \int_{\partial\Omega} \beta'\left(\textstyle\sum_{k=1}^N u_k e_k\right) e_i e_j \, ds. \end{aligned}$$

Note also that, since the e_i's have small support, the Jacobian $F'(u)$ becomes a band matrix.

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