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REVIEW

History of the stiffness method

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SUMMARY

This paper presents a brief history of the development of the stiffness method. We start by tracing the evolution of the method to solve discrete-type problems such as trusses and frames composed of two node members. We then describe the method as it is applied to solve continuum problems modelled by finite-difference and finite-element methods. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Some aspects of the history of the stiffness method have been discussed elsewhere [1–4]. The essence of the method lies in establishing a relationship between forces applied at certain nodes of a component of a structure, and the displacements occurring there. For linear problems such a relationship can easily be presented for each structural component in matrix form. The equations governing the behaviour of the assembled system are obtained simply by summing all the forces acting at each node and equating the sum to any external force and may be written as

$$K_{ij}u_j = f_i \tag{1}$$

where K_{ij} is the stiffness coefficient for a unit displacement u_j in the j-direction for equation i, and f_i is the resultant force in the i-direction. For the assembled system the nodal displacements become the unknowns and, by imposing sufficient boundary restraints on the system, these equations can readily be solved. Generally, the system matrices are obtained by summing the contributions from each component as

$$K_{ij} = \Sigma_e k_{ij} \tag{2}$$

where k_{ij} defines the component stiffness coefficient.

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An alternative to such treatment is the use of various internal forces as the unknowns. These are chosen in such a way that the whole structure becomes statically determinate. The magnitudes of these forces are calculated from continuity requirements. This approach is known as the 'force method' of analysis; however, force methods are nowadays seldom used due to their greater complexity compared to the stiffness approach.

A similar treatment is possible for many other engineering systems where the whole assembly can be considered as a sum of individual components or 'elements'. Typical examples are electrical networks, pipe flows, and heat conduction. In what follows we shall use the terminology of structural engineering, since that was the first field of application of the stiffness method, as the name itself indicates.

2. DISCRETE PROBLEMS

In 1862 Alfred Clebsch, who was professor of theoretical physics at Göttingen, gave in his famous book *Theorie der Elasticität fester Körper* [5] an elegant and concise algorithm for the linear analysis of a moment-free 3D truss using what we now call the stiffness method. For each moment-free (or pinned) joint he introduced three unknown Cartesian displacement components. A typical structural bar has two nodes and thus links six displacement components. For a stable structural configuration the set of equations can be established using equilibrium, geometry and Hooke's law.

In 1883 Barré de Saint Venant (at the age of 86) published a French translation of Clebsch's book with extensive commentary [6]. In connection with the truss algorithm he gives the history of the stiffness method before Clebsch. He notes that Clebsch's problem harks back to an unsolvable problem—that of finding by means of rigid body mechanics a unique solution to the decomposition of a force in more than three directions. To find the unique solution the deformation of the body must be taken into account. According to Saint Venant the first example of such a solution, the decomposition of a force along three bars coplanar with the force, was given by Louis Navier in lecture notes from 1826. The example is shown by Timoshenko [1].

Saint Venant claimed that in his lecture notes from 1838 he gave 'the general principle of that method for determination of the intensities and moments of the reactions'. He also claimed that in 1843 he showed that 'for these types of problems there are always the same number of equations as unknowns when the members of the structure are bent, extended or compressed and twisted'.

2.1. Clebsch's algorithm for 3D pinned trusses

Let us return to Clebsch's algorithm. He treats a 3D bar structure with loads only at the joints connecting the bars. All joints coincide with the ends of the bars, and the bars are completely free to rotate at the joints. This means that there is no bending in the bars of the structure. Clebsch also assumes that the displacements of the joints are very small and thus linear terms will give a close description of the deformation. Hence, the relationship between the forces and the displacements remains linear. With n joints, $3n$ equations are obtained in $3n$ unknowns. In general, the structure is fixed at some nodes in such a way that it cannot be translated or rotated. These conditions reduce the number of unknown displacement components so that the set of equations has a unique solution. Clebsch's algorithm for hinged trusses leads to a large system of equations but he gives no hint as to how to solve it rationally by hand. Furthermore, most trusses have stiff joints, so the algorithm is of limited value.

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2.2. *Joint rotations—secondary moments*

If the joints between components of a structure are rigid, then clearly joint rotations can be imposed and bending moments will be transmitted. The development of general equations including moments was slow. However, the construction of railways and tall buildings provided much of the motivation for finding the necessary formulations. In particular, for simple trusses, many engineers attempted first to solve the pin-jointed behaviour and in a recalculation to add the effect of joint rotations, calling these 'secondary stresses'. Such a calculation is rather complex to perform. It was first presented in 1880 by Heinrich Manderla [7], a young assistant in Munich, for which he was awarded a special prize by the Polytechnicum.

Manderla's paper also broke new ground in proposing an iterative solution to the equations set up with joint rotations as unknowns. Manderla presented results from solutions in tabular form (a similar set was also published in 1919 by Berry [8]). This paper formed an important link in the development of solution methodologies. In 1892 Otto Mohr wrote a paper on secondary stresses [9] in which he gave a modified version of Manderla's work and made the method known to engineers.

The calculation of secondary moments is of course an approximation in which it is assumed that the presence of bending moments does not affect the displacement of nodes in co-ordinate directions. Thus, the process is only justified if bending effects are small.

2.3. *Frames with side-sway*

For rigid joined frames which occur primarily in building structures the secondary moment approach is, of course, untenable. The beginning of the 20th century saw increased use of reinforced concrete for various types of bridges and tall buildings. In such structures bending effects dominate, with purely axial forces causing insignificant nodal displacements. A method for dealing with such problems was presented in 1915 by Wilson and Maney [10] in a *Bulletin* from the University of Illinois in Urbana. They called it the 'slope deflection' method. This is fully described in the book by Matheson [11]. A similar method was published in 1914 in a book written in German by the Danish engineer Axel Bendixsen [12], of which Wilson and Maney were apparently unaware.

Both Bendixsen and Wilson and Maney considered 2D frames with side-sway. Bendixsen described a method for splitting the system of equations into small sets and thus substantially reducing the effort of computation. Nevertheless, with the splitting of the full set of equations as proposed by Bendixsen, a lot of work remained before the solution to the full set of equations could be obtained.

Bendixsen's book does not seem to have been much read, but his ideas were taken up and developed considerably by Ostenfeld in 1926 [13]. The problem still remained that very little attention had been paid to the practical hand-solution of the final set of equations.

2.4. *Successive approximations*

As already mentioned, Manderla suggested an iterative method, based on physical arguments, for solving the set of equations in unknown joint rotations which he established for the calculation of secondary moments. The slope deflection method for frames with side-sway is also well suited to the same type of solution. However, neither Bendixsen nor Wilson and Maney used iterative methods. In 1922 Čališev published a method of successive approximation for frames without side-sway [14]. Like Manderla, he treated joint rotations as unknowns. However, instead of setting up the full set of equations, he solved a series of equations with only one unknown each. He first

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locked all joints, calculated the fixed end moments and summed these for each joint. He then unlocked the joint with the highest unbalanced moment. He calculated the rotation angle and the moments that arose in the neighbour nodes and added these to the moment sums at such nodes. Repeating this procedure for all joints reduced the imbalance in the joints considerably. A second round over all joints gave further rotations. The end moments and rotations were now mostly of sufficient accuracy, according to Čališev—but if not, then residual joint moments could again be calculated and a third round undertaken.

Almost the same method was proposed by Hardy Cross [15] at the University of Illinois at Urbana in a famous paper published in 1930 and then in the *Transactions of ASCE* in 1932 [16] together with 146 pages of discussion. Cross did not, however, calculate the joint rotations. Instead he distributed the unbalanced moments at the joints in proportion to the stiffness of the connecting beams. His method has been called the 'moment distribution' method or simply the Cross method. This method gives the engineer directly what he wants, the end moment in the beams, and he can easily judge how many rounds he needs by comparing the equilibrium imbalance with his data. Cross writes that the procedure should be stopped after each distribution, however, and a check made to see that statics ($\Sigma M = 0$) is satisfied.

The Cross method is primarily designed for frames without side-sway. He showed that side-sway in multistorey buildings can be handled by giving each storey a horizontal unit displacement and then calculating the shear forces obtained in all stories. By combining them the true shear and moments can be calculated in all stories. However, Cross does not demonstrate this approach on a frame with side-sway.

Grinter, of Texas A&M College, proposed in the discussion following the paper by Cross [17] a method for multistorey frames with side-sway which he calls the method of 'successive corrections'. Grinter starts by locking all joints for rotation but leaving the frame free to translate. All fixed end moments caused by vertical and horizontal load are then calculated. Then the frame is fixed against translations and all joint moments are balanced by the Cross method. The shear force for each storey is now calculated and compared to the true one. The difference of these forces in each storey is a restraining force. Forces of opposite sign are now applied to the frame with rotation-free joints and the first procedure is repeated. After a few rounds it is found that the restraining forces can be neglected. The final moments are obtained by addition of the moments from all steps. In a later paper Grinter [18] discussed accelerating the procedure in different ways and demonstrated the algorithm on two multistorey frames.

2.5. The relaxation method

The relaxation method was developed by Southwell and his coworkers at Imperial College in London. He later used it mainly in connection with the finite-difference method for 2D problems in elasticity and other branches of mathematical physics. However, in 1935 he introduced the method in a paper for solution of 3D trusses [19], the structure Clebsch analysed theoretically, but did not indicate a feasible way to solve the large set of linear equations. Southwell and Cross motivated their methods with the advantage of not having to establish the set of equations for the whole structure and having only one unknown to solve in each step.

The 3D truss has three unknown displacement components for each joint. Southwell starts by locking all joints by imaginary constraints. He then proceeds to unlock or relax the displacement component in the joint with the largest imbalance. He calculates the forces and their components in the connecting bars and adds them to the neighbour joint constraints. He then unlocks a new joint component with the largest imbalance and so on. He thus calls the method 'systematic relaxation of

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restraints'. Slowly the imbalances go down. It is interesting to note that Southwell in essence applies the iterative process directly to the same equations as used by Čališev and treats the distribution of forces in a simple manner. In 1940 Southwell gave a clear picture of his method in book form [20] and applied it to trusses and frames as well as to electrical networks. He also used the relaxation method for solution of vibration and stability problems.

It was soon realized that groups of nodes could be released simultaneously, changing only unbalanced forces on the boundary of the group. Such block relaxation formed an essential part of an efficient iteration procedure. Of course, when applied to problems with rotational degrees of freedom, care has to be taken to rotate the whole group in a rigid body manner. In systems with only nodal displacement the convergence is relatively slow. However, in problems in which rotations are the major nodal degrees of freedom the convergence is much more rapid. This accounts for the popularity of the Cross method where such degrees of freedom are the major feature.

It should be noted that the major difference between the methodologies of Southwell and Cross is that in the former the basic equations are not lost, whereas in the Cross method they are. Thus only the Southwell method is self-checking. Indeed, the reader will recognize that Southwell's relaxation method is simply a form of the well-known Jacobi–Gauss iteration method.

2.6. Matrix formulation

In all of the previous examples we have encountered problems where linear equations have to be solved and these linear equations are derived from assembly of component elements. In the 1930s, under the it became fashionable to use matrix forms to illustrate the whole process systematically [21–23].

Similar equations occur in the establishment of electrical networks, and these follow a similar formulation to problems of structural engineering. Significant work was done in this field by the electrical engineer Gabriel Kron at General Electric. Like Kirchhoff and Maxwell before him, he exploited the close analogy between electrical and mechanical networks (3D frames). He published in 1944 a complete algorithm in matrix form for the solution of 3D frames [24]. He first considered in detail the force–displacement relation for each separate member (a 12×12 matrix), then arranged these matrices along the diagonal of a large square matrix. He regarded this as a representation of a tensor. The corresponding representation for the connected frame was obtained by a transformation that expressed the connection properties. This is a more complex way of representing Equation (1).

In structural engineering considerable advances in the use of matrices were made by Argyris [25], who again used a similar formulation to that introduced by Kron. Argyris listed all the component stiffnesses as a large diagonal matrix $\mathbf{k}$ and introduced a transformation $\mathbf{A}$ to construct the assembled matrix $\mathbf{K}$, replacing the summation shown in Equation (2) with the form

$$\mathbf{K} = \mathbf{A}^{\mathrm{T}}\mathbf{k}\mathbf{A} \qquad (3)$$

2.7. The aircraft industry and the transition to continuum problems

In the 1930s aircraft designers found they needed a detailed analysis of metal aircraft structures built from rigs, spars and panels. In the analysis described by Levy in 1937 [26], all normal stresses are assumed to be taken up by the spars and ribs and all shear stresses by the panels. The shear stress was assumed to be constant in each panel. Similar analyses were also used later by Falkenheiner [27], Lang and Bisplinghoff [28], and Langefors [29].

In these early works very crude approximations were introduced to deal with the continuum components represented by the shear panels.

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3. CONTINUUM ANALYSIS

Continua involving the solution of governing differential equations can be analysed by many different procedures. The early work in this area included the analogies between continuum behaviour and lattices of suitably defined discrete components. A method of solving problems in linear elasticity by means of a lattice analogy was presented by Hrenikoff at the University of British Colombia in Vancouver in 1941 [30] and by McHenry at the U.S. Bureau of Reclamation in 1943 [31]. Both authors show that the equations for a rectangular arrangement of trusses model the finite-difference equations of plane elasticity in displacement form. For some arrangements of the bars the Poisson ratio has to be restricted to $1/3$, but with suitable modification any value can be modelled. Clearly, the methodologies by which trusses and frames are solved could again be used to solve elasticity problems.

Hrenikoff also developed other arrangements of lattice elements in triangular and rectangular forms. He also found a lattice for a 3D body, for plates in bending and for cylindrical shells.

McHenry stated that displacement analysis has many advantages compared with the solution of the Airy stress function which had earlier dominated the field. He particularly mentioned the famous paper by Richardson from 1910 [32] in which the stresses in a dam are analysed using the Airy stress function, finite differences, and successive approximations followed by extrapolation.

3.1. Torsion problems

In many problems, such as torsion, stress functions are introduced which behave like continuous membranes. Here again it is easy to show that a membrane can be modelled by a net of tensioned strings. Such an analogy was used by Southwell in his early work on solving torsion problems. This was published in book form in 1946 [33] only after wartime restrictions on its distribution had been lifted.

Indeed, it can be shown that the string analogy represents the finite-difference equation approximating the governing differential equation. Southwell showed that this analogy extends to both rectangular and triangular meshes [33]. Relaxation methods introduced for discrete problems were used here again to solve many continuum problems. Indeed, the problem solved earlier by Richardson [32] was revisited by Zienkiewicz [34] who in 1945 succeeded in solving an almost identical problem with almost 900 variables using Southwell's relaxation methods rather than the 200 variables of Richardson.

Argyris published in 1954–1955 a series of articles on 'Energy theorems and structural analysis' giving a comprehensive presentation of both the force and the displacement method for complicated airplane structures [25]. In the April 1955 issue he considered 'rectangular flat elements of constant thickness'. Together with a frame of stiffeners he modelled the element as a shear panel with unknowns placed at the vertices. He assumed that the displacements varied bilinearly between the nodal points and determined the 8×8 stiffness matrix (k_{ij}) from unit displacements in the co-ordinate directions at the four vertices.

3.2. Direct approaches to continuum analysis—the finite-element method

In May 1941, Richard Courant gave an address before a meeting of the American Mathematical Society in Washington D.C. with the title 'Variational Methods for the Solution of Problems of Equilibrium and Vibrations'. It was published in a 1943 issue of the *Bulletin* of the

Society [35]. He discussed the question of solving variational problems numerically. As it was obviously very difficult to find continuous global functions, which satisfied the boundary conditions required by the Rayleigh–Ritz method, he suggested an alternative procedure in which he subdivided the whole domain into triangles in which the unknown function was defined by vertex parameters. This clearly pre-dated modern concepts of the finite-element method. Not being an engineer, Courant did not realize the direct connection with the structural stiffness concept so important in establishing the equations of the whole procedure. Furthermore, he was not interested in the solution methods but was satisfied in dealing with a very small number of elements. He illustrated the procedure and its convergence using a simple torsion problem with four consecutively decreasing meshes and making use of the so-called Prandtl stress function as the unknowns. In all his solutions a very small number of equations were used. Courant considered that other variational problems in which only first-order derivatives occur could be solved by a similar process.

In the laboratory of the Structural Dynamics Unit at Boeing in Seattle a 45° sweptback box beam had been tested in the laboratory in order to obtain influence coefficients needed for prediction of aeroelastic effects. In the summer of 1952 Ray Clough, from the University of California at Berkeley, joined the Boeing Summer Faculty Program. He talked about this in a book published in honour of Ivar Holand [36].

Clough was asked by the head of the Unit, M. J. Turner, to try to calculate the influence coefficients for the low-aspect box beam tested. He modelled the beam, according to Levy, as an assemblage of shear fields in the skin and direct stresses in the caps, ribs and stringers. The calculated deflections, however, exceeded the measured values by 13–65%.

When Clough returned the next summer, Turner suggested a better way to model the skin panels. He proposed a Ritz-type analysis employing a combination of simple strain fields.

The procedure for what we now call the finite-element method was published in a 1956 paper by Turner, Clough and others [37]. Three finite elements were developed, one triangular, one quadrilateral and one rectangular. The triangular element was based on three constant strain components and three rigid body displacements. The six components are uniquely expressed in the six node displacement components of the triangle. By Hooke's law the stress components can now be calculated from the nodal displacements. Finally, the boundary stresses can be replaced by equivalent nodal forces giving the stiffness matrix. This element can be seen as Courant's triangle deduced directly without use of Ritz's method and applied to a more complex problem (2D elasticity rather than torsion).

In the Turner *et al.* paper a quadrilateral element of general form was obtained by introducing an extra interior node and dividing the quadrilateral into four triangles with the interior node in common. The stiffness matrix was obtained by adding the stiffness matrices for the four triangles. The resulting 10×10 matrix could be reduced to 8×8 by static condensation. The quadrilateral element was tested for convergence on two examples, one a simple square plate and one a box beam.

Finally, a rectangular element was deduced in a similar way to the triangular element based on five stress states. The displacement field then became quadratic so with only four nodes the total test function could not be continuous, meaning that convergence was not to be expected according to the Ritz method. The element was reinvented in other ways by Pian *et al.* in 1969 [38] and by Wilson *et al.* in 1973 [39]. Strang and Fix called such elements variational crimes [40], but found that due to the fact that the energy errors at the sides between the elements summed to zero the solution with the Turner element always converged.

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4. CONCLUDING REMARKS

In this short paper we have described the main steps in the development of the stiffness method from its birth in the 1820s as a method for analysis of statically indeterminate bar structures with joint displacements as unknowns. In the late 19th century joint rotations were introduced as unknowns. In the 1930s effective solution methods for the set of linear equations were established and in the 1940s 2D elasticity problems were solved by truss analogies. The finite-element methodology was developed around 1955 from the displacement method for bar and beam structures. The method rapidly advanced following the appearance of work by Turner and coworkers, with the first comprehensive book (coauthored by the second author of the present paper) appearing in 1967 [41].

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