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Historical Overview of the Kepler Conjecture

Thomas C. Hales

Department of Mathematics, University of Pittsburgh,
Pittsburgh, PA 15217, USA
hales@pitt.edu

Abstract. This paper is the first in a series of six papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. After some preliminary comments about the face-centered cubic and hexagonal close packings, the history of the Kepler problem is described, including a discussion of various published bounds on the density of sphere packings. There is also a general historical discussion of various proof strategies that have been tried with this problem.

1. Introduction

1.1. *The Face-Centered Cubic Packing*

A packing of spheres is an arrangement of nonoverlapping spheres of radius 1 in Euclidean space. Each sphere is determined by its center, so equivalently it is a collection of points in Euclidean space separated by distances of at least 2. The density of a packing is defined as the lim sup of the densities of the partial packings formed by spheres inside a ball with fixed center of radius R. (By taking the lim sup, rather than lim inf, as the density, we prove the Kepler conjecture in the strongest possible sense.) Defined as a limit, the density is insensitive to changes in the packing in any bounded region. For example, a finite number of spheres can be removed from the face-centered cubic packing without affecting its density.

Consequently, it is not possible to hope for any strong uniqueness results for packings of optimal density. The uniqueness established by this work is as strong as can be hoped for. It shows that certain local structures (decomposition stars) attached to the face-centered cubic (fcc) and hexagonal-close packings (hcp) are the only structures that maximize a local density function.

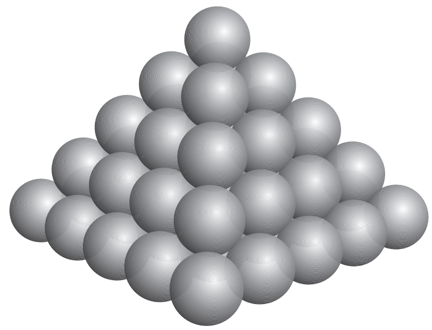

Fig. 1.1. The fcc packing.

Although we do not pursue this point, Conway and Sloane develop a theory of tight packings that is more restrictive than having the greatest possible density [CS1]. An open problem is to prove that their list of tight packings in three dimensions is complete.

The fcc packing appears in Fig. 1.1.

The following facts about packings are well known. However, there is a popular and persistent misconception in the popular press that the fcc packing is the only packing with density $\pi/\sqrt{18}$. The comments that follow correct this misconception.

In the fcc packing, each ball is tangent to twelve others. For each ball in the packing, this arrangement of twelve tangent balls is the same. We call it the fcc pattern. In the hcp, each ball is tangent to twelve others. For each ball in the packing, the arrangement of twelve tangent balls is again the same. We call it the hcp pattern. The fcc pattern is different from the hcp pattern. In the fcc pattern there are four different planes through the center of the central ball that contain the centers of six other balls at the vertices of a regular hexagon. In the hcp pattern there is only one such plane. We call the arrangement of balls tangent to a given ball the *local tangent arrangement* of the ball.

There are uncountably many packings of density $\pi/\sqrt{18}$ that have the property that every ball is tangent to twelve others and such that the tangent arrangement around each ball is either the fcc pattern or the hcp pattern.

By *hexagonal layer*, we mean a translate of the two-dimensional lattice of points M in the A_2 arrangement. That is, M is a translate of the planar lattice generated by two vectors of length 2 and angle $2\pi/3$. The fcc packing is an example of a packing built from hexagonal layers.

If M is a hexagonal layer, a second hexagonal layer M' can be placed parallel to the first so that each lattice point of M' has distance 2 from three different vertices of M. When the second layer is placed in this manner, it is as close to the first layer as possible. Fix M and a unit normal to the plane of M. The normal allows us to speak of the second layer M' as being "above" or "below" layer M. There are two different positions in which M' can be placed closely above M and two different positions in which M' can be placed closely below M. As we build a packing, layer by layer (M, M', M'', and so forth), there are two choices at each stage of the close placement of the layer above the previous layer. Running through different sequences of choices gives uncountably many packings. In each of these packings the tangent arrangement around each ball is that of the twelve spheres in the fcc or the twelve spheres in the hcp.

Let Λ be a packing built as a sequence of close-packed hexagonal layers in this fashion. If P is any plane parallel to the hexagonal layers, then there are at most three different orthogonal projections of layers M to P. Call these projections A, B and C. Each hexagonal layer has a different projection than the layers immediately above and below it. In the fcc packing, the successive layers are $A, B, C, A, B, C, \ldots$. In the hcp packing, the successive layers are $A, B, A, B, \ldots$. If we represent A, B, and C as the vertices of a triangle, then the succession of hexagonal layers can be described by a walk along the vertices of the triangle. Different walks through the triangle describe different packings.

In fact, the different walks through a triangle give all packings of infinitely many equal balls in which the tangent arrangement around every ball is either the fcc pattern of twelve balls or the hcp pattern of twelve balls.

We justify the fact that different walks through a triangle give all such packings. Assume first that a packing Λ contains a ball (centered at v_0) in the hcp pattern. The hcp pattern contains a uniquely determined plane of symmetry. This plane contains v_0 and the centers of six others arranged in a regular hexagonal. If v is the center of one of the six others in the plane of symmetry, its local tangent arrangement of twelve balls must include v_0 and an additional four of the twelve balls around v_0. These five centers around v are not a subset of the fcc pattern. They can be uniquely extended to twelve centers arranged in the hcp pattern. This hcp pattern has the same plane of symmetry as the hcp pattern around v_0. In this way, as soon as there is a single center with the hcp pattern, the pattern propagates along the plane of symmetry to create a hexagonal layer M.

Once a packing Λ contains a single hexagonal layer, the condition that each ball be tangent to twelve others forces a hexagonal layer M' above M and another hexagonal layer below M. Thus, a single hexagonal layer forces a sequence of close-packed hexagonal layers in both directions.

We have justified the claim under the hypothesis that Λ contains at least one ball with the hcp pattern.

Assume that Λ does not contain any balls whose local tangent arrangement is the hcp pattern. Then every local tangent arrangement is the fcc pattern, and Λ itself is then the face-centered cubic packing. This completes the proof.

1.2. *Early History, Hariot, and Kepler*

The study of the mathematical properties of the fcc packing can be traced back to a Sanskrit work composed around 499 CE [P].

The modern mathematical study of spheres and their close packings can be traced to T. Hariot. Hariot's work—unpublished, unedited, and largely undated—shows a preoccupation with sphere packings. He seems to have first taken an interest in packings at the prompting of Sir Walter Raleigh. At the time, Hariot was Raleigh's mathematical assistant, and Raleigh gave him the problem of determining formulas for the number of cannonballs in regularly stacked piles. In 1591 he prepared a chart of triangular numbers for Raleigh. Shirley, Hariot's biographer, writes that the charts prepared for Raleigh led

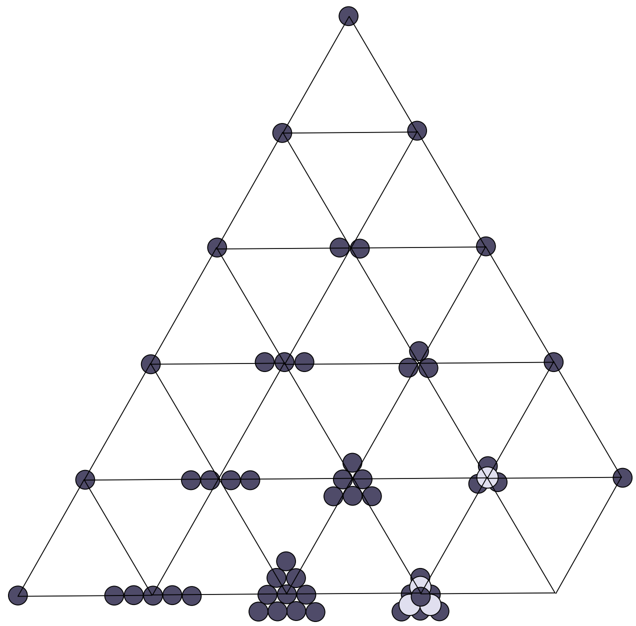

Fig. 1.2. Hariot's view of Pascal's triangle.

to the study of the sums of squares, "a study which led inevitably to the corpuscular or atomic theory of matter orginally deriving from Lucretius and Epicurus" [Sh, p. 242].

Hariot connected sphere packings to Pascal's triangle long before Pascal introduced the triangle. See Fig. 1.2.

Hariot was the first to distinguish between the fcc and hcp [Ma, p. 52].

Kepler became involved in sphere packings through his correspondence with Hariot in the early years of the 17th century. Despite Kepler's initial reluctance to adopt an atomic theory, he was eventually swayed, and in 1611 he published an essay that explores the consequences of a theory of matter composed of small spherical particles. Kepler's essay was the "first recorded step towards a mathematical theory of the genesis of inorganic or organic form" [W, p. v].

Kepler's essay describes the fcc packing and asserts that "the packing will be the tightest possible, so that in no other arrangement could more pellets be stuffed into the same container." This assertion has come to be known as the Kepler conjecture. The purpose of this collection of papers is to give a proof of this conjecture.

1.3. *History*

The next episode in the history of this problem is a debate between Isaac Newton and David Gregory. Newton and Gregory discussed the question of how many spheres of equal radius can be arranged to touch a given sphere. This is the three-dimensional analogue of the simple fact that in two dimensions six pennies, but no more, can be arranged to touch a central penny. This is the kissing-number problem in n-dimensions.

In three dimensions Newton said that the maximum was twelve spheres, but Gregory claimed that thirteen might be possible.

Newton was correct. In the 19th century, the first papers claiming a proof of the kissing-number problem appeared in [Ben], [Gu], and [Ho]. Although some writers cite these papers as a proof, they are hardly rigorous by today's standards. Another incorrect proof appears in [Bo]. The first proper proof was obtained by Schütte and van der Waerden in 1953 [SW]. An elementary proof appears in Leech [Le]. The influence of van der Waerden, Schütte, and Leech upon the papers in this collection is readily apparent. Although the connection between the Newton–Gregory problem and Kepler's problem is not obvious, L. Fejes Tóth in 1953, in the first work describing a strategy to prove the Kepler conjecture, made a quantitative version of the Gregory–Newton problem the first step [Fej6].

The two-dimensional analogue of the Kepler conjecture is to show that the honeycomb packing in two dimensions gives the highest density. This result was established in 1892 by Thue, with a second proof appearing in 1910 [T1], [T2]. Szpiro's book on the Kepler conjecture calls Thue's proofs into question [Sz]. C. Siegel said that Thue's original proof is "reasonable, but full of holes" [Sz]. A number of other proofs have appeared since then. Three are particularly notable. Rogers's proof generalizes to give a bound on the density of packings in any dimension [Ro1]. A proof by L. Fejes Tóth extends to give bounds on the density of packings of convex disks [Fej5]. A third proof, also by L. Fejes Tóth, extends to non-Euclidean geometries [Fej6]. Another early proof appears in [SM].

In 1900, Hilbert made the Kepler conjecture part of his 18th problem [Hi]. Milnor, in his review of Hilbert's 18th problem, breaks the problem into three parts [Mi]. The third part asks for the densest arrangements of congruent solids such as "spheres with given radii." Milnor writes that it is a "scandalous situation" that the sphere packing "problem in 3 dimensions remains unsolved."

1.4. *The Literature*

Past progress toward the Kepler conjecture can be arranged into four categories:

- bounds on the density,
- descriptions of classes of packings for which the bound of $\pi/\sqrt{18}$ is known,
- convex bodies other than spheres for which the packing density can be determined precisely,
- strategies of proof.

1.4.1. *Bounds*. Various upper bounds have been established on the density of packings:

0.884 (Blichfeldt) [Bl1],
0.835 (Blichfeldt) [Bl2],
0.828 (Rankin) [Ra],
0.7797 (Rogers) [Ro1],
0.77844 (Lindsey) [Li],
0.77836 (Muder)[Mu1],
0.7731 (Muder) [Mu2].

Rogers's is a particularly natural bound. As the dates indicate, it remained the best available bound for many years. His monotonicity lemma and his decomposition of Voronoi cells into simplices have become important elements in the proof of the Kepler conjecture. We give a new proof of Rogers's bound in "Sphere Packings, III." A function τ, used throughout this collection, measures the departure of various objects from Rogers's bound.

Muder's bounds, although they appear to be rather small improvements of Rogers's bound, are the first to make use of the full Voronoi cell in the determination of densities. As such, they mark a transition to a greater level of sophistication and difficulty. Muder's influence on the work in this collection is also apparent.

A sphere packing admits a Voronoi decomposition: around every sphere take the convex region consisting of points closer to that sphere center than to any other sphere center. L. Fejes Tóth's dodecahedral conjecture asserts that the Voronoi cell of smallest volume is a regular dodecahedron with inradius 1 [Fej4]. The dodecahedral conjecture implies a bound of 0.755 on sphere packings. L. Fejes Tóth actually gave a complete proof except for one estimate. A footnote in his paper documents the gap, "In the proof, we have relied to some extent solely on intuitive observation [Anschauung]." As L. Fejes Tóth pointed out, that estimate is extraordinarily difficult, and the dodecahedral conjecture has resisted all efforts until now [Mc].

The missing estimate in L. Fejes Tóth's paper is an explicit form of the Newton–Gregory problem. What is needed is an explicit bound on how close the thirteenth sphere can come to touching the central sphere, or, more generally, to minimize the sum of the distances of the thirteen spheres from the central sphere. No satisfactory bounds are known. Boerdijk has a conjecture for the arrangement that minimizes the average distance of the thirteen spheres from the central sphere. Van der Waerden has a conjecture for the closest arrangement of thirteen spheres in which all spheres have the same distance from the central sphere. Bezdek has shown that the dodecahedral conjecture would follow from weaker bounds than those originally proposed by L. Fejes Tóth [Bez2].

A proof of the dodecahedral conjecture has traditionally been viewed as the first step toward a proof of the Kepler conjecture, and if little progress has been made until now toward a complete solution of the Kepler conjecture, the difficulty of the dodecahedral conjecture is certainly responsible to a large degree.

1.4.2. *Classes of Packings.* If the infinite-dimensional space of all packings is too unwieldy, we can ask if it is possible to establish the bound $\pi/\sqrt{18}$ for packings with special structures.

If we restrict the problem to packings whose sphere centers are the points of a lattice, the packings are described by a finite number of parameters, and the problem becomes much more accessible. Lagrange proved that the densest lattice packing in two dimensions is the familiar honeycomb arrangement [La]. Gauss proved that the densest lattice packing in three dimensions is the fcc [G]. In dimensions 4–8 the optimal lattices are described by their root systems, A_2, A_3, D_4, D_5, E_6, E_7, and E_8. Korkine and Zolotareff showed that D_4 and D_5 are the densest lattice packings in dimensions 4 and 5 [KZ1], [KZ2]. Blichfeldt determined the densest lattice packings in dimensions 6–8 [Bl3]. Cohn and Kumar solved the problem in dimension 24 [CK]. With the exception of dimension 24, beyond dimension 8, there are no proofs of optimality, and yet there are many excellent candidates for the densest lattice packings. For a proof of the existence of optimal lattices, see [O].

Although lattice packings are of particular interest because they relate to so many different branches of mathematics, Rogers has conjectured that in sufficiently high dimensions, the densest packings are not lattice packings [Ro2]. In fact, the densest known packings in various dimensions are not lattice packings. The third edition of [CS2] gives several examples of nonlattice packings that are denser than any known lattice packings (dimensions 10, 11, 13, 18, 20, 22). The densest packings of typical convex sets in the plane, in the sense of Baire categories, are not lattice packings [Fej1].

Gauss's theorem on lattice densities has been generalized by Bezdek, Kuperberg, and Makai, Jr. [BKM]. They showed that packings of parallel strings of spheres never have density greater than $\pi/\sqrt{18}$.

1.4.3. *Other Convex Bodies*. If the optimal sphere packings are too difficult to determine, we might ask whether the problem can be solved for other convex bodies. To avoid trivialities, we restrict our attention to convex bodies whose packing density is strictly less than 1.

The first convex body in Euclidean 3-space that does not tile for which the packing density was explicitly determined is an infinite cylinder [BK]. Here Bezdek and Kuperberg prove that the optimal density is obtained by arranging the cylinders in parallel columns in the honeycomb arrangement.

In 1993 Pach exposed the humbling depth of our ignorance when he issued the challenge to determine the packing density for some bounded convex body that does not tile space [MP]. (Pach's question is more revealing than anything I can write on the subject of discrete geometry.) This question was answered by Bezdek [Bez1], who determined the packing density of a rhombic dodecahedron that has one corner clipped so that it no longer tiles. The packing density equals the ratio of the volume of the clipped rhombic dodecahedron to the volume of the unclipped rhombic dodecahedron.

1.4.4. *Strategies of Proof.* In 1953 L. Fejes Tóth proposed a program to prove the Kepler conjecture [Fej6]. A single Voronoi cell cannot lead to a bound better than the dodecahedral conjecture. L. Fejes Tóth considered weighted averages of the volumes of collections of Voronoi cells. These weighted averages involve up to thirteen Voronoi cells. He showed that if a particular weighted average of volumes is greater than the volume of the rhombic dodecahedron, then the Kepler conjecture follows. The Kepler conjecture is an optimization problem in an infinite number of variables. L. Fejes Tóth's weighted-average argument was the first indication that it might be possible to reduce the Kepler conjecture to a problem in a finite number of variables. Needless to say, calculations involving the weighted averages of the volumes of several Voronoi cells will be significantly more difficult than those involved in establishing the dodecahedral conjecture.

To justify his approach, which limits the number of Voronoi cells to thirteen, L. Fejes Tóth needs a preliminary estimate of how close a thirteenth sphere can come to a central sphere. It is at this point in his formulation of the Kepler conjecture that an explicit version of the Newton–Gregory problem is required. How close can thirteen spheres come to a central sphere, as measured by the sum of their distances from the central sphere?

L. Fejes Tóth made another significant suggestion in [Fej7]. He was the first to suggest the use of computers in the Kepler conjecture. After describing his program, he mentions that his approach reduces the problem to an optimization problem in a finite number of variables. He suggests that computers might be used to approximate the minimum to this optimization problem.

The most widely publicized attempt to prove the Kepler conjecture was that of Hsiang [Hs1]. (See also [Hs2], [Hs3], and [Hs5].) Hsiang's approach can be viewed as a continuation and extension of L. Fejes Tóth's program. Hsiang's paper contains major gaps and errors [CHMS]. The mathematical arguments against his argument appear in my debate with him in the *Mathematical Intelligencer* [Ha3], [Hs4]. There are now many published sources that agree with the central claims of [Ha3] against Hsiang. Conway and Sloane report that the paper "contains serious flaws." G. Fejes Tóth feels that "the greater part of the work has yet to be done" [Fej1]. Bezdek concluded, after an extensive study of Hsiang's work, "his work is far from being complete and correct in all details" [Bez2]. Muder writes, "the community has reached a consensus on it: no one buys it" [Mu3].

2. Overview of the Proof

2.1. *Experiments with other Decompositions*

The following two sections (added January 2003) describe some of the motivation behind the partitions of space that have been used in the proof of the Kepler conjecture. This discussion includes various ideas that were tried, found wanting, and discarded. However, this discussion provides motivation for some of the choices that appear in the proof of the Kepler conjecture.

Let S be a regular tetrahedron of side length 2. If we place a unit ball at each of the four vertices, the fraction of the tetrahedral solid occupied by the part of the four balls within the tetrahedron is $\delta_{\text{tet}} \approx 0.7797$. Let O be a regular octahedron of side length 2. If we place a unit ball at each of the four vertices, the fraction of the octahedral solid occupied by the four balls is $\delta_{\text{oct}} \approx 0.72$. The fcc packing can be obtained by packing eight regular tetrahedra and six regular octahedra around each vertex. The density $\pi/\sqrt{18}$ of this packing is a weighted average of δ_{tet} and δ_{oct}:

$$\frac{\pi}{\sqrt{18}} = \tfrac{1}{3}\delta_{\text{tet}} + \tfrac{2}{3}\delta_{\text{oct}}.$$

My early conception (around 1989) was that for every packing of congruent balls, there should be a corresponding partition of space into regions of high density and regions of low density. Regions of high density should be defined as regions having density between δ_{oct} and δ_{tet}, and regions of low density should be defined as those regions of density at most δ_{oct}. It was my intention to prove that all regions of high density had to be confined to a set of nonoverlapping tetrahedra whose vertices are centers of the balls in the packing.

Thus, the question naturally arises of how much a regular tetrahedron of edge length 2 can be deformed before its density drops below that of a regular octahedron δ_{oct}.

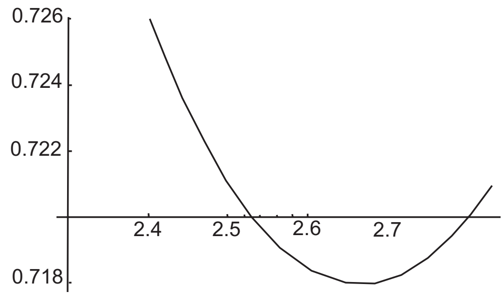

Fig. 2.1. The origin of the constant 2.51.

The graph in Fig. 2.1 shows the density of a tetrahedron with five edges of length 2 and a sixth edge of length x. Numerically, we see that the density drops below δ_{oct}, when $x = x_0 \approx 2.504$. To achieve the design goal of confining regions of high density to tetrahedra, we want a tetrahedron of edge lengths $2, 2, 2, 2, 2, x$, for $x \le x_0$, to be counted as a region of high density. Rounding upward, this example led to the cutoff parameter of 2.51 that distinguishes the tetrahedra (in the high density region) from the rest of space. This is the origin of the constant 2.51 that appears in the proof.

Since the tetrahedra are chosen to have vertices at the centers of the balls in the packing, it was quite natural to base the decomposition of space on the Delaunay decomposition. According to this early conception, space was to be partitioned into Delaunay simplices. A Delaunay simplex whose edge lengths are at most 2.51 is called a quasi-regular tetrahedron. These were the regions of presumably high density. According to the strategy in those early days, all other Delaunay simplices were to be shown to belong to regions of density at most δ_{oct}.

The following problem occupied my attention for a long period.

Problem. Fix a saturated packing. Let $X(oct)$ be the part of space of a saturated packing that is occupied by the Delaunay simplices having at least one edge of length at least 2.51. Let $X(tet)$ be the union of the complementary set of Delaunay simplices. Is it always true that the density of $X(oct)$ is at most δ_{oct}?

Early on, I viewed the positive resolution of this problem as crucial to the solution of the Kepler conjecture. Eventually, when I divided the proof of the Kepler conjecture into a five-step program, a variant of this problem became the second step of the program. See [Ha7].

To give an indication of the complexity of this problem, consider the simplex with edge lengths $2, 2, 2, 2, \ell, \ell$, where $\ell = \sqrt{2(3+\sqrt{6})} \approx 3.301$. Assume that the two longer edges meet at a vertex. This simplex can appear as the Delaunay simplex in a saturated packing. Its density is about 0.78469. This constant is not only greater than δ_{oct}; it is even greater than δ_{tet}, so that the problem is completely misguided at the level of individual Delaunay simplices in $X(oct)$. It is only when the union of Delaunay simplices is considered that we can hope for an affirmative answer to the problem.

By the summer of 1994, I had lost hope of finding a partition of the set $X(oct)$ into small clusters of Delaunay simplices with the property that each cluster had density at most δ_{oct}. Progress had ground to a halt. The key insight came in the fall of 1994 (on Nov. 12, 1994 to be precise). On that day, I introduced a hybrid decomposition that relied on the Delaunay simplices in the regions $X(tet)$ formed by quasi-regular tetrahedra, but that switched to the Voronoi decomposition in certain regions of $X(oct)$. By April 1995 I had reformulated the problem, worked out a proof of the problem [Ha7] in its new form, and submitted it for publication. I submitted a revised version of [Ha6] that same month. The revision mentions the new strategy: "The rough idea is to let the score of a simplex in a cluster be the compression $\Gamma(S)$ [a function based on the Delaunay decomposition] if the circumradius of every face of S small, and otherwise to let the score be defined by Voronoi cells (in a way that generalizes the definition for quasi-regular tetrahedra)." See p. 6 of [Ha6].

The situation is somewhat more complicated than the previous paragraph suggests. Consider a Delaunay simplex S with edge lengths $2, 2, 2, 2, 2, 2.52$. Such a simplex belongs to the region $X(oct)$. However, if we break it into four pieces according to the Voronoi decomposition, the density of two of the pieces is about $0.696 < \delta_{\text{oct}}$ and the density of the other two is about $0.7368 > \delta_{\text{oct}}$. It is desirable not to have any separate regions in $X(oct)$ of density greater than δ_{oct}. Hence it is preferable to keep the four Voronoi regions in S together as a single Delaunay simplex. A second reason to keep S together is that the proof of the local optimality of the fcc packing and hcp seems to require it. A third reason was to treat pentahedral prisms. (This is a thorny class of counterexamples to a pure Delaunay simplex approach to the proof of the Kepler conjecture. See [Ha1], [Ha2], and [Fer].) For these reasons, we identify a class of Delaunay simplices in $X(oct)$ (such as S) that are to be treated according to a special set of rules. They are called *quarters*. As the name suggests, they often occur as the four simplices comprising an octahedron that has been "quartered."

One of the great advantages of a hybrid approach is that there is a tremendous amount of flexibility in the choice of the details of the decomposition. The details of the decomposition continued to evolve during 1995 and 1996. Finally, during a stay in Budapest following the Second European Congress in 1996, I abandoned all vestiges of the Delaunay decomposition, and adopted definitions of quasi-regular tetrahedra and quarters that rely only on the metric properties of the simplices (as opposed to the Delaunay criterion based on the position of other sphere centers in relation to the circumscribing sphere of the simplex). This decomposition of space is essentially what is used in the final proof.

The hybrid construction depends on certain choices of functions (satisfying a rather mild set of constraints). To solve the Kepler conjecture appropriate functions had to be selected, and an optimization problem based on those functions had to be solved. This function is called *the score*. Samuel Ferguson and I realized that every time we encountered difficulties in solving the minimization problem, we could adjust the scoring function σ to skirt the difficulty. The function σ became more complicated, but with each change we cut months—or even years—from our work. This incessant fiddling was unpopular with my colleagues. Every time I presented my work in progress at a conference, I was minimizing a different function. Even worse, the function was mildly incompatible with what I did in earlier papers [Ha6], [Ha7], and this required going back and patching the earlier papers.

The definition of the scoring function σ did not become fixed until it came time for Ferguson to defend his thesis, and we finally felt obligated to stop tampering with it. The final version of the scoring function σ is rather complicated. The reasons for the precise form of σ cannot be described without a long and detailed description of dozens of sphere clusters that were studied in great detail during the design of this function. However, a few general design principles can be mentioned. These comments assume a certain familiarity with the design of the proof.

(1) Simplices (with vertices at the centers of the balls in the packing) should be used whenever careful estimates of the density are required. Voronoi cells should be used whenever crude estimates suffice. For Voronoi cells, it is clear what the scoring function should be $\mathrm{vor}(R)$ (and its truncated versions $\mathrm{vor}_0(R)$, and so forth).
(2) The definition of the scoring function for quasi-regular tetrahedra was fixed in [Ha6] and this definition had to remain fixed to avoid rewriting that long paper.

Because of these first two points, most of the design effort for the function σ was focused on quarters.

(3) The decision to make the scoring for a quarter change when the circumradius of a face reaches $\sqrt{2}$ is to make the proof of the local optimality of the fcc and hcp run smoothly. From [Ha7], we see that the cutoff value $\sqrt{2}$ is important for the success of that proof. The cutoff $\sqrt{2}$ is also important for the proof that standard regions (other than quasi-regular tetrahedra) score at most 0 *pt*.
(4) The purpose of adding terms to the scoring function σ that depend on the truncated Voronoi function vor_0 is to make interval arithmetic comparisons between σ and vor_0 easier to carry out. This is useful in arguments about "erasing upright quarters."

2.2. *Contents of the Papers*

In [Ha6] a five-step program was described to prove the Kepler conjecture. It was planned that there would be five papers, each proving one step in the program. The papers [Ha6] and [Ha7] carry out the first two steps in the program. Because of the changes in the scoring function, it was necessary to issue a short paper [FH] mid-stream whose purpose was to give some adjustments to the five-step program. This paper adjusts the definitions from [Ha6] and checks that none of the results from [Ha6] and [Ha7] are affected in an essential way by these changes. Following this, the papers [Ha9] and [Fer] appeared in preprint form, completing the third and fifth steps of the program. The fourth step turned out to be particularly difficult. It occupies two separate papers [Ha10] and [Ha11].

The original series of papers suffers from the defect of being written over a span of several years. Some shifts in the conceptual framework of the research are evident. Based on comments from referees, a revision of these papers was prepared in 2002. The revisions were small, except for the paper [Ha11], which was completely rewritten. The structure of the proof remains the same, but it adds a substantial amount of introductory material that lessens the dependence on [Ha6] and [Ha7].

The papers were reorganized again in 2003. The series of papers is no longer organized along the original five steps with a mid-stream correction. Instead, the proof is now arranged according to the logical development of the subject matter. Only minor modifications have been made to the original proof. (The earlier versions are still available from [arXiv].) In the 2003 revision the exposition of the proof is entirely independent of the earlier papers [Ha6] and [Ha7].

An introduction to the ideas of the proof can be found in [Ha12]. An introduction to the algorithms can be found in [Ha14]. Speculation on a second-generation design of a proof can be found in [Ha14] and [Ha13].

2.3. *Complexity*

Why is this a difficult problem? There are many ways to answer this question.

This is an optimization problem in an infinite number of variables. In many respects the central problem has been to formulate a good finite-dimensional approximation to the density of a packing. Beyond this, there remains an extremely difficult problem in global optimization, involving nearly 150 variables. We recall that even very simple classes of nonlinear optimization problems, such as quadratic optimization problems, are NP-hard [HPT]. A general highly nonlinear program of this size is regarded by most researchers as hopeless (at least as far as rigorous methods are concerned).

There is a considerable literature on many closely related nonlinear optimization problems (the Tammes problem, circle packings, covering problems, the Lennard–Jones potential, Coulombic energy minimization of point particles, and so forth). Many of our expectations about nonlattice packings are formed by the extensive experimental data that have been published on these problems. The literature leads one to expect a rich abundance of critical points, and yet it leaves one with a certain skepticism about the possibility of establishing general results rigorously.

The extensive survey of circle packings in [Me] gives a broad overview of the progress and limits of the subject. Problems involving a few circles can be trivial to solve. Problems involving several circles in the plane can be solved with sufficient ingenuity. With the aid of computers, various problems involving a few more circles can be treated by rigorous methods. Beyond that, numerical methods give approximations but no rigorous solutions. Melissen's account of the 20-year quest for the best separated arrangement of ten points in a unit square is particularly revealing of the complexities of the subject.

Kepler's problem has a particularly rich collection of (numerical) local maxima that come uncomfortably close to the global maximum [Ha1]. These local maxima explain in part why a large number (around 5000) of planar maps are generated as part of the proof of the conjecture. Each planar map leads to a separate nonlinear optimization problem.

2.4. *Computers*

As this project has progressed, the computer has replaced conventional mathematical arguments more and more, until now nearly every aspect of the proof relies on computer verifications. Many assertions in these papers are results of computer calculations. To make the proof of Kepler's conjecture more accessible, I have posted extensive resources [arXiv].

Computers are used in various significant ways. They will be mentioned briefly here, and then developed more thoroughly elsewhere in the collection, especially in the final paper.

1. *Proof of inequalities by interval arithmetic*. "Sphere Packings, I" describes a method of proving various inequalities in a small number of variables by computer by interval arithmetic.
2. *Combinatorics*. A computer program classifies all of the planar maps that are relevant to the Kepler conjecture.
3. *Linear programming bounds*. Many of the nonlinear optimization problems for the scores of decomposition stars are replaced by linear problems that dominate the original score. They are solved by linear programming methods by computer. A typical problem has between 100 and 200 variables and 1000 and 2000 constraints. Nearly 100,000 such problems enter into the proof.
4. *Branch and bound methods*. When linear programming methods do not give sufficiently good bounds, they have been combined with branch and bound methods from global optimization.
5. *Numerical optimization*. The exploration of the problem has been substantially aided by nonlinear optimization and symbolic mathematical packages.
6. *Organization of output*. The organization of the few gigabytes of code and data that enter into the proof is in itself a nontrivial undertaking.

Acknowledgments

I am indebted to G. Fejes Tóth's survey of sphere packings in the preparation of this overview [Fej3]. For a much more comprehensive introduction to the literature on sphere packings, I refer the reader to that survey and to standard references on sphere packings such as [CS2], [PA], [GO], [Ro2], [Fej7], and [Fej8].

A detailed strategy of the proof was explained in lectures I gave at Mount Holyoke and Budapest during the summer of 1996 [Ha4]. See also the 1996 preprint, "Recent Progress on the Kepler Conjecture" [Ha4].

I owe the success of this project to a significant degree to S. Ferguson. His thesis solves a major step of the program. He has been highly involved in various other steps of the solution as well. He returned to Ann Arbor during the final 3 months of the project to verify many of the interval-based inequalities appearing in the appendices of "Sphere Packings, IV" and "The Kepler Conjecture." It is a pleasure to express my debt to him.

Sean McLaughlin has been involved in this project through his fundamental work on the dodecahedral conjecture. By detecting many of my mistakes, by clarifying my arguments, and in many other ways, he has made an important contribution.

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