History of the Nine-point Circle.

By J. S. MACKAY.

The earliest author to whom the discovery of the nine-point circle has been attributed is Euler, but no one has ever given a reference to any passage in Euler's writings where the characteristic property of this circle is either stated or implied. The attribution to Euler is simply a mistake, and the origin of the mistake may, I think, be explained. It is worth while doing this, in order that subsequent investigators may be spared the labour and chagrin of a fruitless search through Euler's numerous writings.

Catalan in his *Théorèmes et Problèmes de Géométrie Élémentaire*, 5th ed. p. 126 (1872), or 6th ed. p. 170 (1879), says that the learned Terquem attributed the theorem of the nine-point circle to Euler, and refers to the *Nouvelles Annales de Mathématiques*, I. 196 (1842). If the first volume of the *Nouvelles Annales* be consulted, it will be found that Terquem has two articles on the rectilineal triangle. The first (pp. 79–87) is entitled *Considérations sur le triangle rectiligne, d'après Euler;* the second (pp. 196–200) has the same title, but *d'après Euler* is omitted. In the first article Terquem mentions that Euler discovered certain properties * of the triangle, and refers to the place where they are to be found (*Novi Commentarii Academiae . . . Petropolitanae*, xi. 114, 1765). He says he thinks it useful to reproduce them with some developments, and this is exactly what he does, for the first article is a synopsis of Euler's results, and the second article, which begins with the property of the nine-point circle, contains the developments.

Who, then, is the discoverer of the nine-point circle?

The fact is that there have been several independent discoverers of it, English, French, German, Swiss. Their researches will be treated of in the order of publication; and it will conduce to brevity of statement if the following notation be laid down:

ABC	is the fundamental triangle
I, I_1, I_2, I_3	are the incentre and the three excentres
O, M	are the circumcentre and the nine-point centre
X, Y, Z	are the feet of the perpendiculars from A, B, C on the opposite sides.

* An abstract of Euler's paper will be found in the *Proceedings of the Edinburgh Mathematical Society*, IV. 51-55 (1886).

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In Leybourn's *Mathematical Repository*, new series, I. 18 (pagination of the Questions to be answered) Benjamin Bevan proposed in 1804 the following:

"*In a plane triangle let O_0 be the centre of a circle passing through I_1, I_2, I_3, then will $OO_0 = OI$ and be in the same right line, and*

$$O_0I_1 = O_0I_2 = O_0I_3 = 2R$$

or the diameter of the circumscribing circle."

When it is remembered that triangle $I_1I_2I_3$ has I for orthocentre and ABC for orthic triangle, and that the circumcircle of ABC is the nine-point circle of $I_1I_2I_3$, it will be seen that Bevan's theorem establishes the conclusions:

(1) *That the nine-point centre bisects the distance between the circumcentre and the orthocentre.*

(2) *That the radius of the nine-point circle is half the radius of the circumcircle.*

The proof of Bevan's theorem given in the *Mathematical Repository*, Vol. I., Part I., p. 143, is by John Butterworth of Haggate.

In the *Gentleman's Mathematical Companion for the year* 1807 (which was published in 1806) John Butterworth proposes the question:

When the base and vertical angle are given, what is the locus of the centre of the circle passing through the three centres of the circles touching one side and the prolongation of the other two sides of a plane triangle?

In the *Mathematical Companion* for 1808 (pp. 132–3) two solutions of the question are given, the first by the proposer, and the second by John Whitley. Whitley's solution shows that the circumcircle of ABC goes through seven points connected with the triangle $I_1I_2I_3$, namely, the feet of its perpendiculars, the mid points of two of its sides, and the mid points of two of the segments intercepted between the orthocentre and the vertices. It is evident from the tenour of his proof that he must have been aware that the circumcircle of ABC passed through the other two points which make up the well-known nine, but for the purpose he had in hand he did not happen to require them, and they are consequently not mentioned.

Butterworth's solution of his own question shows that the circumcircle of ABC bisects the lines II_1, II_2, II_3, and that the circumcentre O bisects IO_0.

The next step in the history of the nine-point circle is the discussion not of the relations between the circumcircles of $I_1I_2I_3$ and ABC, but of those between the circumcircles of ABC and XYZ.

The nine points are explicitly mentioned in Gergonne's *Annales de Mathématiques* in an article by Brianchon and Poncelet. The article appeared on the 1st January 1821 in vol. xi., and the theorem establishing the characteristic property of the circle in question occurs at p. 215. As this article is reprinted by Poncelet in his *Applications d'Analyse et de Géométrie*, II. 504–516 (1864) I infer that it owes its origin rather to Poncelet than to Brianchon. Poncelet does not draw attention to the easy inference that the radius of the nine-point circle is half the radius of the circumcircle, nor to the position of the nine-point centre. This is natural enough, for the title of his article is *Recherches sur la détermination d'une hyperbole équilatère, au moyen de quatre conditions données*, and the existence of the nine-point circle is noticed incidentally. It may be worth mentioning that in Poncelet's demonstration there occurs the theorem which Mr R. F. Davis discovered some years ago and applied to the triplicate-ratio and Taylor circles, namely:

If on each side of a triangle ABC a pair of points are taken so that any two pairs are concyclic, then all the six points are concyclic.

Karl Wilhelm Feuerbach's *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks* appeared at Nürnberg in 1822. In § 56 of it occurs the theorem that "the circle which goes through the feet of the perpendiculars of a triangle meets the sides at their mid points," but nothing is said of the other three points. The radius of the circle, $\frac{1}{2}$R, is mentioned, and the position of the centre is given, midway between the orthocentre and the circumcentre. In § 57 Feuerbach proves that the circle which goes through the feet of the perpendiculars of a triangle touches the incircle and the excircles, and this is the first enunciation of that interesting property of the nine-point circle. The proof consists in showing that the distance between the nine-point centre and the incentre is equal to $\frac{1}{2}R - r$.

In the *Philosophical Magazine*, II. 29–31 (1827) T. S. Davies proves the characteristic property of the nine-point circle. At the outset of his article, which is entitled *Symmetrical Properties of Plane Triangles*, and dated Janr. 15th, 1827, he says: "The following properties . . . do not appear to have been noticed by mathematicians." Davies was very scrupulous in giving to his predecessors the credit of their discoveries; hence he is another discoverer of the nine-point circle. In the fourth of his propositions and in the corollaries thereto, Davies, besides mentioning the length of the nine-point radius and the position of the nine-point centre, remarks that the centroid is also situated on the line which contains the orthocentre, the nine-point centre, and the circumcentre.

In Gergonne's *Annales de Mathématiques*, xix. 37–64 there is an article* by Steiner entitled *Développement d'une série de théorèmes relatifs aux sections coniques.* The article appeared in 1828, and in the course of it Steiner shows, among other things, that the nine-point circle property is only a particular case of a more general theorem. He remarks also that the centroid is on the line joining the orthocentre and the circumcentre (attributing this property to Carnot; it was discovered by Euler); he states that the four points, the circumcentre, the centroid, the nine-point centre, and the orthocentre, form a harmonic range, and that the orthocentre and the centroid are the centres of similitude of the nine-point circle and the circumcircle; and lastly he adds, without proof, the statement that the nine-point circle touches the incircle and the excircles.

In a long note to § 12 of his tractate† *Die geometrischen Constructionen, ausgeführt mittelst der geraden Linie und eines festen Kreises*, which appeared in 1833, Steiner discusses the nine-point circle and the circumcircle in connection with their centres of similitude, and he enunciates the theorem that twelve points associated with the triangle lie on one and the same circle. At the end of the note Steiner states that when he announced the theorem that the nine-point circle touched the incircle and the excircles he was not aware that it had been previously made known by Feuerbach.

Hitherto the circle had received no special name. The designation "nine-point circle" (*le cercle des neuf points*) was bestowed on it in 1842 by Terquem, one of the editors of the *Nouvelles Annales*; see that journal, Vol. I., p. 198. It has also been called the six-points circle, the twelve-points circle, the n-point circle, Feuer-

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* Republished in Steiner's *Gesammelte Werke*, I. 191-210 (1881).
† Republished in Steiner's *Gesammelte Werke* I. 489-492.

bach's circle, Euler's circle, Terquem's circle, *il circolo medioscritto*, the medioscribed circle, the mid circle, the circum-midcircle.

There are other demonstrations of the characteristic property of the nine-point circle quite distinct from those given in the articles already spoken of. One by Terquem will be found in *Nouvelles Annales*, I. 196 (1842); a second by C. Adams in *Die Lehre von den Transversalen*, p. 37 (1843), and a third, also by Adams, in *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, pp. 14–16 (1846); a fourth by T. T. Wilkinson in the *Lady's and Gentleman's Diary* for 1855, p. 67; a fifth by William Godward in *Mathematical Questions from the Educational Times*, VII. 86–7 (1867). Besides these there are several other proofs not essentially different. Those who are curious in such matters may refer to proofs by Rev. Joseph Wolstenholme in the *Quarterly Journal of Mathematics*, II. 138–9 (1858); by W. H. B[esant] in the *Oxford, Cambridge and Dublin Messenger of Mathematics*, III. 222–3 (1866); by Desboves in *Questions de Géométrie Élémentaire*, 2nd ed., pp. 146–7 (1875); by Captain Mennesson in *Nouvelle Correspondance Mathématique*, IV. 241–2 (1878); by Rev. John Wilson in *Proceedings of the Edinburgh Mathematical Society*, VI. 38–40 (1888).

Of the theorem that the nine-point circle touches the incircle and the excircles the first two published proofs, namely, Feuerbach's in 1822 and Terquem's in 1842, were analytical. Steiner merely enunciated the theorem. The first geometrical proof appeared in the *Nouvelles Annales*, IX. 401–3 (1850) in an article entitled *Note sur le triangle rectiligne* by J. Mention. The next appeared in the *Lady's and Gentleman's Diary*. In the *Diary* for 1853, p. 77, W. H. Levy proposes the question:

In any plane triangle the sum of the four distances from the point of bisection of the line joining the centre of the circumscribing circle and the point of intersection of the perpendiculars from the opposite angles upon the sides, to the centres of the inscribed and escribed circles is equal to three times the diameter of the circumscribing circle.

A geometrical solution is given in the *Diary* for 1854, p. 56, where the following results are established:

$$\mathrm{MI} = \tfrac{1}{2}\mathrm{R} - r$$

$$\mathrm{MI}_1 = \tfrac{1}{2}\mathrm{R} + r_1 \quad \mathrm{MI}_2 = \tfrac{1}{2}\mathrm{R} + r_2 \quad \mathrm{MI}_3 = \tfrac{1}{2}\mathrm{R} + r_3;$$

and hence

$$\Sigma(\mathrm{MI}) = 2\mathrm{R} + (r_1 + r_2 + r_3 - r) = 3\mathrm{D}.$$

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Notwithstanding Davies's article in the *Philosophical Magazine* in 1827, the nine-point circle seems to have been almost unknown in the United Kingdom; and hence it is not so curious as it might appear at first sight that among the 23 mathematicians who answered Mr Levy's question only one should have drawn any further inference from it.

In the same number of the *Diary* on p. 72, T. T. Wilkinson of Burnley, who was a friend of Davies, and was acquainted* with his paper in the *Philosophical Magazine*, proposes the question:

Let ABC be any triangle; AD, BE, CF the perpendiculars drawn from A, B, C to the opposite sides, mutually intersecting at P: then the circle described through D, E, F, the feet of the perpendiculars will be tangential to the sixteen inscribed and escribed circles of the triangles ABC, APB, BPC, and CPA.

The solutions of Wilkinson's question given in the *Diary* for 1855, pp. 67–9, all depend on the four results established in the *Diary* for 1854.

In the *Diary* for 1857, pp. 86–9, John Joshua Robinson enunciates and proves the theorem:

The circle described through the middle points of the sides of any triangle is tangential to several infinite systems of circles inscribed and escribed to triangles drawn according to a given law.

Or the theorem may be expressed more fully:

If the radical centres of the inscribed and escribed circles of any triangle be taken, and circles be inscribed and escribed to the triangles formed by joining these radical centres, and the radical centres of the latter system of circles be again taken and circles inscribed and escribed to the triangles thus formed, and so on ad infinitum, *the infinite number of circles thus formed, as well as the original system of inscribed and escribed circles, always touch the circle drawn through the middle points of the first triangle.*

The editor of the *Diary* in a note to Robinson's theorem mentions that Wilkinson first announced his beautiful theorem under this general form of enunciation.

In the *Diary* for 1858, pp. 86–7, Wilkinson has a short article

* I am in possession of a collection of printed mathematical papers which belonged to Wilkinson. The paper of Davies's referred to is imperfect, but is completed in Wilkinson's handwriting.

entitled *Notae Geometricae,* from which the following extracts are taken.

"In the extended solution to which allusion is made [Robinson's solution] I noticed that the property became porismatic when *any three points* are taken for the feet of the perpendiculars, and the triangles thence resulting are constructed according to an almost obvious law. All the triangles thus formed are evidently those of *least* perimeter to the *primitives* obtained by bisecting the angles formed by joining the feet of the perpendiculars in each instance, and hence connect themselves immediately with the many beautiful and curious properties known to result from this view of the subject. . . .

"But the property admits of a more general enunciation . . . by projecting the system upon a plane . . . ; for the perpendiculars then become lines drawn conjugate to the opposite sides, the contacts are preserved, and the circles become conics similar and similarly placed. Hence if ABC be a triangle inscribed in a conic, and if through each vertex there be drawn a transversal respectively conjugate to the opposite side, then

(1) These three transversals will intersect in the same point O.

(2) The middle points of the lines OA, OB, OC, the middle points of the sides AB, BC, CA, and the points *a*, *b*, *c*, where the transversals meet the sides, are nine points situated in a second conic, similar and similarly placed with respect to the first.

(3) This second conic is also tangential to the sixteen conics inscribed and escribed to the triangles AOB, BOC, COA, and similar and similarly placed with respect to the two first conics.

In this connection, reference may be made to a paper of Professor Eugenio Beltrami (read 12th March 1863), *Intorno alle coniche di nove Punti e ad alcune quistioni che ne dipendono,* printed in the *Memorie della Accademia delle Scienze dell'Istituto di Bologna,* 2nd series, Vol. II., pp. 361–395 (1862); and also to a paper by Schröter (dated October 1867), *Erweiterung einiger bekannten Eigenschaften des ebenen Dreiecks,* in Crelle's *Journal* LXVIII. 203–234 (1868).

In an article,* dated July 17, 1860, the Rev. George Salmon called attention to Feuerbach's theorem. He says:

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* *Quarterly Journal of Mathematics,* IV. 152-4 (1861).

"The following elementary theorems may interest some of the readers of the *Quarterly Journal.*

(1) The distance between the point of intersection of the perpendiculars of a triangle and the centre of the circumscribing circle is given by the equation

$$D^2 = R^2 - 8R^2\cos A\cos B\cos C.$$

(2) The distance between the point of intersection of perpendiculars and the centre of the inscribed circle is given by the equation

$$d^2 = 2r^2 - 4R^2\cos A\cos B\cos C.$$

Hence $$D^2 - 2d^2 = R^2 - 4r^2.$$

(3) It follows that if the two circles be fixed, the locus of the intersection of perpendiculars is a circle whose radius is $R - 2r$, and whose centre is found by producing the line joining the centres to a distance equal to itself, and so that the centre of the inscribed circle may lie in the middle.

From the preceding theorems Dr Hart, to whom I had happened to mention them, drew the following inferences:

Consider the circle passing through the middle points of the sides of the triangle; its radius $= \frac{1}{2}R$, and its centre is the middle point of the line joining the centre of the circumscribing circle to the intersection of perpendiculars. The line then joining this middle point to the centre of the inscribed circle is the line joining the middle points of the sides of a triangle whose base has been proved to be $R - 2r$. Its length is therefore $\frac{1}{2}R - r$.

(4) Hence, when the inscribed and circumscribing circles are given, the locus of the centre of the circle passing through the middle points of sides is a circle having for its centre the centre of the inscribed circle.

Further: the distance between the centres of the inscribed circle and of the circle through the middle points of sides has been proved to be exactly the difference between their radii; and the same argument applies to any of the four circles which touch the three sides of the given triangle; hence

(5) The circle which passes through the middle points of the sides of a triangle touches the four circles which touch the three sides.

This theorem was new both to Dr Hart and myself,* but I have lately learned from a friend that it belongs to M. Terquem, who has given it in his *Annales*, T. I., p. 196. Dr Hart has since made other more direct demonstrations of it, and Sir William R. Hamilton, to whom he mentioned the theorem, was sufficiently interested by it to take the trouble of investigating it algebraically, when he obtained very simple constructions for the common point and common tangent of the two circles."

Dr Salmon then sums up Sir William Hamilton's results in three theorems, which he states and proves.

In the same volume of the *Quarterly Journal* (pp. 245–252), Mr John Casey has an article, dated Nov. 27, 1860, in which he proves not only Feuerbach's theorem, but also, without knowing the results that had been previously published in the *Diary*, extends the contact to an indefinite number of circles. See Casey's *Sequel to Euclid*, 6th ed., pp. 105–6 (1892), where a proof is also given of the following extension (due to Dr Hart) of Feuerbach's theorem :

If the three sides of a plane triangle be replaced by three circles, then the circles touching these, which correspond to the inscribed and escribed circles of a plane triangle, are all touched by another circle.

See also Lachlan's *Elementary Treatise on Modern Pure Geometry*, pp. 251–7 (1893).

There are other demonstrations of Feuerbach's theorem besides those already spoken of. The following references may be given.

Mr J. M'Dowell in the *Quarterly Journal of Pure and Applied Mathematics*, V. 269–271 (1862).

Mr W. F. Walker in the same periodical, VIII. 47–50 (1867).

Herr J. Lappe in Crelle's *Journal*, LXXI. 387–392 (1870).

Herr Binder in 1872 communicated to Dr Richard Baltzer a proof which will be found in the latter's *Elemente der Mathematik*, II. 92–3 (1883).

Mr J. P. Taylor in the *Quarterly Journal*, XIII. 197 (1875).

Mr E. M. Langley in 1876 discovered the proof given in the *Harpur Euclid*, p. 489 (1890).

* Might it not be that Dr Salmon had forgotten it and rediscovered it? This conjecture is made because Mr J. J. Robinson begins an article in the *Diary* for 1858, p. 88, by saying : "My best thanks are due to the Rev. George Salmon of Trinity College, Dublin, for having called my attention to two errors which somehow have crept into my former paper," that is, the paper in which Feuerbach's theorem was extended.

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M. Chadu of Bordeaux in *Nouvelle Correspondance Mathématique*, V. 230–2 (1879).

Mr W. F. M'Michael in the *Messenger of Mathematics*, XI. 77–8 (1882).

Mr Morgan Jenkins in *Mathematical Questions from the Educational Times*, XXXIX. 88–91 (1883); and the Rev. G. Richardson in the same volume, p. 100.

Mr William Harvey, in a letter dated September 11, 1883, communicated to me a proof which will be found in the *Proceedings of the Edinburgh Mathematical Society*, V. 102–3 (1887). In a letter, dated May 6, 1888, the Rev. G. Richardson somewhat abbreviates Mr Harvey's proof.

In the *Messenger of Mathematics*, XIII. 116–120 (1884), Mr C. Leudesdorf has an article, dated Nov. 7, 1883, and entitled "Proofs of Feuerbach's Theorem." He there discusses several of the published proofs, and shows how some of them may be simplified.

Mr Samuel Roberts in the *Messenger*, XVII. 57–60 (1887).

In Milne's *Companion to the Weekly Problem Papers*, pp. 187–8 (1888), will be found a proof by Mr R. F. Davis; another, by Mr W. S. M'Cay, occurs in M'Lelland's *Treatise on the Geometry of the Circle*, p. 183 (1891); and still another, due to Professor Purser, in Nixon's *Euclid Revised*, pp. 350–1, or in Lachlan's *Elementary Treatise on Modern Pure Geometry*, pp. 206–7 (1893).

FIRST DEMONSTRATION (Whitley, 1807).

When the base and vertical angle are given, what is the locus of the centre of the circle passing through the three centres of the circles touching one side and the prolongation of the other two sides of a plane triangle?

FIGURE 1.

Let ABC be a plane triangle, AVCW'UBU' the circumscribing circle, and I_1, I_2, I_3 the centres of the circles specified in the question.

Then by known properties the lines joining the angles A, B, C of the triangle and the centres I_1, I_2, I_3 respectively will bisect those angles, and meet in I the centre of the inscribed circle. Also the lines joining I_1, I_2, I_3 will pass through the angular points A, B, C of the triangle, and be perpendicular to AI_1, BI_2, CI_3; and if AI_1, BI_2 meet the circumscribing circle BU'CU again in U and V, and I_2I_3, I_1I_2 meet it also in U', W' respectively, then will

$$IU = I_1U, \; IV = I_2V, \; I_2U' = I_3U', \; I_1W' = I_2W'.$$

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Draw $U'O_0$, $W'O_0$ perpendicular to I_2I_3, I_1I_2 and their intersection O_0 will evidently be the centre of the circle passing through I_1, I_2, I_3. The rest being drawn as per figure, it is obvious that $W'VU'O_0$ is a parallelogram, and also that UU' is perpendicular to BC and a diameter of the circle $BU'CU$.

Now the base BC and vertical angle BAC being given, EF will be given, as will also

$$UC = UI = W'V = O_0U';$$

therefore O_0U' is given, and the point U' being given, the locus of O_0 is consequently a given circle of which U' is the centre and radius O_0U' equal to UC.

SECOND DEMONSTRATION (Poncelet, 1821).

The circle which passes through the feet of the perpendiculars let fall from the vertices of any triangle on the opposite sides passes also through the mid points of these three sides as well as through the mid points of the distances which separate the vertices from the point of concourse of the perpendiculars.

FIGURE 2.

Let X, Y, Z be the feet of the perpendiculars let fall from the vertices of the triangle ABC on the opposite sides, and let A′, B′, C′ be the mid points of these sides.

The right-angled triangles CBZ and ABX being similar, we have

$$BC : BZ = AB : BX;$$

whence, since A′ and C′ are the mid points of BC and AB,

$$BA' \cdot BX = BC' \cdot BZ;$$

that is to say, the four points A′, X, C′, Z belong to one and the same circumference.

It could be proved in a similar way that the four points A′, X, B′, Y are on one circle, as well as the four points B′, Y, C′, Z.

Now, if it were possible that the three circles in question were not one and the same circle, it would be necessary that the directions of the chords which are common to them two and two should meet in a single point; but these chords are precisely the sides of the triangle ABC, which cannot meet in the same point; therefore it is equally impossible to suppose that the three circles are different; therefore they are one and the same circle.

Let now U, V, W be the mid points of the distances HA, HB, HC which separate H, the point of concourse of the perpendiculars of the triangle ABC, from the respective vertices.

The right-angled triangles CHX, CBZ being similar, we have

$$CH : CX = CB : CZ ;$$

whence, since the points W and A′ are the mid points of the distances CH and CB

$$CW \cdot CZ = CX \cdot CA' ;$$

that is to say, the circle which passes through A′, X, Z passes also through W.

It could be proved in the same way that this circle passes through the two other points, U, V; therefore it passes at the same time through the nine points X, Y, Z, A′, B′, C′, U, V, W.

Poncelet's proof may be somewhat simplified in the following manner.*

FIGURE 2.

Because	B, Z, Y, C	are concyclic ;
therefore		$AB \cdot AZ = AC \cdot AY$;
therefore		$AC' \cdot AZ = AB' \cdot AY$;
therefore	B′, C′, Y, Z	are concyclic.
Because	A, Z, H, Y	are concyclic ;
therefore		$BA \cdot BZ = BH \cdot BY$;
therefore		$BC' \cdot BZ = BV \cdot BY$;
therefore	C′, Y, Z, V	are concyclic,
that is,	V is on the circle through B′, C′, Y, Z.	

Similarly W is on the circle through B′, C′, Y, Z.

that is B′, C′, Y, Z, V, W are concyclic.

Hence C′, A′, Z, X, W, U are concyclic.

Now there are three points C′, Z, W common to these two sets of six points;

therefore the two sets of six points lie all on one circle; and the nine points A′, B′, C′, X, Y, Z, U, V, W are concyclic.

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* One of the principal features of this simplification has been given by Mr R. D. Bohannan in the *Annals of Mathematics*, I. 112 (1884).

Note.—When the diagram of a triangle and its nine-point circle has to be constructed, it will be found convenient to begin by describing the nine-point circle; then to choose three points on its circumference for the vertices of the median triangle A′B′C′; then through A′, B′, C′ to draw parallels to B′C′, C′A′, A′B′. These parallels will intersect the circle again at the feet of the perpendiculars X, Y, Z, and will intersect each other at the vertices of the fundamental triangle ABC.

THIRD DEMONSTRATION (Feuerbach, 1822).

The radius of the circle circumscribed about the triangle XYZ, which is made by the feet of the perpendiculars in the triangle ABC, is half as large as the radius of the circle circumscribed about the triangle ABC, and its centre M bisects the distance between O, the centre of the circle ABC, and H the point of intersection of the perpendiculars.

FIGURE 3.

Join OA, OH, and draw OA′ perpendicular to BC. Through A′ and M the mid point of OH draw A′M and produce it to meet the perpendicular AX in U.

Because $MH = MO$
$\angle HMU = \angle OMA'$
$\angle MHU = \angle MOA'$
therefore $HU = OA'$ and $UM = A'M$.

But because $OA' = \text{half } AH$
therefore $HU = \text{half } AH$, and $AU = OA'$.

Again, because AU is parallel to OA′,
therefore $UA' = AO$ and $A'M = \text{half } AO$.

From M draw MD perpendicular to BC, and join MX.

Then because the straight lines HX, MD, OA′ are all parallel to each other $HM : OM = XD : A'D$;
and since $HM = OM$, therefore $XD = A'D$,
and consequently $MA' = MX$.

And since $MA' = \text{half } AO$, therefore $MX = \text{half } AO$.

Similarly $MY = \text{half } BO$, $MZ = \text{half } CO$.

Now since $AO = BO = CO$
therefore $MX = MY = MZ$

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and M, the mid point of OH, is the centre of the circle circumscribed about the triangle XYZ.

Feuerbach's proof may be simplified as follows.*

FIGURE 3.

Let ABC be a triangle, H its orthocentre, O its circumcentre.

Draw OA′ perpendicular to BC and therefore bisecting BC. Bisect AH at U ; join OA, OH, A′U.

Because AH is twice OA′, therefore AU = OA′ ;
therefore AOA′U is a parallelogram, and A′U = OA = R.

Because UH is equal and parallel to OA′,
therefore A′U and OH bisect each other at M.

Now since M is the mid point of the hypotenuse of the right-angled triangle A′XU, the circle described with M (the mid point of OH) as centre and radius equal to half A′U, that is, equal to $\frac{1}{2}$R, will pass through the three points A′, X, U.

Hence also the same circle will pass through B′, Y, V, and C′, Z, W.

FOURTH DEMONSTRATION (Davies, 1827).

PROPOSITION I.

Let ABC be any plane triangle, and let perpendiculars AX, BY, CZ be demitted from each angle upon its opposite side, and prolonged to meet the circumscribing circle in R, S, T; then if the triangle RST be formed, its angles will be bisected by the said perpendiculars.

FIGURE 4.

For, since the lines CZ, BY are perpendicular to the lines AB, AC, and the angle BAC common to the two triangles BYA, CZA, the angle ABY is equal to the angle ACZ. Hence they stand on equal arcs AS, AT. But the angles ARS, ART stand on the same two arcs, and therefore they are equal; or the angle SRT is bisected by AR.

In the same manner the angles RST, STR are proved to be bisected by BS, CT respectively.

* This simplification is given in Dr Th. Spieker's *Lehrbuch der ebenen Geometrie*, 15th ed., p. 216, or § 220 (1881).

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Cor. 1. The angles of the triangle XYZ are also bisected by the same perpendiculars.

For each side of this triangle is manifestly parallel to a corresponding side of RST.

Cor. 2. Each of the triangles AZY, BXZ, CYX is similar to the original triangle.

For the angles AXZ, ACZ being equal, their complements BXZ, BAC are equal. In like manner it may be shown that BZX is equal to ACB; and therefore the triangle BXZ is similar to ABC. And so of the others.

Or this corollary may be thus deduced:

Because BZC, BYC are right angles, a circle will pass through B, Z, Y, C; and therefore the angles AZY, AYZ are equal respectively to ACB, ABC. And so of the others.

Proposition II.

A circle described through the feet of the perpendiculars, X, Y, Z will also bisect the sides of the triangle.

Figure 5.

For, let the circle cut HA in U, HB in V, HC in W; BC in A′, CA in B′, and AB in C′.

Also, join UA′, VB′, and WC′.

Then, since A′XU, B′YV, C′ZW are right angles, the lines A′U, B′V, C′W are diameters of the circle XYZ. They also pass through the middles of the arcs YZ, ZX, XY; and are, consequently, perpendicular to the middles of the chords YZ, ZX, XY which respectively subtend those arcs. But YZ is also a chord of the semicircle BZYC; and as UA′ is a chord perpendicular to the middle of it, it passes through the centre of the semicircle, and therefore bisects the diameter BC. Hence A′ is the middle of BC.

In the same manner it may be proved that CA, AB are bisected in B′ and C′.

Proposition III.

Let H be the point of intersection of the perpendiculars AX, BY, CZ; then the distance of H from each of the angles A, B, C is bisected by the circle XYZ.

Figure 5.

For join C′V.

Then VC′ZY is a quadrilateral in a circle, and the angle BC′V is equa to the opposite angle ZYV. But ZAYH is also a quadrilateral in a circle, and therefore the angle ZYH is equal to ZAH. Hence the angle BC′V is equal to ZAH, or C′V is parallel to AH. Consequently we have

$$BV : BH = BC' : BA = 1 : 2;$$

or BH is bisected in the point V.

In like manner it appears that U and W are the middles of AH and CH.

Cor. Let O be the centre of the circumscribing circle, and the perpendiculars OA′, OB′, OC′ drawn; we shall have AH equal to twice the perpendiculars OA′, BH to twice OB′, and CH to twice OC′.

For by the above demonstration A′V, C′V are parallel to CH and AH respectively, and consequently to C′O and A′O respectively; whence C′V is equal to OA′. But C′V is half AH, or AH is equal to twice C′V, or to twice OA′.

The same reasoning applies to the other stated equalities.

Proposition IV.

Let M be the centre of the circle XYZ, O that of the circle ABC, and H the intersection of the perpendiculars; these three points M, O, H are in one straight line.

Figure 5.

For, since OA′ is parallel and equal to HU, the lines A′U, OH bisect each other in their point of intersection, or OH passes through the middle of UA′, the diameter of the circle XYZ, and therefore through its centre M.

Cor. 1. The centre M of the circle XYZ is midway between H and O.

Cor. 2. It is known that the centre of gravity of the triangle is also in HO. Whence *four* important points belonging to the triangle are in one line.

Cor. 3. The diameter of the circle XYZ is half the diameter of the circle circumscribing the triangle ABC.

For, join A′B′, B′C′, C′A′.

Then this triangle is similar to the triangle ABC, and has half its linear dimensions. Hence the diameter of a circle about A′B′C′ (viz., the circle XYZ by Proposition III.) is half the diameter of that about ABC.

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Davies's proof may be somewhat simplified in the following manner.

FIGURE 5.

Let AX, BY, CZ be the perpendiculars of the triangle ABC intersecting in H. Join YZ, ZX, XY; and let the circle circumscribed about XYZ cut BC at A′ and AX at U.

Then A′U is a diameter of the circle XYZ.

But since AX bisects the angle YXZ,
therefore U is the mid point of the arc YZ;
therefore A′U bisects perpendicularly the chord YZ.

Now the circle BZYC, whose diameter is BC, has also YZ for a chord;
therefore A′U passes through the centre of the circle BZYC,
that is, A′ is the mid point of BC.

Hence also the circle XYZ passes through the mid points of CA and AB; that is, the circle through the feet of the perpendiculars of a triangle bisects the sides of the triangle.

Now X, Y, Z are also the feet of the perpendiculars of the triangles HCB, CHA, BAH;
therefore the circle XYZ bisects the sides HA, HB, HC.

FIFTH DEMONSTRATION (Steiner, 1828).

If from any point O in the plane of the triangle ABC there be drawn OA′, OB′, OC′ respectively perpendicular to BC, CA, AB, then

$$AB'^2 + BC'^2 + CA'^2 = BA'^2 + CB'^2 + AC'^2\ ;$$

and this is the necessary and sufficient condition that the perpendiculars to BC, CA, AB at the points A′, B′, C′ should be concurrent.

FIGURE 6.

Through A′, B′, C′ the feet of the three perpendiculars let there be described a circle whose centre is M and which cuts the sides of the triangle again at X, Y, Z.

Join OM, and produce it to H so that MH is equal to MO.

Because the perpendiculars from M on the three sides of the triangle would pass through the mid points of the intercepted chords A′X, B′Y, C′Z, it follows that the perpendiculars to the three sides at the points X, Y, Z are concurrent at the point H. Hence the theorem :

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If from any point O in the plane of a triangle ABC there be drawn OA′, OB′, OC′ respectively perpendicular to BC, CA, AB, and if through A′, B′, C′ the feet of these perpendiculars there be described a circle whose centre is M and which cuts the sides again at X, Y, Z, the perpendiculars to the sides at the last three points will be concurrent at a point H such that M is the mid point of OH.

Join B′C′, YZ, OA, HA.

Then $$\angle AB'C' = \angle AZY,$$

because they stand on the same arc YC′.

But on account of the cyclic quadrilaterals B′OC′A, ZHYA

$$\angle AB'C' = \angle AOC' \quad \text{and} \quad \angle AZY = \angle AHY;$$

therefore $$\angle AOC' = \angle AHY.$$

But $$\angle B'OC' = \angle YHZ,$$

because each is supplementary to $\angle A$;

therefore $$\angle AOB' = \angle AHZ, \text{ by subtraction};$$

therefore $$\angle OAB' = \angle HAZ,$$

because these angles are complementary to the former two.

But $$\angle OAB' = \angle OC'B',$$

on account of the cyclic quadrilateral B′OC′A;

therefore $$\angle OC'B' = \angle HAZ.$$

Now since C′O is perpendicular to AZ,

therefore C′B′ is perpendicular to AH;

and the same is the case with C′A′ and BH, with A′B′ and CH.

Let a be the mid point of the chord B′C′;

then the straight line Ma will be perpendicular to B′C′, and consequently parallel to HA.

For the same reasons if b and c be the mid points of C′A′ and A′B′ respectively, the straight lines Mb and Mc will be respectively perpendicular to C′A′ and A′B′.

Hence the theorem :

If from any point O in the plane of a triangle ABC there be drawn OA′, OB′, OC′ respectively perpendicular to BC, CA, AB, and if from the vertices of the triangle other perpendiculars be drawn to B′C′, C′A′, A′B′ the sides of the triangle A′B′C′, these last three perpendiculars

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will be concurrent at one point H. And further, if from this last point there be drawn to the sides of the triangle ABC the perpendiculars HX, HY, HZ, the six points A', B', C', X, Y, Z will belong to one circle having its centre M at the mid point of OH.

From the preceding there is easily deduced the solution of the problem:

Straight lines OA, OB, OC being drawn from any point O in the plane of a triangle ABC to its three vertices, to inscribe in this triangle another triangle XYZ whose three sides YZ, ZX, XY may be respectively perpendicular to these straight lines.

It has been proved that $\angle OAC = \angle HAB$,

and since the same relation should hold good for the three vertices of the triangle ABC, therefore

$$\angle OAC = \angle HAB, \quad \angle OBA = \angle HBC, \quad \angle OCB = \angle HCA.$$

Hence the theorem:

Through any point O in the plane of a triangle ABC let there be drawn to its vertices the straight lines OA, OB, OC; if through the same vertices there be drawn three new straight lines making with the sides AB, BC, CA angles respectively equal to the angles OAC, OBA, OCB these last three straight lines will be concurrent at a point H; and if from the points O, H perpendiculars OA', OB', OC', HX, HY, HZ be drawn to BC, CA, AB, their feet A', B', C', X, Y, Z will belong all six to one circle whose centre M is the mid point of OH.

Among various particular cases we shall call attention only to the following:

Suppose the point O to be the centre of the circle circumscribed about the triangle ABC, the feet A′, B′, C′ of the perpendiculars OA′, OB′, OC′ will be the mid points of BC, CA, AB, and consequently the straight lines B′C′, C′A′, A′B′ will be respectively parallel to the sides BC, CA, AB. Now since AH is perpendicular to B′C′, it will be perpendicular also to BC, and consequently the point H will be the point of concourse of the perpendiculars drawn from the vertices of the triangle ABC to the opposite sides. Hence the theorem:

The mid points A', B', C' of the sides of a triangle ABC and X, Y, Z the feet of the perpendiculars drawn from the vertices to the

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opposite sides are six points situated on the circumference of a circle whose centre M is the mid point of the straight line which joins O the circumscribed centre to H the point of concourse of the perpendiculars of the triangle ABC. Further, the three radii OA, OB, OC are respectively perpendicular to the sides YZ, ZX, XY of the triangle XYZ ; and finally these radii are so situated that the angles OAB, OBC, OCA are respectively equal to the angles HAC, HBA, HCB, or XAC, YBA, ZCB.

On the straight line OH there exists a fourth point G (*Carnot*), the intersection of the straight lines AA′, BB′, CC′ which join the vertices of the triangle ABC to the mid points of its opposite sides, and these four points O, G, M, H are situated harmonically, that is to say, so that

$$GM : GO = HM : HO$$

which is the same as

$$1 : 2 = 3 : 6.$$

Besides, the points H, G are the centres of similitude of the two circles which have their centres at M and O ; therefore the circle which has its centre at M passes through the middle of the straight lines HA, HB, HC ; and the points X, Y, Z are respectively the mid points of the straight lines HR, HS, HT, the prolongations of HX, HY, HZ to the circumference of the circle whose centre is O.*

The circle whose centre is M possesses, in particular, this property well worthy of remark : it touches each of the four circles inscribed and escribed to the triangle ABC.

This demonstration of Steiner's contains some of the fundamental propositions relating to the subject of Isogonals.

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* Hence this theorem is easily inferred :

If on the circumference of the circle whose centre is O, four points A, B, C, D be taken arbitrarily, these four points will be, three and three, the vertices of four inscribed triangles to which will correspond four H points, four M points, and four G points. Now, these four points of each kind will belong to one circle whose radius will be equal to that of the given circle for the four H points, half of this radius for the four M points, and one third of it for the four G points. Besides, the centres of these three new circles will be with the point O harmonically situated on one straight line, as are the four points H, M, G, O ; in such a way that the centre O will be the common centre of similitude of these three new circles.

For example ·

(1) $\left.\begin{array}{l}OA,\ HA\\ OB,\ HB\\ OC,\ HC\end{array}\right\}$ are isogonals with respect to $\left\{\begin{array}{l}\angle A\\ \angle B\\ \angle C\end{array}\right.$

(2) O, H are isogonals with respect to ABC.

(3) If O, H be isogonals with respect to ABC, the mid point of OH is equidistant from the feet of the perpendiculars drawn from O, H to the sides of ABC. Or, in other words,

The projections on the sides of a triangle of two isogonal points furnish six concyclic points.

(4) If O, H be isogonals with respect to ABC, the sides of the pedal triangle corresponding to O are perpendicular to HA, HB, HC; and the sides of the pedal triangle corresponding to H are perpendicular to OA, OB, OC.

(5) If three lines drawn from the vertices of a triangle be concurrent, their isogonals with respect to the angles of the triangle are also concurrent.

(6) Since the radius of the circumcircle drawn to any vertex is isogonal to the perpendicular from that vertex to the opposite side, therefore the three perpendiculars of a triangle are concurrent.

SIXTH DEMONSTRATION (Terquem, 1842).

FIGURE 7.

Let ABC be a triangle, A′, B′, C′ the mid points of the sides, X, Y, Z the feet of the perpendiculars which intersect at H, and U, V, W the mid points of AH, BH, CH.

Join B′C′, C′A′, A′B′, C′X.

Then $$C'X = \tfrac{1}{2}AB = A'B';$$

therefore B′, C′, A′, X are the four vertices of a trapezium having equal diagonals.

This trapezium is therefore inscriptible in a circle; and therefore the three feet of the perpendiculars and the three mid points of the sides are concyclic.

If B′U be joined, it will be parallel to CZ, and therefore perpendicular to A′B′ which is parallel to AB.

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Similarly C′U is perpendicular to A′C′;
therefore the three mid points and U are the vertices of a quadrilateral inscriptible in a circle;
and therefore the nine points mentioned are on a circumference whose radius is $\frac{1}{2}$R.

SEVENTH DEMONSTRATION (Adams, 1843).

The circle described through the feet of the perpendiculars of a given triangle passes through the mid points of the sides.

FIGURE 5.

Let the circle described through X, Y, Z the feet of the perpendiculars cut BC, CA, AB at A′, B′, C′.

Then, by Carnot's theorem

$$\frac{CA'}{BA'}\cdot\frac{AB'}{CB'}\cdot\frac{BC'}{AC'}=\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ},$$

$$= -1,$$

since AX, BY, CZ are concurrent.

Now $\qquad \angle AC'B' = \angle AYZ,$

since B′, C′, Z, Y are concyclic;

and $\qquad \angle AYZ = \angle ABC,$

since B, Z, Y, C are concyclic;

therefore $\qquad \angle AC'B' = \angle ABC.$

Hence C′B′ is parallel to BC;

therefore $\qquad \dfrac{AB'}{CB'}=\dfrac{AC'}{BC'};$

therefore $\qquad \dfrac{AB'}{CB'}\cdot\dfrac{BC'}{AC'}= 1;$

therefore $\qquad \dfrac{CA'}{BA'}= -1,$

that is, A′ is the mid point of BC.

Similarly B′, C′ are the mid points of CA, AB.

[Adams does not mention the other three points through which the circle passes. They are mid points of sides of the triangles HCB, CHA, BAH, and X, Y, Z are the feet of the perpendiculars of these triangles.]

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EIGHTH DEMONSTRATION (Adams, 1846).

FIGURE 9.

Let ABC be a triangle, I, I_1, I_2, I_3 the incentre and excentres; then I is the orthocentre of the triangle $I_1I_2I_3$ and ABC is the orthic triangle.

About the triangle ABC circumscribe a circle, and let it meet AI_1 at U and I_2I_3 at U′. Join UU′, and draw ID, I_1D_1, I_2D_2, I_3D_3 perpendicular to BC.

Because AU and AU′ bisect adjacent angles at A,
therefore $\angle$ UAU′ is right;
therefore UU′ is a diameter of the circle ABC.

And because the arc BU = the arc CU,
therefore UU′ passes through A′, the mid point of BC
and is perpendicular to BC.

Again since $BD_2 = s = CD_3$,
therefore D_2 and D_3 are equidistant from A.
And since $BD = s_2 = CD_1$,
therefore D and D_1 are equidistant from A′.

Lastly, since A′ is the mid point of D_2D_3,
and since I_2D_2, I_3D_3 and U′A′ are parallel,
therefore U′ is the mid point of I_2I_3.

And since A′ is the mid point of DD_1,
and since ID, I_1D_1 and UA′ are parallel,
therefore U is the mid point of II_1.

That is, the circle ABC passes through the mid points of I_2I_3 and II_1.

Hence also it passes through the mid points of I_3I_1, II_2, and I_1I_2, II_3; in other words, it is the nine-point circle of the triangle $I_1I_2I_3$.

NINTH DEMONSTRATION (Wilkinson, 1855).

FIGURE 8.

Let ABC be the triangle; X, Y, Z the feet of the perpendiculars; A′, B′, C′ the bisections of the sides; U, V, W the bisections of the lines AH, BH, CH.

Then taking the four points U, C′, A′, W,
we have UC′ and WA′ each parallel and equal to $\frac{1}{2}$BH
also UW and C′A′ each parallel and equal to $\frac{1}{2}$AC

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hence $UWA'C'$ is a rectangle,
and the four points lie in a circle upon UA' or WC' as diameter.

Similarly $UVA'B'$ is a rectangle,
and the four points U, V, A', B' are in a circle upon UA' or VB' as diameter.

But $\angle UXA' = \angle VYB' = \angle WZC' =$ a right angle;
hence the points X, Y, Z lie in the same circle as the points U, A', V, B', W, C'.

TENTH DEMONSTRATION (Godward, 1878).

FIGURE 10.

If the following lemma be assumed:

Let H be any point within or without a circle whose centre is O, and let AR be any chord passing through H, then the locus of the mid point of HA or HR is a circle whose centre is the mid point of HO, and whose radius is half the radius of the given circle,
the characteristic property of the nine-point circle follows at once.

$$\text{For}\quad \left.\begin{matrix} X,\ Y,\ Z \\ A',\ B',\ C' \\ U,\ V,\ W \end{matrix}\right\}\ \text{are the mid points of}\ \left\{\begin{matrix} HR,\ HS,\ HT \\ HL_1,\ HL_2,\ HL_3 \\ HA,\ HB,\ HC. \end{matrix}\right.$$

FEUERBACH'S THEOREM.

The nine-point circle of a triangle touches the incircle and the three excircles.

FIRST DEMONSTRATION (J. Mention, 1850).

The late M. Richard having asked me several times for a geometrical demonstration of the contact of the nine-point circle, here is the mode* in which I arrived at it, some time ago.

I propose to draw, through the middle of one of the sides BC, a circle tangent to the system (r, r_1) or (r_2, r_3).

Let N be the foot of the interior bisector of angle A; this point is the internal centre of similitude of the system (r, r_1). Let D, D_1 be the points where (r), (r_1) touch BC; A' the middle point of BC, and X the foot of the perpendicular on BC from A.

* M. Mention's notation has been slightly changed.

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This proposition is easily proved

$$ND \cdot ND_1 = NA' \cdot NX \quad (\textit{Nouvelles Annales}, \text{ III. } 496);$$

thus the required circle passes through the foot of the perpendicular.

Now I choose the middle of the side BC because it is a point on the radical axis of each of the systems (r, r_1), (r_2, r_3), and I am brought to this special question :

To find the position of a circle tangent to two given circles and passing through a point on their radical axis.

This position is fixed very clearly by making use of a solution, as elegant as it is little known, given for the general case by M. *Cauchy* when he was a pupil of the Polytechnic School, which leads* to the following theorem :

A′ is a point on the radical axis of two circles O, O′ ; B, B′ the points of contact of one of the common tangents. The points C, C′, where the lines A′B, A′B′ cut the circles, are the points where the circle tangent to O, O′ and passing through A′ touches the circles.

The centre of the circle is situated on the perpendicular let fall from A′ on the common tangent ; and if δ denotes the distance of A′ from this tangent, its radius is equal to $t^2/2\delta$, where t is the common length of the tangents drawn from A′ to the two circles.

Hence denoting by tang (r, r_1) the common tangent to the circles (r), (r_1), and coming back to the original triangle, the perpendicular let fall from A′ on tang (r, r_1) contains the centre of the circle passing through A′ tangentially to the system (r, r_1).

But tang (r, r_1) is perpendicular to the radius of the circumscribed circle issuing from the vertex A ; therefore this perpendicular is a radius of the nine-point circle.

That is more than enough to establish the identity of the required circle and the nine-point circle.

SECOND DEMONSTRATION (1854).

The nine-point circle of a triangle touches the incircle.

FIGURE 11.

Let ABC be a triangle, H the orthocentre, I the incentre, O the circumcentre, and M, the mid point of HO, the nine-point centre.

Join IM, and draw ID, ML perpendicular to BC.

* *Correspondance sur l'École Polytechnique*, I. 193 (1804).

Through O draw UU′ a diameter of the circumcircle perpendicular to BC;
from A draw AK perpendicular to UU′.

Join AU, AU′, AO, MA′.

Then UU′ bisects BC and the arc BUC;
therefore AU bisects $\angle$ BAC, and passes through I.

Now
$$\begin{aligned} OK &= AX - OA', \\ &= AH + HX - OA' \\ &= 2OA' + HX - OA', \\ &= OA' + HX, \\ &= 2ML. \end{aligned}$$

But the triangles AKU′, IDN, having their sides mutually perpendicular, are similar;

$$\begin{aligned} &\text{therefore} & U'K : AK &= ND : ID; \\ &\text{therefore} & U'K : A'X &= ND : ID; \\ &\text{therefore} & ID \cdot U'K &= A'X \cdot ND, \\ &&&= A'D \cdot DX. \end{aligned}$$

Hence if from M a perpendicular be drawn to ID,

$$\begin{aligned} MI^2 &= (ID - ML)^2 + (A'D - A'L)^2, \\ &= (ID - \tfrac{1}{2}OK)^2 + (A'D - \tfrac{1}{2}A'X)^2, \\ &= ID^2 - ID \cdot OK + \tfrac{1}{4}OK^2 + A'D^2 - A'D \cdot A'X + \tfrac{1}{4}A'X^2, \\ &= \tfrac{1}{4}OK^2 + \tfrac{1}{4}A'X^2 + ID^2 - ID \cdot OK - A'D(A'X - A'D), \\ &= \tfrac{1}{4}OA^2 + ID^2 - ID \cdot OK - A'D \cdot DX, \\ &= \tfrac{1}{4}OA^2 + ID^2 - ID \cdot OK - ID \cdot U'K, \\ &= \tfrac{1}{4}OA^2 + ID^2 - ID \cdot OU', \\ &= \tfrac{1}{4}R^2 + r^2 - Rr, \\ &= (\tfrac{1}{2}R - r)^2; \end{aligned}$$

or, $MI = \tfrac{1}{2}R - r.$

Lastly, since the distance between the centres of the nine-point and inscribed circles is equal to the difference of their radii, therefore the nine-point circle touches the incircle.

The nine-point circle of a triangle touches the three excircles.

FIGURE 12.

Let ABC be a triangle, H the orthocentre, I_1 an excentre, O the circumcentre, and M, the mid point of HO, the nine-point centre.

Join I_1M, and draw I_1D_1, ML perpendicular to BC.

Through O draw UU′ a diameter of the circumcircle perpendicular to BC;
from A draw AK perpendicular to UU′.

Join AU, AU′, AO, MA′.

Then UU′ bisects BC and the arc BUC;
therefore AU bisects $\angle$ BAC, and passes through I_1.

Now

$$\begin{aligned} OK &= AX - OA', \\ &= AH + HX - OA', \\ &= 2OA' + HX - OA', \\ &= OA' + HX, \\ &= 2ML. \end{aligned}$$

But the triangles AKU′, I_1D_1N, having their sides mutually perpendicular, are similar;

therefore $U'K : AK = ND_1 : I_1D_1$;
therefore $U'K : A'X = ND_1 : I_1D_1$;
therefore

$$\begin{aligned} I_1D_1 \cdot U'K &= A'X \cdot ND_1, \\ &= A'D_1 \cdot D_1X. \end{aligned}$$

Hence if from M a perpendicular be drawn to I_1D_1 produced,

$$\begin{aligned} MI_1^2 &= (I_1D_1 + ML)^2 + (A'D_1 + A'L)^2, \\ &= (I_1D_1 + \tfrac{1}{2}OK)^2 + (A'D_1 + \tfrac{1}{2}A'X)^2, \\ &= I_1D_1^2 + I_1D_1 \cdot OK + \tfrac{1}{4}OK^2 + A'D_1^2 + A'D_1 \cdot A'X + \tfrac{1}{4}A'X^2, \\ &= \tfrac{1}{4}OK^2 + \tfrac{1}{4}A'X^2 + I_1D_1^2 + I_1D_1 \cdot OK + A'D_1(A'X + A'D_1), \\ &= \tfrac{1}{4}OA^2 + I_1D_1^2 + I_1D_1 \cdot OK + A'D_1 \cdot D_1X, \\ &= \tfrac{1}{4}OA^2 + I_1D_1^2 + I_1D_1 \cdot OK + I_1D_1 \cdot U'K, \\ &= \tfrac{1}{4}OA^2 + I_1D_1^2 + I_1D_1 \cdot OU', \\ &= \tfrac{1}{4}R^2 + r_1^2 + Rr_1, \\ &= (\tfrac{1}{2}R + r_1)^2; \end{aligned}$$

or, $MI_1 = \frac{1}{2}R + r_1$.

Lastly, since the distance between the centres of the nine-point and any one of the escribed circles is equal to the sum of their radii, therefore the nine-point circle touches all the excircles.

THIRD DEMONSTRATION (J. M'Dowell, 1862).

FIGURE 13.

Let ABC be a triangle, H, I, O, the orthocentre, the incentre and the circumcentre;
let AX, ID, OA′ be perpendicular to BC.

Bisect AH in U; join A′U, OH intersecting in M;
and join I with A′, M, U.

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Produce AI to N, and AX to L making UL equal to UA′, and join A′L.

Then A′ is the mid point of BC;
therefore A′U is a diameter of the nine-point circle, and equal to R.

But OH passes through the centre of the nine-point circle;
therefore M is the nine-point centre,

and $\qquad MA' = MU = \frac{1}{2}R.$

Again $\qquad \angle A'UX = \angle OAX = C - B;$

therefore $\qquad \angle A'LX$ is the complement of $\frac{1}{2}(C-B)$;

therefore $\qquad \angle LA'X = \frac{1}{2}(C-B) = NID.$

Hence the triangles NID, LA′X are similar;*

therefore $\qquad A'X : XL = ID : ND;$

therefore $\qquad ID \cdot XL = A'X \cdot ND;$

therefore $\qquad r \cdot XL = A'D \cdot DX;$

therefore $\qquad r \cdot XL + r \cdot UX = A'D \cdot DX + r \cdot UX;$

therefore $\qquad Rr = A'D \cdot DX + r \cdot UX.$

Now if from I a perpendicular be drawn to UX,

$$\begin{aligned} A'D^2 + IU^2 &= A'D^2 + DX^2 + (UX - r)^2, \\ &= A'X^2 - 2A'D \cdot DX + UX^2 - 2r \cdot UX + r^2, \\ &= A'X^2 + UX^2 - 2(A'D \cdot DX + r \cdot UX) + r^2, \\ &= R^2 - 2Rr + r^2; \end{aligned}$$

therefore $\qquad IA'^2 + IU^2 = R^2 - 2Rr + 2r^2.$

But $\qquad IA'^2 + IU^2 = 2MA'^2 + 2MI^2,$

$$= \tfrac{1}{2}R^2 + 2MI^2;$$

therefore $\qquad \frac{1}{2}R^2 + 2MI^2 = R^2 - 2Rr + 2r^2;$

therefore $\qquad MI^2 = \frac{1}{4}R^2 - Rr + r^2,$

$$= (\tfrac{1}{2}R - r)^2;$$

or, $\qquad MI = \frac{1}{2}R - r.$

Lastly, since the distance between the centres of the nine-point and inscribed circles is equal to the difference of their radii, therefore the nine-point circle touches the inscribed circle.

Suppose now I_1 to be the centre of an escribed circle, and r_1 its radius; by changing the sign of r in $MI = \frac{1}{2}R - r$

we have $\qquad MI_1 = \frac{1}{2}R + r_1;$

---

* I have supplied the proof that the triangles NID, LA′X are equiangular, and have omitted the proof that A′X·ND=A′D·DX. It should be added that Mr M'Dowell does not use the terms orthocentre, incentre, circumcentre or nine-point circle.

therefore the circle through middle points of sides and the escribed circle touch one another externally. Hence the theorem is proved, but as this last principle, viz., the change of r into $-r_1$ is not recognised by Euclid, I shall proceed to give a legitimate geometrical proof that the circle with centre M and radius $\frac{1}{2}$R also touches the three escribed circles.

First, I may remark that only *one* circle can be described through the points A′ and X touching the inscribed circle.

For suppose DX less than A′D, produce A′X through X to a point Y such that $A'Y \cdot YX = YD^2$; the tangents from Y to the inscribed circle give the points of contact of the required circle with the inscribed, but one of these points is D, and the circle through A′, X and D is therefore infinite; thus only *one finite circle* can be described through the points A′ and X to touch the inscribed circle, This circle is therefore the one with centre M and radius $\frac{1}{2}$R.

Take $A'D_1 = A'D$, then D_1 is the point of contact of the circle escribed to BC, and N is clearly a centre of similitude of this escribed circle and the inscribed circle.

By a known geometrical property

$$A'X \cdot A'N = A'D^2;$$

therefore taking away $A'N^2$ from these equals we have

$$A'N \cdot NX = D_1N \cdot ND;$$

therefore also by another known geometrical property the circle through A′ and touching the inscribed circle and the circle escribed to BC must also pass through X; but by what has just been proved, this is the circle with centre M and radius $\frac{1}{2}$R.

FOURTH DEMONSTRATION (Binder, 1872).

FIGURE 14.

If U be the mid point of the arc BC, then AU bisects not only $\angle$ BAC but also $\angle$ OAX,

because $$\angle OAU = \angle AUO = \angle UAX.$$

Let the circle with centre U and radius UC cut AU at I and I_1, then I and I_1 are the centres of the incircle and first excircle of ABC.

These circles touch BC at D and D_1 and A′ is the mid point of DD_1.

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Now $\angle BCU = \angle BAU = \angle UAC$;
therefore triangle UCN is similar to UAC,

and $$UN \cdot UA = UC^2,$$
$$= UI^2;$$
and hence $$A'N \cdot A'X = A'D^2.$$

If the radius IK of the incircle be drawn in the direction MA′, then $\angle KID = \angle OAX$, and the triangles KIN, DIN are equal and similar.

Join A′K and let it meet the incircle at T;

then $$A'K \cdot AT = A'D^2,$$
$$= A'N \cdot A'X;$$

therefore the points K, T, N, X are concyclic.

Hence $$2\angle A'TX \text{ or } 2\angle KTX = 2\angle KNX \text{ or } 2\angle KND$$
$$= 2\angle KID$$
$$= \angle A'MX;$$

therefore T lies on Feuerbach's circle.

From the similarity of the isosceles triangles A′MT and KIT, the points M, I, T lie on a straight line, and Feuerbach's circle is touched internally by the incircle.

In like manner Feuerbach's circle is touched externally by the circle whose centre is I_1, because $A'D_1{}^2 = A'D^2$, etc.

FIFTH DEMONSTRATION (J. P. Taylor, 1875).

FIGURE 15.

Let A′, C′ be the middle points of BC, AB; AN the bisector of A; AX perpendicular on BC; I centre of inscribed circle; D its point of contact with BC; D_1 the point of contact of escribed circle; A′U diameter of nine-point circle.

It is easy to prove that $A'X \cdot A'N = A'D^2$ (see M'Dowell's *Exercises on Euclid*, Art. 86). Hence if A′ be centre, and $A'D^2$ constant of inversion, the inscribed circle will invert into itself, as will also the escribed touching at D_1, since $A'D_1 = A'D$;
while the nine-point will invert into a straight line perpendicular to A′U, making therefore with BC an angle

$$= A'UX = A'C'X = BC'X - BC'A'$$
$$= 2BAX - BAC = BAX - CAX = C - B.$$

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Now NS the tangent from N to inscribed circle which also touches the escribed circle makes with BC an angle

$$= DIS = 2DIN = 2XAN = C - B.$$

Therefore this line is the inverse of the nine-point circle. And as it touches the inverse of the inscribed circle and the inverse of the escribed circle, the nine-point circle touches the inscribed and the escribed circles.

If T be the point of contact of the inscribed and nine-point circles, the tangent to the inscribed circle at that point can readily be proved to touch the nine-point circle without using the theory of inversion.

[This is done in Johnson's *Treatise on Trigonometry*, p. 139 (1889).]

SIXTH DEMONSTRATION (E. M. Langley, 1876).

FIGURE 16.

Lemma.

A′ is any fixed point; B_1C_1 a fixed straight line touching a fixed circle at P; K is any other point on B_1C_1.

If along A′K there be taken A′B′ such that

$$A'K \cdot A'B' = \text{square of tangent from D to fixed circle}$$

then B′ lies on another fixed circle touching the first and passing through A′.

Let A′P cut the first circle again in Q,
and let QR be the tangent at Q.

[R must always be taken on the opposite side of QA′ to B′, when the circle is on the opposite side of B_1C_1 to A′, and always on the same side of QA′ as B′ when the circle is on the same side of B_1C_1 as A′.]

Then $$A'K \cdot A'B' = A'P \cdot A'Q,$$
therefore B′, K, P, Q are concyclic;
therefore $\angle A'B'Q = \angle A'PK = \angle RQP$;
therefore B′ lies on a fixed circle through A′ and Q touching RQ, and therefore the first fixed circle at Q.

The nine-point circle of a triangle touches the incircle and the excircles.

Consider the incircle and that excircle which touches BC between B and C.

P. 49: For tangent from D *read* tangent from A′.

The sides of the triangle are three of the common tangents to these two circles. Let the fourth common tangent B_1NC_1 be drawn cutting AC, CB, BA in B_1, N, C_1. Let A′, B′, C′ be mid points of BC, CA, AB.

Then evidently $AB_1 = AB$, $AC_1 = AC$,
and AN is the internal bisector of $\angle BAC$;
therefore AN bisects CC_1 at right angles;
therefore A′B′ passes through W, the point where AN cuts CC_1.

If D, D_1 be the points of contact with BC,

$$A'D_1 = A'D = \tfrac{1}{2}(AB - AC) = \tfrac{1}{2}BC_1 = A'W.$$

Let A′B′, A′C′ cut B_1C_1 in K, L.

Then $\quad A'K : A'W = BC_1 : BA = A'W : A'B'$;

therefore $\quad A'K \cdot A'B' = A'W^2 = A'D^2 = A'D_1^2$.

Similarly $\quad A'L \cdot A'C' = A'D^2 = A'D_1^2$;

therefore B′, C′ lie on the circle through A′ touching the incircle and the first excircle.

If "external" be written for "internal" and

$$AB + AC \text{ for } AB - AC$$

the preceding investigation applies to the remaining excircles.

SEVENTH DEMONSTRATION (Chadu, 1879).

FIGURE 17.

ABC is a triangle; D_2, D_3 are the points of contact with BC of the excircles I_2, I_3; D_2', D_3' are the points of contact of these circles with the other exterior common tangent; N′ is the point of intersection of D_2D_3 and $D_2'D_3'$; A′ is the mid point of BC.

1°. The perpendicular AO drawn from A to $D_2'D_3'$ passes through the circumcentre of the triangle ABC.*

2°. If U be the point of intersection of the straight lines AX, A′K respectively perpendicular to BC, $D_2'D_3'$,
the circle described on A′U as diameter is the nine-point circle of the triangle ABC.

That being premised, we have

$$A'U \cdot A'K = A'X \cdot A'N'.$$

* Because BC $D_2'D_3'$ are antiparallel, $\angle BAO$ is equal to the complement of $\angle ACB$.

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But N′ being the point of intersection of the side BC and the bisector of the exterior angle A of triangle ABC,

$$A'X \cdot A'N' = \left(\frac{AB+AC}{2}\right)^2 = \frac{D_2D_3^2}{4} = A'D_2^2.$$

If $A'D_2'$ cut the circumference I_2 at L_2

$$A'L_2 \cdot A'D_2' = A'D_2^2 = A'U \cdot A'K;$$

therefore $\angle UL_2D_2'$ is right.

And since the lines I_2D_2', A′K are parallel, the circumference described on A′U as diameter is tangent at L_2 to the circumference I_2.

In the same way, the point L_3 where $A'D_3'$ cuts the circumference I_3 is the point of contact of this circumference with the nine-point circle.

In the same way again, the nine-point circle touches the incircle I and the excircle I_1.

Let D, D_1 be the points of contact with BC of the circles I, I_1; D′, D_1' the points of contact of these circles with the other interior common tangent; N the point of intersection of DD_1 and $D'D_1'$; K′ the point of intersection of A′U and $D'D_1'$.

We have $A'U \cdot A'K = A'X \cdot A'N$.

But N being the point of intersection of the side BC and the bisector of $\angle A$,

$$A'X \cdot A'N = \left(\frac{AB-AC}{2}\right)^2 = \frac{DD_1^2}{4} = A'D^2.$$

If A′D′ cut the circumference I at L,

$$A'L \cdot A'D' = A'D^2 = A'U \cdot A'K';$$

therefore $\angle ULD'$ is right.

And since the lines ID′, A′K′ are parallel, the circumference described on A′U as diameter is tangent at L to the circumference I.

In the same way, the point L_1 where $A'D_1'$ cuts the circumference I_1 is the point of contact of this circumference with the nine-point circle.

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EIGHTH DEMONSTRATION (W. Harvey, 1883).

FIGURE 18.

Of the triangle ABC, O is the circumcentre, H the orthocentre, and A′ is the mid point of BC.

OA′ produced bisects the arc BC in U; I the incentre lies on AU and is so situated that $AI \cdot IU = 2Rr$; also $\angle XAU = \angle AUO = \angle OAU$.

M, the centre of the nine-point circle, bisects the distance HO, and the circumference passes through A′, X, and K the mid point of AH. Hence M bisects both A′K and HO, and $OA' = HK = AK$; therefore A′K is parallel to OA.

MLP is a radius of the nine-point circle, bisecting the chord XA′ in L and the arc XA′ in P; ID is a radius of the incircle.

Since the arc XPA′ is bisected at P,

$$\begin{aligned} \text{therefore} \qquad \angle XA'P &= \text{half } \angle XKA', \\ &= \text{half } \angle XAO, \\ &= \angle XAU \text{ or } \angle AUO. \end{aligned}$$

Hence if through I we draw a straight line (not shown in the figure) parallel to BC to meet AX and OU, the segments of this line are respectively equal to XD and A′D, and we have by similar triangles

$$\begin{aligned} XD : IA &= LP : PA', \\ A'D : IU &= LP : PA'; \\ \text{therefore} \qquad XD \cdot A'D : AI \cdot IU &= LP^2 : A'P^2. \\ \text{But} \qquad AI \cdot IU &= 2Rr, \text{ and } A'P^2 = R \cdot LP; \\ \text{therefore} \qquad XD \cdot A'D &= 2r \cdot LP; \\ \text{hence} \qquad A'L^2 - LD^2 = XD \cdot A'D &= 2r \cdot LP. \\ \text{Now} \qquad IM^2 &= (ML - r)^2 + LD^2, \\ &= (ML - r)^2 + A'L^2 - 2r \cdot LP, \\ &= ML^2 + A'L^2 - 2r(ML + LP) + r^2, \\ &= \tfrac{1}{4}R^2 - rR + r^2; \\ \text{therefore} \qquad IM &= \tfrac{1}{2}R - r, \end{aligned}$$

and the tangency of the circles is evident.

NINTH DEMONSTRATION (G. Richardson, 1888).

FIGURE 19.

In triangle ABC, H, O, I, M are orthocentre, circumcentre incentre, nine-point centre. UU′, QQ′ are diameters of the circum-circle and nine-point circle perpendicular to BC. I is situated on

AU, and QQ′ bisects the chord A′X at L and the arc at Q. Through I a parallel to BC is drawn meeting UU′ at S and AH at T; the rest of the construction is obvious from the figure.

Since C′ is the mid point of the hypotenuse of ABX,

therefore $\angle XC'B = 180° - 2B.$

Now $\angle A'C'B = A$;

therefore $\angle XC'A' = 180° - 2B - A$;

therefore
$$\angle XQ'Q = 90° - B - \tfrac{1}{2}A = \angle BAX - \angle BAI = \angle IAX.$$

Hence triangles QQ′X, QXL, IAT, IUS are similar;

therefore
$$\frac{A'D}{BU} = \frac{IS}{IU} = \frac{QX}{QQ'} = \frac{QX}{R}$$
$$\frac{DX}{AI} = \frac{IT}{AI} = \frac{QL}{QX}.$$

But
$$\frac{BU}{2R} = \frac{BU}{UU'} = \frac{IF}{AI} = \frac{r}{AI};$$

therefore
$$\frac{A'D}{BU}\cdot\frac{DX}{AI}\cdot\frac{BU}{2R} = \frac{QX}{R}\cdot\frac{QL}{QX}\cdot\frac{r}{AI};$$

therefore
$$A'D\cdot DX = 2r\cdot QL;$$

therefore
$$\tfrac{1}{4}R^2 - MD^2 = 2r\cdot QL.$$

Now
$$MD^2 + ID^2 - MI^2 = 2ID\cdot ML;$$

therefore, by addition,
$$\tfrac{1}{4}R + r^2 - MI^2 = 2r\cdot MQ = Rr;$$

therefore
$$MI^2 = (\tfrac{1}{2}R - r)^2.$$

ADDITIONAL PROPERTIES.

(1) The nine-point circle of ABC is the nine-point circle of HCB, CHA, BAH, for it passes through the mid points of their sides; hence the circumcircles of these triangles are equal.

In naming these triangles the order of the letters is such that X is the foot of the perpendicular from the first named vertex, Y the foot of that from the second, and Z the foot of that from the third. This is a matter of much more importance than appears at first sight.

(2) If P be any point on the circumcircle of ABC, and H the

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orthocentre, the locus of the mid point of PH when P moves along the circumference is the nine-point circle.

See Godward's demonstration of the characteristic property of the nine-point circle.

(3) The nine-point circle bisects all straight lines drawn from the orthocentre H to the circumcircle ABC; hence the nine-point circle bisects all straight lines drawn from A to the circle HCB, from B to the circle CHA, from C to the circle BAH.

(4) If straight lines be drawn from the incentre or any one of the excentres of a triangle to the circumference of the circle passing through the other three centres, they will be bisected by the circumcircle.

(5) *Since the nine-point circle touches the incircle and the excircles of ABC*, it touches also the incircle and the excircles of HCB, CHA, BAH.

(6) If A′, B′, C′, the mid points of the sides of ABC, be taken as the feet of perpendiculars of a second triangle $A_1B_1C_1$, the nine-point circle of ABC will be the nine-point circle of $A_1B_1C_1$, and hence will touch another set of 16 circles.

Again take the mid points of the sides of $A_1B_1C_1$ and make them the feet of perpendiculars of a third triangle $A_2B_2C_3$. Another set of 16 circles will thus be obtained which are all touched by the nine-point circle And this process may be carried indefinitely far.

It will be found that these successive triangles $A_1B_1C_1$, $A_2B_2C_2$, and so on, approximate more and more to an equilateral triangle; and consequently that the nine-point circle of ABC will not only be the nine-point circle of the limiting triangle, but also the incircle of it.

(7) Instead of taking the mid points of the sides of ABC and making them the feet of the perpendiculars of a second triangle, take the feet of the perpendiculars of ABC and make them the mid points of the sides of a second triangle. There is thus obtained another set of 16 circles all touched by the nine-point circle; and this process also may be carried indefinitely far.

(8) Thirdly take the U, V, W points and make them either the

mid points of the sides, or the feet of the perpendiculars, of a second triangle, and other sets of circles are obtained all touched by the nine-point circle.

(9) Lastly, take a circle whose centre is O, radius R, and any point H inside it. It will be seen that H may be the orthocentre of an indefinite number of triangles inscribed in ABC. The nine-point circles of these triangles are all equal since their radii are $\frac{1}{2}$R, and their centres are at the mid point of OH; hence this indefinite number of triangles have all the same nine-point circle, and their incircles and excircles are all touched by it.

(10) If through the vertices of ABC straight lines be drawn parallel to the opposite sides, a new triangle $A_1B_1C_1$ is formed, and the nine-point circle of ABC touches the nine-point circles of the triangles A_1BC, B_1CA, C_1AB at the mid points of BC, CA, AB.

(11) If the perpendicular AX of ABC be produced to A_1 so that A_1X is equal to AX, and if through A_1 there be drawn A_1B_1 parallel to AB, and A_1C_1 parallel to AC, and these parallels meet BC at B_1, C_1, the nine-point circles of ABC, $A_1B_1C_1$ touch each other at X.

(12) If through A′, B′, C′, and U, V, W two sets of three lines are drawn parallel to the external bisectors of the angles A, B, C respectively, two new triangles will be formed having the same nine-point circle as ABC.

(13) If I′ denote the centre of any one of the excircles of the triangles ABC, HCB, CHA, BAH, the nine-point circle of ABC touches the common tangent of the circle I′ and the circle described on MI′ as diameter.

(14) If H be the orthocentre of ABC, and on HA, HB, HC three points U′, V′, W′ be taken such that

$$HU' = HA/n, \quad HV' = HB/n, \quad HW' = HC/n;$$

and on HA′, HB′, HC′ three other points A″, B″, C″ such that $HA'' = 2HA'/n$, $HB'' = 2HB'/n$, $HC'' = 2HC'/n$, then

(*a*) The lines U′A″, V′B″, W′C″ intersect on the line HO in a point M′ such that $HM' = HO/n$.

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(*b*) The six points U′, V′, W′, A″, B″, C″ lie on a circle whose centre coincides with M′, and whose radius is R/n.

(15) If A, B, C, D, E be five points on a circle, the consecutive intersections of the nine-point circles of the triangles ABC, BCD, CDE, DEA, EAB lie on another circle whose radius is one half that of the first.

(16) In triangle ABC the circles described on AG, B′C′, BC as diameters are coaxal. If G′ be a point on A′H such that $A'G' = \frac{1}{3}A'H$, and A″ a point on AA′ produced such that $AA'' = 2AA'$, the nine-point circle of the triangle is coaxal with the circles described on AG′ and HA″ as diameters.*

(17) If OA′ be produced to A_1 so that $A'A_1 = OA'$, and similar constructions be made with OB′, OC′, a new triangle $A_1B_1C_1$ is obtained of which O is the orthocentre, H the circumcentre, and the nine-point circle coincides with the nine-point circle of ABC.†

(18) If from a point P on the circumcircle of a triangle, whose orthocentre is H, perpendiculars PD, PE are drawn to two of the sides, then HP, DE intersect on the nine-point circle.‡

(19) From the ends of a diameter of a given circle perpendiculars are drawn on the sides of an inscribed triangle; the two Wallace lines thus obtained intersect at right angles on the nine-point circle of the triangle.§

(20) Let AA′, BB′, CC′ be the medians of triangle ABC, intersecting in G; let AX, BY, CZ be the perpendiculars from the vertices on the opposite sides intersecting in H; and let B′C′, YZ intersect in U; C′A′, ZX in V; and A′B′, XY in W. Then AU,

* (10)-(16).—Mr J. Griffiths in *Mathematical Questions from the Educational Times*, II. 69 (1864); III. 102, 76 (1865); IV. 60 (1865); V. 72 (1866); VII. 57-8, 76 (1867).

† (17) T. T. Wilkinson in *Mathematical Questions from the Educational Times*, VI. 25 (1866).

‡ (18) Dr C. Taylor in *Mathematical Questions from the Educational Times*, XVII. 92 (1872).

§ (19) Mr R. Tucker in *Mathematical Questions from the Educational Times* III. 58 (1865).

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BV, CW will all be perpendicular to GH; and the triangle UVW will circumscribe the triangle ABC.

Let N, P, Q be the feet of the interior bisectors of the angles A, B, C, and N′, P′, Q′ the feet of the exterior bisectors; then the six straight lines UN, VP, WQ, UN′, VP′, WQ′ pass three and three through four points which are the points of contact of the nine-point circle with the inscribed and escribed circles.*

Geometrical Note.

By R. Tucker, M.A.

If in a triangle ABC, points are taken on the sides such that

$$BP : CP = CQ : AQ = AR : BR = m : n = CP' : BP'$$
$$= AQ' : CQ' = BR' : AR'$$

then the radical axis of the circles PQR, P′Q′R′ passes through the centroid and "S." points of ABC; and if QR, Q′R′ cut in 1, RP, R′P′ in 2, PQ, P′Q′ in 3, then the equation to the circle 123 is

$$abc\Sigma a\beta\gamma = mn\Sigma a\alpha.\Sigma a\alpha\{-mna^2 + (m^2+mn+n^2)(b^2+c^2)\}.$$

Figure 20.

The points P, Q, R are given by

$$(0,\ nc,\ mb),\ (mc,\ 0,\ na),\ (nb,\ ma,\ 0),$$

i.e., P, in trilinear co-ordinates, is (0, nc sinA, mb sinA), etc.; and P′, Q′, R′ by

$$(0,\ mc,\ nb),\ (nc,\ 0,\ ma),\ (mb,\ na,\ 0).$$

It is hence evident that the pairs of triangles are concentroidal with each other and with ABC.

It is also evident that PQ′, P′Q are parallel to AB, and so on; also that P′Q, PR′ intersect on the median through A; and so on.

The triangle PQR $= (m^2 - mn + n^2)\Delta =$ the triangle P′Q′R′.

The equation to the circle PQR is

$$(m^2-mn+n^2)abc.\Sigma(a\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha.-mna^2+m^2b^2+n^2c^2),$$

and to P′Q′R′ is

$$(m^2-mn+n^2)abc.\Sigma(a\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha.-mna^2+n^2b^2+m^2c^2).$$

* (20) Rev. W. A. Whitworth in *Mathematical Questions from the Educational Times*, X. 51 (1868).

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