Histogram Measurement of ADC Nonlinearities Using Sine Waves

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***Abstract*—This paper gives results concerning the measurement of differential and integral nonlinearity of ADC's using the histogram method with a sine wave input signal. We specify the amount of overdrive required as a function of the noise level and the desired accuracy and the number of samples required as a function of the desired accuracy, the desired confidence level, and the noise level. An analysis of the effect on the results of harmonic distortion of the applied signal is given. The error analysis assumes a mixture of coherent and random sampling rather than pure random sampling.**

I. INTRODUCTION

THE use of sine wave histogram tests for the determination of the nonlinearities of analog-to-digital converters (ADC's) has become quite common and is described in [1] and [2]. Our purpose is to extend the results of [1] and [2] in several ways.

First, we consider the effect of additive random noise on the measurement results. When a triangle wave is used for histogram tests (as in [3]), additive noise has no effect on the results; however, it is difficult to guarantee the accuracy of a triangle wave. When a sine wave is used, an error is produced which becomes larger near the peaks. This error can be made as small and desired by sufficiently overdriving the ADC, and we give formulas for the required amount of overdrive as a function of the desired accuracy.

Additionally, we consider the situation in which the sample points are taken from records with a fixed sampling frequency, where this frequency is chosen to minimize the errors. We specify how to select the input signal frequency and the accuracy required of this frequency. We give formulas giving the required minimum number of samples that must be taken to guarantee a given accuracy. It is shown that the number of samples required with this approach is smaller than the number required with the random sampling studied in [2] and [3].

We give results for guaranteeing a specified accuracy for integral nonlinearity (INL) as well as differential nonlinearity (DNL), and we consider the effect of harmonic distortion of the input signal. Finally, we give formulas based on a specified confidence level for worst case deviations. The latter is especially important when measuring DNL. For example, if we measure the DNL of a 10-bit ADC in such a manner as to achieve the desired accuracy in individual DNL values with 99% confidence, we expect 1% of the DNL values (or approximately 10 of them) to be out of tolerance. This means that there is a very high probability that the worst case DNL will be measured with an error greater than the tolerance.

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II. BACKGROUND AND NOTATION

A. General Notation

The following general notation and definitions will be used throughout:

N = number of bits of the ADC. The output codes of the ADC are integers between 0 and $2^N - 1$ (inclusive).

$T[k]$ = k^{th} transition level. The voltage level at which the ADC will produce an output code of $k - 1$ or less 50% of the time and an outputof k or more 50% of the time.

$W[k] = T[k+1] - T[k]$ = the k^{th} code bin width.

V = $T[2^N - 1] - T[1]$ = the reduced full-scale voltage of the ADC.

Q = $V/(2^N - 2)$ = the average code bin width.

The reduced full-scale voltage, V, is the difference between the last and first transition levels and is one code bin width smaller than what is commonly called the full-scale voltage. Additional parameters relating to the ADC depend on the *Gain* and *Offset*, which have nonunique definitions. The gain and offset are parameters of a straight-line fit to $T[k]$ versus k, and will have different values depending on how the fit is done. The general relation used to define gain and offset is

$$G \cdot T[k] + V_{os} + \epsilon[k] = (k-1) \cdot Q + T[1]$$

where

G = the gain, nominally 1,

V_{OS} = the offset voltage, nominally 0,

$\epsilon[k]$ = the residual error.

The fit might be done to minimize the sums of the squares of the residuals, to minimize the maximum residual, to make the residuals zero at the end points, or by some other method. Different methods yield slightly different values for the gain, the offset, and the residuals.

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B. Differential and Integral Nonlinearity

For any particular values for the gain and the offset, the differential nonlinearity (DNL) and integral nonlinearity (INL) are defined as follows:

$$DNL[k] = \frac{G \cdot W[k] - Q}{Q}, \tag{1}$$

$$DNL = \max |DNL[k]|, \tag{2}$$

$$INL[k] = \frac{\epsilon[k]}{Q} \quad \text{and} \tag{3}$$

$$INL = \max |INL[k]|. \tag{4}$$

We have defined both INL and DNL in fractions of a code bin width, though other units are frequency used.

C. Histogram Measurements and Calculations

A sine wave that slightly overdrives the ADC is sampled many times. The data is collected as a series of R records each of which contains M samples. Each record is taken with the same constant sampling rate. The record length and ratio of the sampling rate to the signal frequency are chosen so that the phases of the samples are uniformly distributed between 0 and 2π. The rules for selecting the signal frequency and the record length are given in Section III-B. The phase of the first sample point of each record is assumed to be randomly and uniformly distributed between 0 and 2π with the phases of different records being independent. Formulas are given in Sections III-A and III-C for determining the amount of overdrive required and the number of samples required. Results in Section IV relate harmonic distortion of the sine wave source to errors in transition levels, INL and DNL. If the range of the ADC is not symmetrical about 0 v, a constant, approximately equal to the mid-scale voltage of the ADC, must be added to the sine wave.

Let

$h[i]$ = the total number of samples received in code bin i,

and let

$$ch[k] = \sum_{i=0}^{k} h[i], \tag{5}$$

and

$$S = M \cdot R = \text{the total number of samples.}$$

The applied signal is of the form

$$v[t] = A \sin[\omega t + \phi] + d.$$

The frequency and phase of the sine wave are not used in the data analysis. The values of A and d are assumed to be known but they need not be. Errors in the values for A and d will affect the values calculated for the gain and the offset of the ADC but will not affect values for DNL or INL at all. The transition levels are estimated from the data ([1], [2]) by

$$T[k] = d - A\,\mathrm{Cos}\left[\frac{\pi ch[k-1]}{S}\right]. \tag{6}$$

The code bin widths are given by

$$W[k] = T[k+1] - T[k]. \tag{7}$$

If the values of A and d are unknown, approximate values can be obtained from (6) and approximate values for the first and last (or any two) transition levels. Values for gain and offset may then be determined by any desired method, and INL and DNL can be determined from (1) through (4).

D. Tolerance and Confidence Level

The histogram approach is based on the assumption that the relative number of counts occurring in that code bin is equal to the probability of a measurement occurring in that code bin. This is only true in the limiting case of an infinite number of samples. For any finite number of samples there is a statistical error, and the number of samples must be chosen large enough to make this error sufficiently small.

Two quantities are used to describe the errors—the *tolerance* and the *confidence level.* We follow the convention here of measuring tolerances in fractions of a code bin width. We say that a code bin width, W, is measured with tolerance, B, and confidence $1 - u$ if the probability is equal to or greater than $1 - u$ that

$$W_T/(1+B) \le W_M \le W_T(1+B)$$

where W_M is the measured value and W_T is the true value. We say that a transition level, T, is measured with tolerance, B, and confidence $1 - u$ if the probability is equal to or greater than $1 - u$ that

$$T_T - BQ \le T_M \le T_T + BQ$$

where T_T and T_M are the true and measured values, and Q is the average code bin width.

III. Determining the Test Parameters

This section contains the main results of the paper. The proofs of these results will be postponed until the next sections.

The first step is to determine the desired tolerance. Frequently, there is a different required tolerance for code bin widths (and DNL) than there is for transition levels (and INL). For example, one may want the DNL to $\pm 5\%$ ($B = 0.05$), but the INL may only be needed to ± 1 code bin width ($B = 1$). Having specified the desired tolerance, one next determines the required amount of overdrive. The amount of overdrive required depends on the combined noise level of the signal source and the ADC. The only information required about the noise is an upper limit on the rms noise level. Note that the amount of overdrive required is the same whether one is using the more commonly studied random sampling method or the method proposed here.

Next, the required minimum number of samples is determined. This depends on the tolerance, the confidence level and on the overdrive. For given values of these parameters, the number of samples required depends whether one is specifying the tolerance for an individual measurement or for the worst case. The total number of samples required also depends on the record length chosen, the longer the record the fewer samples that are required. However, the accuracy required of the input signal frequency increases with increasing record length.

A. Required Overdrive Versus Noise Level

The positive overdrive voltage is the difference between the maximum voltage of the applied signal and the largest transition level of the ADC. The negative overdrive voltage is the difference between the smallest (most negative) transition level of the ADC and the minimum of the applied signal. The overdrive voltage, V_{OD}, is the smallest of the positive and negative overdrives.

To obtain a tolerance, B, in the code bin widths choose the overdrive voltage to satisfy

$$V_{OD} \geq \sigma \times \max\left(3, \sqrt{\frac{3}{2B}}\right) \tag{8}$$

where σ is the combined rms noise level (in volts) of the signal source and the ADC. Note that the value of σ only includes the rms value of the *random* noise (i.e., errors that are not repeatable); it does not include distortion or quantization error.

To obtain a tolerance, B, in the transition levels choose the overdrive voltage to satisfy

$$V_{OD} \geq \sigma \times \max\left(2, \frac{\sigma 2^N}{VB}\right). \tag{9}$$

Note that, to first order, the effect of noise on the results is not random but systematic. The error is largest near the peaks of the sine wave; the overdrive keeps the measurements far enough from the peaks to make the error as small as desired. The values of overdrive in (8) and (9) are adequate to keep the errors due to noise to $\leq B/3$ code bin widths so that these errors are negligible when added to the statistical errors due to taking a finite number of samples.

The amount of overdrive also affects the errors in DNL due to harmonic distortion of the signal source. This is covered in Section IV.

B. Determining the Frequency and Record Length

To obtain meaningful measurements of transition levels and of integral and differential nonlinearity, it is important to choose the signal frequency low enough that dynamic errors are negligible.

The frequency of the input signal and the record length of the data collected must be carefully selected for the error estimates of the following section to apply. There must be an exact integer number of cycles in a record, and the number of cycles in a record must be relatively prime to the number of samples in the record. This guarantees that the samples in each record are uniformly distributed in phase from 0 to 2π.

A frequency that meets the above requirements can be selected as follows. Choose the number of cycles per record, D, and a record length, M, such that $M + 1$ is an integer multiple of D. Choose the ratio of the signal frequency to the sampling frequency by the following formula:

$$\frac{f}{f_s} = \frac{D}{M} \tag{10}$$

where

f = the signal frequency, and

f_s = the sampling frequency.

This selection of signal frequency is described in [4] where it is shown that, with rearrangement of the data, it produces an equivalent time sampling of the input signal with an equivalent sampling frequency of Df_s. For the rest of our analysis we assume that the ratio, ρ, of the signal frequency to the sampling frequency satisfies (10) with an error, $\Delta\rho$, which satisfies

$$\frac{\Delta\rho}{\rho} \leq \begin{cases} \dfrac{1}{4(D-1)M} & \text{for } D > 1 \\ \dfrac{1}{4M} & \text{for } D = 1. \end{cases} \tag{11}$$

For $D > 1$ this, by the results in [4], is the condition that the error in equivalent sampling time is equal to or less than 1/4 of the interval between equivalent time samples. The decision to derive the results with precisely this restriction was somewhat arbitrary. Note that larger values of M or D require more accurate frequencies. We will see in the next section that fewer total samples are required with larger values of M, so the best approach is to choose the largest value of M compatible with the frequency accuracy obtainable.

C. Required Number of Records

The relation giving the number of records required for a given tolerance and confidence level contains a constant that depends on whether the confidence level is for individual values or for worst case values. For worst case values the constant depends on the number of bits, N, of the ADC. In both cases we use values from Table I. For values of N between those in the table, linear interpolation may be used. The values Z_α and $Z_{N,\alpha}$ in the table are defined as follows. If x is a random variable with a Gaussian distribution with a mean of zero and a standard deviation of one, then the probability that $|x| \geq Z_\alpha$ is 2α. If x is the maximum of the absolute values of 2^N independent random variables with mean zero and standard deviation 1, then the probability that $|x| \geq Z_{N,\alpha}$ is 2α. The values of $Z_{N,u/2}$ were calculated by the formula

$$Z_{N,u/2} = \sqrt{2}\,\mathrm{erfc}^{-1}[1 - (1-u)^{2^{-N}}],$$

where erfc is the complimentary error function. Values of $Z_{u/2}$ were calculated with the same formula using $N = 0$. The standard statistical package from *Mathematica* version 2.1 was used to perform the calculations.

The number of records required also depends on whether the tolerances are specified for transition levels (INL) or for code bin widths (DNL). Having chosen the record length, M, the number of records required to obtain a tolerance of B and a confidence of $1 - u$ is given by

$$R = C\left[\frac{2^{N-1}K_u}{B}\right]^2\left[\alpha\frac{\pi}{M}\right] \cdot\left\{1.13\left(\frac{\sigma^*}{V}\right) + .2\left[\alpha\frac{\pi}{M}\right]\right\}, \tag{12}$$

TABLE I

u	$Z_u/2$	$Z_{4,u/2}$	$Z_{8,u/2}$	$Z_{12,u/2}$	$Z_{16,u/2}$	$Z_{20u/2}$
0.2	1.28	2.46	3.33	4.04	4.64	5.19
0.1	1.64	2.72	3.53	4.21	4.80	5.33
0.05	1.96	2.95	3.72	4.37	4.94	5.46
0.02	2.33	3.22	3.95	4.57	5.12	5.62
0.01	2.58	3.42	4.11	4.71	5.25	5.74
0.005	2.81	3.60	4.27	4.85	5.38	5.85
0.002	3.09	3.84	4.47	5.03	5.54	6.01
0.001	3.29	4.00	4.62	5.16	5.66	6.12

where

- R = minimum required number of records,
- C = 1 for INL, and $C = 2$ for DNL,
- M = the number of samples per record,
- α = $1 + 2V_{OD}/V$,
- V = the reduced full-scale voltage of the instrument in volts,
- V_{OD} = overdrive voltage,
- K_u = $Z_{u/2}$ for obtaining the specified confidence in an individual transition level or code bin width,
- K_u = $Z_{N,u/2}$ for obtaining the specified confidence in the worst case transition level or code bin width,
- σ^* = σ, the rms noise level in volts, for INL,
- σ^* = $\min[\sigma, Q/1.13]$ (Q = code bin width) for DNL,
- B = desired test tolerance as a fraction of a code bin width, and
- N = number of bits of the ADC.

Note that if the noise level is sufficiently large, then the first term in the curly braces dominates, and the total number of samples, RM, obtained from (12) for DNL is the same as that obtained from (A15) of [2]. As the noise level gets smaller and the second term in the braces dominates, the total number of samples required becomes inversely proportional to the record length. The condition for the required number of samples given by (12) to be smaller than that given in [2] is $M \geq 0.63 \times \alpha^2 \times 2^N$. (Note: the result in [2] is only valid with $\alpha = 1$.)

IV. Effects of Harmonic Distortion

In this section we examine the effect of harmonic distortion in the sine wave signal used for the histogram measurements. We obtain upper bounds for the errors in transition levels and in code bin widths due to a given amount of harmonic distortion. We assume that the input signal satisfies

$$v[t] = A\sin[\omega t + \phi] + d + r(t), \tag{13}$$

where $r(t)$ is the harmonic distortion which we assume satisfies

$$|r(t)| \leq \epsilon. \tag{14}$$

The result for transition levels is that if e_k^T is the error due to harmonic distortion in the kth transition level, then

$$|e_k^T| \leq \epsilon. \tag{15}$$

This result, which will be proved shortly, says that the maximum error in a transition level is bounded by the maximum error in the input signal. If the harmonic distortion is in phase with the signal then an error of magnitude ϵ will occur, so the bound in (15) cannot be improved.

Since a code bin width is the difference between two transition levels, we have for the error, e_k^W, in the kth code bin

$$|e_k^W| \leq 2\epsilon. \tag{16}$$

We will later give a much smaller bound than (16) for the typical case where the distortion consists of low-order harmonics.

To prove the result (15) we define, for any voltage value, v, the set of time values $S(v) = \{t : v(t) \leq v\}$. We also define the real-valued functions, ψ and g, by

$$\psi(v) = \mu[S(v)]/\tau, \quad \text{and} \quad g = \psi^{-1} \tag{17}$$

where $\mu[S]$ is the measure of the set, S (the sums of the lengths of the intervals comprising S), τ is the period of the signal, and g is the inverse function of ψ. Thus, $\psi(v)$ is the fraction of time that the signal spends with voltage $\leq v$. We define S_0, ψ_0, and g_0 in the same manner with $v(t)$ replaced with $v(t) - r(t)$ (the undistorted signal). Since for $v_2 \geq v_1, S(v_2) \supseteq S(v_1)$, it follows that ψ and g are nondecreasing functions of their arguments. From (14) and the definition of S it follows that $S_0(v - \epsilon) \subseteq S(v) \subseteq S_0(v + \epsilon)$ and, therefore, that

$$\psi_0(v - \epsilon) \leq \psi(v) \leq \psi_0(v + \epsilon). \tag{18}$$

If T_k is the kth transition level, then the expected fraction of counts, u_k, in the kth cumulative histogram bin is given by $u_k = \psi(T_k)$, and the calculated transition level, T_k', is given by $T_k' = g_0(u_k) = g_0(\psi(T_k))$. But from (18), $\psi(T_k)$ is between $\psi_0(T_k - \epsilon)$ and $\psi_0(T_k + \epsilon)$, so T_k' is between $g_0(\psi_0(T_k - \epsilon)) = T_k - \epsilon$ and $g_0(\psi_0(T_k + \epsilon)) = T_k + \epsilon$ which yields (15).

Although (16) gives the smallest possible bound on DNL errors for general harmonic distortion, a smaller bound can be obtained for low-order harmonic distortion. Harmonic distortion induces a relative error in the DNL value of a particular code bin equal to the derivative of the distorting signal divided by the derivative of the undistorted signal. However, with small amounts of overdrive the derivative of the undistorted signal approaches zero for the code bins near the peaks of the signal, so the analysis must be done very carefully.

Using the same notation as that following (18) and letting $h(x) = g_0(x) - g(x)$, we have

$$\begin{aligned} e_k^W &= e_{k+1}^T - e_k^T = h(u_{k+1}) - h(u_k) \\ &= h'(\overline{u}_k)(u_{k+1} - u_k) \end{aligned} \tag{19}$$

where $\overline{u}_k$ is some value between u_k and u_{k+1}. We will develop upper bounds for each of the terms on the right of (19).

We first obtain a bound for $h'(\overline{u}_k)$ under the condition of nth harmonic distortion of magnitude ϵ. The peak magnitude of $h(u)$ and its derivative will depend on the relative phases of the harmonics and the fundamental and will be maximum if the peaks of the distortion are at the peaks of the fundamental. In this case we have

$$h(u) = \epsilon \cos[n\pi u], \quad \text{and} \quad |h'(u)| \le n\pi\epsilon. \tag{20}$$

The value of $u_{k+1} - u_k$ in (19) is the fraction of time spent by the signal in the kth code bin, so this is largest for the code bins nearest the peaks of the signal and depends on the amount of overdrive. We will assume that the amount of overdrive is βQ, where Q is the code bin width. This gives for the upper bound on $u_{k+1} - u_k$,

$$\begin{aligned} u_{k+1} - u_k &\le u_2 - u_1 \\ &= \psi(d - A + \beta Q + Q) - \psi(d - A + \beta Q). \end{aligned} \tag{21}$$

We will approximate the right-hand side of (21) by replacing ψ with ψ_0 and will justify this later. From (6) we have

$$g_0(u) = d - A\cos[\pi u] \cong d - A + \tfrac{1}{2}A\pi^2 u^2 \tag{22}$$

where the right-hand side is valid near the negative peak of the signal. Solving for u as a function of the left-hand side gives for the inverse function

$$\psi_0(v) = \frac{\sqrt{2}}{\pi\sqrt{A}}\sqrt{v - d + a}.$$

Substituting this for ψ in (21) gives

$$u_{k+1} - u_k \le \frac{1}{\pi}\sqrt{\frac{2Q}{A}}(\sqrt{\beta + 1} - \sqrt{\beta}).$$

Combining this with (19) and (20) gives

$$|e_k^W| \le n\epsilon\sqrt{\frac{2Q}{A}}(\sqrt{\beta + 1} - \sqrt{\beta}). \tag{23}$$

If we assume second-harmonic distortion with $\epsilon = Q/2$, an 8-bit A/D converter and $\beta = 3$ (i.e., 2.3% overdrive), then the code bin width error calculated from (23) is $0.033Q$, while that from (16) is Q, a reduction by a factor of 30.

The justification for replacing ψ with ψ_0 in (21) is as follows. Near the peaks of the signal both g_0 and g are given by a second-order expansion as in (22). If the distortion is small relative to the signal, then the relative difference between the coefficients is small, and ψ and ψ_0 also have the same functional form with slightly different coefficients.

V. Derivation of Results for Overdrive Versus Noise Level

We assume that the input signal to the ADC is of the form

$$v[t] = A\sin[\omega t + \phi] + d.$$

It will be more convenient to change variables from v to x, with x given by

$$x = \frac{v - d}{A}. \tag{24}$$

Thus the value of x ranges from -1 to 1 for the input signal, and the range of the ADC will be somewhat less than this. If the noise has a standard deviation of σ in volts, then it will have a standard deviation of σ_x in "x -units" with

$$\sigma_x = \frac{\sigma}{A}. \tag{25}$$

Lemma 1: If $f[x]$ is any function with a continuous second derivative and $p[x]$ is a probability density function with a mean of zero and standard deviation of σ_x then

$$f * p[x] \cong f[x] + \frac{\sigma_x^2}{2}f''[x] \tag{26}$$

where the asterisk denotes the convolution integral.

Proof: Expanding $f[t]$ in a Taylor series about $t = x$ and keeping terms to second order gives

$$f[t] \cong f[x] + f'[x](t - x) + \tfrac{1}{2}f[x](t - x)^2.$$

We then have

$$\begin{aligned} f * p[x] &= \int f[t]\,p[x - t]\,dt \\ &\cong f[x]\int p[x - t]\,dt + f'[x]\int p[x - t](t - x)\,dt \\ &\quad + \tfrac{1}{2}f''[x]\int p[x - t](t - x)^2\,dt. \end{aligned}$$

Since $p[x]$ is a probability density the first integral is one; since its mean value is zero the second integral is zero; and since its standard deviation is σ_x the third integral is σ_x^2. Substituting these values gives (26).

A. Overdrive Result for Code Bin Widths

Let $f[x]$ be the probability density of the input signal and let $p[x]$ be the probability density for the noise. Let $g[x]$ be the probability density for the signal plus noise. We then have

$$g[x] = f * p[x] \cong f[x] + \frac{\sigma_x^2}{2}f''[x] \tag{27}$$

If W is the width of a code bin at some value x, then the measured code bin width, W_m, will satisfy

$$\frac{W_m}{W} \cong \frac{g[x]}{f[x]},$$

because the measured code bin width is proportional to the number of samples in the code bin which, in turn, is proportional to the probability density at the code bin.

This gives for the error

$$E_W \equiv \frac{W_m - W}{W} \cong \frac{g[x]}{f[x]} - 1 \cong \frac{\sigma_x^2}{2}\frac{f''[x]}{f[x]}.$$

For a sine wave of amplitude one the probability density is given by

$$f[x] = \frac{1}{\pi\sqrt{1 - x^2}}, \quad \text{and} \quad f''[x] = \frac{1 + 2x^2}{\pi(1 - x^2)^{5/2}}.$$

This gives the approximation for the error

$$E_W \cong \frac{\sigma_x^2}{2}\frac{1 + 2x^2}{(1 - x^2)^2} = \frac{\sigma_x^2}{2}\frac{1 + 2x^2}{(1 + x)^2(1 - x)^2}.$$

This has its maximum values near $x = \pm 1$. Since it is an even function of x we can examine the error at either end point; near $x = 1$ we have, by substituting $x = 1$ in all terms except $(1 - x)$,

$$E_W \cong= \frac{3\sigma_x^2}{8(1 - x)^2}. \tag{28}$$

To guarantee that the error is $\leq$ some value, ρ, we have

$$(1-x)^2 \geq \frac{3\sigma_x^2}{8\rho}.$$

Letting $\rho = B/4$ and taking the square root, it follows that

$$(1-x) \geq \sigma_x\sqrt{\frac{3}{2B}}. \tag{29}$$

We set the maximum error to $B/4$ to guarantee that the extra error caused by the noise is small compared to the desired error. Substituting (24) and (25) into this gives

$$(A+d-v) \geq \sigma\sqrt{\frac{3}{2B}} \quad \text{or} \quad V_{OD} \geq \sigma\sqrt{\frac{3}{2B}}. \tag{30}$$

The last inequality comes from the fact that $A+d$ is the peak of the signal, and v is an arbitrary voltage within the range of the ADC.

This is an approximation based on the approximation (27) for the convolution of the signal probability density with the noise probability density. This, in turn, is based on the assumption that $f''[x]$ is relatively constant throughout the width of the noise probability density function. In fact, as x approaches 1, $f''[x]$ is rapidly increasing, and the error is larger than that predicted by (27). To determine how much larger the error is as a function of x, the convolutions were evaluated numerically for $\sigma_x = 0.01, 0.001$, and 0.0001 for a Gaussian, a uniform, and a triangular probability density for the noise. In all cases it was found that if the overdrive was $\geq 3\sigma$, then the error was no more than 1.43 times that predicted by (27); in all cases it was the Gaussian density that gave the largest error. This means that if the overdrive is $\geq 3\sigma$, then the error caused by the noise will be $\leq 0.36B$.

B. Overdrive Result for Transition Levels

Let $F[x]$ be the probability that the input signal is $\leq x$, and let $G[x]$ be the probability that the input signal plus noise is $\leq x$. As in the previous section we have

$$G[x] = F * p[x] \cong F[x] + \frac{\sigma_x^2}{2}F''[x] \tag{31}$$

where $p[x]$ is the probability density for the noise (after converting from volts to x-units), and σ_x is the standard deviation of p. For a sine wave of amplitude one we have

$$F[x] = \frac{\text{Cos}^{-1}[-x]}{\pi}, \quad \text{and} \quad F''[x] = \frac{x}{\pi(1-x^2)^{3/2}}. \tag{32}$$

If x is a true transition level, then [from (6)] the calculated transition level, x_m, will be

$$x_m = -\cos[\pi G[x]] \cong -\cos[\pi F[x] + \pi\sigma_x^2 F''[x]/2].$$

Taking a first-order Taylor expansion of the cosine function about $\pi F[x]$ gives

$$x_m \cong -\cos[\pi F[x]] + \frac{\pi\sigma_x^2 F''[x]\sin[\pi F[x]]}{2}.$$

Substituting the expression for F and making the substitutions $-\cos[\pi F[x]] = x$ and $\sin[\pi F[x]] = \sqrt{1-x^2}$ we obtain

$$x_m - x \cong \frac{\sigma_x^2 x}{2(1-x^2)} = \frac{\sigma_x^2 x}{2(1+x)(1-x)}. \tag{33}$$

To guarantee that the error, $x_m - x$, is $\leq\rho$ for x near 1 we have

$$1-x \geq \frac{\sigma_x^2}{4\rho}.$$

Substituting $\rho = (B2^{-(N-1)})/4$ to guarantee that the error is $\leq B/4$ code bin widths gives

$$1-x \geq \frac{\sigma_x^2 2^{N-1}}{B}.$$

Substituting from (24) and (25) and noting that $V_{OD} = A + d - v_{\max}$, we obtain

$$V_{OD} \geq \frac{\sigma^2 2^{N-1}}{AB} \cong \frac{\sigma^2 2^N}{VB} \tag{34}$$

where V is the reduced full-scale voltage of the ADC. To obtain the rightmost expression we have made the substitution $A \cong V/2$.

All of the above is based on the convolution approximation (31). As in the previous section the actual error is larger, because $F''[x]$ is rapidly increasing near $x = \pm 1$. The convolution was calculated numerically with $p[x]$ being a Gaussian distribution and a uniform distribution for values of σ_x of 0.01, 0.001, and 0.0001. In all cases the actual error was ≤ 1.28 times the error predicted from (33) if the overdrive was at least 2σ.

VI. Derivation of Results for Number of Samples

We will estimate the variances in the cumulative histogram values, $ch[k]$ in (5), for a single record, then use (6) and (7) to estimate the variance in the transition levels or the code bin widths. This is done in three parts. First we consider the case where the samples are evenly spaced in time, i.e., the signal and sample frequencies satisfy (10) exactly. We next consider the effect of nonuniform spacing as allowed by (11). We finally consider the effect of noise. These results are next used to determine the number of records required to reduce the variance in any particular transition level or code bin width to any desired level. We finally consider worst case errors.

A. Variance and Covariance for Uniform Sampling

We will associate with each sample a phase angle, ϕ, which we will take to be between $-\pi$ and π and relative to the negative peak of the signal. Associated with any transition level, $T[k]$, is the positive phase angle, ψ_k, which satisfies

$$T[k] = d - A\cos[\psi_k] \tag{35}$$

where d and A are as in Section II-C. The number of counts in the cumulative histogram for this level, $ch[k]$, will be the number of samples with phase angles between $-\psi_k$ and $+\psi_k$.

We assume in this section that the phase angles of the samples are uniformly spaced with a spacing $\Delta\phi = 2\pi/M$ and that the smallest positive phase angle of a sample is equally likely to be any value between 0 and $\Delta\phi$. If the phase length, $2\psi_k$, corresponding to a transition level is of the form $(n+\alpha)\Delta\phi$ with n an integer and α between 0 and 1, then we will have $ch[k] = n+1$ with probability α and $ch[k] = n$ with probability $1-\alpha$. The mean value is easily seen to be $n+\alpha$ and the variance to be $\alpha(1-\alpha)$. Assuming α to be equally likely to be any value between 0 and 1, the average variance, in counts, is given by

$$\sigma_c^2 = \int_0^1 \alpha(1-\alpha)\,d\alpha = \tfrac{1}{6}. \tag{36}$$

Using (6) and the fact that $\psi_k = \pi ch[k]/M$ gives, for the variance in volts,

$$\sigma_v^2 = \frac{1}{6}A^2\frac{\pi^2}{M^2}\sin^2[\psi_k] \le \frac{1}{6}\left(\frac{A\pi}{M}\right)^2. \tag{37}$$

Later we will need the covariances between $ch[j]$ and $ch[k]$. We will show that for $j \ne k$.

$$\operatorname{cov}[ch[j], ch[k]) = \begin{cases} +1/12 \text{ with probability } 0.5 \\ -1/12 \text{ with probability } 0.5. \end{cases} \tag{38}$$

The value of $ch[k]$ is the number of samples that occur in the phase interval between $-\psi_k$ and $+\psi_k$. It will be convenient to scale all phase intervals so that the distance between samples is 1 rather than $2\pi/M$. We let the scaled lengths of the two intervals for which we are calculating the covariance be $n_i+\alpha_i$ for $i = 1, 2$, where n_i is an integer and $0 \le \alpha_i < 1$. The situation differs depending on whether n_1 and n_2 are even or odd. We first consider the case where both are even. We assume that the sample points have x-coordinates given by $m+\xi$, where m is an integer and ξ is uniformly distributed between 0 to 1. Fig. 1 illustrates an interval of length $2+2/3$ with $\xi = 0$. The dots represent the sampling points, and the brackets represent the ends of the interval, which are at $\pm(n_i+\alpha_i)/2$. As ξ varies from 0 to 1, the dots move to the right. We let $c_i'(\xi)$ = the number of dots contained in the interval when the dots are translated to the right by ξ, and $c_i(\xi) = c_i'(\xi) - n_i$. The variance and covariance of c_i and c_i' will be the same, because they differ by constants. Fig. 2 shows a plot of $c_i(\xi)$. Since the mean value of c_i is α_i we have

$$\begin{aligned}\operatorname{cov}(c_1[\alpha_1], c_2[\alpha_2]) &= \int_0^1 c_1(\xi)\,c_2(\xi)\,d\xi - \alpha_1\alpha_2 \\ &= \min(\alpha_1, \alpha_2) - \alpha_1\alpha_2.\end{aligned}$$

We now let α_1 and α_2 range from 0 to 1 and get

$$\begin{aligned}\operatorname{cov}(c_1, c_2) &= \int_0^1\int_0^1 [\min(\alpha_1, \alpha_2) - \alpha_1\alpha_2]\,d\alpha_1\,d\alpha_2 \\ &= \tfrac{1}{12}.\end{aligned} \tag{39}$$

The situation is different when either n_1 or n_2 is odd. Fig. 3 illustrates an interval of length $3+1/2$ with $\xi = 0$. Fig. 4 is a plot of $c_i(\xi)$ for the case where n_i is odd. The result is easily derived by examination of Fig. 3. In the case when n_1 is odd and n_2 is even (or vice versa) we have

$$\begin{aligned}\operatorname{cov}(c_1[\alpha_1], c_2[\alpha_2]) &= \int_0^1 c_1(\xi)\,c_2(\xi)\,d\xi - \alpha_1\alpha_2 \\ &= \max(0, \alpha_1+\alpha_2-1) - \alpha_1\alpha_2.\end{aligned}$$

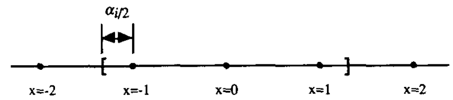

Fig. 1. The brackets represent a phase interval of 2 2/3 sample points. The dots represent the sample points.

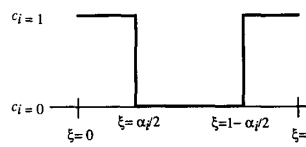

Fig. 2. The number of counts n_i in a phase interval of length $n_i+\alpha_i$ sample points as a function of the location parameter, ξ, for n_i even.

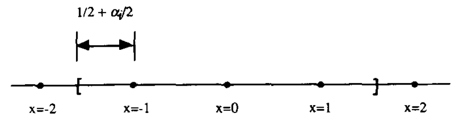

Fig. 3. Representation of a phase interval of length 3 1/2 sample points.

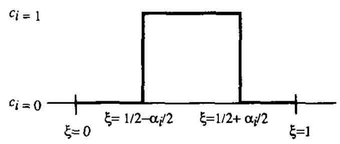

Fig. 4. The number of counts n_i in a phase interval of length $n_i+\alpha_i$ sample points as a function of the location parameter, ξ, for n_i odd.

Averaging over α_1 and α_2 we obtain

$$\begin{aligned}\operatorname{cov}(c_1, c_2) &= \int_0^1\int_0^1 [\max(0, \alpha_1+\alpha_2-1) \\ &\quad - \alpha_1\alpha_2]\,d\alpha_1\,d\alpha_2 \\ &= -\tfrac{1}{12}.\end{aligned} \tag{40}$$

A similar analysis shows that when both n_1 and n_2 are odd, we get the same result as (39). If we let the lengths of the intervals vary over many sample lengths we will get situations with the covariance = +1/12 equally often with situations with the covariance = $-1/12$.

B. Effect of Nonuniform Sampling

In this section we show that the nonuniform sampling resulting from errors in the frequency satisfying (11) will cause an increase in the variance of no more than 20% above what was determined in the previous section. As in Section III-B, we let D be the number of cycles in a record. As shown in [4], if the signal frequency satisfies (10), then the phase angles associated with the samples will have a uniform spacing of $\Delta\phi_0 = 2\pi/M$. If the signal frequency is incorrect, then there will be a nonuniform pattern to the sample phases that will repeat every D samples. We will have $D-1$ sample intervals of length $\Delta\phi_0 + e/(D-1)$ followed by one of length $\Delta\phi_0 - e$. If the signal frequency is too large, e will be positive; if it is too small, e will be negative. Since the maximum error, e, occurs

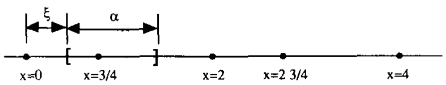

Fig. 5. A phase interval of length, α, and location parameter, ξ, with nonuniform sample points with $e = 1/4$.

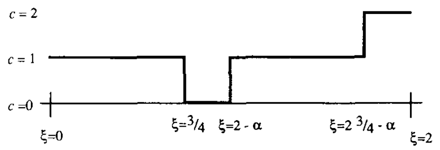

Fig. 6. The number of sample points in the interval of Fig. 5 as a function of ξ.

in only one of every D intervals, its effect on the variance will be greatest, for fixed e, when $D = 2$.

As in the previous section we will stretch out the phase axis so that the nominal distance between samples is 1. If the frequency error satisfies (11), then the error, e, will be $\leq 1/4$ in these units. The locations of the sampling points for $e = 1/4$ are illustrated in Fig. 5. The spacing between sample points alternates between 3/4 and 5/4. We assume that the width of a histogram bin is of the form $n + \alpha$ with n an *even* integer and α between 0 and 2. The situation with $n = 0$ and $\alpha = 1$ is shown in Fig. 5. We let ξ denote the distance between the left of the histogram interval and the beginning of a sample interval of length 3/4. We determine the number of sample points, $c(\xi)$, in a histogram interval as ξ varies from 0 to 2 and calculate the variance of c as a function of α. We can, without loss of generality, assume that $n = 0$.

There are three different situations: $\alpha \leq 1/4, 1/4 \leq \alpha \leq 5/4$, and $5/4 \leq \alpha \leq 2$. We will show the calculations for the second, most complicated, situation and summarize the other two. Fig. 6 shows a graph of $c(\xi)$ for α between 3/4 and 5/4. This graph is easily derived from examination of Fig. 5. The mean value of $c(\xi)$ is easily seen to be α, and the variance is given by

$$\sigma_c^2(\alpha) = \tfrac{1}{2}\int_0^2 c(\xi)^2\, d\xi - \alpha^2 = 2\alpha - 3/4 \quad \text{for } 3/4 \leq \alpha \leq 5/4.$$

A similar analysis gives, for the other two situations,

$$\sigma_c^2(\alpha) = \alpha - \alpha^2 \qquad \text{for } \alpha \leq 3/4, \text{ and}$$
$$\sigma_c^2(\alpha) = 3\alpha - 2 - \alpha^2 \qquad \text{for } 5/4 \leq \alpha \leq 2.$$

Averaging over all values of α gives

$$\sigma_c^2 = \frac{1}{2}\int_0^2 \sigma_c^2(\alpha)\, d\alpha = \frac{19}{16 \cdot 6} = 1.1875 \times \frac{1}{6}.$$

This exceeds the variance, 1/6, of the uniform spacing case by less than 20%.

C. Effect of Noise

The presence of noise will add additional variance to the number of counts that will be sampled in any phase interval. The effect of noise is to cause a sample point that is inside (outside) the phase interval of a histogram bin, but near the edge of that phase interval, to sometimes be recorded in a bin outside (inside) the phase interval. We will assume that the noise has a Gaussian distribution with a mean of zero and a standard deviation of σ (in volts). We will separate the analysis into two situations—depending on whether or not the voltage width of the histogram bin is large compared to σ. We will find that this means the bin width is larger than 1.1σ.

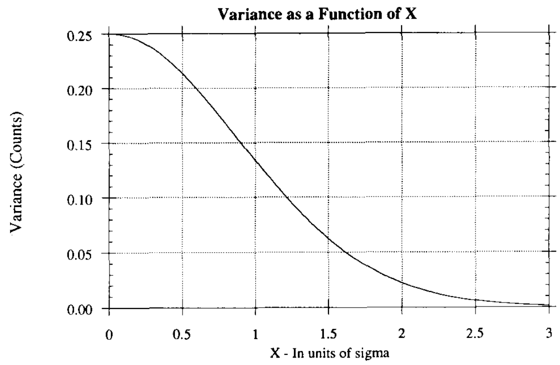

Fig. 7. The variance in the number of counts recorded in a bin due to noise for a single sample point at a distance x from the edge of the bin.

1) Noise Effect on Transition Levels: For this analysis we focus our attention on an individual sample and look at how the noise determines whether or not the sample appears in a particular cumulative histogram bin. Let

$p(x)$ = the probability that a sample that is inside the phase interval for a cumulative histogram bin at a distance x, in volts, from the edge of the interval will be recorded in that interval.

If the sample point is at distance x from the right edge of the interval, then $p(x)$ is the probability that the noise voltage is $\leq x$; if x is the distance from the left edge, then $p(x)$ is the probability that the noise voltage is $\geq -x$. In either case we have

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-(v^2/2\sigma^2)}\, dv = 1 - \frac{1}{2} erfc\left(\frac{x}{\sigma\sqrt{2}}\right).$$

Let c_j be the random variable that takes the value one if the jth sample point is recorded in the cumulative histogram bin in question and takes the value zero if it is recorded outside. If the sample point is inside the phase interval for the bin at a distance x from the edge of the interval, then the mean value of c_j will be $p(x)$ and the variance will be $v(x) = p(x)(1-p(x))$. If the sample point is outside the phase interval for the bin at a distance x from the edge of the interval, then the mean value of c_j will be $1 - p(x)$ and the variance will also be $v(x)$. A plot of $v(x)$ for x between 0 and 3σ is shown in Fig.7. The variance, due to the noise, in the counts in a cumulative histogram bin is given by

$$\sigma_c^2 = 4\lambda \int_0^\infty v(x)\, dx \tag{41}$$

where λ is the density of sample points per volt near the end points of the phase interval for the bin. The factor of 4 appears, because there are four separate regions that contribute to the variance—the regions immediately inside and outside of both ends of the interval. The corresponding variance, in volts, is given by

$$\sigma_v^2 = \frac{\sigma_c^2}{(2\lambda)^2} = \frac{1}{\lambda}\int_0^\infty v(x)\,dx. \quad (42)$$

The density of sample points per unit phase is $M/2\pi$; by (35) the derivative of the voltage of a transition level with respect to the phase, ϕ, is $A\sin(\phi)$, where A is the magnitude of the signal. This gives

$$\lambda = \frac{M}{2\pi}\frac{1}{A\sin(\phi)} \geq \frac{M}{2\pi A}. \quad (43)$$

Note also that $v(x) = v_1(x/\sigma)$, where $v_1(x)$ is the value of $v(x)$ with $\sigma = 1$. This gives

$$\int_0^\infty v(x)\,dx = \sigma\int_0^\infty v_1(x)\,dx = 1.13\frac{\sigma}{4} \quad (44)$$

where the value of the constant was determined by numerical integration. Combining (42), (43), and (44) we obtain

$$\sigma_v^2 \leq \frac{1.13}{2}\frac{\sigma A\pi}{M}. \quad (45)$$

2) Noise Effect on Code Bin Widths: A code bin width is the difference between two adjacent transition levels. If the code bin width is larger than twice the noise level, then the errors in the two transition levels will be nearly independent. This means that the variance in the code bin width will be twice that for the transition levels. When the bin width is small compared to the noise level, the errors in adjacent transition levels are highly correlated, and the variance for the bin width is smaller than twice the variance for the transition levels. One could determine the variance in the bin width by calculating the covariance between adjacent transition levels, but we choose the simpler approach of separately analyzing the variance in the case of small bin width-to-noise ratio.

We focus our attention on a particular code bin of width Q centered at voltage v_b. The probability that any particular sample point, with nominal sample voltage v_i, will be recorded in this code bin is $p = QP(v_i - v_b)$, where $P(v)$ is the probability density for the noise. As in the previous section we let c_i be the random variable that is 1 if this sample point is recorded in this code bin and that is 0 otherwise. The variance of c_i is given by $p(1-p) \leq p$. The total variance σ_c^2 in the number of counts in the code bin is given by

$$\sigma_c^2 \leq \sum QP(v_i - v_b) = \frac{Q}{\Delta v}\sum P(v_i - v_b)\Delta v \cong \frac{Q}{\Delta v} \quad (46)$$

where Δv is the average voltage difference between samples near the code bin. Note that the second sum is approximately the integral of a probability density, which is equal to one. Now, $1/\Delta v = 2\lambda$, where λ is the constant used in (41) (the factor of 2 appears here because there are sample points on both the increasing and decreasing sections of the input signal). This gives for small bin width-to-noise ratio

$$\sigma_c^2 \leq 2\lambda Q. \quad (47)$$

For large bin width-to-noise ratio we have, by combining (41), (44) and the fact that the variance for code bins is twice that for transition levels,

$$\sigma_c^2 \leq 2\lambda 1.13\sigma. \quad (48)$$

Combining these results gives for all cases

$$\sigma_c^2 \leq 2\lambda \min[Q, 1.13\sigma]. \quad (49)$$

D. Combining the Results and Worst Case Errors

Combining the results of the previous sections, the variance in the voltage for a transition level is given by

$$\sigma_v^2 \leq \frac{1.2}{6}\left(\frac{\alpha V\pi}{2M}\right)^2 + \frac{1.13}{4}\frac{\sigma\alpha V\pi}{M}. \quad (50)$$

The first term comes from (37), with the factor of 1.2 being the extra 20% added to the variance due to the allowed errors, (11), in the frequency. The second term comes from (45). The substitution $A = \alpha V/2$ was made in both terms. When R records are taken, the variance is reduced by a factor of R. The variance in code bin widths for R records is then

$$\sigma_Q^2 = \frac{\sigma_v^2}{(2^{-N}V)^2R} \leq \frac{2^{2(N-1)}}{R} \cdot\left\{0.2\left[\frac{\alpha\pi}{M}\right]^2 + 1.13\frac{\sigma}{V}\left[\frac{\alpha\pi}{M}\right]\right\}. \quad (51)$$

We make the approximation that the transition levels have a Gaussian distribution with this variance. This is a reasonable approximation, because the counts in any cumulative histogram bin are the sum of many random variables with a binomial distribution. To the extent that the distribution is not Gaussian, our results for confidence intervals are conservative, because the Gaussian distribution has a larger probability of deviations of several σ than do the actual distributions. To meet the requirement that $K\sigma_Q \leq B$, for any constant, K, we have

$$R \geq \left[\frac{2^{(N-1)}K}{B}\right]^2\left[\frac{\alpha\pi}{M}\right]\left\{0.2\left[\frac{\alpha\pi}{M}\right] + 1.13\frac{\sigma}{V}\right\}. \quad (52)$$

This is the result, (12), for transition levels for appropriate values of K. The relationship of the value of K to the confidence levels is given in [2].

We showed in Section VI-A that the covariance between transition levels due to sampling errors is equally likely to be +1/2 of the variance or −1/2 of the variance and that the covariance of the errors due to noise, for large code bin width-to-noise ratio, is zero. If we assume, for large code bin width-to-noise ratio, that this covariance is zero then the variance for code bin widths is twice that given by (51); hence, the number of required records is twice that given by (52). Combining this with (49) gives (12) for code bin widths and DNL. Although it is not technically correct to assume zero covariance, simulation results (see the following section) indicate that the predictions are accurate.

The constants in Table I are based on the assumption that the errors in individual transition levels or code bin widths are independent. For Gaussian random variables independence is

TABLE II

Parameter Name	Variable	Value
Samples per Record	M	250
Cycles per Record	D	4
Sampling Frequency	fs	1
Signal Frequency	f	0.016005
Signal Offset	d	32
Tolerance	B	0.2
Confidence Level	$1-u$	0.9

TABLE III
THE NUMBER OF RECORDS AND THE SIGNAL AMPLITUDE USED FOR EACH OF THE FOUR TYPES OF SIMULATIONS

	Without Noise	With Noise
INL	$R=3$ $A=31.25$	$R=10$ $A=32.5$
DNL	$R=5$ $A=31.25$	$R=21$ $A=32.5$

TABLE IV
HISTOGRAM OF THE INL AND DNL ERRORS OBSERVED IN THE TWELVE SIMULATIONS. THE MEASUREMENT PARAMETERS WERE CHOSEN TO GUARANTEE THAT LESS THAN 10% OF THE VALUES, OR 75 OF THEM, WOULD BE OUT OF TOLERANCE. THE TABLE SHOWS THAT 40 OF THE MEASUREMENTS, OR 5.3% OF THEM, WERE ACTUALLY OUT OF TOLERANCE

Error	INL			DNL		
0–0.02	12	6	19	11	10	20
0.02–0.04	4	4	11	11	10	18
0.04–0.06	14	12	17	5	3	6
0.06–0.08	2	3	3	6	6	7
0.08–0.10	8	12	5	9	9	5
0.10–0.12	4	5	3	1	3	3
0.12–0.14	8	9	3	2	2	1
0.14–0.16	5	6	1	2	5	0
0.16–0.18	3	5	1	3	5	2
0.18–0.20	1	0	0	3	2	0
0.20–0.22	0	0	0	3	3	0
0.22–0.24	0	0	0	2	2	0
0.24–0.26	1	1	0	2	1	0
0.26–0.28	0	0	0	0	1	0
0.28–0.30	0	0	0	2	0	0
0.30–0.32	1	0	0	0	0	0
Total Out of Toler-ance	**2**	**1**	**0**	**9**	**7**	**0**
Total Counts	63	63	63	62	62	62

TABLE V
HISTOGRAM OF THE INL AND DNL ERRORS OBSERVED IN THE TWELVE SIMULATIONS. THE MEASUREMENT PARAMETERS WERE CHOSEN TO GUARANTEE THAT LESS THAN 10% OF THE VALUES, OR 75 OF THEM, WOULD BE OUT OF TOLERANCE. THE TABLE SHOWS THAT 40 OF THE MEASUREMENTS, OR 5.3% OF THEM, WERE ACTUALLY OUT OF TOLERANCE

Error	INL			DNL		
0–0.02	11	11	11	13	8	7
0.02–0.04	6	7	8	8	11	10
0.04–0.06	17	12	7	11	11	13
0.06–0.08	5	6	4	9	13	10
0.08–0.10	5	6	5	6	5	7
0.10–0.12	1	5	3	8	4	4
0.12–0.14	6	7	4	1	1	2
0.14–0.16	5	3	2	2	1	4
0.16–0.18	2	2	6	1	3	1
0.18–0.20	2	2	3	2	2	2
0.20–0.22	2	1	1	0	2	1
0.22–0.24	1	1	4	1	0	1
0.24–0.26	0	0	1	0	0	0
0.26–0.28	0	0	1	0	0	0
0.28–0.30	0	0	2	0	0	0
0.30–0.32	0	0	1	0	1	0
Total Out of Toler-ance	**3**	**2**	**10**	**1**	**3**	**2**
Total Counts	63	63	63	62	62	62

the same as zero covariance. When there is correlation between the errors, either positive or negative, the probability of the worst case value exceeding any particular value is always smaller than the probability of uncorrelated errors exceeding the same value.

VII. SIMULATION RESULTS

All of the simulations reported here were done using a six-bit ADC with no differential or integral nonlinearity. The true transition levels were given by $T[k] = k$ for $k = 1..63$ (i.e., $Q = 1$). The measurement parameters that were common to all simulations are given in Table II. The signal frequency was determined by using (10) then adding the maximum error allowed by (11).

The simulations were divided into four groups depending on whether the specified tolerance was sought for DNL or for INL and depending on whether or not noise was present. For the simulations with noise the rms noise level was $\sigma = 0.5$, the noise level that reduces the number of effective bits of the ADC from 6 to 5. The parameters used for the simulations in each of the four situations are given in Table III.

For the simulations without noise the signal amplitude was set to give an overdrive of $Q/4$; with noise the overdrive was $1.5Q = 3\sigma$. The amounts of overdrive required by (8) and (9) are $\sigma \times \max[3, 2.7] = 3\sigma$ and $\sigma \times \max[2, 2.5] = 2.5\sigma$, respectively. The number of required records, calculated from (12), is 2.2 for INL and 4.4 for DNL. We used the next larger integer. The vaue of K used in (12) was 1.64, giving a probability of 0.1 or less that any particular INL or DNL value will exceed the tolerance. For all of the simulations the amplitude and the offset of the signal were assumed unknown, and their values were calculated from the data using $T[1] = 1$ and $T[63] = 63$ in (6).

Each simulation resulted in estimates for each of the 63 values of $T[k]$ and each of the 62 values of $W[k]$. The 63 values of INL and 62 values of DNL were then calculated as in Section II using $G = 1, Q = 1$, and $V_{OS} = 0$. All of the correct values for INL and DNL are zero. For each simulation a histogram (with a bin width of 0.02) was constructed of the absolute values of the calculated INL and DNL values. The histograms for three simulations of each of the four types are shown in Figs. 4 and 5. The solid line below the error range of 0.18 to 0.20 indicates that data below the line is out of tolerance. With the 0.9 confidence level chosen we expect an average of up to 6.3 counts to fall below this line. Note that the standard deviation in this value is $\sqrt{6.3} = 2.5$; thus, the largest number of out-of-tolerance measurements (10) is only 1.5σ greater than the expected value; the second largest (9) is 1.1σ greater than the expected value. The probability of a random variable with a normal distribution deviating by 1.5σ (or 1.1σ) from its expected value is 0.14 (or 0.28), so the two values out of 12 in the two tables with these deviations are to be expected statistically.

VIII. Conclusions

We have shown that histogram tests using sine wave input signals can be used to determine differential nonlinearity and integral nonlinearity of an ADC to any desired accuracy. We have given formulas for calculating the amount of overdrive required, as a function of the noise level, to obtain any desired level of accuracy. We have shown that if the ratio of sampling frequency to signal frequency is chosen appropriately, the number of records required to obtain any desired tolerance and confidence level is smaller than that required with random sampling. We have shown how to determine the ratio of sampling frequency to signal frequency and have given formulas for calculating the number of samples required. These formulas can be used to obtain a given confidence in individual values of INL or DNL or for the worst case values of INL or DNL. Results were also given relating harmonic distortion in the signal source to errors in INL and DNL results.

References

[1] M. Vanden Bossche, J. Schoukens, and J. Renneboog, "Dynamic testing and diagnostics of A/D converters," *IEEE Trans. Circuits Syst.*, vol. CAS-33, Aug. 1986.

[2] J. Doernberg, H-S. Lee, and D. A. Hodges, "Full speed testing of A/D converters," *IEEE J. Solid-State Circuits,* vol. SC-19, Dec. 1984.

[3] "Dynamic performance testing of A to D converters," *Hewlett Packard Product Note 5180A-2.*

[4] T. M. Souders, D. R. Flach, and J. J. Blair, "Step and frequency response testing of waveform recorders," in *Conf. Rec., IEEE Instrum. Meas. Technol. Conf.*, Feb. 1990, pp. 214–220.

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