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Hirzebruch's Proportionality Theorem in the Non-Compact Case

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Dedicated to Friedrich Hirzebruch

In the conference on Algebraic Topology [7] in 1956, F. Hirzebruch described a remarkable theorem relating the topology of a compact locally symmetric variety:

$X = D/\Gamma$,
D = bounded symmetric domain,
Γ = discrete torsion-free co-compact group of automorphisms of D

with the topology of the extremely simply rational variety $\check{D}$, the "compact dual" of D. (See §3 for full definitions.) His main result is that the Chern numbers of X are proportional to the Chern numbers of $\check{D}$, the constant of proportionality being the volume of X (in a natural metric). This is a very useful tool for analyzing the structure of X. Many of the most interesting locally symmetric varieties that arise however are not compact: they have "cusps". It seems a priori very plausible that Hirzebruch's line of reasoning should give some relation even in the non-compact case between the chern numbers of X and of $\check{D}$, with some correction terms for the cusps. The purpose of this paper is to show that this is indeed the case. We hope that the generalization that we find will have applications.

The paper is organized as follows. In §1, we make a few general definitions and observations concerning Hermitian metrics on bundles with poles and describe an instance where such metrics still enable one to calculate the Chern classes of the bundle. This section is parallel to work of Cornalba and Griffiths [6]. In §2, which is the most technical, we prove a series of estimates for a class of functions on a convex self-adjoint cone. In §3, the results of §1 and §2 are brought together, and the Proportionality Theorem is proven. One consequence is that D/Γ has the property, defined by Iitaka [8], of being of logarithmic general type. Finally, in §4, we analyze the step from logarithmic general type to general type and reprove a Theorem of Tai that D/Γ is of general type if Γ is sufficiently small.

§ 1. Singular Hermitian Metrics on Bundles

In this section we will not be concerned specifically with the locally symmetric algebraic varieties D/Γ, but with general smooth quasi-projective algebraic varieties X. When X is not compact, we want to study the order of poles of differential forms on X at infinity, and when E is moreover a vector bundle on X, we want to study Hermitian metrics on E which also "have poles at infinity". This situation has been studied by Cornalba-Griffiths [6]. The following idea of bounding various forms by local Poincaré metrics on punctured polycylinders at infinite is due to them. More precisely, we choose a smooth projective compactification $\bar{X}$:

$$X \subset \bar{X}$$

where $\bar{X} - X$ is a divisor on $\bar{X}$ with normal crossings.

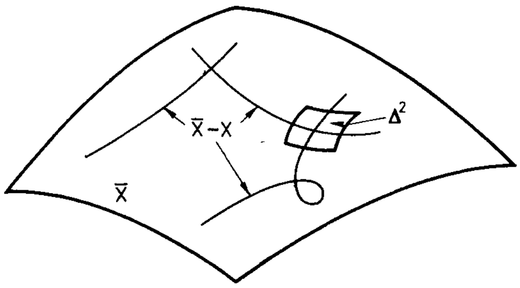
Then we look at polycylinders:

$$\Delta^r \subset \bar{X} \quad \begin{pmatrix} \Delta = \text{unit disc} \\ r = \dim \bar{X} \end{pmatrix}$$

$$\text{where } \Delta^r \cap (\bar{X} - X) = \begin{Bmatrix} \text{union of coordinate hyperplanes} \\ z_1 = 0,\ z_2 = 0, \ldots, z_k = 0 \end{Bmatrix}$$

hence:

$$\Delta^r \cap X = (\Delta^*)^k \times \Delta^{r-k}.$$



In Δ^* we have the Poincaré metric:

$$ds^2 = \frac{|dz|^2}{|z|^2 (\log|z|)^2}$$

and in Δ we have the simple metric $|dz|^2$, giving us a product metric on $(\Delta^*)^k \times \Delta^{r-k}$ which we call $\omega^{(p)}$.

Definition. A complex-valued C^∞ p-form η on X is said to have *Poincaré growth* on $\bar{X} - X$ if there is a set of polycylinders $U_\alpha \subset \bar{X}$ covering $\bar{X} - X$ such that in each U_α, an estimate of the following type holds:

$$|\eta(t_1, \ldots, t_p)|^2 \leqq C_\alpha\, \omega_{U_\alpha}^{(p)}(t_1, t_1) \cdot \cdots \cdot \omega_{U_\alpha}^{(p)}(t_p, t_p)$$

(all $t_1, \ldots, t_p$ tangent vectors to $\bar{X}$ at some point of $U_\alpha \cap X$).

It is not hard to see that this property is independent of the covering U_α of $\bar{X} - X$ (but unfortunately it does depend on the compactification $\bar{X}$). Moreover, if η_1, η_2 both have Poincaré growth on $\bar{X} - X$, then so does $\eta_1 \wedge \eta_2$. This leads to the basic property:

Proposition 1.1. *A p-form η with Poincaré growth on $\bar{X} - X$ has the property that for every $C^\infty(r-p)$-form ζ on $\bar{X}$,*

$$\int_{\bar{X}-X} |\eta \wedge \zeta| < +\infty$$

hence η defines a p-current $[\eta]$ on $\bar{X}$.

Proof. Since ζ has Poincaré growth, we are reduced to checking that if η is an r-form with Poincaré growth, then

$$\int_{\bar{X}} |\eta| < +\infty.$$

In a polycylinder U_α, this amounts to the well-known fact that for all relatively compact $V \subset\subset U_\alpha$, the Poincaré metric volume of $V \cap (\Delta^{*k} \times \Delta^{r-k})$ is finite. QED

Definition. A complex-valued C^∞ p-form η on X is *good on $\bar{X}$* if both η and $d\eta$ have Poincaré growth.

The set of all good forms η is differential graded algebra for which we have the next basic property:

Proposition 1.2. *If η is a good p-form, then*

$$d([\eta]) = [d\eta].$$

Proof.[1] By definition of $d([\eta])$, this means that for all $C^\infty(r-p-1)$-forms ζ on $\bar{X}$,

$$\int_{\bar{X}-X} d\eta \wedge \zeta = -\int_{\bar{X}-X} \eta \wedge d\zeta.$$

This comes down to asserting that if U_ε is a tube of radius ε around $\bar{X} - X$. then

$$\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} (\eta \wedge \zeta) = 0.$$

If we take, for instance, $r = 2$ and set up this integral in local coordinates x, y on $\bar{X}$ near a point where $\bar{X} - X$ has 2 branches $x = 0$ and $y = 0$, then this comes

[1] Compare Cornalba-Griffiths [6], p. 25

down to the assertion

$$\lim_{\varepsilon\to 0} \int_{\substack{|x|\geqq\varepsilon \\ |y|=\varepsilon}} \frac{|dx|^2}{|x|^2(\log|x|)^2} \cdot \frac{|dy|}{|y|\cdot|(\log|y|)|} = 0$$

which is easy to check. The general case is similar. QED

Next, let $\bar{E}$ be an analytic rank n vector bundle on $\bar{X}$, let E be the restriction of $\bar{E}$ to X and let $h: E\to\mathbb{C}$ be a Hermitian metric on E. For such h we define "good" as follows:

Definition. A Hermitian metric h on E is *good on* $\bar{X}$ if for all $x\in\bar{X}-X$ and all bases $e_1, \ldots, e_n$ of $\bar{E}$ in a neighborhood Δ^r of x in which $\bar{X}-X$ is given as above by $\prod_{i=1}^{k} z_i=0$, if $h_{ij}=h(e_i, e_j)$, then

i) $|h_{ij}|, (\det h)^{-1} \leqq C\left(\sum_{i=1}^{k} \log|z_i|\right)^{2n}$, for some $C>0$, $n\geqq 1$,

ii) the 1-forms $(\partial h\cdot h^{-1})_{ij}$ are good on $\bar{X}\cap U$.

The first point about good Hermitian metrics is that given (E, h), there is at most one extension $\bar{E}$ of E to $\bar{X}$ for which h is good. This follows from:

Proposition 1.3. *If h is good, then for all polycylinders $\Delta^r\subset\bar{X}$ in which $\bar{X}-X$ is given by $\prod_{i=1}^{k} z_i=0$,*

$$\Gamma(\Delta^r, \bar{E})=\{s\in\Gamma(\Delta^r\cap X, E)\,|\,h(s,s)\leqq C\cdot(\textstyle\sum\log|z_i|)^{2n}, \text{ for some } C, n\}.$$

Proof. The inclusion "$\subset$" is immediate. As for "$\supset$", if $s=\sum_{i=1}^{n} a_i(z)\,e_i$ is a holomorphic section of E on $\Delta^r\cap X$, for which $h(s,s)$ is bounded as above, then it follows that

$$|a_i(z)|\leqq C'(\textstyle\sum\log|z_i|)^{2m}, \quad \text{for suitable } C', m.$$

Therefore $\left(\prod_{j=1}^{k} z_i\right)\cdot a_i(z)$ is bounded on Δ^r, hence is analytic, hence $a_i(z)$ is meromorphic with simple poles on $\bar{X}-X$. But as no inequality

$$\frac{1}{|z|^2}\leqq C(\log|z|)^{2n}$$

holds, $a_i(z)$ is in fact analytic. QED

The main result of this section is the following:

Theorem 1.4. *If $\bar{E}$ is a vector bundle on $\bar{X}$ and h is a good Hermitian metric on $E=\bar{E}|_X$, then the Chern forms $c_k(E,h)$ are good on $\bar{X}$ and the current $[c_k(E,h)]$ represents the cohomology class $c_k(\bar{E})\in H^{2k}(\bar{X},\mathbb{C})$.*

Proof. Let h^* be a C^∞ Hermitian metric on $\bar{E}$. Define

$$\theta = \partial h \cdot h^{-1}, \qquad \theta^* = \partial h^* \cdot h^{*-1},$$
$$K = \bar{\partial}\theta, \qquad K^* = \bar{\partial}\theta^*.$$

Intrinsically, K and K^* are $\operatorname{Hom}(E, E)$-valued $(1,1)$-forms and $\theta - \theta^*$ is a $\operatorname{Hom}(E, E)$-valued $(1,0)$-form. According to results in Bott-Chern [5], for each k there is a universal polynomial P_k with rational coefficients in the forms K, K^* and $\theta - \theta^*$ such that *on* X:

$$c_k(E, h) - c_k(E, h^*) = d(\operatorname{Tr} P_k(K, K^*, \theta - \theta^*)).$$

Now K, K^* and $\theta - \theta^*$ are forms good on $\bar{X}$, hence the (k, k)-form $\operatorname{Tr} P_k(K, K^*, \theta - \theta^*)$ is good on $\bar{X}$. It follows that $c_k(E, h)$ is good on $\bar{X}$ and that

$$[c_k(E, h)] = d[\operatorname{tr} P_k] + \underbrace{[c_k(E, h^*)]}_{\text{represents } c_k(\bar{E})}. \quad \text{QED}$$

§ 2. Estimates on Cones

The results of this section are purely preliminary. We have isolated all the inequalities needed for the general proportionality theorem which involve only the cone variables (cf. § 3, definition of Siegel Domain), and worked these out in this section.

The object of study then is a real vector space V and

$$C \subset V,$$

C an open, convex, non-degenerate ($\bar{C} \not\supset$ (pos. dim. subspace of V), or equivalently, $\exists\, l \in V$, $l > 0$ on $\bar{C}$) cone. Most of our results relate only to those C which are homogeneous and self-adjoint; for any C, we let $G \subset GL(V)$ be the group of linear maps which preserve C, and say C is *homogeneous* if G acts transitively. If, moreover, there is a positive-definite inner product $\langle\, , \rangle$ on V for which

$$\bar{C} = \{x \in V \mid \langle x, y \rangle \geqq 0, \text{ all } y \in C\}$$

we say C is *self-adjoint*. The classification of these is well known (see [1], p. 63), as in the fact that all such arise by considering formally real Jordan algebras V and setting

$$C = \{x^2 \mid x \in V, \ x \text{ invertible}\}.$$

All convex non-degenerate cones C carry several canonical metrics on them. First of all, there is a canonical Finsler metric on C, which is analogous to the Caratheodory metrics on complex manifolds ([10], p. 49):

$\forall x \in C$, $t \in T_{x,C} \cong V$, let:

$$\rho_x(t) = \sup_{l \in \hat{C}} \frac{|l(t)|}{l(x)}.$$

(Another canonical Finsler metric, analogous to Kobayashi's metric in the complex case, can also be defined. First introduce on the positive quadrant $\mathbb{R}_+ \times \mathbb{R}_+$ the metric $\frac{dx^2}{x^2}+\frac{dy^2}{y^2}$; then we have on any cone the definition:

$$\rho'_x(t)=\left\{\begin{matrix}\text{canonical length in}\\ \text{the cone } C\cap(\mathbb{R}x+\mathbb{R}t)\end{matrix}\right\}$$
$$=\sqrt{\frac{1}{a_1^2}+\frac{1}{a_2^2}}$$

if $a_1>0$ and $a_2<0$ are determined by $x+a_1 t, x+a_2 t\in\bar{C}, \notin C$. We won't need this second metric however.)

The advantage of the Finsler metric is that (as in the complex case) it behaves in a monotone way when you replace C by a smaller (or bigger) cone:

Proposition 2.1. i) *If C is an open convex non-degenerate cone in V, and $a\in\bar{C}$, $x\in C$, $t\in V$, then*

$$\rho_{x+a}(t)_C\leqq\rho_x(t)_C.$$

ii) *If $C_1\subset C_2$ are 2 open convex non-degenerate cones in V, then for all $x\in C_1$, $t\in V$,*

$$\rho_x(t)_{C_1}\geqq\rho_x(t)_{C_2}.$$

(The proofs are easy.)

Now suppose C is homogeneous and self-adjoint. Then one can introduce a Riemannian metric on C as follows. Chose a base point $e\in V$ which we take as the identity for the Jordan algebra and let $\langle\ ,\ \rangle$ be an inner product on V in terms of which $G={}^tG$. Then ([1], p. 62), C is self-adjoint with respect to $\langle\ ,\ \rangle$. Moreover, $K=\mathrm{Stab}(e)$ is a maximal compact subgroup of G and if $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition with respect to K and $*: G\to G$ the Cartan involution, then:

1) $\langle gx, g^*y\rangle=\langle x,y\rangle$, hence $K=\exp(\mathfrak{k})$ acts by orthogonal maps while $P=\exp(\mathfrak{p})$ acts by self-adjoint maps,

2) $(gx)^{-1}=g^*(x^{-1})$ (here x^{-1} is the Jordan algebra inverse).

Now identifying $T_{e,C}$ with V, we use $\langle\ ,\ \rangle$ to define a Riemannian metric on C at e: since it is K-invariant, it globalizes to a unique G-invariant Riemannian metric on C, which we write ds_C^2. For later use, we need the following formula:

Lemma. *Take $t_1,t_2\in T_{x,C}$, and let $f(x)=\langle t_2,x^{-1}\rangle$ where x^{-1} is the Jordan algebra inverse. Then*

$$ds_C^2(t_1,t_2)=-D_{t_1}f$$

where D_{t_1} is the derivative of functions on V in the direction t_1.

Proof. Let $g\in\exp(\mathfrak{p})$ carry e to x. Then by G-invariance:

$$ds_{C,x}^2(t_1,t_2)=ds_{C,e}^2(g^{-1}t_1,g^{-1}t_2)=\langle g^{-1}t_1,g^{-1}t_2\rangle$$

and

$$D_{t_1}f(x)=D_{g^{-1}t_1}(f\circ g)(e).$$

But

$$\begin{aligned} f\circ g(x) &= \langle t_2,(gx)^{-1}\rangle \\ &= \langle t_2, g^{-1}\cdot(x^{-1})\rangle \\ &= \langle g^{-1}t_2, x^{-1}\rangle. \end{aligned}$$

If $\delta x\in V$ is small, then $(e+\delta x)^{-1}=e-\delta x+$(terms of lower order), hence

$$D_{g^{-1}t_1}(f\circ g)(e)=-\langle g^{-1}t_2, g^{-1}t_1\rangle.\quad \text{QED}$$

On a homogeneous cone C, any $2G$-invariant metrics are necessarily comparable, so we can deduce, from the monotone behavior of the Finsler metric ρ, a weaker monotonicity for ds_C^2:

Proposition 2.2. i) *If C is a self-adjoint homogeneous cone in V and $a\in\bar{C}$, then there is a constant $K>0$ such that*

$$ds^2_{C,x+a}(t,t)\leqq K\cdot ds^2_{C,x}(t,t)\qquad \textit{all } t\in V,\ x\in C.$$

ii) *If $C_1\subset C_2$ are 2 self-adjoint homogeneous cones in V, then there is a constant $K>0$ such that*

$$ds^2_{C_2,x}(t,t)\leqq K\cdot ds^2_{C_1,x}(t,t),\qquad \textit{all } t\in V,\ x\in C_1.$$

(The proofs are easy.)

The main estimates of this section deal with the following situation:
C a self-adjoint homogeneous cone,
$G=\mathrm{Aut}^\circ(V,C)$ ($^\circ$ means connected component),
$C_n=$cone of positive definite $n\times n$ Hermitian matrices,
$\rho: G\to GL(n,\mathbb{C})$ a representation,
$H: C\to C_n$ an equivariant, symmetric map with respect to ρ.

Here (ρ,H) *equivariant* means:

$$H(gx)=\rho(g)H(x)\,{}^t\overline{\rho(g)},\qquad \text{all } x\in C,\ g\in G, \tag{1}$$

while (ρ,H) *symmetric* means:

$$\rho(g^*)=H(e)\cdot{}^t\overline{\rho(g)}^{-1}\cdot H(e)^{-1},\qquad \text{all } g\in G. \tag{2}$$

Note that $A\mapsto H(e)\cdot{}^t\bar{A}^{-1}\cdot H(e)^{-1}$ is just the Cartan involution of $GL(n,\mathbb{C})$ with respect to the maximal compact subgroup given by the unitary group of $H(e)$: calling this $\natural$, we can rewrite the symmetry condition (2):

$$\rho(g^*)=\rho(g)^\natural. \tag{2'}$$

Condition (2) is actually independent of the choice of e: if $e' \in C$ is any other point, and $g \mapsto g^{*\prime}$ is the corresponding Cartan involution then one can check that (1)+(2) imply:

$$\rho(g^{*\prime}) = H(e')^t\overline{\rho(g)}^{-1} H(e')^{-1}. \tag{2''}$$

We will also need for applications the slightly more general situation where ρ is fixed but H depends on some extra parameters $t \in T$ *with* T *compact*. In this case, we ask that (1) and (2) hold for each H_t. Note that we can change coordinates in $\mathbb{C}^n$, to get a new pair:

$$\rho'(g) = a\rho(g)a^{-1},$$
$$H_t'(g) = aH_t(g)^t\bar{a},$$

satisfying the same identities. In this way, we can, for instance, normalize the situation so that

$$H_{t_0}(e) = I_n$$

hence:

$$\rho(K) \subset U(n),$$
$$\rho(\exp \mathfrak{p}) \subset \left\{\begin{matrix}\text{self-adjoint matrices, commuting} \\ \text{with } H_t(e), \text{ all } t\end{matrix}\right\}.$$

This normalization will not affect our estimates. The first of these is:

Proposition 2.3. *For all* $\lambda > 0$, *there is a constant* $K > 0$ *and an integer* N *such that*

$$\|H_t(x)\| \quad \textit{and} \quad |\det H_t(x)|^{-1} \leqq K \cdot \langle x, x \rangle^N, \quad \textit{all } x \in (C + \lambda \cdot e).$$

Proof. Let $A \subset \exp(\mathfrak{p})$ be a maximal $\mathbb{R}$-split torus. Then $C = K \cdot A \cdot e$ and $C + \lambda e = K \cdot (Ae + \lambda e)$. Write $x = k(a(e))$. Then

$$\begin{aligned}\|H_t(x)\| &= \|H_t(ae)\| \\ &= \|\rho(a) \cdot H_t(e) \cdot \rho(a)\| \\ &\leqq \|\rho(a)\|^2 \cdot \|H_t(e)\|.\end{aligned}$$

We may change coordinates in $\mathbb{C}^n$ by a unitary matrix so that all the matrices $\rho(a)$ are diagonalized. Write then $\rho(a) = \chi_i(a) \cdot \delta_{ij}$ where χ_i: $A \to \mathbb{R}^*$ are characters. Now we may coordinatize A by

$$A \xrightarrow{\approx} A \cdot e \cong R_+^r \subset V, \quad e \mapsto (1, \ldots, 1).$$

Let $a_1, \ldots, a_r$ be coordinates in $\mathbb{R}_+^r$. Then

$$\chi_i(a) = \prod_{j=1}^r a_j^{s_{ij}}, \quad s_{ij} \in \mathbb{R}.$$

Note that if $ae \in C+\lambda e$, then $ae \in \mathbb{R}^n_+ + (\lambda, \ldots, \lambda)$, i.e., $a_i \geqq \lambda$, all i. Hence

$$\begin{aligned}\|\rho(a)\|^2 &\leqq \max_{1\leqq i\leqq n} (\chi_i(a))^2\\ &\leqq K_1 \left(\sum_{l=1}^{r} |a_l|^2\right)^{(\max s_{ij})r}\\ &\leqq K_1 \cdot K_2 (\langle ae, ae\rangle)^{(\max s_{ij})r}\end{aligned}$$

hence

$$\|H_t(kae)\| \leqq K_1 \cdot K_2 \cdot K_3 (\langle kae, kae\rangle)^{(\max s_{ij})r}.$$

The same proof works for $|\det|^{-1}$. QED

Next, let $\xi \in V$, and let D_ξ be the derivative on V in the direction ξ. We wish to estimate the matrix-valued functions

$$(D_\xi H_t) \cdot H_t^{-1}\colon C \to M_n(\mathbb{C}).$$

To do this, we prove first:

Proposition 2.4. *For all* $1 \leqq \alpha, \beta \leqq n$, *let* $(D_\xi H_t \cdot H_t^{-1})_{\alpha\beta}$ *be the* $(\alpha,\beta)^{\text{th}}$ *entry in this matrix. There is a linear map*

$$C_{\alpha\beta,t}\colon V \to V$$

depending continuously on t such that

$$(D_\xi H_t \cdot H_t^{-1})_{\alpha\beta}(x) = \langle C_{\alpha\beta,t}(\xi), x^{-1}\rangle.$$

Moreover $C_{\alpha\beta,t}$ *has the property:*

$$\left.\begin{matrix}\xi, \eta \in \bar{C}\\ \langle \xi, \eta\rangle = 0\end{matrix}\right\} \Rightarrow \langle C_{\alpha\beta,t}(\xi), \eta\rangle = 0.$$

For some reason, I can't prove this by a direct calculation, but must resort to the following trick:

Lemma. *Let* $C \subset V$ *be a convex open cone, let* $e \in C$, *and let* $f\colon C \to \mathbb{C}$ *be a differentiable function. Suppose that for all* $W \subset V$, $\dim W = 2$, *and* $e \in W$, f *is linear on* $C \cap W$. *Then* f *is linear.*

Proof. The hypothesis means that

$$f(ax+be) = af(x) + bf(e), \quad \text{all } x \in C,\ a, b \in \mathbb{R}_+.$$

So we may extend f to all of V by the formula

$$f(x) = f(x+ae) - af(e), \quad \text{provided } x + ae \in C.$$

Note that

$$f(tx)=f(tx+tae)-atf(e)$$
$$=t(f(x+ae)-af(e))$$
$$=tf(x).$$

Thus for all $x\in V$, $n\geqq 1$:

$$f(x)=\frac{f(x/n)-f(0)}{n}$$
$$\therefore f(x)=D_x f(0)$$

and the right hand side is linear in x. QED

We prove $(D_\xi H_t\cdot H_t^{-1})_{\alpha\beta}(x^{-1})$ is bilinear in ξ, x. Since it is linear in ξ, it suffices to find a basis $\{\xi_k\}$ of V such that $(D_{\xi_k} H_t\cdot H_t^{-1})_{\alpha\beta}(x^{-1})$ is linear in x. In fact, we will show that for every $\xi\in C$, $(D_\xi H_t\cdot H_t^{-1})_{\alpha\beta}(x^{-1})$ is linear in x. To do this, by the lemma, it suffices to show that for every $W\subset V$ with $\dim W=2$, $\xi\in W$, $(D_\xi H_t\cdot H_t^{-1})_{\alpha\beta}(x^{-1})$ is linear on $C\cap W$. We will do this for all α, β at once, so in verifying this we can change coordinates in $\mathbb{C}^n$.

What we do is this: we let ξ be a new base point of C and we change coordinates so that $H_{t_0}(\xi)=I_n$. This reduces us to verifying $(D_e H_t\cdot H_t^{-1})_{\alpha\beta}(x^{-1})$ is linear on $C\cap W$ when $\dim W=2$, $e\in W$. Now any such W is part of a subspace of V of the form $A\cdot e$, where $A\subset\exp(\mathfrak{p})$ is a maximal $\mathbb{R}$-split torus (of course, this $\mathfrak{p}$ corresponds to the *new* choice of e). Moreover, as above, we can diagonalize ρ on A:

$$\rho(a_1,\dots,a_n)=(\chi_i(a)\,\delta_{ij}),\qquad \chi_i(a)=\prod_{j=1}^{r} a_j^{s_{ij}}.$$

Then

$$H_t(ae)=\rho(a)^2 H_t(e),\qquad \rho(a)H_t(e)=H_t(e)\rho(a).$$

Since $e=(1,\dots,1)$, it follows:

$$D_e(H_t(ae))=\sum_{i=1}^{r}\frac{\partial}{\partial a_i}H_t(ae).$$

Using this, one calculates:

$$(D_e H_t\cdot H_t^{-1})_{\alpha\beta}(ae)=\left(\sum_{j=1}^{r}\frac{2s_{\alpha j}}{a_j}\right)\delta_{\alpha\beta}$$

and since on Ae, x^{-1} is given by $(a_1,\dots,a_r)\mapsto(a_1^{-1},\dots,a_r^{-1})$, this proves that $(D_e H_t\cdot H_t^{-1})_{\alpha\beta}$ is linear in x^{-1} on every $W\subset Ae$.

Now say $\xi,\eta\in\bar{C}$, $\langle\xi,\eta\rangle=0$. For a suitable coice of maximal torus A, we have $\xi,\eta\in\overline{A\cdot e}$. In the coordinates $(a_1,\dots,a_r)$ on $A\cdot e$, let

$$\xi=(\xi_1,\dots,\xi_r),$$
$$\eta=(\eta_1,\dots,\eta_r).$$

Since $\langle\ ,\ \rangle$ on Ae makes A self-adjoint, it is a quadratic form of the type $\langle a e, b e\rangle = \sum \lambda_i a_i b_i$, so $\langle \xi, \eta\rangle = 0$ means that for every i, $\xi_i = 0$ or $\eta_i = 0$. A calculation like that just made shows:

$$(D_\xi H_t \cdot H_t^{-1})_{\alpha\beta}(a e) = \left(\sum_{j=1}^{r} \frac{2 s_{\alpha j} \xi_j}{a_j}\right) \cdot \delta_{\alpha\beta}$$

i.e.,

$$(D_\xi \cdot H_t \cdot H_t^{-1})_{\alpha\beta}(x^{-1})|_{A \cdot e} = \left(\sum_{j=1}^{r} 2 s_{\alpha j} \xi_j a_j \cdot \delta_{\alpha\beta}\right).$$

This is clearly zero if $x = \eta$. QED

For the next result, suppose δ is any vector field in the manifold of values of t. Then

Proposition 2.5. *For all vector fields δ on T,*

$$(\delta H_t \cdot H_t^{-1})(x)$$

is independent of x, i.e., *depends on t alone.*

Proof. By equivariance:

$$\delta H_t(g x) = \rho(g) \cdot \delta H_t(x) \cdot {}^t\overline{\rho(g)}$$

hence

$$(\delta H_t \cdot H_t^{-1})(g \cdot e) = \rho(g) \cdot (\delta H_t \cdot H_t^{-1})(e) \cdot \rho(g)^{-1}.$$

By symmetry:

$$\rho(g^*) \cdot \delta H_t(e) = \delta H_t(e) \cdot {}^t\overline{\rho(g)}^{-1}$$

hence

$$\rho(g^*) \cdot (\delta H_t \cdot H_t^{-1})(e) \cdot \rho(g^*)^{-1} = (\delta H_t \cdot H_t^{-1})(e).$$

Together, these imply the Proposition. QED

Proposition 2.4 gives us estimates on $\|D_\xi H_t \cdot H_t^{-1}\|$. To work these out, we fix a maximal flag of boundary components of C. In the notation of [1], p. 109, choosing this flag and the base point $e \in C$ is equivalent to choosing in the Jordan algebra V, a maximal set of orthogonal idempotents:

$$e = \varepsilon_1 + \cdots + \varepsilon_r.$$

Let

$$C_i = \text{boundary component containing } \varepsilon_{i+1} + \cdots + \varepsilon_r.$$

Then

$$\bar{C} \supsetneqq \bar{C}_1 \supsetneqq \bar{C}_2 \supsetneqq \cdots \supsetneqq \bar{C}_r$$

is the flag. Also let

$$\tilde{C} = C \cup C_1 \cup C_2 \cup \cdots \cup C_r \cup (0)$$

and let

$$A = \sum_{i=1}^{r} \mathbb{R} \cdot \varepsilon_i.$$

Let P be the parabolic group which stabilizes the flag $\{\bar{C}_i\}$. Our estimates are based on:

Proposition 2.6. (1) *Let $\xi_1 \in \tilde{C}$ and let $\xi_1' \in V$ satisfy*

$$\left.\begin{array}{l}\langle \xi_1, \eta \rangle = 0\\ \eta \in \bar{C}\end{array}\right\} \Rightarrow \langle \xi_1', \eta \rangle = 0.$$

Then for every compact set $\omega \subset P$, there is a $K > 0$ such that:

$$|\langle \xi_1', x^{-1} \rangle| \leqq K \sqrt{ds^2_{C,x}(\xi_1, \xi_1)}, \qquad \text{all } x \in \omega \cdot A \cdot e.$$

(2) *Let $\xi_1 \in \tilde{C}$, $\xi_1' \in V$ be as above. Let $\xi_2 \in \tilde{C}$. Then for every compact set $\omega \subset P$, there is a $K > 0$ such that:*

$$|ds^2_{C,x}(\xi_1', \xi_2)| \leqq K \sqrt{ds^2_{C,x}(\xi_1, \xi_1)} \cdot \sqrt{ds^2_{C,x}(\xi_2, \xi_2)}.$$

Proof. We will use the Peirce decomposition of V defined by the idempotents ε_i:

$$V = \bigoplus_{i \leqq j} V_{ij}$$

where

$$x \in V_{ij} \Rightarrow \sum a_i \varepsilon_i \cdot x = \frac{a_i + a_j}{2} x.$$

This decomposition is orthogonal with respect to $\langle\ ,\ \rangle$ and C_k is an open cone in the subspace $\bigoplus_{k < i \leqq j} V_{ij}$. If $\xi_1 \in C_k$, then note that

$$\xi_1 \perp \bigoplus_{i \leqq j \leqq k} V_{ij}$$

and

$$\bigoplus_{i \leqq j \leqq k} V_{ij} \supset (\text{the boundary component of } C \text{ corresponding to } \varepsilon_1 + \cdots + \varepsilon_k).$$

This boundary component is open in $\bigoplus_{i \leqq j \leqq k} V_{ij}$. Thus

$$\xi_1' \perp \bigoplus_{i \leqq j \leqq k} V_{ij}$$

or

$$\xi_1' \in \bigoplus_{\substack{i \leqq j \\ k < j}} V_{ij}.$$

To prove (1), let $x=gae$, $g\in\omega$, $ae=\sum a_i\varepsilon_i$. Then

$$\begin{aligned}\langle\xi_1',x^{-1}\rangle&=\langle g^{-1}\xi_1',(ae)^{-1}\rangle\\&=\sum_{i=1}^{r}\frac{1}{a_i}\langle(g^{-1}\xi_1')_{ii},\varepsilon_i\rangle\end{aligned}$$

where $(g^{-1}\xi_1')_{ii}$ is the component of $(g^{-1}\xi_1')$ in V_{ii}. But ω preserves the flag, so

$$g^{-1}\xi_1'\in\bigoplus_{\substack{i\leq j\\k<j}}V_{ij}$$

too. Thus

$$\langle\xi_1',x^{-1}\rangle=\sum_{i=k+1}^{r}\frac{1}{a_i}\langle(g^{-1}\xi_1')_{ii},\varepsilon_i\rangle.$$

As g varies in ω, $\langle(g^{-1}\xi_1')_{ii},\varepsilon_i\rangle$ is bounded, hence

$$|\langle\xi_1',x^{-1}\rangle|\leq K_1\sum_{i=k+1}^{r}\frac{1}{a_i}.$$

Let $e^{(k)}=\varepsilon_{k+1}+\cdots+\varepsilon_r$ be the base point in C_k and note that, again by compactness of ω, there is a $K_2>0$ such that

$$g^{-1}\xi_1\in C_k+K_2e^{(k)},\qquad \text{all } g\in\omega,$$

hence

$$ds^2_{C,x}(K_2e^{(k)},K_2e^{(k)})\leq ds^2_{C,x}(g^{-1}\xi_1,g^{-1}\xi_1),\qquad \text{all } x\in C.$$

Thus

$$\begin{aligned}ds^2_{C,gae}(\xi_1,\xi_1)&=ds^2_{C,ae}(g^{-1}\xi_1,g^{-1}\xi_1)\\&\geq K_2^2ds^2_{C,ae}(e^{(k)},e^{(k)})\\&=K_2^2ds^2_{C,e}(a^{-1}e^{(k)},a^{-1}e^{(k)})\\&=K_2^2\sum_{i=k+1}^{r}\frac{1}{a_i^2}\langle\varepsilon_i,\varepsilon_i\rangle\\&\geq K_2^2K_3\left(\sum_{i=k+1}^{r}\frac{1}{a_i}\right)^2.\end{aligned}$$

The same procedure proves (2). If $\xi_1\in C_k$, $\xi_2\in C_l$, the argument goes like this:

$$\begin{aligned}ds^2_{C,gae}(\xi_1',\xi_2)&=\langle a^{-1}(g^{-1}\xi_1'),a^{-1}(g^{-1}\xi_2)\rangle\\&=\sum_{\left(\substack{i\leq j\\l<i\\k<j}\right)}\frac{1}{a_ia_j}\langle(g^{-1}\xi_1')_{ij},(g^{-1}\xi_2)_{ij}\rangle\\&\leq K_1\sqrt{\sum_{i=l+1}^{r}\frac{1}{a_i^2}}\cdot\sqrt{\sum_{j=k+1}^{r}\frac{1}{a_j^2}}\\&\leq K_1K_2\sqrt{ds^2_{gae}(\xi_1,\xi_1)}\cdot\sqrt{ds^2_{gae}(\xi_2,\xi_2)}.\end{aligned}\qquad \text{QED}$$

We can put everything we have said together as follows: Suppose $N=\dim V$ and $\xi_1,\ldots,\xi_N\in\tilde{C}$ span V. Define a simplicial cone $\sigma\subset C$ by

$$\sigma=\sum_{i=1}^{N}\mathbb{R}_+\cdot\xi_i.$$

Let $l_i: V\to\mathbb{R}$ be a dual basis: $l_i(\xi_j)=\delta_{ij}$. Then for all ρ, H_t as above, we have the estimates:

Proposition 2.7. *For all vector fields δ,δ' to the T-space, and $a\in\bar{C}$ there is a constant $K>0$ such that for all $x\in\operatorname{Int}(\sigma+a)$:*

$$\|D_{\xi_i}H_t\cdot H_t^{-1}(x)\|\leqq\frac{K}{l_i(x)-l_i(a)},$$

$$\|\delta H_t\cdot H_t^{-1}(x)\|\leqq K,$$

$$\|D_{\xi_i}(D_{\xi_j}H_t\cdot H_t^{-1})(x)\|\leqq\frac{K}{(l_i(x)-l_i(a))\cdot(l_j(x)-l_j(a))},$$

$$\|D_{\xi_i}(\delta H_t\cdot H_t^{-1})(x)\|=0,$$

$$\|\delta(D_{\xi_i}H_t\cdot H_t^{-1})(x)\|\leqq\frac{K}{l_i(x)-l_i(a)},$$

$$\|\delta(\delta' H_t\cdot H_t^{-1})(x)\|\leqq K.$$

Proof. Combine (2.4), (2.5) and (2.6) and the formula $ds_C^2(t_1,t_2)=-D_{t_1}(\langle t_2,x^{-1}\rangle)$ to get estimates in terms of $ds_C^2(\xi_i,\xi_i)$ on sets $\omega\cdot A\cdot e$. Then apply (2.2)(i) and (ii) to the inclusion $\operatorname{Int}(\sigma+a)\to C$, plus Ash's theorem that $\operatorname{Int}(\sigma+a)\subset\omega\cdot A\cdot e$ if ω is large enough ([1], Ch. II, §4), plus the formula

$$ds_\sigma^2=\sum_{i=1}^{n}\frac{dl_i^2}{l_i^2}$$

for the canonical metric on the homogeneous convex cone σ. QED

There is one final estimate that we need. For this, we first make a definition:

Definition. A linear map $T:\mathbb{C}^n\to\mathbb{C}^n$ is called *ρ-upper triangular* if the following holds: For all maximal $\mathbb{R}$-split tori $A\subset G$, let $X(A)=\operatorname{Hom}(A,\mathbb{G}_m)$ be the character group of A. As is well known, there is a basis $\gamma_1,\ldots,\gamma_r$ of $X(A)$ such that the weights of A acting on V are contained in $\gamma_i+\gamma_j$, $i\leqq j$ (and contain $2\gamma_i$, $1\leqq i\leqq r$). Partially order the characters by defining

$$\sum n_i\gamma_i\geqq\sum m_i\gamma_i \quad \text{if } n_i\geqq m_i, \quad \text{all } i.$$

Diagonalize $\rho(A)$:

$$\mathbb{C}^n=\bigoplus_{\lambda\in X(A)}V_\lambda.$$

Then T is ρ-upper triangular if for all A, all $\lambda_0\in X(A)$,

$$T\Big(\bigoplus_{\lambda\geqq\lambda_0}V_\lambda\Big)\subseteq\bigoplus_{\lambda\geqq\lambda_0}V_\lambda.$$

The estimate we need is:

Proposition 2.8. *If T is ρ-upper triangular, then for all $a \in C(F)$, there is a constant $K>0$ such that*

$$\|H_t(x)\cdot {}^t\bar{T}\cdot H_t(x)^{-1}\| \leqq K, \qquad all \ x \in C(F)+a.$$

Proof. Take a as a base point of $C(F)$ and pick any maximal torus A so that $Ae = \mathbb{R}^n_+$, $a=(1,\dots,1)$, $C(F)=K\cdot a\cdot e$. Therefore

$$C(F)+a=\{kae \mid k \in K, a=(a_1\dots,a_n), a_i \geqq 1, \text{ all } i\}.$$

Change coordinates in $\mathbb{C}^n$ so that

$$\rho(a)=\begin{pmatrix}\lambda_1(a) & & 0\\ & \ddots & \\ 0 & & \lambda_n(a)\end{pmatrix}.$$

Now

$$H_t(x)\cdot {}^t\bar{T}\cdot H_t(x)^{-1}=\rho(k)H_t(e)\rho(a)^2\cdot {}^t\overline{\rho(k)}\cdot {}^t\bar{T}\cdot {}^t\overline{\rho(k)}^{-1}\rho(a)^{-2}H_t(e)^{-1}\rho(k)^{-1}$$

so it suffices to bound $\|\rho(a)^2\cdot {}^t\overline{(\rho(k)^{-1}\cdot T\cdot\rho(k))}\cdot\rho(a)^{-2}\|$. This means we wish to bound:

$$|\lambda_i(a)^2\lambda_j(a)^{-2}(\rho(k)^{-1}\cdot T\cdot\rho(k))_{ji}|$$

when k ranges over K, and $a=(a_1,\dots,a_n)$ satisfies $a_i \geqq 1$, all i. This is equivalent to:

$$(\rho(k)^{-1}\cdot T\cdot\rho(k))_{ji} \neq 0 \Rightarrow \lambda_j \geqq \lambda_i \qquad (*)$$

(weights of A being partially ordered as in the definition). But for all i define

$$W_i=\rho(k)\Big(\bigoplus_{\lambda_j \geqq \lambda_i}\mathbb{C}\cdot e_j\Big)$$

(where $e_i \in \mathbb{C}^n$ is the i^{th} unit vector). Note that kAk^{-1} maps W_i into itself and that W_i is one of the sums of weight spaces referred to in the definition. Therefore $T(W_i) \subseteq W_i$, hence

$$(\rho(k)^{-1}T\rho(k))e_i \in \bigoplus_{\lambda_j \geqq \lambda_i}\mathbb{C}\cdot e_j$$

which is precisely $(*)$. QED

§ 3. The Proportionality Principle

Let D be an r-dimensional bounded symmetric domain and let Γ be a neat[2] arithmetic group acting on D. Then $X=D/\Gamma$ is a smooth quasi-projective

[2] Recall that following a definition of Borel, a "neat" arithmetic subgroup Γ of an algebraic group $\mathscr{G} \subset GL(n,\mathbb{C})$ is one such that for every $x \in \Gamma$, $x \neq e$, the group generated by the eigenvalues of x is torsion-free. Every arithmetic group Γ' has neat arithmetic subgroups of finite index

variety, called a *locally symmetric variety*, or an *arithmetic variety*. In [1], Ash, Rapoport, Tai and I have introduced a family of smooth compactifications $\bar{X}$ of X such that $\bar{X}-X$ has normal crossings. We must first recall how $\bar{X}$ is described locally. At the same time, we will need various details from the whole cumbersome appartus used to manipulate D so we will rapidly sketch these too. All results stated without proof can be found in [1].

By definition, $D \cong K \backslash G$, where G is a semi-simple adjoint group and K is a maximal compact subgroup. Inside the complexification $G_{\mathbb{C}}$ of G, there is a parabolic subgroup of the form $P_+ \cdot K_{\mathbb{C}}$ (P_+ its unipotent radical which is, in fact, abelian and $K_{\mathbb{C}}$ the complexification of K) such that $K = G \cap (P_+ \cdot K_{\mathbb{C}})$ and $G \cdot (P_+ \cdot K_{\mathbb{C}})$ open in $G_{\mathbb{C}}$. This induces an open G-equivariant immersion

$$\begin{array}{ccc} D & \hookrightarrow & \check{D} \\ \| & & \| \\ K \backslash G & & P_+ \cdot K_{\mathbb{C}} \backslash G_{\mathbb{C}}. \end{array}$$

Here $\check{D}$ is a rational projective variety known as a flag space and $G_{\mathbb{C}}$ is an algebraic group acting algebraically on $\check{D}$. Let $\bar{D}$ be the closure of D in D. The maximal analytic submanifolds $F \subset \bar{D} - D$ are called the boundary components of D. For each F, we set

$N(F) = \{g \in G \mid gF = F\}$,

$W(F) =$ unipotent radical of $N(F)$,

$U(F) =$ center of $W(F)$, a real vector space of dimension k, say,

$V(F) = W(F)/U(F)$: known to be abelian, centralizing $U(F)$. Via "exp", we get a section and write $W(F)$ set-theoretically as $V(F) \cdot U(F)$. Also $\dim V$ is even – let it be $2l$.

Next splitting $N(F)$ into a semi-direct product of a reductive part and its unipotent radical, we decompose $N(F)$ further:

$$N(F) = \underbrace{(G_h(F) \cdot G_l(F) \cdot M(F))}_{\substack{\text{direct product mod finite} \\ \text{central group}}} \cdot V(F) \cdot U(F),$$

where

a) $G_l \cdot M \cdot V \cdot U$ acts trivially on F, G_h mod a finite center being $\mathrm{Aut}^\circ(F)$,

b) $G_h \cdot M \cdot V \cdot U$ commutes with $U(F)$, G_l mod a finite central group acts faithfully on $U(F)$ by inner automorphisms

c) M is compact.

Here F is said to be rational if $\Gamma \cap N(F)$ is an arithmetic subgroup of $N(F)$. Mod Γ there are only finitely many such F, and if $F_1, \ldots, F_k$ are representatives:

$$X \cup \bigcup_{i=1}^{k} (F_i/\Gamma \cap N(F_i)),$$

with suitable analytic structure is Satake-Baily-Borel's compactification of X.

Next, for each F, we define an open subset $D_F \subset \check{D}$ by

$$D_F = \bigcup_{g \in U(F)_{\mathbb{C}}} g \cdot D.$$

The embedding of D in D_F is Pjatetski-Shapiro's realization of D as "Siegel Domain of 3rd kind". In fact, there is an isomorphism:

$$D_F \cong U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$$

such that not only $N(F)$ but even the bigger group $G_h \cdot (M \cdot G_l)_{\mathbb{C}} \cdot V_{\mathbb{C}} \cdot U_{\mathbb{C}}$ acts by "semi-linear transformations":

$$(x, y, t) \mapsto (Ax + a(y, t), B_t y + b(t), g(t))$$

(A, B_t matrices, a, b vectors) and

$$D = \{(x, y, t) \mid \operatorname{Im} x + l_t(y, y) \in C(F)\}$$

where $C(F) \subset U(F)$ is a self-adjoint convex cone homogeneous under the G_l-action on $U(F)$ and $l_t: \mathbb{C}^l \times \mathbb{C}^l \to U(F)$ is a symmetric $\mathbb{R}$-bilinear form.

Moreover

$U(F) \cong$ group of automorphisms of D: $(x, y, t) \mapsto (x + a, y, t)$, $a \in U(F)$,

$U(F)_{\mathbb{C}} \cong$ group of automorphisms of $D(F)$: $(x, y, t) \mapsto (x + a, y, t)$, $a \in U(F)_{\mathbb{C}}$,

$W(F) \cong$ group of automorphisms of D: $(x, y, t) \mapsto (x + a(u, t), y + b(t), t)$ and the group $V(F)$ acts, for each t, simply transitively on the space $\mathbb{C}^l$ of possible y-values.

There is a technical lemma which we will need about this action:

Lemma. *Let $t_0 \in F$, $e_0 \in C(F)$, and let $u_0 \in U(F)_{\mathbb{C}}$ be the map $(x, y, t) \mapsto (x + ie_0, y, t)$. Let $e = (ie_0, 0, t_0)$ be a base point of D, so that* $\operatorname{Stab}_G(e) = K$, *a maximal compact of G and* $\operatorname{Stab}_{G_{\mathbb{C}}}(e) = K_{\mathbb{C}} \cdot P_+$. *Moreover,* $\operatorname{Stab}_{G_l}(e_0) = K_l$ *is a maximal compact in G_l. Since $G_l \subset \operatorname{Stab}(0, 0, t_0)$, $u_0(G_l)u_0^{-1} \subset \operatorname{Stab}(e)$ and we may look at*

$$\alpha: G_l \xrightarrow{\text{conj. by } u_0} \operatorname{Stab}_{G_{\mathbb{C}}}(e) \xrightarrow{\text{mod } P_+} K_{\mathbb{C}}.$$

If $*$ is the Cartan involution of G_l with respect to K_l, then:

$$\alpha(g^*) = \overline{\alpha(g)}.$$

Proof. This is a straightforward calculation, for instance, using the fundamental decomposition of $\mathfrak{g} = \operatorname{Lie} G$ via $\mathrm{sl}(2)^r \subset \mathfrak{g}$ (cf. [1], p. 182) and the description of $\operatorname{Lie}(M \cdot G_l)$ in this decomposition for the standard boundary components F_S given in [1], p. 226.

We now describe local coordinates on $\bar{X}$. Recall that $\bar{X}$ is not unique but depends on the choice of certain auxiliary simplicial decompositions. We need not recall these in detail. The chief thing is that each $\bar{X}$ is covered by a finite set of coordinate charts constructed as follows:

1) take a rational boundary component F of D,

2) take $\{\xi_1, \ldots, \xi_k\}$ a basis of $\Gamma \cap U(F)$ such that $\xi_i \in \overline{C(F)} \subset U(F)$ and in fact, $\xi_i \in C(F) \cup C_1 \cup C_2 \cup \cdots \cup C_l = \tilde{C}$, where $\overline{C(F)} \supseteq \bar{C}_1 \supseteq \bar{C}_2 \supseteq \cdots \supseteq \bar{C}_l$ is a flag of

boundary components (cf. §2) and at least one ξ_i is in $C(F)$: say

$$\xi_1, \ldots, \xi_m \in C(F), \qquad \xi_{m+1}, \ldots, \xi_k \in \overline{C(F)} - C(F),$$

3) let l_i: $U(F)_{\mathbb{C}} \to \mathbb{C}$ be dual to $\{\xi_i\}$, i.e., $l_i(\xi_j) = \delta_{ij}$,

4) consider the exponential:

$$\begin{array}{ccc}
D & \subset & (U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F) \\
\downarrow & & \downarrow {\scriptstyle (e^{2\pi i l_1(x)}, \ldots, e^{2\pi i l_k(x)}, y, t)} \\
D/\Gamma \cap U(F) & \subset & (\mathbb{C}^{*k} \times \mathbb{C}^l \times F) \\
\downarrow & & \\
X & &
\end{array}$$

5) Define $(D/\Gamma \cap U(F))^{\sim}$ to be the set of $P \in \mathbb{C}^k \times \mathbb{C}^l \times F$ which have a neighborhood U such that

$$U \cap (\mathbb{C}^{*k} \times \mathbb{C}^l \times F) \subset (D/\Gamma \cap U(F)).$$

Note that

$$(D/\Gamma \cap U(F))^{\sim} \supset \bigcup_{i=1}^{m} \{(z, y, t) \mid z = (z_1, \ldots, z_k), z_i = 0\} = S(F, \{\xi_i\}).$$

6) The basic property of $\bar{X}$ is that for suitable F, $\{\xi_i\}$, the covering map p extends to a local homeomorphism

$$\bar{p}: (D/\Gamma \cap U(F))^{\sim} \to \bar{X}$$

and that every point of $\bar{X}$ is equal to $\bar{p}((z, y, t))$, where $z_i = 0$, some $1 \leqq i \leqq m$, for some such F, $\{\xi_i\}$.

We now come to the main results of this paper. Let E_0 be a G-equivariant analytic vector bundle of rank n on D. E_0 is defined by the representation

$$\sigma: K \to GL(n, \mathbb{C})$$

of the stabilizer K of the base point $e_+ \in D$ in the fibre $\mathbb{C}^n$ of E_0 over e_+. We complexify σ and extend it to $P_+ \cdot K_{\mathbb{C}}$ by letting it kill P_+. Then σ defines a $G_{\mathbb{C}}$-equivariant analytic vector bundle $\check{E}_0$ on $\check{D}$ also. In the other direction, we can divide E_0 by Γ obtaining a vector bundle E on X. Since K is compact, E_0 carries a G-invariant Hermitian metric h_0, which induces a Hermitian metric h on E. We claim:

Main Theorem 3.1. *E admits a unique extension $\bar{E}$ to $\bar{X}$ such that h is a singular Hermitian metric good on $\bar{X}$.*

These various bundles are all linked as in this diagram:

$$\begin{array}{ccccccc}
\check{E}_0 & \overset{\text{restriction}}{\dashrightarrow} & E_0 & \overset{\text{descent}}{\dashrightarrow} & E & \overset{\text{extension}}{\dashrightarrow} & \bar{E} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D & \longleftarrow & D & \longrightarrow & X & \longrightarrow & \bar{X}.
\end{array}$$

Proof. We saw in § 1 that $\bar{E}$, if it existed, has as its sections the sections of E with growth $O\left(\left(\sum_{i=1}^{m} \log|z_i|\right)^{2N}\right)$ along $\bar{X}-X$. To see that the set of these sections defines an analytic vector bundle on $\bar{X}$, it suffices to check this locally, e.g., on $(D/\Gamma \cap U(F))^{\sim}$. But now the bundle $\check{E}_0$ restricts to a bundle E_F on D_F with $N(F) \cdot U(F)_{\mathbb{C}}$ acting equivariantly. Now note that the subgroup $U(F)_{\mathbb{C}}$ acts simply transitively and holomorphically on the first factor $U(F)$ of D_F in its Siegel Domain presentation. Since $\mathbb{C}^l \times F$ is contractible and Stein (F is another bounded symmetric domain), it follows that E_F has a set of n holomorphic sections $e_1, \ldots, e_n$ such that

i) e_i is $U(F)_{\mathbb{C}}$-invariant,

ii) $e_1(x), \ldots, e_n(x)$ are a basis of $E_F(x)$, all $x \in D_F$.

Dividing by $\Gamma \cap U(F)$, E_F descends to a vector bundle E'_F on $\mathbb{C}^{*k} \times \mathbb{C}^l \times F$, which is also globally trivial via the same basic sections $e_1, \ldots, e_n$. We can then extend E'_F to $\mathbb{C}^k \times \mathbb{C}^l \times F$ so as to be trivial with these basic sections. We must show that along $S(F, \{\xi_i\})$ the sheaf of sections of this extension is exactly the sheaf of sections of E'_F on $(D/\Gamma \cap U(F))$ with growth $O\left(\left(\sum_{i=1}^{k} \log|z_i|\right)^{2N}\right)$ on the coordinate hyperplanes $\bigcup_{i=1}^{k} (z_i = 0)$. Equivalently, this means that $h(e_i, e_j)$ and $(\det h(e_i, e_j))^{-1}$ have this growth. To do this, it is convenient to use a 2nd basis of E'_F, which is C^∞ but not analytic. Note that $V(F) \cdot U(F)_{\mathbb{C}}$ acts simply transitively on the 1st 2 factors of D_F. So we can find $e'_1, \ldots, e'_n \in \Gamma(D_F, E_F)$ such that

i') e'_i is $V(F) \cdot U(F)_{\mathbb{C}}$-invariant,

ii') $e'_1(x), \ldots, e'_n(x)$ are a basis of $E_F(x)$, all $x \in D_F$.

iii') On $(0, 0) \times F$, $e'_i = e_i$, hence are holomorphic sections.

$\{e_i\}$ and $\{e'_i\}$ are related by an invertible $U(F)_{\mathbb{C}}$-invariant matrix S; so that $|S_{ij}|$ and $|\det S|^{-1}$ are uniformly bounded on subsets

$$(\mathbb{C}^*)^k \times \begin{pmatrix} \text{compact subset} \\ \text{of } \mathbb{C}^l \times F \end{pmatrix} \subset (\mathbb{C})^k \times \mathbb{C}^l \times F.$$

Therefore it is enough to check that $h(e'_i, e'_j)$, $(\det h(e'_i, e'_j))^{-1}$ have the required growth.

Now if $g \in G_l(F)$, note that because g normalizes $V \cdot U_{\mathbb{C}}$, $g e'_i$ is another $V \cdot U_{\mathbb{C}}$-invariant section of E'_F. Therefore

$$g \cdot (e'_i) = \sum_{j=1}^{n} r_{ij}(t) \cdot e'_j \qquad \text{(here } t \text{ is the coordinate on } F).$$

Since $g(U(F)_{\mathbb{C}} \times \mathbb{C}^l \times \{t\}) = U(F)_{\mathbb{C}} \times \mathbb{C}^l \times \{t\}$, it follows that for each t,

$$g \mapsto \rho_t(g) = (r_{ij}(t))$$

is an n-dimensional representation of $G_l(F)$. In fact, as $g(0, 0, t) = (0, 0, t)$, this is just the representation of the stabilizer of $(0, 0, t)$ restricted to $G_l(F)$. This shows that ρ_t is a holomorphic family of algebraic representations of G_l (this is not trivial because G_l has a positive dimensional center). Since G_l is reductive, we may change our basis $\{e'_i\}$ so that ρ_t is in fact independent of t.

Now consider the functions $h_{ij} = h(e'_i, e'_j)$ on D. Since h, e'_i and e'_j are $U \cdot V$-invariant, so is h_{ij}, hence in Siegel domain notation, it is a function of $u = \operatorname{Im} x + l_t(y, y)$ and t, i.e., is a function on $C(F) \times F$. For each fixed $t \in F$, and variable $u \in C(F)$,

$$H_t(u) = (h_{ij}(u, t))$$

is a map

$$C(F) \to C_n = \begin{pmatrix} \text{cone of pos. def. } n \times n \\ \text{Hermitian matrices} \end{pmatrix}.$$

I claim that (ρ, H_t) satisfy the hypothesis of §2. In fact,

$$\begin{aligned} H_t(gu)_{ij} &= h(ge'_i, ge'_j) \\ &= \sum_{k,l} r_{ik}(g)\, h(e'_k, e'_l)\, \overline{r_{jl}(g)} \\ &= (\rho(g) \cdot H_t(u) \cdot {}^t\overline{\rho(g)})_{ij}. \end{aligned}$$

Let $e = (ie_0, 0, t_0)$ be a base point of D. Since h is a K-invariant metric on E_0,

$$h(ke'_i, ke'_j)(e) = h(e'_i, e'_j)(e), \quad \text{all } k \in K.$$

Complexifying, we get too:

$$h(ke'_i, \bar{k}e'_j)(e) = h(e'_i, e'_j)(e), \quad \text{all } k \in K_{\mathbb{C}}.$$

Let $u_0 \in U_{\mathbb{C}}$ be given by $(x, y, t) \mapsto (x + ie_0, y, t)$. Then for all $g \in G_l(F)$, $u_0 g u_0^{-1}(e) = e$, so $u_0 g u_0^{-1} = k \cdot p$, where $k \in K_{\mathbb{C}}$, $p \in P_+$. By the lemma above, $u_0 g^* u_0^{-1} = \bar{k} \cdot p'$ for some other element $p' \in P_+$. Therefore:

$$\begin{aligned} h(e'_i, e'_j)(e) &= h(pe'_i, p'e'_j)(e) \quad \text{(since } P_+ \text{ acts trivially on } E_0(e)) \\ &= h(kpe'_i, \bar{k}p'e'_j)(e) \\ &= h(u_0 g u_0^{-1}(e'_i), u_0 g^* u_0^{-1}(e'_j))(e) \\ &= \sum r_{ik}(g)\, h(e'_k, e'_l)(e)\, \bar{r}_{jl}(g^*) \quad \text{(since } u_0 e'_i = e'_i, \text{ all } i) \end{aligned}$$

or

$$H_{t_0}(e_0) = \rho(g)\, H_{t_0}(e_0)\, {}^t\overline{\rho(g^*)}.$$

In fact, the same holds if t_0 is replaced by any $t \in F$ as follows easily using the $G_h(F)$-invariance of h and the fact that $G_l(F)$ and $U(F)_{\mathbb{C}}$ commute with $G_h(F)$.

Thus we have the full situation of § 2. In particular, we have available all the bounds of § 2.

As above, to describe local coordinates near boundary points of $\bar{X}$, choose a simplicial cone:

$$\sigma = \sum_{i=1}^{k} \mathbb{R}_+ \cdot \xi_i \subset \overline{C(F)}; \qquad \xi_i \in C(F) \Leftrightarrow 1 \leqq i \leqq m$$

and let l_i be dual linear functionals on $U(F)_{\mathbb{C}}$. Then if (x, y, t) are Siegel domain coordinates on $D(F)$, and $z_i = e^{2\pi i\, l_i(x)}$, then (z, y, t) at points where at least one z_i is $0\,(1 \leqq i \leqq m)$ are local coordinates on $\bar{X}$. Moreover each point $P \in S(F, \{\xi_i\}) \subset \bar{X}$ has an open neighborhood whose intersection with X is contained in the image of

$$\{(x, y, t) \mid u = \operatorname{Im} x + l_t(y, y) \in \sigma + a,\ y \in Y',\ t \in F'\}$$

for some σ as above, $a \in C(F)$, Y' (resp. F') a relatively compact subset of Y (resp. F). Note that $\log |z_i| = -2\pi l_i(\operatorname{Im} x)$. At all points $P \in S(F, \{\xi_i\})$, we have to estimate $h(s_i', s_j')^{\pm 1}$ in terms of

$$\left(\sum \log \left|\frac{z_i}{C}\right|\right)^{2N}$$

$\left(\text{choose } C \text{ large enough so that } \left|\frac{z_i}{C}\right| < 1 \text{ in a neighborhood of } P\right)$. This is the same as estimating $h(s_i', s_j')^{\pm 1}$ in terms of

$$\left(\sum l_i(u) + C\right)^{2N}$$

(choose C large enough so that $l_i(u) + C > 0$ in a neighborhood of P). But if $l_i(u) + C > 0$, $(\sum l_i(u) + C)^{2N}$ is comparable with $(\sum l_i(u)^2)^N$, hence with $\langle u, u\rangle^N$. This is exactly the estimate that Proposition 2.3 gives us. Next we have to estimate the connection and the curvature. Now in terms of a *holomorphic* trivialization of E_0, the connection is given by $\partial h \cdot h^{-1}$. What we have is good control of $\delta_1 h \cdot h^{-1}$ and $\delta_2(\delta_1 h \cdot h^{-1})$ for all vector fields δ_1, δ_2 in terms of the *real analytic* trivialization given by $\{e_i'\}$. Write

$$e_i = \sum_{j=1}^{n} a_{ij}\, e_j',$$
$$A = \text{matrix } (a_{ij}),$$
$$H^{an} = \text{matrix } h(e_i, e_j).$$

Then

$$H^{an} = A \cdot H \cdot {}^t\bar{A}.$$

From this you calculate:

$$\begin{pmatrix}\text{Connexion in} \\ \text{trivialization} \\ \{e_i\}\end{pmatrix} = \partial H^{an} \cdot (H^{an})^{-1}$$
$$= \partial A \cdot A^{-1} + A \cdot \partial H \cdot H^{-1} \cdot A^{-1} + A \cdot H \cdot (\partial^t\bar{A} \cdot {}^t\bar{A}^{-1})\, H^{-1} A^{-1},$$

$$d\begin{pmatrix}\text{connexion in}\\ \text{trivialization}\\ \{e_i\}\end{pmatrix} = d(\partial H^{an}\cdot(H^{an})^{-1})$$
$$\begin{aligned}
&= d(\partial A\cdot A^{-1}) + d(A\cdot\partial H\cdot H^{-1}\cdot A^{-1})\\
&\quad + d(A\cdot H\cdot(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})\cdot H^{-1}\cdot A^{-1})\\
&= d(\partial A\cdot A^{-1}) + dA\cdot(\partial H\cdot H^{-1})\cdot A^{-1}\\
&\quad + A\cdot d(\partial H\cdot H^{-1})\cdot A^{-1} + A\cdot\partial H\cdot H^{-1}\cdot A^{-1}\cdot dA\cdot A^{-1}\\
&\quad + dA\cdot[H\cdot(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})H^{-1}]\cdot A^{-1}\\
&\quad + A(dH\cdot H^{-1})[H\cdot(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})\cdot H^{-1}]\cdot A^{-1}\\
&\quad + A\cdot[H\cdot d(\partial{}^t\bar{A}\cdot{}^tA^{-1})\cdot H^{-1}]\cdot A^{-1}\\
&\quad + A\cdot[H\cdot(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})\cdot H^{-1}]\cdot(dH\cdot H^{-1})\cdot A^{-1}\\
&\quad + A\cdot[H\cdot(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})\cdot H^{-1}]\cdot A^{-1}\cdot dA\cdot A^{-1}.
\end{aligned}$$

Therefore, since A is a C^∞ metric on $\bar{X}$, to show that the connexion and its differential have Poincaré growth on $\bar{X}$ it suffices to prove that the 4 forms:

$$\begin{aligned}
&\partial H\cdot H^{-1},\\
&d(\partial H\cdot H^{-1}),\\
&H\cdot(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})\cdot H^{-1},\\
&H\cdot d(\partial{}^t\bar{A}\cdot{}^t\bar{A}^{-1})\cdot H^{-1}
\end{aligned}$$

have Poincaré growth on $\bar{X}$. To check this for the first two, note that H is a function on $C(F)\times F$, hence it suffices to bound $\delta_1 H\cdot H^{-1}$, $\delta_2(\delta_1 H\cdot H^{-1})$ for all vector fields δ_1, δ_2 on $C(F)\times F$. But the Poincaré metric is given in Siegel coordinates by:

$$\begin{aligned}
ds^2 &= \sum\frac{|dz_i|^2}{|z_i|^2\left(\log\left|\frac{z_i}{C}\right|\right)^2} + \sum|dy_i|^2 + \sum|dt_i|^2\\
&= \sum\frac{|dl_i(x)|^2}{(l_i(\operatorname{Im}x)+C')^2} + \sum|dy_i|^2 + \sum|dt_i|^2
\end{aligned}$$

(choose C large enough so that $l_i(\operatorname{Im}x)+C>0$ near the boundary point in question). Therefore the bounds of Proposition 2.7 imply that $\partial H\cdot H^{-1}$ and $d(\partial H\cdot H^{-1})$ have Poincaré growth.

To check the result for the last two, we need to know what sort of a function A is. Firstly, since e_i and e_i' are both $U_{\mathbb{C}}$-invariant, A is $U_{\mathbb{C}}$-invariant, i.e., is a function of y and t alone. Therefore it has derivatives only in the y and t directions, so to say these have Poincaré growth is just to say they are bounded along $\bar{X}$. Next, for all $t_0\in F$, the action of V on vector space of points (y, t_0) puts a complex structure on V. Thus it defines a splitting $V_{\mathbb{C}} = V_{t_0}^+\oplus V_{t_0}^-$, where $V_{t_0}^\pm$ are complex subspaces and $V_{t_0}^-$ acts trivially on the points (y, t_0), while $V_{t_0}^+$ acts simply transitively. If we fix one t_0, then $V_{t_0}^+$ still acts simply transitively and holomorphically on the vector spaces of points (y, t) for t near t_0. Thus it is natural to choose our holomorphic basis e_i to be in fact $V_{t_0}^+\cdot U_{\mathbb{C}}$-invariant.

Moreover, it is easy to see that $G_l \cdot V$ normalizes $V_{t_0}^+ \cdot U_{\mathbb{C}}$ for all t_0, hence that the action of $G_l \cdot V$ in terms of the holomorphic basis is given by

$$g(e_i) = \sum_{j=1}^{n} \tilde{r}_{ij}(t) \cdot e_j$$

where

$$g \mapsto \tilde{\rho}_t(g) = (matrix\ \tilde{r}_{ij}(t))$$

is a representation of $G_l \cdot V$. Comparing ρ and $\tilde{\rho}$, we get:

$$\rho(g) = g^* A^{-1} \cdot \tilde{\rho}_t(g) \cdot A.$$

Since $e_i = e_i'$ on $(0,0) \times F$, $\rho(g) = \tilde{\rho}_t(g)$ for all $g \in G_l$ and $A(0,t) = I_n$. Thus if $v(0,t) = (y,t)$,

$$I_n = \rho(v) = A(y,t)^{-1} \cdot \tilde{\rho}_t(v) \cdot I_n,$$

or

$$A(y,t) = \tilde{\rho}_t(v).$$

Now we use the simple:

Lemma. *Let σ be an algebraic representation of $G_l \cdot V$, and let σ_0 be the restriction of σ to G_l. Then for all $v \in V$, $\sigma(v)$ is σ_0-upper triangular.*

Proof. Let $A \subset G_l$ be a maximal $\mathbb{R}$-split torus. As in §2, there is a basis $\gamma_1, \ldots, \gamma_r$ of the character group of A such that A acts on the vector space $U(F)$ containing the cone $C(F)$ through the weights $\gamma_i + \gamma_j$. Then its action on $V(F)$ is through the weights γ_i (cf. [1], p. 224). Now if $V_i(F) \subset V(F)$ is the root space corresponding to γ_i, and if we diagonalize $\sigma(A)$:

$$\mathbb{C}^n = \bigoplus_{\lambda \in X(A)} W_\lambda,$$

then

$$V_i(F)(W_\lambda) \subset W_{\lambda + \gamma_i}$$

hence $V(F)$ acts in a σ_0-upper triangular fashion. QED

Thus $A(y,t)$ is ρ-upper triangular for all y, t, hence so are

$$A^{-1} \cdot \bar{\partial} A \quad \text{and} \quad d(A^{-1} \cdot \bar{\partial} A).$$

Applying Proposition 2.8, it follows that

$$H \cdot {}^t\overline{(A^{-1} \cdot \delta A)} \cdot H^{-1} \quad \text{and} \quad H \cdot {}^t\overline{(d(A^{-1} \cdot \bar{\partial} A))} \cdot H^{-1}$$

are bounded in a neighborhood of every point of $\bar{X}$, as required. This completes the proof of the Main Theorem.

A natural question is whether these vector bundles $\bar{E}$ are in fact pull-backs of vector bundles on less blown up compactifications of D/Γ. Thus Baily and Borel defined in [2] a "minimal" but usually highly singular compactification $(D/\Gamma)^*$ of D/Γ. Unfortunately $\bar{E}$ is only rarely a vector bundle on $(D/\Gamma)^*$ (we will see below one case where it is however). However, in [1], Ash, Rapoport, Tai and I defined not only smooth compactifications of D/Γ but also a bigger class of compactifications with toroidal singularities (cf. [9]). These are important because when you try to resolve $(D/\Gamma)^*$, often there is a $\overline{D/\Gamma}$ with relatively simple structure on the boundary but still with some toroidal singularities. It is easy to see that the construction above of $\bar{E}$ goes through equally well on all of these compactifications: it gives vector bundles on all of them such that whenever compactification a dominates compactification b, then extension a is the pull-back of extension b.

The Main Theorem, plus Hirzebruch's original proof of his proportionality theorem for compact locally symmetric varieties X, gives us easily the proportionality theorem in the general case:

Proportionality Theorem 3.2. *As above, fix:*

$X=$ *an arithmetic variety* D/Γ, $D=K\setminus G$,

$\bar{X}=$ *a smooth compactification as in* [1],

$\check{D}=$ *compact dual of* D.

Then there is a constant K, *which in terms of a natural choice of metric on* D *is the volume of* X, *such that for all:*

$\check{E}_0=G_{\mathbb{C}}$*-equivariant analytic rank* n *vector bundle on* $\check{D}$ *defined by a representation of* $\mathrm{Stab}_{G_{\mathbb{C}}}(e)$ *trivial on* P_+,

$\bar{E}=$ *corresponding vector bundle on* $\bar{X}$,

the following formula holds:

$$c^{\alpha}(\bar{E})=(-1)^{\dim X}\cdot K\cdot c^{\alpha}(\check{E}_0), \qquad \textit{all } \alpha=(\alpha_1,\dots,\alpha_n),\ \textstyle\sum\alpha_i=\dim X.$$

Proof. As above, choose a G-invariant Hermitian metric h_0 on E_0. By the Main Theorem, h_0 defined a "good" Hermitian metric h on E, hence its Chern forms $c_k(E,h)$ represent the Chern classes of $\bar{E}$. But on D, $c_k(E,h)$ are G-invariant forms, so:

$$\begin{aligned} c^{\alpha}(\bar{E}) &= \int_X c^{\alpha}(E,h) \\ &= \int_{\substack{F\text{' al Domain}\\ F\subset D}} c^{\alpha}(E_0,h_0) \\ &= \mathrm{vol}(F)\cdot c^{\alpha}(E_0,h_0)(e). \end{aligned}$$

Now if G^c is a compact form of G:

$$\mathrm{Lie}\,G=\mathfrak{k}\oplus\mathfrak{p},$$
$$\mathrm{Lie}\,G^c=\mathfrak{k}\oplus i\mathfrak{p}$$

then $\check{E}_0$ has a unique G^c-invariant Hermitian metric $\check{h}$ equal to h_0 at e. So

$$\begin{aligned} c^\alpha(\check{E}_0) &= \int_{\check{D}} c^\alpha(\check{E}_0, \check{h}) \\ &= \mathrm{vol}(\check{D}) \cdot c^\alpha(\check{E}_0, \check{h})(e). \end{aligned}$$

Then – and this is the essence of Hirzebruch's remarkable proof – a simple local calculation shows (cf. [7]):

$$c_k(E, h)(e) = (-1)^k\, c_k(\check{E}_0, \check{h}).$$

This proves the result.

To apply this result, it is important to describe the bundles $\bar{E}$ as closely as possible. Firstly, we can characterize their sections, precisely as a special case of the general definition of automorphic forms given by Borel [3]. Let $\rho: K \to GL(n, \mathbb{C})$ be a representation of K, and let

$$E_0 = G \times^K \mathbb{C}^n$$

be the associated G-equivariant vector bundle over $D = K \setminus G$ (i.e., E_0 = set of pairs (g, a), $\mathrm{mod}(g, a) \sim (kg, \rho(k)a)$). E_0 has a complex structure as follows: complexify ρ and extend it to $K_{\mathbb{C}} \cdot P_+$ to be trivial on P_+. Then E_0 is the restriction to D to the bundle

$$\check{E}_0 = G_{\mathbb{C}} \times^{(K_{\mathbb{C}} \cdot P_+)} \mathbb{C}^n$$

on $\check{D}$, and in the definition of $\check{E}_0$, everything is analytic. Borel introduces a measure of size on G by:

$$\|g\|_G = \mathrm{tr}(\mathrm{Ad}\, g^{*-1} \cdot g)$$
$$* = \text{Cartan involution on } G \text{ w.r.t. } K,$$

and defines holomorphic ρ-automorphic form f to be a function

$$f: G \to \mathbb{C}^n$$

such that

(1) $f(kg\gamma) = \rho(k) f(g)$, all $k \in K$, $\gamma \in \Gamma$,
(2) f induces a holomorphic section of E_0,
(3) $|f(g)| \leqq C \cdot \|g\|_G^n$, some $n \geqq 1$, $C > 0$.

Then one can show:

Proposition 3.3. *In the above notation:*

$$\Gamma(\bar{X}, \bar{E}) \cong \{\textit{vector space of holomorphic } \rho\textit{-automorphic forms}\}.$$

Sketch of Proof. The problem is to check that the bound (3) is equivalent to requiring that the corresponding section of $\bar{E}$ over X has growth $O((\sum \log |z_i|)^{2n})$

along $\bar{X}$. But $\|g\|_G$ defines a measure of size on D and on X by:

$\forall x \in D$: $\|x\|_D = \|g\|_G$ if x corresponds to the coset $K \cdot g$,

$\forall \bar{x} \in X$, image of x: $\|\bar{x}\|_X = \min_{\gamma \in \Gamma} \|\gamma(x)\|_D = \min_{\gamma \in \Gamma} \|g\gamma\|_G$.

Then holomorphic ρ-automorphic forms are clearly holomorphic sections s of $\bar{E}$ *over* X, such that

$$h(s,s)(x) \leqq C_1 \|x\|_X^n, \quad \text{some } n \geqq 1,\ C_1 > 0.$$

But if d_D is a G-invariant distance function on D, then it is easy to see (using $G = K \cdot A \cdot K$) that $d_D(x,e)$ and $\log\|x\|$ are bounded with respect to each other. In another paper [4], Borel has proven that if x is restricted to a Siegel set $\mathfrak{S} = \omega \cdot A_t \cdot e \subset D$, then

$$\min_{\gamma \in \Gamma} d_D(x\gamma, e) \approx d_D(x,e) \approx d_A(a(x), e)$$

(here $x = \omega(x)\, a(x) \cdot e$ and $\approx$ means the differences are bounded). Applying this to a subset of a Siegel Domain of 3rd kind of the type $\{(x,y,t) \mid y \in V', t \in F', \operatorname{Re} x \in U', \operatorname{Im} x - l_t(y,y) \in \sigma + a\}$ where $U' \subset U$, $V' \subset V$, $F' \subset F$ are compact subsets and $\sigma \subset \overline{C(F)}$ is a simplicial cone and $a \in C(F)$, we see that

$$\min_{\gamma \in \Gamma} d_D((x,y,t)\,\gamma, e)$$

can be bounded above and below by expressions

$$C_2 \log(\langle \operatorname{Im} x - l_t(y,y), e \rangle), \quad e \in C(F),\ C_2 > 0$$

hence $\|x\|_X$ can be bounded above and below by expressions

$$\langle \operatorname{Im} x - l_t(y,y), e \rangle^n, \quad e \in C(F),\ n \geqq 1.$$

Describing σ as $l_i \geqq 0$ as above (l_i linear functionals on U), this is of the same size as

$$\left(\sum l_i(\operatorname{Im} x) + C_3\right)^n$$

and as $z_i = e^{2\pi i\, l_i(x)}$, this is equal to

$$\left(-\sum \log(|z_i|/C_4)\right)^n. \quad \text{QED}$$

Next, there are 2 particular equivariant bundles where we can describe $\bar{E}$ more completely:

Proposition 3.4. a) *If* $E_0 = \Omega_D^1$, *the cotangent bundle, with canonical G-action, then* $\bar{E} = \Omega_{\bar{X}}^1(\log)$, *the bundle on* $\bar{X}$ *whose sections in a polycylinder* $\Delta^n \subset \bar{X}$ *such that*

$$\Delta^n \cap (\bar{X} - X) = \bigcup_{i=1}^{k} \begin{pmatrix} \text{coordinate hyperplanes} \\ z_i = 0 \end{pmatrix}$$

are given by

$$\sum_{i=1}^{k} a_i(z)\frac{dz_i}{z_i}+\sum_{i=k+1}^{n} b_i(z)dz_i.$$

b) *If E_0 is the canonical line bundle Ω_D^n, then $\bar{E}$ is the pull-back of an ample line bundle $\mathcal{O}(1)$ on the Baily-Borel compactification X^* of X. The sections of $\mathcal{O}(n)$ are the modular forms with respect to the* nth *power of the canonical automorphy factor given by the jacobian, hence $\mathcal{O}(n)$, $n \gg 0$, is the very ample bundle used by Baily and Borel to embed in X^* in $\mathbb{P}^N$.*

Proof. Using Siegel Domain coordinates (x, y, t) on $D(F)$, a $U(F)_{\mathbb{C}}$-invariant basis of $\Omega^1_{\underline{D}(F)}$ is given by $\{dx_i, dy_j, dt_k\}$. Therefore these span the corresponding bundle on $\bar{X}$ near the boundary F. But here

$$\{z_i = e^{2\pi i\, l_i(x)},\ y_j, t_k\}$$

are coordinates and these differentials are $\left\{\frac{dz_i}{z_i},\ dy_j,\ dt_k\right\}$. This proves (a).

To prove (b), recall that X^* is set-theoretically the union of X and of $F/\Gamma \cap N(F)$ for all rational boundary components F. Moreover, if $P \in F/\Gamma \cap N(F) \subset X^*$, then there exists a neighborhood $U \subset X^*$ and an open set $V \subset D$ such that V maps to $U \cap X$ and V is a $(G_l(F)\cdot V(F)\cdot U(F)) \cap \Gamma$-bundle over $U \cap X$. Now say $\{s_i\}$ is the $U(F)_{\mathbb{C}}$-invariant holomorphic basis of $\check{E}_0$ on $D(F)$ used to extend E over the F-boundary points of $\bar{X}$. If we verify that each s_i is $(G_l(F)\cdot V(F)\cdot U(F)) \cap \Gamma$-invariant it follows that E is trivial on $U \cap X$ and moreover, if $\pi\colon \bar{X} \to X^*$ is the canonical birational map, then $\{s_i\}$ are a basis of $\pi_* \bar{E}$ over U. Thus $\pi_* \bar{E}$ is a vector bundle which pulls-back to $\bar{E}$ on $\bar{X}$. Now in the case in question, E is a line bundle. s_1 can be identified with the differential form

$$(\bigwedge_i dx_i) \wedge (\bigwedge_j dy_j) \wedge (\bigwedge_k dt_k)$$

on $D(F)$, and $G_l \cdot V \cdot U$ acts on it by multiplication by the Jacobian determinant in the Siegel Domain coordinates. But Baily-Borel ([2], Prop. 3.14) showed that the Jacobian on $(G_l \cdot V \cdot U) \cap \Gamma$ was a root of unity. Since Γ is neat, it is one and $(G_l \cdot V \cdot U) \cap \Gamma$ indeed fixes s_1. The last assertion is just a restatement of Proposition 3.3 for this special case. QED

The following consequence of the proportionality principle seems to be more or less well known to experts, but does not seem to be contained in any published articles:

Corollary 3.5. *Let $L = (\Omega^n_{\check{D}})^{-1}$ be the ample line bundle on $\check{D}$, and let*

$$P(l) = \chi(L^{\otimes l})$$

be the Hilbert polynomial of $\check{D}$. Let $\pi\colon \bar{X} \to X^*$ map a smooth compactification of X onto Baily-Borel's compactification. Let $n_1 = \dim(X^* - X)$. Then there exists a *polynomial* $P_1(l)$ *of degree at most* n_1 *such that for all* $l \geqq 2$:

$$\dim \begin{bmatrix} \text{cusp forms on } D \\ \text{w.r.t. } \Gamma \text{ of weight } l \end{bmatrix} = \operatorname{vol}(X) \cdot P(l-1) + P_1(l).$$

Proof. The Riemann-Roch theorem gives us a "universal polynomial" Q such that if L is any line bundle on a smooth projective variety W, then

$$\chi(L) = Q(c_1(L); c_1(\Omega^1_W), \ldots, c_n(\Omega^1_W)).$$

Therefore if $n = \dim D$,

$$\begin{aligned}(-1)^n \operatorname{vol}(X) \cdot P(-l) &= (-1)^n \operatorname{vol}(X) \cdot \chi((\Omega^n_{\check{D}})^{\otimes l}) \\ &= (-1)^n \operatorname{vol}(X) \cdot Q(l c_1(\Omega^1_{\check{D}});\ c_1(\Omega^1_{\check{D}}), \ldots, c_n(\Omega^1_{\check{D}})) \\ &= Q(l c_1(\Omega^1_{\bar{X}}(\log));\ c_1(\Omega^1_{\bar{X}}(\log)), \ldots, c_n(\Omega^1_{\bar{X}}(\log)))\end{aligned}$$

by Proportionality Theorem 3.2.

Consider a typical term

$$[l c_1(\Omega^1_{\bar{X}}(\log))]^k \cdot c^\alpha(\Omega^1_{\bar{X}}(\log)), \quad |\alpha| + k = n.$$

Now by Proposition 3.4b:

$$c_1(\Omega^1_{\bar{X}}(\log)) = \pi^* H,$$

H an ample divisor on X^*. Let $n_1 = \dim(X^* - X)$. If $k > n_1$, the cycle class H^k on X^* is represented by a cycle supported on X alone, hence so is $\pi^* H_k$. Thus if $k > n_1$

$$(l \cdot \pi^* H)^k \cdot c^\alpha(\Omega^1_{\bar{X}}(\log)) = (l \cdot \pi^* H)^k \cdot c^\alpha(\Omega^1_{\bar{X}}).$$

Therefore

$$\begin{aligned}&Q(l c_1(\Omega^1_{\bar{X}}(\log));\ c_1(\Omega^1_{\bar{X}}(\log)), \ldots, c_n(\Omega^1_{\bar{X}}(\log))) \\ &\quad = Q(l c_1(\Omega^1_{\bar{X}}(\log));\ c_1(\Omega^1_{\bar{X}}), \ldots, c_n(\Omega^1_{\bar{X}})) + (\text{polyn. of degree} \leqq n_1) \\ &\quad = \chi(\Omega^n_{\bar{X}}(\log)^{\otimes l}) + (\text{polyn. of degree} \leqq n_1) \\ &\quad = (-1)^n \chi(\Omega^n_{\bar{X}}(\log)^{\otimes(-l)} \otimes \Omega^n_{\bar{X}}) + (\text{polyn. of degree} \leqq n_1)\end{aligned}$$

by Serre duality.

Thus for suitable P_1 of degree at most n_1:

$$\operatorname{vol}(X) \cdot P(l-1) = \chi(\Omega^n_{\bar{X}}(\log)^{\otimes(l-1)} \otimes \Omega^n_{\bar{X}}) - P_1(l).$$

But since $(\Omega^n_{\bar{X}}(\log))^{\otimes N}$ is generated by its sections and maps $\bar{X}$ to X^* of the same dimension for $N \gg 0$, Kodaira Vanishing (cf. [13]) applies if $l \geqq 2$ and we have

$$h^0(\Omega^n_{\bar{X}}(\log)^{l-1} \otimes \Omega^n_{\bar{X}}) = \operatorname{vol}(X) \cdot P(l-1) + P_1(l).$$

The left-hand side is exactly the space of sections of $\Omega^n_{\bar{X}}(\log)^l$ which vanish on the boundary. By Proposition 3.3, these are exactly the cusp forms of weight l. QED

§ 4. Applications: General Type and Log General Type

The purpose of this section is to consider the application of the preceding theory to the question of when D/Γ is of general type, and to reprove as a consequence of our theory the following theorem of Y.-S. Tai ([1], Ch. IV, § 1).

Tai's Theorem 4.1. *If Γ is any arithmetic variety acting on a bounded symmetric domain D, then there is a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that for all $\Gamma_1 \subset \Gamma_0$ of finite index, the variety D/Γ_1 is of general type.*

We recall that if X is any variety of dimension n, we say that X is of general type, if for one (and hence all) smooth complete varieties $\bar{X}$ birational to X, the transcendence degree of the ring

$$\bigoplus_{N=0}^{\infty} \Gamma(\bar{X}, (\Omega^n_{\bar{X}})^{\otimes N})$$

is $(n+1)$. More generally, the transcendence degree of this ring minus one is called the Kodaira dimension of X.

Recall that Iitaka [8] has recently introduced a complementary theory of "logarithmic Kodaira dimension" for arbitrary varieties Y. In fact, he first chooses a smooth blow-up Y' of Y and then a smooth compactification $\bar{Y}$ of Y' such that $\bar{Y} - Y'$ has normal crossings and defines $\Omega^1_{\bar{Y}}(\log)$ as the complex of 1-forms

$$\sum_{i=1}^{k} a_i(z) \frac{dz_i}{z_i} + \sum_{i=k+1}^{n} a_i(z)\, dz_i$$

if, locally, $\bar{Y} - Y'$ is given by $\prod_{i=1}^{k} z_i = 0$. By definition $\Omega^k_{\bar{Y}}(\log) = \Lambda^k(\Omega^1_{\bar{Y}}(\log))$. He then looks at the "logarithmic canonical ring":

$$R = \bigoplus_{N=0}^{\infty} \Gamma(\bar{Y}, \Omega^n_{\bar{Y}}(\log^{\otimes N}).$$

He shows that this ring, as well as all other vector spaces of global forms with logarithmic poles (obtained from decomposing $\Omega^1_{\bar{Y}}(\log) \otimes \cdots \otimes \Omega^1_{\bar{Y}}(\log)$ under the symmetric group and taking global sections) are independent of the choice of Y' and $\bar{Y}$. He then defines the logarithmic Kodaira dimension of Y to be the transcendence degree of R minus 1. We may restate Proposition 3.4(b) in this language as follows:

Proposition 4.2. *If Γ is a neat[3] arithmetic group, then D/Γ is a variety of logarithmic general type, i.e., its logarithmic Kodaira dimension equals its dimension.*

---

[3] Some hypothesis on elements of finite order is needed because $H/SL_2(\mathbb{Z}) \cong \mathbb{A}^1$ which is *not* of log general type!

Proof. In fact, by Proposition 3.4, R is just the homogeneous coordinate ring of the Baily-Borel compactification of D/Γ.

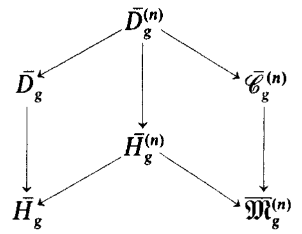
Note that D/Γ of logarithmic general type is weaker than saying D/Γ is general type.

I would like to add one comment to his theory which, in some cases, makes it easier to apply: one does not need smooth compactifications, but merely a toroidal compactification $\bar{Y}$ of Y' (cf. [9], p. 54). This means that locally $Y' \subset \bar{Y}$ is isomorphic to $(\mathbb{C}^*)^n \subset X_\sigma$, where X_σ is an affine torus embedding (i.e., $(\mathbb{C}^*)^n$ is Zariski-open in X_σ, X_σ is normal affine and translations by $\mathbb{C}^{*n}$ extend to an action of $(\mathbb{C}^*)^n$ on X_σ). On X_σ, define $\Omega^1_{X_\sigma}(\log)$ to be the sheaf generated by the $(\mathbb{C}^*)^n$-invariant 1-forms. Carrying these over, we define $\Omega^1_{\bar{Y}}(\log)$ to be the coherent sheaf of 1-forms on $\bar{Y}$, regular on Y', isomorphic locally to $\Omega^1_{X_\sigma}(\log)$. If $\bar{Y}' \to \bar{Y}$ is an "allowable" modification of toroidal embedding of Y ([9], p. 87), then $p^*(\Omega^1_{\bar{Y}}(\log)) \cong \Omega^1_{\bar{Y}'}(\log)$. In particular, there is always a smooth allowable modification $\bar{Y}'$ ([9], p. 94). So Iitaka's spaces of forms with log poles can be calculated equally well on a smooth $\bar{Y}$ or a toroidal $\bar{Y}$.

This extension is helpful in checking the analog of the above Proposition for the moduli space of curves:

Proposition 4.3. *Let $\mathfrak{M}_g^{(n)}$ be the moduli space of smooth curves of genus g with level n structure. If $n \geq 3$, then $\mathfrak{M}_g^{(n)}$ is of log general type.*

Sketch of Proof. The proof follows the ideas of [12], § 5 very closely. Let H_g be the Hilbert scheme of e-canonically embedded smooth curves of genus g. Let $H_g^{(n)} \to H_g$ be the covering defined by the set of level n structures on these curves. Let $\bar{H}_g, \bar{\mathfrak{M}}_g$ be the compactified spaces allowing stable singular curves as well. Let $\bar{H}_g^{(n)}$, $\bar{\mathfrak{M}}_g^{(n)}$ be the normalization of $\bar{H}_g, \bar{\mathfrak{M}}_g$ in the coverings $H_g^{(n)}, \mathfrak{M}_g^{(n)}$. The group $G = PGL(v)$ $(v = (2e-1)(g-1))$ acts on $\bar{H}_g$ and on $\bar{H}_g^{(n)}$, freely on the latter, so that $\bar{\mathfrak{M}}_g \cong \bar{H}_g/G$, $\bar{\mathfrak{M}}_g^{(n)} \cong \bar{H}_g^{(n)}/G$. We have the diagram:



where D and $\mathscr{C}$ are the universal curves. Recall from [12] the notation: whenever $p: C \to S$ is a flat family of stable curves,

$$\lambda = \Lambda^g p_*(\omega_{C/S}),$$
$$\omega_{C/S} = \text{relative dualizing sheaf}$$

and if p is smooth over all points of S of depth zero, then $\Delta \subset S$ is the divisor of singular curves and

$$\delta = \mathcal{O}_S(\Delta).$$

Now on all 3 families above, we wish to show $\lambda^{13}\otimes\delta^{-1}$ and the sheaf of top logarithmic forms $\Omega^n(\log)$ are isomorphic line bundles. Firstly, for $p\colon \bar{D}_g\to\bar{H}_g$, we proceed like this: a) a simple modification of the proof of Theorem 5.10 [12] shows:

$$\lambda^{13}\otimes\delta^{-2}\cong\Lambda^{3g-3}(p_*(\Omega^1_{\bar{D}/\bar{H}}\otimes\omega_{\bar{D}/\bar{H}})).$$

b) Since $\bar{H}_g$ represents the functor of e-canonical stable curves, $T_{\bar{H}_g,[C]}$ is canonically isomorphic to the vector space of deformations of C. This has a subspace consisting of deformations of the e-canonical embedding where C doesn't change, and a quotient space of the deformations of C alone:

$$0\to\operatorname{Lie}G\to T_{\bar{H}_g,[C]}\to\operatorname{Ext}^1(\Omega^1_C,\mathcal{O}_C)\to 0$$

or dually:

$$0\to H^0(\Omega^1_C\otimes\omega_C)\to\Omega^1_{\bar{H}_g}\otimes\mathbb{K}(C)\to(\operatorname{Lie}G)'\to 0.$$

Therefore globally, we get

$$0\to p_*(\Omega^1_{\bar{D}/\bar{H}}\otimes\omega_{\bar{D}/\bar{H}})\to\Omega^1_{\bar{H}_g}\to(\operatorname{Lie}G)'\otimes\mathcal{O}_{\bar{H}_g}\to 0$$

hence if $m=\dim\bar{H}_g$,

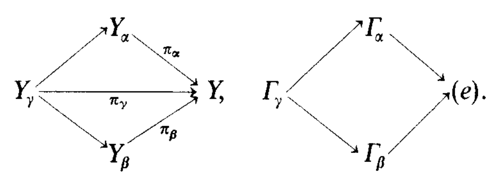
$$\Omega^m_{\bar{H}_g}\cong\Lambda^{3g-3}p_*(\Omega^1_{\bar{D}/\bar{H}}\otimes\omega_{\bar{D}/\bar{H}})\cong\lambda^{13}\otimes\delta^{-2}$$
$$\therefore\ \Omega^m_{\bar{H}_g}(\log)\cong\lambda^{13}\otimes\delta^{-1}.$$

Secondly, for $p\colon \bar{D}^{(n)}_g\to\bar{H}^{(n)}_g$, $\lambda^{13}\otimes\delta^{-1}$ pulls back to the analogous sheaf on $\bar{H}^{(n)}_g$. Moreover, because $\bar{H}^{(n)}_g\to\bar{H}_g$ is ramified only along Δ which has normal crossings, $\bar{H}^{(n)}_g$ has toroidal singularities and $\Omega^m_{\bar{H}_g}(\log)$ pulls back to $\Omega^m_{\bar{H}^{(n)}_g}(\log)$. Finally both bundles descend to $\overline{\mathfrak{M}}^{(n)}_g$ and by Proposition 1.4 [11], are still isomorphic. Finally, it is proven in [12] (Th. 5.18 and 5.20: cf. diagram in § 5) that $\lambda^{13}\otimes\delta^{-1}$ is sample on $\overline{\mathfrak{M}}^{(n)}_g$. QED

In certain cases, there is a way of deducing that coverings of a variety of log general type are actually of general type. To explain this, suppose we are given a smooth quasi-projective variety Y, and a tower of connected étale Galois coverings:

$$\pi_\alpha\colon Y_\alpha\to Y,\quad \text{group } \Gamma_\alpha.$$

We assume that any 2 covers π_α,π_β are dominated by a third one π_γ:



Let $\bar{Y}$ be a smooth compactification of Y with normal crossings at infinity. Extend the covering Y_α to a finite covering

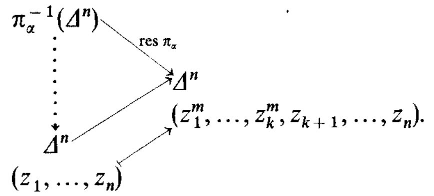
$$\pi_\alpha: \bar{Y}_\alpha \to \bar{Y}$$

be defining $\bar{Y}_\alpha$ to be the normalizations of $\bar{Y}$ in the function field of Y_α. We now make the definition:

Definition. *The tower $\{\pi_\alpha\}$ is locally universally ramified over $\bar{Y}-Y$ if for all $x\in\bar{Y}-Y$, we take a nice neighborhood of x:*

$$\Delta^n \subset \bar{Y},$$
$$\Delta^n \cap (\bar{Y}-Y) = \begin{pmatrix}\text{union of coordinate hyperplanes}\\ z_1=0,\dots,z_k=0\end{pmatrix}$$

then for all m, there is an α and a commutative diagram:



In other words, $\pi_\alpha^{-1}(\Delta^n\cap(\bar{Y}-Y))$ is cofinal in the set of all unramified coverings of $\Delta^n\cap(\bar{Y}-Y)$.

Then we assert:

Proposition 4.4. *Let $Y\subset\bar{Y}$ be as above and let π_α: $\bar{Y}_\alpha\to\bar{Y}$ be a tower of coverings unramified over Y and locally universally ramified over $\bar{Y}-Y$. Then if Y is logarithmically of general type, there is an α_0 such that for all α_1 such that the covering π_{α_1} dominates π_{α_0}, Y_{α_1} is of general type.*

Proof. Let $\Delta=\bar{Y}-Y$, $n=\dim Y$ and let $\omega=\Omega^n_{\bar{Y}}(\log)$. Then we know that

a) for some N, there are differentials $\eta_0,\dots,\eta_n\in\Gamma(\bar{Y},\omega^{\otimes N})$ such that $\eta_1/\eta_0,\dots,\eta_n/\eta_0$ are a transcendence base of the function field $\mathbb{C}(Y)$,

b) $h^0(\bar{Y},\omega^{\otimes N})\geqq C_1N^{n+1}$ if $N\geqq N_0$.

From (b) it follows that

$$\Gamma(\bar{Y},\omega^{\otimes N})\to\Gamma(\Delta,\omega^{\otimes N}\otimes\mathcal{O}_\Delta)$$

has a non-zero kernel for some N: let ζ be in the kernel. Replacing η_i by $\eta_i\otimes\zeta$, we may assume that all η_i are zero on Δ. Now let's examine locally what happens to η_i when lifted to a covering of $\bar{Y}$: let $\Delta^n\subset\bar{Y}$ be a polycylinder such that $\Delta^n\cap(\bar{Y}-Y)=V\left(\prod_{i=1}^k z_i\right)$. Write out η_i:

$$\eta_1=a_i(z)\cdot\prod_{i=1}^k z_i\cdot\left(\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\dots z_k}\right)^N,$$

a_i holomorphic. Let $w_i = z_i^{1/m}$, $1 \leqq i \leqq k$, $w_i = z_i$, $k+1 \leqq i \leqq n$ and let

$$\pi_m\colon \Delta^n \to \Delta^n$$

be the covering of the z-polycylinder by the w-polycylinder. Then

$$\frac{dw_i}{w_i} = \frac{1}{m}\frac{dz_i}{z_i}, \qquad 1 \leqq i \leqq k,$$

$$dw_i = dz_i, \qquad k+1 \leqq i \leqq n$$

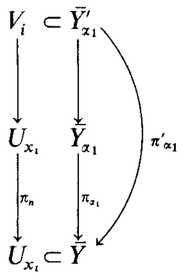
hence

$$\pi_m^*(\eta_i) = m^{kN} a_i \cdot \prod_{i=1}^{k} w_i^m \cdot \left(\frac{dw_1 \wedge \cdots \wedge dw_n}{w_1 \ldots w_k}\right)^N.$$

So if $m \geqq N$, $\pi_m^*(\eta_i)$ is a holomorphic differential form on Δ^n. Now for each $x \in \bar{Y} - Y$, fix a neighborhood $U_x \subset \bar{Y}$ of this type and a covering

$$\pi_{\alpha(x)}\colon \bar{Y}_{\alpha(x)} \to \bar{Y}$$

which, over U_x, dominates π_N. $\bar{Y} - Y$ is covered by a finite number $\{U_{x_i}\}$ of U_x's, so we can find one cover π_{α_0} which dominates all the covers $\pi_{\alpha(x_i)}$. I claim that if π_{α_1} dominates π_{α_0}, hence $\pi_{\alpha(x_i)}$, then $\pi_{\alpha_1}^*(\eta_i)$ has no poles on a desingularization $\bar{Y}'_{\alpha_1}$ over $\bar{Y}_{\alpha_1}$. This is clear because it has an open covering by open sets V_i sitting in a diagram



so $\pi_m^*(\eta_i)$ regular $\Rightarrow \pi'_{\alpha_1}(\eta_i)$ regular on V_i. But now $\pi'^*_{\alpha_1}(\eta_i/\eta_0)$ are a transcendence base of the function field of $\bar{Y}_{\alpha_1}$ so $\bar{Y}_{\alpha_1}$ is of general type. QED

Let's consider the case $Y = D/\Gamma$ again. If Γ is an arithmetic group, then for every positive integer n, we have its level n subgroup $\Gamma(n)$, i.e., if

$$\Gamma = \mathscr{G}(\mathbb{Z}), \qquad \mathscr{G} \text{ algebraic group over Spec } \mathbb{Z}$$

then

$$\Gamma(n) = \operatorname{Ker}[\mathscr{G}(\mathbb{Z}) \to \mathscr{G}(\mathbb{Z}/n\mathbb{Z})].$$

It's easy to see that

$$\pi_n\colon D/\Gamma(n) \to D/\Gamma$$

is locally universally ramified over $\overline{D/\Gamma} - D/\Gamma$. In fact, let F be a rational boundary component. Then near boundary points associated to F, the pair $D/\Gamma \subset \overline{D/\Gamma}$ is isomorphic to

$$\begin{array}{ccc} D(F)/U(F) \cap \Gamma & \subset & (D(F)/U(F) \cap \mathbb{Z})_\sigma \\ \| & & \| \\ C^{*n} \times \mathbb{C}^m \times F & & \mathbb{C}^n \times \mathbb{C}^m \times F. \end{array}$$

Thus if $U \subset \overline{D/\Gamma}$ is a small neighborhood of a point corresponding in the above chart to $(0, y, t)$, $\pi_1(U \cap (D/\Gamma))$ is isomorphic to $U(F) \cap \Gamma$. Thus we must check that for all n, there is an m such that

$$U(F) \cap \Gamma(m) \subset n \cdot U(F) \cap \Gamma.$$

But if F is rational, $U(F)$ is an algebraic subgroup of the full group $\mathcal{G}$ which is defined over $\mathbb{C}$, so this is clear. So now Proposition 4.2, Proposition 4.4 and this remark altogether imply Tai's theorem.

It is now known that this same method will show that at least some high-non-abelian levels of $\mathfrak{M}_g$ are varieties of general type too. It is not simple however to check that the Teichmuller tower is locally universally ramified at infinite. This has recently been proven by a ingenious use of dihedral level by T.-L. Brylinski.

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