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HITTING AND RETURN TIMES IN ERGODIC DYNAMICAL SYSTEMS

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Given an ergodic dynamical system (X, T, μ), and $U \subset X$ measurable with $\mu(U) > 0$, let $\mu(U)\tau_U(x)$ denote the normalized hitting time of $x \in X$ to U. We prove that given a sequence (U_n) with $\mu(U_n) \to 0$, the distribution function of the normalized hitting times to U_n converges weakly to some subprobability distribution F if and only if the distribution function of the normalized return time converges weakly to some distribution function $\tilde{F}$, and that in the converging case,

$$F(t) = \int_0^t \big(1 - \tilde{F}(s)\big)\, ds, \qquad t \geq 0. \tag{$\diamond$}$$

This in particular characterizes asymptotics for hitting times, and shows that the asymptotics for return times is exponential if and only if the one for hitting times is also.

1. Introduction. In recent years there has been an interest in the statistics of hitting and return times. Typically a neighborhood of a point is considered which can be either a metric ball or a "cylinder set" associated with a measurable partition. In accordance with a theorem due to Kac one then looks at the return times which are normalized by the measure of the return set. A number of recent papers (e.g., [1–6, 9–16, 20, 21]) have provided conditions under which this distribution converges to the exponential distribution if the set is shrunk so that its measure converges to zero. On a different note, Lacroix and Kupsa have shown that with a suitable choice of return set one can realize any arbitrarily chosen limiting return time distribution [19] and hitting time distribution [18] within some class (see Theorem 1).

The purpose of this paper is to show that limiting distributions for hitting and return times are intimately linked by the transformation ($\diamond$).

Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable transformation that preserves μ, that is, $T^*\mu = \mu$. We also assume the dynamical system (X, T, μ) to be ergodic.

For (measurable) $U \subset X$ with $\mu(U) > 0$ we define the *return/hitting time function* τ_U by

$$\tau_U(x) = \inf\{k \geq 1 : T^k x \in U\}.$$

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For $x \in U$, $\tau_U(x)$ denotes the *return time*. On the other hand, if we refer to $\tau_U(x)$ as a function on all of X, then we call it the *hitting time function*. Poincaré's recurrence theorem ([17], Theorem 1′), together with ergodicity, then shows that τ_U is a.s. finite. We also have Kac's theorem ([17], Theorem 2′) according to which

$$\int_U \tau_U(x)\, d\mu(x) = \sum_{k=1}^{\infty} k\mu(U \cap \{\tau_U = k\}) = 1.$$

Finer statistical properties of the variable $\mu(U)\tau_U$ have been investigated, in a rather large number of recent papers, where particular attention was given to the study of weak convergence of $\mu(U_n)\tau_{U_n}$ as $\mu(U_n) \to 0$. See [3] for a recent survey in the mixing case.

We say a sequence of distribution functions F_n, $n = 1, 2, \dots$, *converges weakly* to a function F (which might not be a distribution itself) if F is (nonstrictly) increasing right-continuous and satisfies $\lim_{n\to\infty} F_n(t) = F(t)$ at every point t of continuity of F. We will write $F_n \Rightarrow F$ if F_n converges weakly to F.

Given a $U \subset X$ measurable with $\mu(U) > 0$, we define

$$\tilde{F}_U(t) = \frac{1}{\mu(U)} \mu\big(U \cap \{\tau_U \mu(U) \le t\}\big),$$
$$F_U(t) = \mu\big(\{\mu(U)\tau_U \le t\}\big).$$

Define

$$\mathcal{F} = \{F : \mathbb{R} \to [0, 1], F \equiv 0 \text{ on }]-\infty, 0], F \text{(nonstrictly) increasing, continuous, concave on } [0, +\infty[, F(t) \le t \text{ for } t \ge 0\},$$

$$\tilde{\mathcal{F}} = \left\{\tilde{F} : \mathbb{R} \to [0, 1], \tilde{F} \text{ (nonstrictly) increasing, right-continuous, } \tilde{F} \equiv 0 \text{ on }]-\infty, 0], \int_0^{+\infty} \big(1 - \tilde{F}(s)\big)\, ds \le 1\right\}.$$

These functional classes appear in the following (U_n is always assumed to be measurable):

THEOREM 1. *Let (X, T, μ) be an ergodic and aperiodic dynamical system. Then*:

(a) [19] *for any $\tilde{F} \in \tilde{\mathcal{F}}$ there exists a sequence $\{U_n \subset X : n = 1, 2, \dots\}$ such that $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$.*

(b) [19] *if $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$, then $\tilde{F} \in \tilde{\mathcal{F}}$.*

(c) [18] *for any $F \in \mathcal{F}$, there exists $\{U_n \subset X : n = 1, 2, \dots\}$ such that $\mu(U_n) \to 0$ and $F_{U_n} \Rightarrow F$.*

(d) [18] *if $\mu(U_n) \to 0$ and $F_{U_n} \Rightarrow F$, then $F \in \mathcal{F}$.*

In this paper we prove the following rather unexpected and surprisingly unknown result:

MAIN THEOREM. *Let* (X, T, μ) *be ergodic, and let* $\{U_n \subset X : n \geq 1\}$ *be a sequence of positive measure measurable subsets. Then the sequence of functions* $\tilde{F}_{U_n}$ *converges weakly if and only if the functions* F_{U_n} *converge weakly.*

Moreover, if the convergence holds, then

$$(\diamondsuit) \qquad F(t) = \int_0^t \bigl(1 - \tilde{F}(s)\bigr)\, ds, \qquad t \geq 0,$$

where $\tilde{F}$ *and* F *are the corresponding limiting (subprobability) distributions.*

The only previous result in this direction was obtained in [16] where it is shown that $\tilde{F}_{U_n} \to \tilde{F}$ and $\tilde{F}(t) = 1 - e^{-t}$ for $t \geq 0$ if and only if $F_{U_n} - \tilde{F}_{U_n} \to 0$ in the supremum norm on the real line. Our Main Theorem shows that the exponential distribution is the only distribution which can be asymptotic to both return and hitting times, as it is clearly the only fixed point of ($\diamondsuit$). The equation ($\diamondsuit$) also gives an equivalence between parts (a) and (b) of Theorem 1. We state a corollary in this direction:

COROLLARY 2. (i) *The asymptotic distribution for hitting times, if it exists, is positive exponential with parameter* 1 *if and only if the one for return times is also.*

(ii) *Parts* (a) *and* (b) *of Theorem* 1 *are equivalent.*

Before we prove the Main Theorem in Section 3, and Corollary 2, we present an application to irrational rotations on the torus.

2. Homeomorphisms of the circle. In the paper [8] (see also [7] for further developments), Coelho and De Faria were able to characterize the limiting laws for the distribution of hitting time for orientation-preserving homeomorphisms of the circle without periodic points, provided the set U is chosen in a descending chain of renormalization intervals. We could therefore apply our theorem to compute the corresponding distribution for the first return time.

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle without periodic points and let $\alpha = \alpha(f) \in [0, 1)$ be its irrational rotation number; we consider here its continued fraction expansion: $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$. It is well known that $a_n = [H^n(\alpha)]$, where $[\cdot]$ denotes now the integer part and $H : [0, 1[\to [0, 1[$ is the Gauss transformation defined by $H(0) = 0$ and $H(\alpha) = \{1/\alpha\}$ for $\alpha \neq 0$, denoting $\{\cdot\}$ the fractional part. The truncated expansion of order n of α is given by $p_n/q_n = [a_0, a_1, \ldots, a_{n-1}]$, where p_n and q_n verify the recursive relations

$$q_{k+1} = a_k q_k + q_{k-1},$$

$$p_{k+1} = a_k p_k + p_{k-1},$$

with $p_0 = 1$, $p_1 = 1$ and $q_0 = 0$, $q_1 = 1$. Then, for any number $\beta \in [0, 1]$ and by setting $b_j = [H^j(\beta)]$ for $j \geq 0$, we can construct the following quantities for $n \geq 1$ (natural extension of H):

$$\Gamma^n(\alpha, \beta) = \bigl(H^n(\alpha), [a_{n-1}, a_{n-2}, \ldots, a_0, b_0, b_1, \ldots]\bigr).$$

Notice that the convergent subsequences of $\Gamma^n(\alpha, \beta)$ for $n \geq 0$ do not depend on β. The sets U, where the points of the circle enter, are constructed in the following way. Let us take any point z on the circle and define J_n as the closed interval of endpoints $f^{q_{n-1}}(z)$ and $f^{q_n}(z)$; the sequence of sets U shrinking to z is taken as the descending chain of intervals J_n. We call $F_n(t)$ the distribution function of the hitting time into the sets $U = \{J_n\}$. Coelho and De Faria proved the following theorem:

THEOREM ([8]). *For each subsequence $\sigma = \{n_i\}$ of $\mathbb{N}$, the corresponding distribution functions F_{n_i} converge (pointwise or uniformly) if and only if either*:

(a) $H^{n_i}(\alpha) \to 0$, *in which case the limit distribution is the uniform distribution on the unit interval*; *or*

(b) $\Gamma(\alpha, \cdot) \to (\theta, \omega)$ *for some* $\theta > 0$ *and* $\omega < 1$, *in which case the limit distribution is the continuous piecewise linear function* F_σ *given by*

$$F_\sigma(t) = \begin{cases} t, & \text{if } 0 \leq t < \dfrac{(1+\theta)\omega}{1+\theta\omega}, \\ \dfrac{1}{1+\omega}t + \dfrac{\omega^2(1+\theta)}{(1+\theta\omega)(1+\omega)}, & \text{if } \dfrac{(1+\theta)\omega}{1+\theta\omega} \leq t < \dfrac{1+\theta}{1+\theta\omega}, \\ 1, & \text{if } t \geq \dfrac{1+\theta}{1+\theta\omega}. \end{cases}$$

For the second, more interesting case, we get immediately the distribution of the first return time by applying our result. It reads:

$$\tilde{F_\sigma}(t) = \begin{cases} 0, & \text{if } 0 \leq t < \dfrac{(1+\theta)\omega}{1+\theta\omega}, \\ \dfrac{\omega}{1+\omega}, & \text{if } \dfrac{(1+\theta)\omega}{1+\theta\omega} \leq t < \dfrac{1+\theta}{1+\theta\omega}, \\ 1, & \text{if } t \geq \dfrac{1+\theta}{1+\theta\omega}. \end{cases}$$

As a concrete example we could take $\alpha = \frac{\sqrt{5}-1}{2}$, the golden number, which exhibits the continued fraction expansion $[1, 1, 1, \ldots]$. In this case it is easy to check that $\Gamma^n(\alpha, \cdot)$ converges, when $n \to \infty$, to $(\theta = \alpha, \omega = \alpha)$.

3. Proof of the Main Theorem. For $k \geq 1$, let us denote $V_k = \{x \in U : \tau_U = k\}$. Then up to a zero measure set X is the disjoint union of the sets $\bigcup_{j=0}^{k-1} T^j V_k$, $k = 1, 2, \ldots$. (This is Kac's Theorem 1′; no invertibility of the transformation is needed.)

Let us also introduce $U_k = \{x : \tau_U(x) = k\}$. Then

$$U_k = \bigcup_{j=0}^{+\infty} T^j V_{k+j} \quad \text{and} \quad \mu(U_k) = \sum_{j \geq k} \mu(V_j).$$

Also for $t \geq 0$, $F_U(t) = \mu(\{\tau_U \leq t/\mu(U)\}) = \sum_{k \leq t/\mu(U)} \mu(U_k)$; notice that F_U is constant on intervals $[k\mu(U), (k+1)\mu(U)[$, with a jump $\mu(U_k)$ at $k\mu(U)$.

We next define $\bar{F}_U$ by:

(a) $\bar{F}_U(k\mu(U)) = F_U(k\mu(U))$,
(b) $\bar{F}_U$ is linear on $[k\mu(U), (k+1)\mu(U)]$.

Then because the discontinuity jumps of F_U decrease, the function $\bar{F}_U$ is (nonstrictly) increasing, concave on the positive axis, piecewise linear, continuous, and the right-hand side derivative satisfies

$$\bar{F}_U'^{+}(t) = \frac{\mu(U_{k+1})}{\mu(U)} \qquad \text{for } t \in [k\mu(U), (k+1)\mu(U)[.$$

On the other hand, $\tilde{F}_U(t) = \frac{1}{\mu(U)} \sum_{k \leq t/\mu(U)} \mu(V_k)$, whence

$$\tilde{F}_U \text{ is constant on } [k\mu(U), (k+1)\mu(U)[\text{ and has jump } \frac{\mu(V_k)}{\mu(U)} \text{ at } k\mu(U).$$

Putting the above together yields

$$(\star) \qquad \bar{F}_U'^{+}(t) = 1 - \tilde{F}_U(t), \qquad t \geq 0.$$

Notice also that

$$(\star\star) \qquad \|F_U - \bar{F}_U\|_\infty \leq \mu(U)$$

since F_U has its discontinuities located at points $\mu(U), 2\mu(U), \ldots$.

We continue with the proof of the Main Theorem:

(I) Let us assume there is a sequence of subsets $U_n \subset X$ so that $\mu(U_n) \to 0$ and $\tilde{F}_{U_n} \Rightarrow \tilde{F}$ (then $\tilde{F} \in \tilde{\mathcal{F}}$). Since $\tilde{F}$ is (nonstrictly) increasing, this implies that $\tilde{F}_{U_n} \to \tilde{F}$ Lebesgue almost surely on $[0, +\infty[$. Whence, for given $t \geq 0$, by the Lebesgue dominated convergence theorem on $[0, t]$ ($\tilde{F} \in [0, 1]$), combining with $(\star)$ one has

$$\bar{F}_{U_n}(t) = \int_0^t \bigl(1 - \tilde{F}_{U_n}(s)\bigr)\, ds \to \int_0^t \bigl(1 - \tilde{F}(s)\bigr)\, ds =: F(t).$$

We put $F(t) = 0$ for $t < 0$. Since $\tilde{F} \in \tilde{\mathcal{F}}$, it follows that $F \in \mathcal{F}$.

Moreover, by ($\star\star$), $F_{U_n}(t) \to F(t)$ for all $t \in \mathbb{R}$. (The convergence is in fact uniform on compact subsets of $\mathbb{R}$ by [22], Theorem 10.8.)

Hence if $\tilde{F}_{U_n} \Rightarrow \tilde{F}$, then (F_{U_n}) converges weakly to the F associated to $\tilde{F}$ by formula ($\diamond$).

Proving the converse for the Main Theorem, we need the following:

LEMMA 3. *Let f_n, $n = 1, 2, \ldots,$ be a sequence of concave functions defined on a nonempty open interval I and assume that f_n converges pointwise to a limit function f. Then off an at most countable subset of I the sequence of derivatives f_n' converges pointwise to the derivative f' of f.*

PROOF. By [22], Theorem 25.3, off an at most countable subset of I, the functions f_n, and f, are differentiable, as concave functions.

Next, using the argument for the proof of [22], Theorem 25.7, but for a fixed $x \in I$ rather than along a sequence of point x_i or points x_i in a closed bounded subset of I, the convergence of the derivatives, when all defined, follows at once. $\square$

(II) Let us now assume that $F_{U_n} \Rightarrow F$. Then [18] implies that $F \in \mathcal{F}$. Whence by ($\star$) and ($\star\star$), we have, for $t \geq 0$,

$$\bar{F}_{U_n}(t) = \int_0^t \bar{F}_{U_n}'^{+}(s)\,ds = \int_0^t \bigl(1 - \tilde{F}_{U_n}(s)\bigr)\,ds \to F(t)\biggl(= \int_0^t F'^{+}(s)\,ds\biggr).$$

It now follows from Lemma 3 that off an at most countable subset Ω of $]0, +\infty[$, the functions $1 - \tilde{F}_{U_n}(s)$ converge pointwise to $F'^{+}(s)$. We a priori have $F'^{+}(s)$ nonstrictly decreasing. We then set $\tilde{F}(s) = 1 - F'^{+}(s)$ where F'^{+} is continuous; otherwise we ask $\tilde{F}$ to be right-continuous which sets automatically its values at eventual discontinuity points of F'^{+}.

It remains to show that $\tilde{F}_{U_n}(s) \to \tilde{F}(s)$ at points s of continuity of F'^{+}. Clearly if $s \notin \Omega$ or $s < 0$, there is nothing to do. Otherwise, for any s_1 and s_2 not in Ω such that $s_1 < s < s_2$, we have

$$\tilde{F}(s_1) \leq \liminf_n \tilde{F}_{U_n}(s) \leq \limsup_n \tilde{F}_{U_n}(s) \leq \tilde{F}(s_2),$$

and since Ω is dense in $[0, +\infty[$, the conclusion follows. So $\tilde{F}_{U_n} \Rightarrow \tilde{F} = 1 - F'^{+}$, which ends the proof.

PROOF OF COROLLARY 2. (i) The proof is standard.

(ii) We have a map $\tilde{F} \mapsto F$ defined by ($\diamond$) which clearly maps $\tilde{\mathcal{F}} \to \mathcal{F}$ in a one-to-one way.

To prove this map is surjective we may proceed as follows: given $F \in \mathcal{F}$ we have for any $s \geq 0$, $F(s) = \int_0^s F'^{+}(t)\,dt$, where F'^{+} is the right-hand side derivative of F. Since $F \in \mathcal{F}$, it follows quite easily that F'^{+} is positive (nonstrictly) decreasing and takes on values in $[0, 1]$. By integrability $[F(s) \to 1$ when $s \to +\infty]$, $\lim_{+\infty} F'^{+} = 0$.

We can set $\tilde{F}$ to be equal to $1 - F'^{+}$ on $]0, +\infty[$, at continuity points of F'^{+}. We otherwise set $\tilde{F} \equiv 0$ on $]-\infty, 0]$, and we eventually complete the definition of $\tilde{F}$ on $]0, +\infty[$ by requiring that it is right-continuous.

Then one has $\tilde{F} \mapsto F$ by $(\diamond)$, and $\tilde{F} \in \tilde{\mathcal{F}}$. $\square$

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