Ann. Rev. Fluid Mech. 1977. 9:87–111

HISTORY OF BOUNDARY-LAYER THEORY

Itiro Tani
National Aerospace Laboratory, 1880 Jindaiji, Chofu, Tokyo, Japan

GENESIS AND EARLIER DEVELOPMENTS

Introduction

The boundary-layer theory began with Ludwig Prandtl's paper *On the motion of a fluid with very small viscosity,* which was presented at the Third International Congress of Mathematicians in August, 1904, at Heidelberg and published in the Proceedings of the Congress in the following year. This paper marked an epoch in the history of fluid mechanics, opening the way for understanding the motion of real fluids. Nevertheless, the genesis of the boundary-layer theory stood in sublime isolation: nothing similar had ever been suggested before, and no publications on the subject followed except a small number of papers due to Prandtl's students for almost two decades.

The equations of motion of a viscous fluid were established in the first half of the last century by Navier (1823), Poisson (1831), Saint-Venant (1843), and Stokes (1845), having attained the form that is now called the Navier-Stokes equations. Stokes used the equations to consider the small oscillations of a sphere in a viscous fluid by assuming that there is no slip, that is, no relative tangential velocity, at the surfacc of the sphere. Confusion had prevailed before as to the conditions to be satisfied by the fluid at the wall of the solid boundary: Stokes (1845) seems to have been initially inclined to the hypothesis of no slip, but when calculations on flow through a pipe gave results at variance with experiments known to him at that time, he hesitated between the no-slip and slip hypotheses. In his 1851 paper, however, he decided to adopt the former on the grounds that this would mean regarding the friction between solid and fluid as of the same nature as the friction between fluids, and also that this would lead to satisfactory agreement with experiments. Later, it was found that the calculations based on the same hypothesis for flow through a pipe, begun by Stokes (1845) and repeated by various authors, also gave good agreement with subsequent experimental results.

The solutions obtained by Stokes, however, were confined to rather special cases, where it was possible to solve the Navier-Stokes equations exactly, because the nonlinear terms were either negligibly small or identically vanishing. This having not been the case in the majority of the problems met in practice, it was necessary to introduce some approximations for solution. The simplest was, of course, to neglect the viscosity of the fluid, but this brought about nothing but the d'Alembert paradox, according to which a solid body of any shape placed in a uniform stream experiences no resistance. This failure was particularly disturbing since the viscosity was considered to produce only small effects in the motion of such fluids as air or water. According to the 1888 edition of the *Encyclopaedia Britannica,* "hydrodynamics" was the branch of science that dealt with the mathematical theory of the motion of fluids, neglecting viscosity, while it was in the branch of "hydraulics" that hydrodynamical questions of practical application were investigated.

The mathematical difficulties of integrating the equations of a viscous fluid made it compelling to neglect the nonlinear terms. This approximation, justified only for slow motions, was made unavoidably also for faster motions, but with the optimistic hope that these solutions might give a better representation of the flow than those obtained by neglecting the viscosity (Bassett 1888). It was a relief to find that the solutions predicted at least nonzero resistance, although far too small in magnitude. It was almost universally agreed that there is no slip at the solid wall in the case of slow motions. The views divided, however, on fast motions. Some authors adopted the no-slip condition also for fast motions, but do not seem to have thought about the necessarily continuous variation of velocity starting from zero at the wall (Lighthill 1963). Other authors
and that the slip is resisted by a frictional force depending on the relative velocity. A number of attempts were made to express the law of friction in the form of an empirical formula applicable to fast as well as slow motions (Unwin 1888).

Prandtl's Paper

In the paper of 1905, Prandtl started from the clear recognition that the most important question concerning the flow of a fluid of small viscosity is the behavior of the fluid at the wall of the solid boundary. It appears that the flow is almost irrotational until comparatively close to the wall, so that the variation of velocity from the value corresponding to irrotational motion to the zero velocity demanded by the condition of no slip at the wall takes place within a thin layer adjacent to the wall. The smaller the viscosity, the thinner is the transition layer. But the steep velocity gradient, in spite of the small viscosity, produces marked effects, which are comparable in magnitude with those due to the inertia force, if the thickness of the transition layer is proportional to the square root of the kinematic viscosity. Thus, the effects of viscosity are significant only within a thin transition layer, which is called the *boundary layer.*[1] Outside this layer, the flow is essentially free of viscosity and is described by an irrotational motion to a high degree of accuracy.

[1] It may be noted that Prandtl used the term *Grenzschicht* (boundary layer) only once and the term *Übergangsschicht* (transition layer) several times in the paper. The term Grenzschicht has come into more definite use since the paper of Blasius (1908). Later, Prandtl (1925a) wrote that the term Grenzschicht, usually employed by specialists, appears by no means to be happy, but should be continued since it has already been introduced.

The small thickness of the boundary layer permits certain approximations for the governing equations within the boundary layer: the variation of pressure normal to the wall is negligibly small, and the variation of velocity along the wall is much smaller than its variation normal to it. In the case of flow in two dimensions the effect of moderate curvature of the wall is negligibly small, so that x and y may be taken as the distances along and normal to the wall, respectively, and u and v as the corresponding velocity components. The x-component of the Navier-Stokes equations is then simplified to the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},$$

where t is the time, p the pressure, ρ the density, and ν the kinematic viscosity. The pressure p is regarded as a function of x and t and prescribed by the irrotational motion outside the boundary layer. The equation is parabolic, although the original Navier-Stokes equations are elliptic. Thus it can be integrated step by step in the direction of x when u is known at a fixed value of x for all values of y and t, the upstream influence being suppressed to the order of approximation. Prandtl considered the solution of the equation for the simple case $p =$ constant, that is, the case of a semi-infinite thin flat plate placed parallel to a stream of uniform velocity U, and obtained a rough estimate $1.1\rho\nu^{1/2}l^{1/2}U^{3/2}$ for the frictional resistance exerted on the two sides of unit width of a plate of length l. This was the first theoretical analysis of the frictional resistance, although the numerical coefficient 1.1 was later corrected by Blasius (1908) to 1.33.

A remarkable consequence of the investigation from the standpoint of application was, according to Prandtl, that "in certain cases, *the flow separates from the surface* at a point entirely determined by external conditions. A fluid layer, which is set in rotation by the friction on the wall, is thus forced into the free fluid and, in accomplishing a complete transformation of the flow, plays the same role as the Helmholtz separation layers. A change in the viscosity constant μ simply changes the thickness of the transition[2] layer (proportional to the quantity $\sqrt{\mu l/\rho U}$), everything else remaining unchanged. It is therefore possible to pass to the limit $\mu = 0$ and still retain the same flow figure" (1928 translation of Prandtl 1905). Without going into a mathematical analysis, Prandtl explained the plausible reason for flow separation with the increase of pressure in the streamwise direction. He also deduced that "the treatment of a given flow process is resolved into two components mutually related to one another. On the one hand, we have the free fluid, which can be treated as nonviscous according to the Helmholtz vortex laws, while, on the other hand, we have the transition layers on the solid boundaries, whose motion is determined by the free fluid, but which, in their turn, impart their characteristic impress to the free flow by the emission of vortex[2] layers" (1928 translation of Prandtl 1905). Prandtl closed the paper with confirmation of the theory by photographs of flows obtained in a small hand-operated water tank.

[2] Here the translation of *Wirbelschicht* has been corrected.

Prototype of Concept

Prandtl's paper is an extraordinary paper in at least three aspects. First, it is extraordinary in the sense that an unprecedentedly novel, but fully matured, idea emerged in a single paper. Of course, brief mention of the existence of a boundary layer and its connection with frictional resistance had already been scattered in the literature up to that time, but it had amounted to very little compared with Prandtl's achievement. There had been no boundary-layer equations and no explanation of flow separation, as is seen below.

In a paper on the prediction of the required engine power of proposed ships, Rankine (1864) considered the frictional resistance to be due to "the direct and indirect effects of the adhesion between the skin of the ship and the particles of water which glide over it; which adhesion, together with the stiffness of the water, occasions the production of a vast number of small whirls or eddies in the layer of water immediately adjoining the ship's surface. The velocity with which the particles of water whirl in those eddies bears some fixed proportion to that with which those particles glide over the ship's surface: hence the actual energy of the whirling motion impressed on a given mass of water at the expense of the propelling power of the ship, being proportional to the square of the velocity of whirling motion, is proportional to the square of the velocity of gliding." Thus, Rankine visualized the formation of a boundary layer adjacent to the ship's surface. However, the argument leading to the quadratic law of resistance is relevant only to a surface of appreciable roughness. According to Loitsianskii (1970), the Russian chemist Dmitrii Mendeleyev clearly distinguished between smooth and rough surfaces in his monograph entitled *On the Resistance of Fluids and the Problem of Flight* (St. Petersburg, 1880). He recognized the important role played by "a thin layer of fluid adjacent to the solid surface and carrying along the neighboring layers" (translated) in generating frictional resistance of a smooth surface. He also considered the resistance of a rough surface to be of the same nature as the resistance experienced by a plate at right angles to the stream.

Experiments of Froude (1872) on a thin flat plate towed through still water made it clear that the frictional resistance does not vary as the length but at a smaller rate. This result was considered to be due to the fact that the rear portions of the surface are in contact with water that has been set in motion by the front portions, and therefore cannot experience as much frictional force as the front portions. Froude thus anticipated the existence of a boundary layer growing in thickness with the distance downstream. In a subsequent paper, Froude (1874) pointed out that the frictional force must have its counterpart in the loss of momentum of the fluid that has passed along the surface of the plate. Prandtl (1927b) quoted Froude as "the first English author to refer the frictional resistance of a flat plate to the layers of fluid in intense shear near the surface." Judging from the summary of his lecture delivered at the British Association for the Advancement of Science in 1869, Froude seems to have had arrived at some concept of the boundary layer before carrying out the systematic towing experiments.

In considering the free convection from a heated vertical plate in still air, Lorenz

(1881) assumed that the flow is parallel to the plate ($v = 0$) and that the velocity u and temperature T depend only on y, where x and y are distances along and normal to the plate, respectively, with the origin at the lower edge of the plate. The momentum equation was simplified to $0 = g(T - T_0)/T_0 + \nu d^2u/dy^2$, and the energy equation to $u(T - T_0)/x = \kappa d^2T/dy^2$, where T_0 is the ambient temperature, g the acceleration due to gravity, and κ the thermometric conductivity. The transformation of variables $y = \alpha y'$, $u = \beta u'$, $T - T_0 = (T_w - T_0)\theta'$ was made so as to reduce the equations to a form expressed purely in terms of the nondimensional variables y', u', and θ'. This led to the result $\alpha^4 = \nu\kappa x T_0/g(T_w - T_0)$, $\beta^2 = \kappa g x(T_w - T_0)/\nu T_0$, indicating that the thickness of the boundary layer increases as $x^{1/4}$, while the maximum velocity increases as $x^{1/2}$. In spite of the inconsistent approximation for the governing equations (in which the convection terms were wholly neglected in the momentum equation, but approximately taken into account in the energy equation), the results are in agreement with those obtained by the subsequent, more consistent treatment (Schmidt & Beckmann 1930). Lorenz expressed the solution of the nondimensional equations as power series in the variable $z = 1 - \exp(-y')$ and obtained the rate of heat transfer from the plate, which turned out to be 36% larger than the correct value for the case $\kappa = \nu$.

The paper[3] thus contains the prototype of the boundary-layer concept in that the conduction term is considered to be of the same order of magnitude as the convection term within the temperature boundary layer. Because of the linear behavior of temperature in the energy equation, it might have been a little easier to think of its boundary layer, rather than that relevant to velocity. Prandtl (1949) referred to this paper as "the first paper on free heat convection and at the same time the first on boundary layers! However, the dependence on x of the thickness and maximum velocity is not given in it" (translated). Calling it the first paper on free heat convection might be appropriate, but the first paper on boundary layers seems to be excessive praise. Prandtl's additional comment that the dependence on x of the thickness and maximum velocity is not given in the paper seems to be due to his oversight.

It appears to the writer that Prandtl did not notice this paper before publishing his boundary-layer theory (1905). If there were any opportunity for him to read the paper of Lorenz, he might have mentioned it in his paper, as is readily imaginable from his strict fairness in history, particularly in priority. His oversight just mentioned appears to support this conjecture. It is the writer's guess that Prandtl found the paper of Lorenz after the publication of his own paper and felt surprise at a thread of connection between the two papers.

Slow Acceptance

The second aspect that makes Prandtl's paper extraordinary is its very slow acceptance and growth. The statement has often been repeated that the paper occupied less than eight pages. In reply to Goldstein's question as to why he had kept it so short, Prandtl explained "that he had been given ten minutes for his lecture at the Congress and that, being still quite young, he had thought he could

[3] The writer is indebted to Prof. S. Corrsin and Prof. Y. Katto for this reference.

publish only what he had had time to say" (Goldstein 1969). The greater part of the paper was devoted to showing to the assembled mathematicians such items as the d'Alembert paradox, the Helmholtz vortex theorems, diagrams of streamlines involving separation, experimentally obtained photographs of flows past a projection and a circular cylinder, etc. As a result, the essentials of the boundary-layer theory were compressed into two and a half pages, largely descriptive and extremely curtailed in expression. It is quite certain that the paper was very difficult to understand at that time, making its spread rather sluggish.

In 1908 there appeared two papers on boundary layers, one by Blasius and the other by Boltze, both prepared as dissertations at Göttingen under Prandtl's guidance. Blasius applied Prandtl's theory to the detailed study of the flow along a flat plate placed parallel to a uniform stream, as well as of the flow around a circular cylinder that is started moving in a fluid at rest. Boltze investigated flow around a body of revolution, particularly a sphere. Subsequently, Prandtl (1910) applied the boundary-layer concept to the heat-transfer problem, and Hiemenz (1911), also in a Göttingen dissertation, carried out boundary-layer calculations with an experimentally determined pressure distribution on a circular cylinder. Töpfer (1912) refined the numerical computations of Blasius. Prandtl (1914) explained the change in flow pattern around a sphere on passing through the critical Reynolds number, which had been observed by Eiffel (1912), as due to transition of the flow in the boundary layer from laminar to turbulent.

Thus, in the first decade, there were seven papers on boundary layers, due to five authors, all at Göttingen. Through these papers the concept of boundary layers spread out of Göttingen, but only very slowly. Most of the papers were written in a more accessible and conventional form than Prandtl's original paper, but they seem to have escaped the attention they deserved. This may be demonstrated, for example, by reference to Lanchester's *Aerodynamics* (1907). On pages 50–51 of this book, Lanchester found that the frictional resistance of a flat plate would vary as $\rho\nu^{1/2}l^{1/2}U^{3/2}$, without knowing of Prandtl's result $1.1\rho\nu^{1/2}l^{1/2}U^{3/2}$. He arrived at the result by comparing the frictional force with the loss of momentum in the boundary layer. Lanchester gave, in addition, an explanation of flow separation, less detailed than that of Prandtl, and also indications that the flow becomes turbulent at higher speeds. At any rate, this publication aroused the interest of Rayleigh (1911), who made a simple but less accurate estimate $2.26\rho\nu^{1/2}l^{1/2}U^{3/2}$ for the frictional resistance on the basis of the analogous problem of an infinite flat plate that is started impulsively from rest, now referred to as the Rayleigh problem. No reference to Prandtl was made in this paper. A crude calculation was also made by Gümbel (1913) in an attempt to predict the frictional resistance of ships. He assumed the velocity distribution near a flat plate to be of the form $u = U\{1-\exp[-y(U/2\nu x)^{1/2}]\}$ and obtained $2.83\rho\nu^{1/2}l^{1/2}U^{3/2}$ for the frictional resistance. Reference was made to Blasius's result $1.33\rho\nu^{1/2}l^{1/2}U^{3/2}$, which was rejected, however, since Froude's experimental data (1872, 1874) support Gümbel's formula. His comment that there is no information on velocity distribution in Blasius's paper makes one suspect that he did not fully understand the paper.

Mention should also be made of Zhukovskii's *Aérodynamique* (1916), the French edition of his lecture notes of 1911–1912 in Moscow. On pages 119–22 of this edition, Zhukovskii assumed that the fluid velocity is zero at the wall and rapidly increases until it becomes equal to the theoretical velocity of irrotational motion, the transition layer of fluid adjacent to the wall being thin and rotational. He then made a rough estimate of the thickness of the layer by assuming it to vary in inverse proportion to the theoretical velocity. In this connection no mention was made of Prandtl's boundary-layer theory. On page 198, however, there appears a reference to Prandtl's article on fluid motion in *Handwörterbuch der Naturwissenschaften* (1913). This article having contained, among other things, a brief but plain account of boundary layers, it is strange that it was referred to only in connection with vortex formation behind a bluff body, but not in connection with boundary-layer theory.

A blank of six years was caused by World War I in the record of publications on boundary layers. Then, Kármán (1921) proposed the momentum integral equation, obtained by integrating the momentum equation across the boundary layer, for approximate calculation of the development of boundary layers. K. Pohlhausen (1921) applied the method to several cases, using a polynomial approximation for the velocity distribution. E. Pohlhausen (1921) obtained the solution for forced convection in the boundary layer. Tollmien (1924) investigated the growth of the boundary layer on a circular cylinder impulsively set in rotation from rest. Burgers (1925) reported on experimental observations of the velocity distribution across the boundary layer on a flat plate, bringing to light the simultaneous presence of laminar and turbulent regions. The experiments, carried out by Burgers's student van der Hegge Zijnen (Burgers & van der Hegge Zijnen 1924) using a hot-wire anemometer, formed an important achievement in boundary-layer research, not only because it was the first experimental investigation on the subject but also because it was the first direct observation on the boundary layer itself. Up to that time, every experimental result was indirect, inferred from the overall aspects of the flow. For example, Prandtl's explanation (1914) of the critical Reynolds number of a sphere was based on the observation of a reduction in resistance at fairly low Reynolds numbers by inducing turbulence with a wire hoop fixed on a sphere. No direct observation was made on the boundary layer except smoke photographs of flow separation. Of course, to a limited extent measurements had been made before on the velocity of fluid in the neighborhood of the solid wall (Calvert 1893, Kempf 1913, Riabouchinskii 1914), but none of these had been systematic enough to afford a deeper understanding of boundary-layer flows.

Thus, in the second decade, the number of papers (six) was still about the same as in the first decade, but the interest in boundary layers had now spread out of Göttingen. With this momentum, boundary-layer theory became the subject of widespread attention and acceptance. A brief reference to boundary-layer theory appeared in Lamb's *Hydrodynamics* (1924). A suggestion was offered by Mises (1927) regarding the use of the stream function as one of the independent variables, so that the boundary-layer equation was reduced to a form analogous to the heat-conduction equation. In a remark on this paper, Prandtl (1928) expressed his pleasure in observing interest in boundary-layer theory spread outside of his group. He also stated on another occasion (1938) that he had used the same form in 1914 to apply the boundary-layer equation to flow through a two-dimensional channel, the boundary condition at the two walls being simply expressed in terms of the stream function. The result having been unpublished, however, he thought that "the priority in a usual sense should be due to Mises" (translated).

There are seemingly good reasons for the slow acceptance of the boundary-layer theory: favorable growth was hampered by the war; the first paper of Prandtl was so very short and published where no one could appreciate it; most of the practical requirements were more concerned with the gross aspects of the flow like the force and moment, rather than the local structure of the flow, etc. However, none of these reasons is convincing enough. It appears to the writer that the most essential reason is that Prandtl's idea was so much ahead of the times.

Sowing Seeds

The third aspect that makes Prandtl's paper extraordinary is the presentation of subjects to be pursued further, thus forming the source of some lines of subsequent developments in boundary-layer theory. The above-mentioned investigations due to Blasius (1908), Boltze (1908), and Hiemenz (1911) exemplified the outgrowth of the earlier developments. There are other subjects, however, that had to wait for solution until the arrival of new students.

One of the subjects posed in Prandtl's paper is concerned with the algebraic singularities affecting the numerical analysis, in which the velocity profiles are to be calculated starting with a given profile at the initial station, the pressure distribution having been prescribed. The singularities arise from the no-slip condition $u = 0$ at the wall. The problem was first taken up by Goldstein during his stay in Göttingen as a guest. Goldstein (1930) considered, among other things, the conditions to be satisfied by the initial profile for the absence of singularities. This provided a basis for devising a numerical step-by-step method of solution of the boundary-layer equations (Prandtl 1938, Görtler 1939, Hartree 1939).

Another subject posed is concerned with the possibility of controlling the boundary layer. Prandtl's paper contains an experimental demonstration of preventing separation by removing boundary-layer fluid by suction. It was only after World War I, however, that extensive experiments were carried out in Göttingen with the aim of practical utilization of boundary-layer control (Prandtl 1925a, 1927a; Ackeret 1925, 1926; Schrenk, 1928, 1931, 1935). The remaining subject is not concerned with the boundary layer itself, but is closely associated with it. By using the Helmholtz vortex theorem, Prandtl arrived at the result that in a region of closed streamlines in which the vorticity has been established by the action of very small viscosity, the vorticity should be uniform in a two-dimensional flow. The vorticity should be proportional to the radial distance from the axis of symmetry in an axially symmetric flow. After the lapse of half a century the problem was considered in detail by Batchelor (1956), but without reference to Prandtl. It is perfectly astonishing to find the seeds of subjects ranging from basic to practical problems sown in the soil of a single paper.

Genesis of Boundary-Layer Theory

Having received a degree at Munich in 1900 after writing a thesis on the buckling of a deep beam, Prandtl joined Maschinenfabrik Augsburg-Nürnberg as a mechanical engineer. His interest in fluid mechanics was awakened when he arranged a conical diffuser in a large air duct but failed to achieve the expected pressure recovery. The conical angle seems to have been a little too large, resulting in flow separation from the diffuser wall. At that time (1901), however, Prandtl had to leave Nürnberg to receive an appointment as professor of mechanics at the Technische Hochschule Hannover. The loss of pressure was of no serious concern to the engineering works, but the question as to why and how the flow separated from the wall occupied Prandtl's inquiring mind until, after three years, his concept of the boundary layer provided him with the answer. In 1904 he accepted Felix Klein's invitation to take charge of the newly established chair of applied mechanics at the University of Göttingen, which he held until his retirement in 1947, six years before he died.

It is interesting to learn that Prandtl's failure in diffuser design caused him to reflect seriously on the matters underlying the phenomenon. The accidental flow separation gave impetus to the concept of the boundary layer. On the occasion of his being elected an honorary member of the German Physical Society, Prandtl gave a talk entitled *My road to hydrodynamical theories* (Prandtl 1948). In response to Heisenberg's congratulatory address containing the statement that Prandtl had the ability to see through equations, without calculation, what solution they may possess, he said, "I should reply that I admit having no such ability, but I endeavor to gain as clear a conception as possible about the matters forming the basis of the problem and seek to understand the course of events. The equations do not come up until later, when I believe to have understood the problem: they are useful not only to produce quantitative information which cannot certainly be obtained by conception alone, but also to afford good means of adducing proofs for my conclusions, thus winning recognition from others" (translated). In the same talk he also formulated the heuristic principle of solution underlying the boundary-layer theory expressed by the following (translated): "When the complete problem appears hopeless in mathematics, it is advisable to examine what takes place if the relevant parameter of the problem is made to tend to zero. It is thereby assumed that the problem admits an exact solution when the parameter is set equal to zero from the beginning and that a simplified approximate solution is possible for very small values of the parameter. At the same time it is necessary to examine whether, in the limit as the parameter tends to zero, the solution tends to the solution for the case when the parameter is set equal to zero. The boundary conditions must be chosen in order that this is the case. As to the physical trustworthiness of the solution, the classical proposition '*Natura non facit saltus*' gives the guiding principle: in nature the parameter is possibly small, but not equal to zero. Thus, the first way is always the physically correct one."

Spread of Boundary-Layer Theory

As already mentioned, the boundary-layer theory spread very slowly but steadily from Göttingen to other groups within its country of origin and then to other countries of the world. The diffusion of the theory was considerably facilitated by the appearance in the third decade of Prandtl's book *Abriss der Strömungslehre* (1931), Tollmien's article in *Handbuch der Experimentalphysik* (1931), and Prandtl's article in *Aerodynamic Theory* (1935).

In the meanwhile, the concept of the boundary layer turned out to be remarkably fruitful, not only in forming the basis for approximate methods of calculation of practical utility but also in offering clarification of phenomena that were otherwise incomprehensible or at least obscure, thus exerting an enormous, far-reaching influence. It is no exaggeration to say that it paved the way for all modern developments in fluid mechanics. The concept, originally developed for laminar flow along a solid boundary, has been extended to the corresponding case of turbulent flow and also to boundary-free shear flows occurring in wakes and jets. Along with these extensions, the stability of laminar flow was examined as a possible key to understanding the origin of turbulence. According to Dryden's statistics (1955), there were in the third decade five to six papers per year on boundary layers.

The fourth decade produced Goldstein's *Modern Developments in Fluid Dynamics* (1938) and World War II. Goldstein's volumes received a widespread welcome as the most timely compendium of the existing knowledge, making an important contribution to the diffusion of boundary-layer theory. World War II apparently did not check the development, although it confined the spread of information largely to the country of origin. Many of the results of investigations completed during the war remained unknown in other countries until much later. Dryden's statistics indicate that there were in the fourth decade about fourteen papers per year on boundary layers. Before the impetus of the original idea was exhausted in dealing with a laminar boundary layer in incompressible fluids, some of the effort was turned to examining the effects of compressibility, in response to the requirements of high-speed flight. This trend continued to the subsequent decade, in which attention was further directed to real-gas effects. It may safely be said that the boundary-layer theory found its happy hunting ground in the field of aeronautical engineering. The expansion of aeronautical activity stimulated basic research on boundary layers. Along with these developments, the boundary-layer concept began to pervade other fields of engineering—mechanical engineering, chemical engineering, etc. On the other hand, studies of heat and mass transfer in moving fluids were greatly facilitated by the knowledge of boundary layers. These developments have created an almost exponential growth of interest in boundary-layer theory in recent years.

The remainder of this article is devoted to a brief historical review of the major branches of the subject in boundary-layer theory, more detailed for classical branches but less detailed for extended branches. Because of limitations of space and time, however, the period of coverage is restricted to about six decades after the birth of boundary-layer theory, that is, up to about the end of the 1960s. No attempt at an exhaustive survey is made, and the references are quoted sometimes merely by way of example.

DEVELOPMENTS IN MAJOR BRANCHES

Steady Two-Dimensional Laminar Boundary Layers

The form of similarity solution introduced by Prandtl (1905) and Blasius (1908) for flow on a flat plate was extended by Falkner & Skan (1930) to the case in which the free-stream velocity varies in proportion to x^m, representing irrotational flow in a corner formed by two plane boundaries meeting at an angle $\pi/(m+1)$. Subsequent studies by Hartree (1937) and Stewartson (1954) revealed nonuniqueness of the solution for negative values of m. The series solution initiated by Blasius (1908) and Hiemenz (1911) for flow past a blunt-nosed cylinder of arbitrary cross section was extended by Howarth (1934) and Görtler (1952, 1957). The series solution for a linearly retarded free stream was considered by Howarth (1938) and extended to the more general case by Tani (1949). Series solutions for flow past a parabolic cylinder (Van Dyke 1964b) and for flow past a blunt-nosed wedge (Chen et al 1969) are worthy of mention as rare examples provided with convergence considerations. The boundary-layer approximation was also applied by Goldstein (1933) to flow in a wake, and by Schlichting (1933a) to flow in a jet.

The approximate method of solution of Kármán and Pohlhausen (Kármán 1921, K. Pohlhausen 1921), based on the momentum integral equation and a quartic form of the velocity profile, was found to give good results in nonretarded flow but less satisfactory in the retarded region, as first noticed in Schubauer's experimental observations (1935) on flow past an elliptic cylinder. Almost immediately, the approximate method of Kármán & Millikan (1934), in which the boundary layer was divided into inner and outer regions with separate solutions, was applied by Millikan (1936) to Schubauer's ellipse with reasonable success. Attempts were subsequently made to secure improved accuracy of the method of Kármán and Pohlhausen by assuming a more adequate form of the velocity profile (Walz 1941, Mangler 1944, Timman 1949), or by using another integral relation in addition to the momentum integral equation (Wieghardt 1948, Loitsianskii 1949, Truckenbrodt 1952, Tani 1954). Along with these efforts an approximate method of integrating the momentum integral equation was suggested independently by Walz (1941), Hudimoto (1941), Tani (1941), and Thwaites (1949), yielding the relation now commonly referred to as the Thwaites formula. An improved version of the method of Kármán and Millikan was put forward by Stratford (1954).

The numerical solution of Hartree (1939) for a linearly retarded free stream suggested the presence of a singularity at the point of separation, where the wall shear stress vanishes. This led Goldstein (1948) to construct a singular solution containing an arbitrary constant in the neighborhood of separation. Stewartson (1958) reconsidered the problem and obtained the more general solution involving an infinite number of arbitrary constants. Terrill (1960) extended Stewartson's work to include suction and also gave a numerical solution for irrotational flow past a circular cylinder, which again suggested the presence of a singularity. It must be borne in mind that the singularity disappears when all the arbitrary constants are set equal to zero, while the only evidence of a singularity comes from the numerical investigations for prescribed pressure distributions. Without any evidence to the contrary, one may infer that the singular solution would be the most reasonable representation of which the boundary-layer equations are capable and that the solutions of the Navier-Stokes equations would exhibit an abrupt but regular approach to separation. Landau & Lifshitz (1959) gave a discussion on flow near separation by postulating that the normal component of velocity tends to infinity at the separation point.

Unsteady Two-Dimensional Laminar Boundary Layers

The growth of the boundary layer on a body set impulsively from rest into translational motion, first studied by successive approximations (a series in time) by Blasius (1908), was extended by Goldstein & Rosenhead (1936) for a better estimate of the time required for separation, which occurs at the rear stagnation point for a circular cylinder. The method of analysis fails, however, in the case of a semi-infinite flat plate, transition from the time-dependent Rayleigh solution to the space-dependent Blasius solution occurring by way of an essential singularity (Stewartson 1951), which, of course, originates in the use of the boundary-layer approximation. Proudman & Johnson (1962) considered the flow near the rear stagnation point and showed that at large times there is an inner boundary layer of reversed flow.

The flow generated by the small-amplitude oscillation of a body in a fluid at rest was also studied by successive approximations (a series in amplitude), first by Rayleigh (1883) in connection with acoustic phenomena in the Kundt tube but without recourse to the boundary-layer concept, and later by Schlichting (1932) with a boundary-layer formulation. The important result that appears in the second approximation is the occurrence of a steady streaming in addition to the oscillatory flow components. Moore (1951) considered the case in which a semi-infinite flat plate moves with a gradually changing but arbitrary time-dependent velocity. Another problem of practical importance, in which the free-stream velocity exhibits a small-amplitude fluctuation in magnitude, was initiated by Lighthill (1954) and extended by Rott & Rosenzweig (1960) and Lam & Rott (1962). It was pointed out by Rott, Moore, and Sears (Rott 1956, Moore 1958) that the criterion of vanishing wall shear stress does not in general denote flow separation from the wall in the case of unsteady motions.

Three-Dimensional Laminar Boundary Layers

The extension of two-dimensional boundary-layer theory to flows with axial symmetry was considered first by Boltze (1908) and later by Millikan (1932). It was found independently by Stepanov (1947), Mangler (1948), and Hatanaka (1949) that the problem of an axially symmetric boundary layer on a body of revolution can be reduced to that of an equivalent two-dimensional flow past a cylinder. Glauert & Lighthill (1955) and Stewartson (1955) independently investigated the flow at large distances downstream on the outside of a long circular cylinder, where the thickness of the boundary layer is no longer small compared with the radius of the cylinder. The extension of boundary-layer calculation was made by Taylor (1950) and Cooke (1952) to include swirling motion and by Illingworth (1953) and Schlichting (1953) to include rotation of the body.

Flow past a yawed infinite cylinder was considered independently by Prandtl (1945b), Struminskii (1946), Jones (1947), and Sears (1948). This class of flows has the useful feature that the velocity components in planes normal to the generators of the cylinder can be determined independently of the velocity component parallel to the generators. Results of calculation illustrated the deviation of flow in the boundary layer from the direction of the free stream, a characteristic behavior of three-dimensional boundary layers. Formulation of the boundary-layer equations in curvilinear coordinates was given for flow over a general three-dimensional surface (Howarth 1951, Hayes 1951, Watson 1963). The approximate method of solution of Kármán and Pohlhausen was extended to three dimensions by Timman & Zaat (1955), Eichelbrenner & Oudart (1955b), and Cooke (1959). When the streamlines outside the boundary layer have small geodesic curvature, choice of the projections of those streamlines on the solid surface as a family of streamwise coordinate lines causes the velocity component in the crosswise direction to be small. This simplification leads to an equation for the streamwise velocity component that is analogous to that for axially symmetric flows and independent of the crosswise velocity component (Eichelbrenner & Oudart 1955b).

In two-dimensional flows separation occurs at, or very close to, the point where the wall shear stress vanishes, and if it is considered as a three-dimensional flow, there is a separation line of singularities. In truly three-dimensional flows, however, the wall shear stress has two components, and the concept of lines of wall shear stress or limiting streamlines (limits of streamlines as the wall is approached) is found to be useful. Both components of wall shear stress simultaneously vanish, in general, only at isolated singular points, which are either nodal or saddle points of the topographical pattern of limiting streamlines, while flow separation occurs along a line on which the parallel component of wall shear stress is not everywhere zero. Maskell (1955) and Eichelbrenner & Oudart (1955a) defined the separation line as the envelope of the limiting streamlines, but Lighthill (1963) proposed a more comprehensive definition of the separation line as a limiting streamline that issues from both sides of a saddle point of separation and, after embracing the body, disappears into a nodal point of separation.

Instability and Transition to Turbulence

The fact that the laminar-flow solutions of the Navier-Stokes equations are not observed at high Reynolds numbers brought out the question of the stability of flow, in particular, the question as to the existence of infinitesimal disturbances growing with time. Rayleigh (1880) examined the stability of a plane parallel flow by neglecting viscosity and showed that a necessary condition for instability is that the velocity profile has a point of inflection. Stability theory for viscous fluids was formulated by Orr (1907) and Sommerfeld (1909), but calculations (Hopf 1914) indicated complete stability when applied to plane Couette flow generated by parallel walls in relative motion. As regards the effect of viscosity, Prandtl (1921) pointed out its dual role in stabilizing by dissipating energy, but destabilizing by producing phase lags in a layer close to the wall, as illustrated by Tietjens's calculation (1925). Heisenberg (1924) showed that plane Poiseuille flow between parallel walls at rest becomes unstable at high Reynolds numbers, but the result was too incomplete to gain general acceptance. Tollmien (1929) considered the Blasius velocity profile near a flat plate and obtained the critical Reynolds number above which the flow becomes unstable to a traveling-wave type of disturbances in a certain frequency range. Schlichting (1933b) extended Tollmien's calculation to amplified disturbances. Squire (1933) reduced the problem of three-dimensional disturbances of a plane parallel flow to that of equivalent two-dimensional disturbances at a lower Reynolds number, enabling the theory to concentrate attention on two-dimensional disturbances when calculating the critical Reynolds number. Attempts at experimental verification of stability theory met with only little success (Prandtl 1933, Nikuradse 1933b, Schiller 1934). On the other hand, Dryden's experiments (1936) indicated that transition to turbulence in the flow near a flat plate originates in the turbulence of the free stream. Taylor (1936) postulated that transition results from momentary separation of the boundary layer caused by the fluctuating pressure gradient of the free-stream turbulence, and some of the conclusions on the overall aspects were confirmed by measurements of Dryden, Schubauer, Mock & Skramstad (1937) and Hall & Hislop (1938).

In spite of the notable achievement in surmounting mathematical difficulties, Tollmien's theory was disregarded for more than a decade until Schubauer & Skramstad (1943) observed transition preceded by slow oscillations, of the kind predicted by theory, in the boundary layer on a flat plate in a wind tunnel of very weak turbulence. The characteristics of the oscillations agreed so well with the predicted values that the theory was regarded as proven in every particular. Only a little later, Liepmann (1943) independently made a similar observation. It was clear that high levels of free-stream turbulence typical of earlier experiments had masked the existence of amplified waves. Lin (1945) improved Heisenberg's approach and obtained the boundary of neutral stability for plane Poiseuille and Blasius flows. Shen (1954) extended Lin's method of solution to amplified disturbances.

Squire's theorem does not hold for curved flows, where three-dimensional disturbances may grow due to the destabilizing effect of the centrifugal force. Disturbances take the form of cellular toroidal vortices in circular Couette flow between rotating cylinders when the rotation of the inner cylinder dominates, for which Taylor (1923) found extremely close agreement between theoretical prediction and experimental observation. Another example is provided by streamwise vortices produced in the boundary layer on a concave wall (Görtler 1940). Instability similar in form also occurs in a horizontal boundary layer heated from below (Jeffreys 1928), with buoyancy as the destabilizing agent.

Amplification of infinitesimal disturbances is but a prelude to the whole process of transition. Passage to the subsequent stage occurs as a result of disturbances of increased amplitude giving rise to nonlinear interactions. Stability theory including nonlinear effects, first stated by Landau (1944) and developed by Meksyn & Stuart (1951), Stuart (1958, 1960), and Watson (1960, 1962), was successfully applied to circular Couette flow for predicting the equilibrium of disturbances under supercritical conditions. On the other hand, there is experimental evidence for the boundary layer on a flat plate (Klebanoff and associates 1959, 1962) that the nonlinear effect manifests itself as a nearly periodic variation in the spanwise direction of the amplitude of the initially two-dimensional Tollmien waves. The nonlinear theory has not yet gone far enough to deal with three-dimensional disturbances in slowly growing boundary-layer flows, although the observed phenomena were fairly well accounted for by another form of nonlinear theory (Benney & Lin 1960; Benney 1961, 1964) based on some debatable assumptions. The spanwise variation of wave amplitude generates locally unstable velocity profiles possessing an inflection point (Kovasznay, Komoda & Vasudeva 1962), bringing about a rapid collapse into eddies, until eventually random oscillations characteristic of turbulence burst forth in small localized "spots" (Emmons 1951, Schubauer & Klebanoff 1955). The turbulent spots grow as they travel downstream, until they merge into a fully developed turbulent flow. The evolution leading to the formation of turbulent spots in boundary-layer and channel flows is rather abrupt compared with the gradual evolution observed during the transition process in circular Couette flow dominated by rotation of the inner cylinder (Taylor 1923, Coles 1965) and also in boundary-free shear flows in wakes and jets (Sato & Kuriki 1961, Browand 1966).

Since the Tollmien waves were first observed at very low free-stream turbulence levels, it had been thought that at higher turbulence levels transition occurs without the precedence of instability oscillations. The experiments of Bennett (1953) suggested, however, that the evolution leading to transition is not basically different at least up to moderately high turbulence levels. On the other hand, there has been no experimental evidence of momentary separation at or prior to transition, raising some doubt as to the validity of Taylor's postulate (1936). Observations of Tani & Sato (1956) and Klebanoff (1966) indicated that instability oscillations are also induced by the presence of a two-dimensional roughness element.

Boundary-Free Turbulent Shear Flows

The effect of turbulent fluctuations in causing apparent stresses to operate on the mean motion, vaguely anticipated by Saint-Venant (1843), was assumed by Boussinesq (1877) to be simply equivalent to an increase in viscosity, thus introducing the concept of eddy viscosity. Reynolds (1895) showed that the correlations between fluctuating velocity components give rise to apparent stresses, which now bear his name. Taylor (1915), and independently Prandtl (1925b), expressed the Reynolds shear stress in terms of the mean velocity gradient and the mixing length, which represents the mean distance traveled by lumps of fluid before losing their identities. It was tacitly assumed in Prandtl's formulation that the momentum is a transferable property, while the transfer of vorticity formed the basis of Taylor's theory.

The mixing-length approach was first applied to boundary-free shear flows in jets, wakes, etc., with the assumption that the mixing length is constant across the shear layer and proportional to its width (Tollmien 1926, Schlichting 1930). Both transfer theories yielded the same result for the mean velocity profile, but the vorticity-transfer theory predicted the mean temperature profile in the wake of a heated cylinder in better agreement with experiments (Taylor 1932). Later, Prandtl (1942) found that a more satisfactory description for mean velocity is provided by assuming eddy viscosity to be constant across the shear layer. In this formulation, however, one must for heated wakes and jets take the eddy diffusivity for heat greater than the eddy viscosity for momentum (Corrsin 1943).

A striking feature of boundary-free shear flows is that the region of shear is bounded by a relatively sharp but irregularly meandering interface that separates the turbulent motion possessing vorticity fluctuations from the surrounding irrotational motion. This phenomenon was first discovered by Corrsin[4] (1943), and was thoroughly investigated by Townsend (1949, 1950, 1956) and Corrsin & Kistler (1954). In particular, Townsend visualized a double structure of flow consisting of the main body of turbulence having relatively small eddies, loosely termed turbulent fluid and containing most of the turbulent energy, and a superposed system of slowly moving large eddies, responsible for distorting interface and entraining nonturbulent fluid. He advanced the hypothesis that large eddies are gaining energy from the mean flow at nearly the same rate as they are losing energy to the small eddies. Townsend postulated large eddies with streamwise elongation on the basis of his own measurements of velocity correlations in a two-dimensional wake, although the subsequent more comprehensive measurements of Grant (1958) suggested a pair of counter-rotating eddies with axes nearly normal to the center plane of the wake, and planes of circulation roughly normal to the maximum strain rate. Attempts were also made to interpret the motion of large eddies as due to the instability of turbulent fluid (Liepmann 1952, 1962; Landahl 1967). Measurements on pressure correlations in a turbulent jet (Mollo-Christensen 1967) revealed the coherent structure of large eddies, much more coherent than the chaotic randomness that had been thought to be the case.

Wall-Bounded Turbulent Shear Flows

For flow through a two-dimensional channel or a circular pipe, it was found experimentally that in the central region the velocity defect relative to the maximum value at the center depends only on the relative distance from the center for a given wall shear stress. This velocity-defect law, first enunciated by Darcy (1858), was interpreted by Kármán (1930) as suggesting that the mechanism of turbulence is almost independent of viscosity. By postulating that the turbulent fluctuations in the neighborhood of any two points are similar, Kármán derived the velocity profile expressed by a logarithmic function of the distance from the wall. On the other hand, dimensional arguments on the basis of Nikuradse's measurements (1932) led Prandtl (1932) to the law of the wall, in which the velocity in the wall region depends only on the shear stress at the wall, the distance from the wall, and the kinematic viscosity. Except very close to the wall, the velocity profile was found to be logarithmic. Shortly later, Prandtl (1933) showed that the assumption of the mixing length proportional to the distance from the wall yields the logarithmic velocity profile. It is important to note that the regions of validity of the velocity-defect law and the law of the wall overlap. Izakson (1937) and Millikan (1939) independently found that the logarithmic velocity profile is the direct outcome of the existence of a region of overlap, without need for any specific assumption on similarity or mixing length.

Adjacent to the wall the flow is principally viscous, forming a region called the viscous sublayer. The significance of the role of this layer in relation to heat transfer

[4] Corrsin's discovery was made in a subsonic turbulent jet. It is interesting to find that the irregular interface of a wake, clearly visible on the schlieren picture of a projectile in supersonic flight (for example, C. Cranz, 1927, *Lehrbuch der Ballistik*, Vol. 3, 2nd ed., Berlin), had remained unnoticed for so many years.

was noticed by Prandtl (1910) and Taylor (1916). Effects of wall roughness were also discussed with consideration for this layer (Nikuradse 1933a). Measurements of Laufer (1953) indicated that much of the turbulent energy is generated just outside the viscous sublayer. Einstein & Li (1956) visualized an inherently unsteady sublayer, periodically building up and disintegrating. The detailed mechanism involved, however, had not been made clear until Kline and associates (1959, 1967) visually observed the formation of low-speed streaks, which lift up and burst into ejection of low-momentum fluids into the fast-moving outer region.

For flow in boundary layers the velocity profile near the wall was found to be unaffected by the pressure gradient, following the law of the wall of the same form as for pipe flows (Ludwieg & Tillmann 1949). On the other hand, the flow in the outer region resembles more the boundary-free shear flows, and the similarity of the form of the velocity-defect law holds only for a particular type of pressure gradient (Rotta 1950, Clauser 1954). A breakthrough from a practical viewpoint was made by Coles (1956), who described the departure of the velocity profile from the law of the wall by a universal function, which has been called the law of the wake.

An approximate method of predicting boundary-layer growth was initiated by Kármán (1921) on the basis of the momentum integral equation and the velocity profile assumed by reference to pipe flows. The method was extended by taking account of pressure gradients and by employing additional equations (Buri 1931, Gruschwitz 1931, Doenhoff & Tetervin 1943, Head 1958, Rotta 1962, Walz 1966). The difficulty of extending these integral methods to wider classes of flows, coupled with the advent of high-speed computers, has turned general attention toward differential methods, in which the momentum equation is integrated numerically with an eddy viscosity or mixing-length hypothesis. Having recognized the conceptual weakness of the mixing-length formulation in which the eddy viscosity was equal to the product of the mixing length squared and the mean velocity gradient, Prandtl (1945a) made an improved proposal to take the eddy viscosity as the product of the mixing length and the root-mean-square velocity fluctuation, the latter of which was determined from the energy equation of fluctuating motion. This antedated by two decades the upsurge of interest in computing turbulent shear flows (Glushko 1965; Bradshaw, Ferriss & Atwell 1967; Nee & Kovasznay 1969). As noticed by Batchelor (1950), however, the eddy-viscosity hypothesis relating the shear stress directly to the local mean velocity gradient is physically sound provided there is energy equilibrium, a condition only roughly fulfilled in turbulent shear flows in the light of measurements by Townsend (1949), Laufer (1953), and Klebanoff (1954).

Boundary Layers in Compressible Fluids

The introduction of compressibility into boundary-layer theory was first stated by Busemann (1931). The density is now variable and related to pressure and temperature by the state equation of perfect gases, while the temperature is governed by the energy equation of the form simplified by the boundary-layer approximation. Calculations were made for flow on a flat plate by Busemann (1935), Kármán & Tsien (1938), Wada (1944), Crocco (1946), and Chapman & Rubcsin (1949) by specifying the variation of viscosity with temperature but assuming Prandtl number as constant. The results showed a marked increase of the boundary-layer thickness and the temperature near the wall with increase of Mach number of the free-stream velocity. In the meanwhile, attempts were made to transform the equation for a compressible boundary layer into that for an incompressible boundary layer by confining attention to the flow of an idealized fluid, for which Prandtl number is unity and the viscosity is proportional to temperature, along a thermally insulated wall. Transformation of the normal coordinate was first introduced by Dorodnitsyn (1942) and independently by Howarth (1948) to correlate compressible and incompressible boundary layers in zero pressure gradient. This was followed by Illingworth (1949) and Stewartson (1949), who arrived independently at the transformation of both normal and streamwise coordinates for correlation in the more general case of nonzero pressure gradient. The transformation threw open the resources of incompressible boundary-layer theory to the idealized, but by no means unrepresentative, class of compressible flows. It proved useful also for relaxing the idealizing conditions when combined with the approximate method of solution of Kármán and Pohlhausen. Along this line Tani (1954) extended his solution for incompressible fluids to the compressible flow of a more representative fluid, for which the Prandtl number is slightly different from unity and the viscosity varies with temperature according to the Sutherland formula. Poots (1960) extended the solution to include heat transfer at the wall.

The stability of compressible boundary layers was first considered by Lees & Lin (1946) and followed by Lees (1947), Dunn & Lin (1955), Lees & Reshotko (1962), and Mack (1965). Two important results brought about by theory are that the boundary layer could be stabilized by sufficient cooling of the wall (Lees 1947) and that there could be more than one mode of instability, the mode of lowest frequency corresponding to the Tollmien wave for incompressible flows being less amplified than the higher mode at moderate supersonic speeds (Mack 1965).

In incompressible flows the pressure is determined by the velocity field so that the governing feature of turbulence is the fluctuating velocity field, or vorticity field. Compressibility brings in two more fields due to fluctuating pressure and temperature. Chu & Kovasznay (1958) considered small-amplitude fields in a homogeneous flow and found that interaction could be expected to second order in amplitude, the most interesting being the generation of a pressure field from vorticity-vorticity interaction and a vorticity field from temperature-pressure interaction. These correspond to sound generation by turbulence (Lighthill 1952) and vorticity generation by density gradient (Bjerknes 1898), respectively. However, it may be inferred (Morkovin 1962, Laufer 1968) from experimental results on turbulent boundary layers at moderate supersonic speeds that compressibility does not appear to add any substantial source of vorticity, suggesting that the basic mechanism differs little from that for incompressible flows. This afforded a basis for attempts at using a transformation to correlate compressible and incompressible turbulent boundary layers (Mager 1958, Coles 1964).

New phenomena are observed when the boundary layer interacts with the shock wave, which have no counterpart in incompressible flows. For example, when a shock wave impinges on a boundary layer, the pressure rise across the shock wave tends to be diffused in the boundary layer, making its effect felt some distance upstream of the point of impingement. If the shock wave is strong, the boundary layer separates, which in turn reacts upon the formation of the shock wave. The interaction is more spectacular when the boundary layer is laminar (Ackeret, Feldmann & Rott 1946; Liepmann 1946). The second example is provided by hypersonic flow over a flat plate with a sharp leading edge, where a falling pressure gradient is induced by the interaction of the thick boundary layer with the shock wave originating near the leading edge (Becker 1950, Lees & Probstein 1952, Lees 1953). The third example is offered by hypersonic flow near the stagnation point of a blunt-nosed body, where the boundary layer is influenced by vorticity and entropy gradients produced by the shock wave (Hayes & Probstein 1959). Besides these dynamical effects due to high Mach numbers, hypersonic considerations should include the real-gas effects associated with high temperatures, such as ionization, dissociation, and radiation.

Higher Approximations

We recall that Prandtl's boundary-layer theory yields the first approximation to the solution of the Navier-Stokes equations near the solid wall in the limit of small viscosity or large Reynolds number. The approximation having changed the type of the equation and reduced its order, difficulties can be expected to arise when attempts are made to improve on it. Prandtl himself (1935) suggested the possibility of improving the solution for flow on a flat plate by correcting for the effect of displacement thickness. Subsequently, various authors considered the effects of wall curvature, external vorticity, downstream disturbance, etc., in particular cases, bringing about more or less sporadic, but sometimes controversial, results (Van Dyke 1969). It was only in the 1950s that systematic studies were made by Lagerstrom and his associates to establish Prandtl's approximation as the basis of an asymptotic solution of the Navier-Stokes equations, leading to what is now known as the method of matched asymptotic expansions (Kaplun 1954, 1967; Lagerstrom & Cole 1955; Van Dyke 1962, 1964a). The basic idea is to construct two asymptotic expansions, outer and inner expansions, by iterating the Navier-Stokes equations about the inviscid solution and about the boundary-layer solution, respectively, and to match the two expansions in their overlap region of validity.

However, the inviscid solution is not unique for given boundary conditions, and it is difficult in general to select the relevant one that is the limit of the solution of the Navier-Stokes equations. For flow past a certain semi-infinite or streamlined body, one may expect that there is no separation and take the irrotational motion as the relevant inviscid solution. For flow past a bluff body involving separation, the relevant inviscid solution is unknown. Higher approximations have thus been found only for flows without separation. In such cases the first term of the outer expansion is the inviscid irrotational flow, from which the first term of the inner expansion is determined by Prandtl's approximation. The second term of the outer expansion is the irrotational flow due to an apparent source distribution representing the displacement effect of Prandtl's boundary layer. This then determines a correction to the boundary-layer solution, yielding the second term of the inner expansion. As a typical example one may cite the solution to the second approximation by Van Dyke (1964b) for flow past a parabolic cylinder. The wall shear stress was found to be reduced near the stagnation point by both displacement and curvature effects. Calculations were also carried out to higher approximation for flow over a semi-infinite flat plate (Imai 1957; Goldstein 1960; Libby & Fox 1963; Murray 1965, 1967). Contrary to Prandtl's expectations (1935), the second-order displacement effect vanishes and undetermined constants remain that depend on the details of flow near the leading edge, where the boundary-layer approximation fails. For a finite flat plate Kuo (1953) obtained a nonzero second-order displacement correction. Subsequently, however, a slightly more important correction was discovered by Stewartson (1969) and Messiter (1970), originating from a triple-deck structure near the trailing edge.

Even for moderately high Reynolds numbers the second-order correction to the boundary-layer solution is very small so that its calculation is mainly of theoretical interest. It is remarkable, however, to find that the concept of boundary-layer theory was extended and generalized to the method of matched asymptotic expansions, opening the way for treating singular perturbation problems for differential equations. Thus the ideas underlying the boundary-layer theory have been applied to sciences other than fluid mechanics and, in fluid mechanics, to problems other than those associated with small viscosity.

Flow with Separation

When separation occurs, one may not use the inviscid irrotational flow as a basis for setting up a uniformly valid solution, the situation being made even more difficult by the inevitable turbulence and large-scale unsteadiness resulting from instability at high Reynolds numbers. Thus, knowledge of flow with separation has been drawn mostly from experiments. One of the unknown elements is the relevant inviscid solution, although the free-streamline solution due to Helmholtz (1868) and Kirchhoff (1869) is still a likely candidate. When used for high but finite Reynolds numbers, however, the free-streamline solution predicts too small resistance, and various modifications have been suggested for a better description of flow around the body (Zhukovskii 1890, Riabouchinskii 1920, Gilbarg & Rock 1945, Roshko 1955, Woods 1955, Wu 1962).

Theoretical studies of interaction between boundary layer and free stream began first in supersonic flows, where the separated region is more or less localized so that its features depend only on the local properties of the flow, exhibiting what is called free interaction (Chapman, Kuehn & Larson 1957). Such a situation occurs, for example, when a shock wave impinges on the boundary layer. For this problem Lighthill (1953) divided the boundary layer into two layers, treating the outer as virtually inviscid and providing a pressure gradient, and the inner as virtually incompressible and producing changes in displacement thickness. A more detailed calculation along this line was carried out by Gadd (1957) to obtain the pressure distribution across the region of separation induced by the shock wave. Another method of approximate calculation based on the use of integral relations was developed by Lees & Reeves (1964). The method was also applied to the wake behind a bluff body or a backward-facing step (Reeves & Lees 1965). Korst (1956) and Chapman, Kuehn & Larson (1957) independently put forward a simple theory for flow past a concave wall with leading-edge separation by dividing the flow field into a recirculating region, in which the pressure is nearly constant, and a reattachment region, in which the total pressure is nearly constant along the dividing streamline.

In subsonic flows the problem is made rather difficult by the elliptic nature of the flow, although there are some classes of flows in which the separation is localized. In most subsonic flows past a body, however, separation occurs so catastrophically that the problem has so far not given way to theoretical treatment. In fact, there has been little new advance in the theory except Kármán's stability consideration (Kármán 1911, Kármán & Rubach 1912) of the vortex street in the wake of a two-dimensional bluff body. Particular mention should be made of the experimental investigations of Fage & Johansen (1928), Kovasznay (1949), Roshko (1953), Taneda (1959), and Gerrard (1966) as giving insight into the structure of flow downstream of a bluff body.

Literature Cited

Ackeret, J. 1925. *Z. Flugtechn. Motorluftschiff.* 16:44–52

Ackeret, J. 1926. *Z. Ver. Dtsch. Ing.* 70: 1153–58

Ackeret, J., Feldmann, F., Rott, N. 1946. *Mitt. Inst. Aerodyn. Zürich No. 10*

Bassett, A. B. 1888. *A Treatise on Hydrodynamics.* Cambridge: The University Press

Batchelor, G. K. 1950. *J. Aeronaut. Sci.* 17:441–45

Batchelor, G. K. 1956. *J. Fluid Mech.* 1: 177–90

Becker, J. V. 1950. *J. Appl. Phys.* 21:619–28

Bennett, H. W. 1953. *Rep. Kimberly-Clark Corp., Neenah, Wisconsin*

Benney, D. J. 1961. *J. Fluid Mech.* 10:209–36

Benney, D. J. 1964. *Phys. Fluids* 7:319–26

Benney, D. J., Lin, C. C. 1960. *Phys. Fluids* 4:656–57

Bjerknes, V. 1898. *Videnskabsselskabets Skrifter, Kristiania*

Blasius, H. 1908. *Z. Math. Phys.* 56:1–37

Boltze, E. 1908. Göttingen dissertation

Boussinesq, J. 1877. *Mém. Prés. Par Div. Sav. Acad. Sci. Paris* 23:46

Bradshaw, P., Ferriss, D. H., Atwell, N. P. 1967. *J. Fluid Mech.* 28:593–616

Browand, F. K. 1966. *J. Fluid Mech.* 26: 281–307

Burgers, J. M. 1925. *Proc. Int. Congr. Appl. Mech., 1st, Delft, 1924,* pp. 113–28

Burgers, J. M., van der Hegge Zijnen, B. G. 1924. *Verh. Akad. Wetensch. Amsterdam Ser. I* 13: No. 3

Buri, A. 1931. Zürich dissertation

Busemann, A. 1931. In *Handbuch der Experimentalphysik,* ed. W. Wien, F. Harms, 4(I):341–460. Leipzig: Akademische

Busemann, A. 1935. *Z. Angew. Math. Mech.* 15:23–25

Calvert, G. A. 1893. *Trans. Inst. Nav. Archit.* 34:61–77

Chapman, D. R., Kuehn, D. M., Larson, H. K. 1957. *NACA Tech. Note No. 3869*

Chapman, D. R., Rubesin, M. W. 1949. *J. Aeronaut. Sci.* 16:547–65

Chen, K. K., Libby, P. A., Rott, N., Van Dyke, M. D. 1969. *Z. Angew. Math. Phys.* 20:919–27

Chu, B. T., Kovasznay, L. S. G. 1958. *J. Fluid Mech.* 3:494–514

Clauser, F. H. 1954. *J. Aeronaut. Sci.* 21: 91–108

Coles, D. 1956. *J. Fluid Mech.* 1:191–226

Coles, D. 1964. *Phys. Fluids* 7:1403–23

Coles, D. 1965. *J. Fluid Mech.* 21:385–425

Cooke, J. C. 1952. *J. Aeronaut. Sci.* 19:486–90

Cooke, J. C. 1959. *Aeronaut. Res. Coun., Rep. Mem. No. 3201*

Corrsin, S. 1943. *NACA Wartime Rep. W-94*

Corrsin, S., Kistler, A. L. 1954. *NACA Tech. Note No. 3133*

Crocco, L. 1946. *Monogr. Sci. Aeronaut. No. 3*

Darcy, H. 1858. *Mém. Savants Etrangers* 15:265–342

Doenhoff, A. E. von, Tetervin, N. 1943.

NACA Tech. Rep. No. 772
Dorodnitsyn, A. A. 1942. *Prikl. Mat. Mekh.* 6:449–85
Dryden, H. L. 1936. *NACA Tech. Rep. No. 562*
Dryden, H. L. 1955. *Science* 121:375–80
Dryden, H. L., Schubauer, G. B., Mock, W. C., Skramstad, H. K. 1937. *NACA Tech. Rep. No. 581*
Dunn, D. W., Lin, C. C. 1955. *J. Aeronaut. Sci.* 22:455–77
Eichelbrenner, E. A., Oudart, A. 1955a. *Publ. Off. Natl. Etude Rech. Aéron. No. 47*
Eichelbrenner, E. A., Oudart, A. 1955b. *Publ. Off. Natl. Etude Rech. Aéron. No. 76*
Eiffel, G. 1912. *C. R. Acad. Sci. Paris Ser. A* 155:1597–99
Einstein, H. A., Li, H. 1956. *Proc. Am. Soc. Civ. Eng.* 82: Paper No. 945
Emmons, H. W. 1951. *J. Aeronaut. Sci.* 18: 490–98
Fage, A., Johansen, F. C. 1928. *Philos. Mag. Ser. 7* 5:417–41
Falkner, V. M., Skan, S. W. 1930. *Aeronaut. Res. Coun., Rep. Mem. No. 1314*
Froude, W. 1869. *Rep. Br. Assoc. Adv. Sci.*, pp. 211–14
Froude, W. 1872. *Rep. Br. Assoc. Adv. Sci.*, pp. 118–24
Froude, W. 1874. *Rep. Br. Assoc. Adv. Sci.*, pp. 249–55
Gadd, G. E. 1957. *J. Aeronaut. Sci.* 24:759–71
Gerrard, J. H. 1966. *J. Fluid Mech.* 25: 401–13
Gilbarg, D., Rock, D. H. 1945. *Mem. US Nav. Ordnance Lab. No. 8718*
Glauert, M. B., Lighthill, M. J. 1955. *Proc. R. Soc. London Ser. A* 230:188–203
Glushko, G. S. 1965. *Bull. Acad. Sci. USSR, Mech. Ser. No. 4*, pp. 13–23
Goldstein, S. 1930. *Proc. Cambridge Philos. Soc.* 26:1–30
Goldstein, S. 1933. *Proc. R. Soc. London Ser. A* 142:545–62
Goldstein, S. 1938. *Modern Developments in Fluid Dynamics*. Oxford: Clarendon
Goldstein, S. 1948. *Q. J. Mech. Appl. Math.* 1:43–69
Goldstein, S. 1960. *Lectures on Fluid Mechanics*. New York: Interscience
Goldstein, S. 1969. *Ann. Rev. Fluid Mech.* 1:1–28
Goldstein, S., Rosenhead, L. 1936. *Proc. Cambridge Philos. Soc.* 32:392–401
Görtler, H. 1939. *Z. Angew. Math. Mech.* 19:129–40
Görtler, H. 1940. *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.*, pp. 1–26
Görtler, H. 1952. *Z. Angew. Math. Mech.* 32:270–71
Görtler, H. 1957. *J. Math. Mech.* 6:1–66
Grant, H. L. 1958. *J. Fluid Mech.* 4:149–90
Gruschwitz, E. 1931. *Ing. Arch.* 2:321–46
Gümbel, L. 1913. *Jb. Schiffbautechn. Ges.* 14:393–509
Hall, A. A., Hislop, G. S. 1938. *Aeronaut. Res. Coun., Rep. Mem. No. 1843*
Hartree, D. R. 1937. *Proc. Cambridge Philos. Soc.* 33:223–39
Hartree, D. R. 1939. *Aeronaut. Res. Coun., Rep. Mem. No. 2426*
Hatanaka, H. 1949. *Rep. Inst. Sci. Technol., Tokyo Univ.* 3:115–17
Hayes, W. D. 1951. *Rep. US Nav. Ordnance Lab. No. 1313*
Hayes, W. D., Probstein, R. F. 1959. *Hypersonic Flow Theory*. New York: Academic
Head, M. R. 1958. *Aeronaut. Res. Coun., Rep. Mem. No. 3152*
Heisenberg, W. 1924. *Ann. Phys. Leipzig Ser. 4* 74:577–627
Helmholtz, H. v. 1868. *Monatsber. Königl. Akad. Wiss. Berlin*, pp. 215–28
Hiemenz, K. 1911. *Dinglers J.* 326:321–24, 344–48, 357–62, 372–76, 391–93, 407–10
Hopf, L. 1914. *Ann. Phys. Leipzig Ser. 4* 44:1–60
Howarth, L. 1934. *Aeronaut. Res. Coun., Rep. Mem. No. 1632*
Howarth, L. 1938. *Proc. R. Soc. London Ser. A* 164:547–79
Howarth, L. 1948. *Proc. R. Soc. London Ser. A* 194:16–42
Howarth, L. 1951. *Philos. Mag. Ser. 7* 42: 239–43
Hudimoto, B. 1941. *J. Soc. Aeronaut. Sci. Jpn.* 8:279–82
Illingworth, C. R. 1949. *Proc. R. Soc. London Ser. A* 199:533–57
Illingworth, C. R. 1953. *Philos. Mag. Ser. 7* 44:389–403
Imai, I. 1957. *J. Aeronaut. Sci.* 24:155–56
Izakson, A. 1937. *Tech. Phys. USSR* 4:155–62
Jeffreys, H. 1928. *Proc. R. Soc. London Ser. A* 118:195–208
Jones, R. T. 1947. *NACA Tech. Rep. No. 884*
Kaplun, S. 1954. *Z. Angew. Math. Phys.* 5:111–35
Kaplun, S. 1967. *Fluid Mechanics and Singular Perturbations*, ed. P. A. Lagerstrom, L. N. Howard, C. S. Liu. New York: Academic
Kármán, Th. v. 1911. *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.*, pp. 509–17
Kármán, Th. v. 1921. *Z. Angew. Math. Mech.* 1:233–52
Kármán, Th. v. 1930. *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.*, pp. 58–76

Kármán, Th. v., Millikan, C. B. 1934. *NACA Tech. Rep. No. 504*
Kármán, Th. v., Rubach, H. 1912. *Phys. Z.* 13:49–59
Kármán, Th. v., Tsien, H. S. 1938. *J. Aeronaut. Sci.* 5:227–32
Kempf, G. 1913. *Jb. Schiffbautechn. Ges.* 14:504–5
Kirchhoff, G. 1869. *Crelle J. Math.* 70:289–98
Klebanoff, P. S. 1954. *NACA Tech. Note No. 3178*
Klebanoff, P. S. 1966. *Proc. Int. Congr. Appl. Mech., 11th, Munich, 1964*, pp. 803–5
Klebanoff, P. S., Tidstrom, K. D. 1959. *NASA Tech. Note D-195*
Klebanoff, P. S., Tidstrom, K. D., Sargent, L. M. 1962. *J. Fluid Mech.* 12:1–34
Kline, S. J., Reynolds, W. C., Schraub, F. A., Runstalder, P. W. 1967. *J. Fluid Mech.* 30:741–73
Kline, S. J., Runstalder, P. W. 1959. *Trans. ASME, J. Appl. Mech.* 26:166–70
Korst, H. H. 1956. *Trans. ASME, J. Appl. Mech.* 23:593–600
Kovasznay, L. S. G. 1949. *Proc. R. Soc. London Ser. A* 198:174–90
Kovasznay, L. S. G., Komoda, H., Vasudeva, B. R. 1962. *Proc. Heat Transfer Fluid Mech. Inst., Stanford, 1962*, pp. 1–26
Kuo, Y. H. 1953. *J. Math. Phys.* 32:83–101
Lagerstrom, P. A., Cole, J. D. 1955. *J. Ration. Mech. Anal.* 4:817–82
Lam, S. H., Rott, N. 1962. *Proc. Int. Congr. Appl. Mech., 10th, Stresa, 1960*, p. 239
Lamb, H. 1924. *Hydrodynamics.* 5th ed. Cambridge: The University Press
Lanchester, F. W. 1907. *Aerodynamics.* London: Constable
Landahl, M. T. 1967. *J. Fluid Mech.* 29:441–59
Landau, L. D. 1944. *Dokl. Akad. Nauk SSSR* 44:311–14
Landau, L. D., Lifshitz, E. M. 1959. *Fluid Mechanics.* Oxford: Pergamon
Laufer, J. 1953. *NACA Tech. Note No. 2953*
Laufer, J. 1968. In *Compressible Turbulent Boundary Layers*, ed. M. H. Bertram, pp. 1–13. *NASA SP-216*
Lees, L. 1947. *NACA Tech. Rep. No. 876*
Lees, L. 1953. *J. Aeronaut. Sci.* 20:143–45
Lees, L., Lin, C. C. 1946. *NACA Tech. Note No. 1115*
Lees, L., Probstein, R. F. 1952. *Rep. Dep. Aeronaut. Eng.. Princeton Univ., No. 195*
Lees, L., Reeves, B. L. 1964. *J. Am. Inst. Aeronaut. Astron.* 2:1907–20
Lees, L., Reshotko, E. 1962. *J. Fluid Mech.* 12:555–90
Libby, P. A., Fox, H. 1963. *J. Fluid Mech.* 17:433–49
Liepmann, H. W. 1943. *NACA Wartime Rep. W-107*
Liepmann, H. W. 1946. *J. Aeronaut. Sci.* 13:623–37
Liepmann, H. W. 1952. *Z. Angew. Math. Phys.* 3:321–42, 407–26
Liepmann, H. W. 1962. In *Mécanique de la Turbulence*, ed. A. Favre, pp. 211–27. Paris: CNRS
Lighthill, M. J. 1952. *Proc. R. Soc. London Ser. A* 211:564–87
Lighthill, M. J. 1953. *Proc. R. Soc. London Ser. A* 217:478–507
Lighthill, M. J. 1954. *Proc. R. Soc. London Ser. A* 224:1–23
Lighthill, M. J. 1963. In *Laminar Boundary Layers*, ed. L. Rosenhead, pp. 1–45. Oxford: Clarendon
Lin, C. C. 1945. *Q. Appl. Math.* 3:117–42, 218–34, 277–301
Loitsianskii, L. G. 1949. *Prikl. Mat. Mekh.* 13:513–24
Loitsianskii, L. G. 1970. *Mechanics of Liquid and Gas.* Moscow: Nauka. (In Russian)
Lorenz, L. 1881. *Ann. Phys. Leipzig Ser. 2* 13:422–47, 582–606
Ludwieg, H., Tillmann, W. 1949. *Ing. Arch.* 17:288–99
Mack, L. M. 1965. In *Methods in Computational Physics*, ed. B. Alder, 4:247–99. New York: Academic
Mager, A. 1958. *J. Aeronaut. Sci.* 25:305–11
Mangler, W. 1944. *Z. Angew. Math. Mech.* 24:251–56
Mangler, W. 1948. *Z. Angew. Math. Mech.* 28:97–103
Maskell, E. C. 1955. *Aeronaut. Res. Coun., Rep. No. 18063*
Meksyn, D., Stuart, J. T. 1951. *Proc. R. Soc. London Ser. A* 208:517–26
Messiter, A. F. 1970. *SIAM J. Appl. Math.* 18:241–57
Millikan, C. B. 1932. *Trans. ASME, Appl. Mech. Sect.* 54:29–43
Millikan, C. B. 1936. *J. Aeronaut. Sci.* 3:91–94
Millikan, C. B. 1939. *Proc. Int. Congr. Appl. Mech., 5th, Cambridge. Mass.. 1938*, pp. 386–92
Mises, R. v. 1927. *Z. Angew. Math. Mech.* 7:425–31
Mollo-Christensen, E. 1967. *Trans. ASME, J. Appl. Mech.* 34:1–7
Moore, F. K. 1951. *NACA Tech. Note No. 2471*
Moore, F. K. 1958. In *Grenzschichtforschung*, ed. H. Görtler, pp. 296–311. Berlin: Springer
Morkovin, M. V. 1962. In *Mécanique de la Turbulence*, ed. A. Favre, pp. 367–80.

Paris: CNRS
Murray, J. D. 1965. *J. Fluid Mech.* 21:337–44
Murray, J. D. 1967. *J. Math. Phys.* 46:1–20
Navier, C. L. M. H. 1823. *Mém. Acad. R. Sci. Paris* 6:389–416
Nee, V. M., Kovasznay, L. S. G. 1969. *Phys. Fluids* 12:473–84
Nikuradse, J. 1932. *Ver. Dtsch. Ing., Forschungsheft No. 356*
Nikuradse, J. 1933a. *Ver. Dtsch. Ing., Forschungsheft No. 361*
Nikuradse, J. 1933b. *Z. Angew. Math. Mech.* 13:174–76
Orr, W. M. F. 1907. *Proc. R. Irish Acad. Ser. A* 27:9–27, 69–138
Pohlhausen, E. 1921. *Z. Angew. Math. Mech.* 1:115–21
Pohlhausen, K. 1921. *Z. Angew. Math. Mech.* 1:252–68
Poisson, S. D. 1831. *J. Ec. Polytech. Paris* 13:139–66
Poots, G. 1960. *Q. J. Mech. Appl. Math.* 13:57–84
Prandtl, L. 1905. *Verh. Int. Math. Kongr., 3rd. Heidelberg. 1904*, pp. 484–91. Transl. 1928. *NACA Memo. No. 452*
Prandtl, L. 1910. *Phys. Z.* 11:1072–78
Prandtl, L. 1913. In *Handwörterbuch der Naturwissenschaften,* 4:101–40. Jena: Fischer
Prandtl, L. 1914. *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.*, pp. 177–90
Prandtl, L. 1921. *Z. Angew. Math. Mech.* 1:431–36
Prandtl, L. 1925a. *Naturwissenschaften* 13:93–108
Prandtl, L. 1925b. *Z. Angew. Math. Mech.* 5:136–39
Prandtl, L. 1927a. In *Ergeb. Aerodyn. Versuchsanst. Göttingen,* 3:6–9. Munich and Berlin: Oldenbourg
Prandtl, L. 1927b. *J. R. Aeronaut. Soc.* 31:720–43
Prandtl, L. 1928. *Z. Angew. Math. Mech.* 8:249–51
Prandtl, L. 1931. *Abriss der Strömungslehre.* Braunschweig: Vieweg
Prandtl, L. 1932. In *Ergeb. Aerodyn. Versuchsanst. Göttingen,* 4:18–29. Munich and Berlin: Oldenbourg
Prandtl, L. 1933. *Z. Ver. Dtsch. Ing.* 77:105–14
Prandtl, L. 1935. In *Aerodynamic Theory,* ed. W. F. Durand, 3:34–209. Berlin: Springer
Prandtl, L. 1938. *Z. Angew. Math. Mech.* 18:77–82
Prandtl, L. 1942. *Z. Angew. Math. Mech.* 22:241–43
Prandtl, L. 1945a. *Nachr. Akad. Wiss. Göttingen, Math.-phys. Kl.*, pp. 6–19
Prandtl, L. 1945b. In *Festschrift zum 60. Geburtstage von A. Betz,* pp. 134–41. Göttingen
Prandtl, L. 1948. *Phys. Blätter* 4:89–92
Prandtl, L. 1949. *Führer durch die Strömungslehre.* 3rd ed. Braunschweig: Vieweg
Proudman, I., Johnson, K. 1962. *J. Fluid Mech.* 12:161–68
Rankine, W. J. M. 1864. *Trans. Inst. Nav. Archit.* 5:316–33
Rayleigh, Lord 1880. *Proc. London Math. Soc.* 11:57–70
Rayleigh, Lord 1883. *Philos. Trans. R. Soc. London Ser. A* 175:1–21
Rayleigh, Lord 1911. *Philos. Mag. Ser. 6* 21:697–711
Reeves, B. L., Lees, L. 1965. *J. Am. Inst. Aeronaut. Astron.* 3:2061–74
Reynolds, O. 1895. *Philos. Trans. R. Soc. London Ser. A* 186:123–64
Riabouchinskii, D. 1914. *Bull. Inst. Aérodyn. Koutchino* 5:51
Riabouchinskii, D. 1920. *Proc. London Math. Soc.* 19:206–15
Roshko, A. 1953. *NACA Tech. Note No. 2913*
Roshko, A. 1955. *J. Aeronaut. Sci.* 22:124–32
Rott, N. 1956. *Q. Appl. Math.* 13:444–51
Rott, N., Rosenzweig, M. L. 1960. *J. Aeronaut. Sci.* 27:741–47
Rotta, J. C. 1950. *Mitt. Max-Planck-Inst. Strömungsforsch. Göttingen No. 1*
Rotta, J. C. 1962. In *Progress in Aeronautical Sciences,* ed. D. Küchemann, 2:1–219. Oxford: Pergamon
Saint-Venant, B. 1843. *C. R. Acad. Sci. Paris Ser. A* 17:1240–42
Sato, H., Kuriki, K. 1961. *J. Fluid Mech.* 11:321–52
Schiller, L. 1934. *Z. Angew. Math. Mech.* 14:36–42
Schlichting, H. 1930. *Ing. Arch.* 1:533–71
Schlichting, H. 1932. *Phys. Z.* 33:327–35
Schlichting, H. 1933a. *Z. Angew. Math. Mech.* 13:260–63
Schlichting, H. 1933b. *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.*, pp. 181–208
Schlichting, H. 1953. *Ing. Arch.* 21:227–44
Schmidt, E., Beckmann, W. 1930. *Tech. Mech. Thermodyn.* 1:341–49, 391–406
Schrenk, O. 1928. *Luftfahrtforschung* 2:49–62
Schrenk, O. 1931. *Z. Flugtechn. Motorluftschiff.* 22:259–64
Schrenk, O. 1935. *Luftfahrtforschung* 12:10–27
Schubauer, G. B. 1935. *NACA Tech. Rep. No. 527*
Schubauer, G. B., Klebanoff, P. S. 1955.

NACA Tech. Note No. 3489
Schubauer, G. B., Skramstad, H. K. 1943. *NACA Wartime Rep. W-8*
Sears, W. R. 1948. *J. Aeronaut. Sci.* 15: 49–52
Shen, S. F. 1954. *J. Aeronaut. Sci.* 21: 62–64
Sommerfeld, A. 1909. *Atti Congr. Int. Math., 4th, Rome, 1908,* pp. 116–24
Squire, H. B. 1933. *Proc. R. Soc. London Ser. A* 142:621–28
Stepanov, E. I. 1947. *Prikl. Mat. Mekh.* 11:203–04
Stewartson, K. 1949. *Proc. R. Soc. London Ser. A* 200:84–100
Stewartson, K. 1951. *Q. J. Mech. Appl. Math.* 4:182–98
Stewartson, K. 1954. *Proc. Cambridge Philos. Soc.* 50:454–65
Stewartson, K. 1955. *Q. Appl. Math.* 13:113–22
Stewartson, K. 1958. *Q. J. Mech. Appl. Math.* 11:399–410
Stewartson, K. 1969. *Mathematika* 16: 106–21
Stokes, G. G. 1845. *Trans. Cambridge Philos. Soc.* 8:287–305
Stokes, G. G. 1851. *Trans. Cambridge Philos. Soc.* 9(II):8–106
Stratford, B. S. 1954. *Aeronaut. Res. Coun., Rep. Mem. No. 3002*
Struminskii, V. V. 1946. *Dokl. Akad. Nauk SSSR* 54:765–68
Stuart, J. T. 1958. *J. Fluid Mech.* 4:1–21
Stuart, J. T. 1960. *J. Fluid Mech.* 9:353–70
Taneda, S. 1959. *J. Phys. Soc. Jpn.* 14: 843–48
Tani, I. 1941. *J. Aeronaut. Res. Inst., Tokyo Imp. Univ., No. 199*
Tani, I. 1949. *J. Phys. Soc. Jpn.* 4:149–54
Tani, I. 1954. *J. Aeronaut. Sci.* 21:487–95
Tani, I., Sato, H. 1956. *J. Phys. Soc. Jpn.* 11:1284–91
Taylor, G. I. 1915. *Philos. Trans. R. Soc, London Ser. A* 215:1–26
Taylor, G. I. 1916. *Aeronaut. Res. Coun., Rep. Mem. No. 272*
Taylor, G. I. 1923. *Philos. Trans. R. Soc. London Ser. A* 223:289–343
Taylor, G. I. 1932. *Proc. R. Soc. London Ser. A* 135:685–96
Taylor, G. I. 1936. *Proc. R. Soc. London Ser. A* 156:307–17
Taylor, G. I. 1950. *Q. J. Mech. Appl. Math.* 3:129–39
Terrill, R. M. 1960. *Philos. Trans. R. Soc. London Ser. A* 253:55–100
Thwaites, B. 1949. *Aeronaut. Q.* 1:245–80
Tietjens, O. G. 1925. *Z. Angew. Math. Mech.* 5:200–17
Timman, R. 1949. *Rep. Trans. Natl. LuchtvLab. Amsterdam* 15:F29–45
Timman, R., Zaat, J. A. 1955. In *50 Jahre Grenzschichtforschung,* ed. H. Görtler, W. Tollmien, pp. 432–45. Braunschweig: Vieweg
Tollmien, W. 1924. Göttingen dissertation
Tollmien, W. 1926. *Z. Angew. Math. Mech.* 6:468–78
Tollmien, W. 1929. *Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl.,* pp. 21–44
Tollmien, W. 1931. In *Handbuch der Experimentalphysik,* ed. W. Wien, F. Harms, 4(I):243–87. Leipzig: Akademische
Töpfer, C. 1912. *Z. Math. Phys.* 60:397–98
Townsend, A. A. 1949. *Proc. R. Soc. London Ser. A* 197:124–40
Townsend, A. A. 1950. *Philos. Mag. Ser.* 7 41:890–906
Townsend, A. A. 1956. *The Structure of Turbulent Shear Flow.* Cambridge: The University Press
Truckenbrodt, E. 1952. *Ing. Arch.* 20:211–28
Unwin, W. C. 1888. In *Encyclopaedia Britannica,* 12:459–535, 9th ed.
Van Dyke, M. 1962. *J. Fluid Mech.* 14: 161–77
Van Dyke, M. 1964a. *Perturbation Methods in Fluid Mechanics.* New York and London: Academic
Van Dyke, M. 1964b. *J. Fluid Mech.* 19: 145–59
Van Dyke, M. 1969. *Ann. Rev. Fluid Mech.* 1:265–92
Wada, K. 1944. *Rep. Aeronaut. Res. Inst., Tokyo Imp. Univ., No. 302*
Walz, A. 1941. *Ber. Lilienthal Ges. Luftfahrtf. No. 141,* pp. 8–12
Walz, A. 1966. *Strömungs- und Temperaturgrenzschichten.* Karlsruhe: Braun
Watson, J. 1960. *J. Fluid Mech.* 9:371–89
Watson, J. 1962. *J. Fluid Mech.* 14:211–21
Watson, J. 1963. In *Laminar Boundary Layers,* ed. L. Rosenhead, pp. 411–15. Oxford: Clarendon
Wieghardt, K. 1948. *Ing. Arch.* 16:231–42
Woods, L. C. 1955. *Proc. R. Soc. London Ser. A* 227:367–86
Wu, T. Y. 1962. *J. Fluid Mech.* 13:161–81
Zhukovskii, N. E. 1890. *Math. Collect. Moscow Math. Soc.* 15:121–278
Zhukovskii, N. E. 1916. *Aérodynamique.* Paris: Gauthier-Villars