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# Hitting probabilities for the Greenwood model and relations to near constancy oscillation

M. MÖHLE

*Mathematisches Institut, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. E-mail: martin.moehle@uni-tuebingen.de*

We derive some properties of the Greenwood epidemic Galton–Watson branching model. Formulas for the probability $h(i, j)$ that the associated Markov chain $X$ hits state $j$ when started from state $i \geq j$ are obtained. For $j \geq 1$, it follows that $h(i, j)$ slightly oscillates with varying $i$ and has infinitely many accumulation points. In particular, $h(i, j)$ does not converge as $i \to \infty$. It is shown that there exists a Markov chain $Y$ which is Siegmund dual to the chain $X$. The hitting probabilities of the dual Markov chain $Y$ are investigated.

*Keywords:* branching process; duality; epidemic model; Greenwood model; hitting probabilities; leader election; near constancy oscillation

## 1. Introduction and results

Let $\mathbb{N}_0 := \{0, 1, \ldots\}$. Fix $p \in (0, 1)$ and let $X = (X_k)_{k\in\mathbb{N}_0}$ be a Markov chain with state space $\mathbb{N}_0$ and lower left triangular transition matrix $P = (p_{ij})_{i,j\in\mathbb{N}_0}$ having binomial entries

$$p_{ij} = \binom{i}{j} p^j q^{i-j}, \qquad i, j \in \mathbb{N}_0, i \geq j, \tag{1}$$

where $q := 1 - p$. One may interpret $X_k$ as the size of some population in generation $k$. At each time step each individual survives (independently of the other individuals) with success probability $p$. In the epidemics literature (see, for example, [1,7] or [22], page 78), this model is known as the Greenwood model [9]. Obviously, $p_{ij} = \mathbb{P}(K_i = j)$, where $K_i := \sum_{k=1}^{i} \nu_k$ and $\nu_1, \nu_2, \ldots$ are i.i.d. Bernoulli random variables with $\mathbb{P}(\nu_1 = 1) = p$ and $\mathbb{P}(\nu_1 = 0) = q$. Thus, $X$ is a subcritical Bienaymé–Galton–Watson branching process with Bernoulli offspring distribution $q\delta_0 + p\delta_1$. For general information on branching processes we refer the reader to [3,12] and [14]. Note however that in the literature (see, for example, [3], page 3, [11], page 475, or [12], page 5) the relatively simple situation without true branching when the offspring variable $\nu_1$ satisfies $\mathbb{P}(\nu_1 \leq 1) = 1$ is often excluded. Alternatively, the model may be viewed as a game where players throw coins. Players throwing heads (with probability $q$) are losers and have to leave the game; those who throw tails (with probability $p$) remain in the game and flip their coins again. The random variable $X_k$ counts the number of remaining players in the game after $k$ rounds. A slight modification of this game, where rounds in which all remaining players throw heads are disregarded, is called the leader election model [5]. Clearly, the state 0 is absorbing. For some information on the absorption time of the process $X$ we refer the reader to the remark after the proof of Proposition 1.1. Given the process starts in state $i \in \mathbb{N}_0$, the extinction probability is $\mathbb{P}(X_k = 0 \text{ for some } k \in \mathbb{N}_0 | X_0 = i) = 1$. More generally, we are interested in the hitting probabilities

$$h(i, j) := \mathbb{P}(X_k = j \text{ for some } k \in \mathbb{N}_0 | X_0 = i)$$

that the chain $X$, started with initially $i \in \mathbb{N}_0$ individuals, ever visits state $j \in \mathbb{N}_0$. Note that (see, for example, Norris [18], page 145) the hitting probabilities $h(i, j)$ are related to the Green matrix $G$ of the Markov chain $X$. Clearly, $h(i, i) = 1$ for all $i$ and $h(i, j) = 0$ for all $i < j$, since the transition matrix of the Markov chain $X$ is lower left triangular. We can therefore assume without loss of generality that $i > j$ in the following proposition.

**Proposition 1.1 (Hitting probabilities, part 1).** *For the Greenwood model with parameter* $p \in (0, 1)$, *the Markov chain* $X$ *has hitting probabilities* $h(i, 0) = 1$, $i \in \mathbb{N}$, *and*

$$h(i, j) = \left(1 - p^j\right)\binom{i}{j} \sum_{k=j}^{i} \binom{i-j}{k-j} \frac{(-1)^{k-j}}{1 - p^k}, \qquad i, j \in \mathbb{N}, i > j. \tag{2}$$

Proposition 1.1 is useful to compute $h(i, j)$ for moderate values of $i - j$. It is reasonable to ask how the hitting probability $h(i, j)$ behaves when $i$ becomes large. Formula (2) however does not seem to be very helpful to answer this question directly. We therefore provide alternative formulas for $h(i, j)$, which will turn out to be useful to understand the behavior of $h(i, j)$ as $i$ varies. For $n \in \mathbb{N}$ let $M_n$ denote the maximum of $n$ i.i.d. geometric random variables $G_1, \ldots, G_n$ with distribution $\mathbb{P}(G_1 = k) = qp^{k-1}$, $k \in \mathbb{N}$.

**Theorem 1.2 (Hitting probabilities, part 2).** *For the Greenwood model with parameter* $p \in (0, 1)$, *the Markov chain* $X$ *has hitting probabilities*

$$h(i, j) = \left(1 - p^j\right)\binom{i}{j} \sum_{k=1}^{\infty} \left(1 - p^k\right)^{i-j} p^{jk} \tag{3}$$

$$= \binom{i}{j} \sum_{k=1}^{\infty} \left(\left(1 - p^k\right)^{i-j} - \left(1 - p^{k-1}\right)^{i-j}\right) p^{jk} \tag{4}$$

$$= \binom{i}{j} \mathbb{E}\left(p^{jM_{i-j}}\right), \qquad i, j \in \mathbb{N}_0, i > j. \tag{5}$$

***Remark.*** A formula for the generating function $\phi_j(z) := \sum_{i=j+1}^{\infty} h(i, j) z^i / i!$, $z \in \mathbb{C}$, is provided in (14). For some information on (hitting probabilities for) the more general class of $\theta$-linear fractional branching processes we refer the reader to the recent work [10].

Let us now turn to the behavior of $h(i, j)$ as $i$ varies. In the literature, several examples of Markov chains are known (see, for example, [8], pages 85–86, and [17], Theorem 1.1) where the associated hitting probability $h(i, j)$ converges as $i \to \infty$.

Let us start with the following heuristics. Fix $j \in \mathbb{N}$ and define $\mu := -\log p \in (0, \infty)$. The sum on the right-hand side in (3) is approximately equal to $\int_0^\infty p^{jt}(1-p^t)^{i-j}\,\mathrm{d}t = \mu^{-1}\int_0^1 x^{j-1}(1-x)^{i-j}\,\mathrm{d}x = \mu^{-1}B(j, i-j+1)$, where $B$ denotes the beta function. Multiplying with $(1-p^j)\binom{i}{j}$ it follows that $h(i, j)$ can be roughly approximated by $(1-p^j)\binom{i}{j}\mu^{-1}B(j, i-j+1) = \mu^{-1}(1-p^j)/j$, a value which surprisingly does not depend on $i$ anymore. It is hence tempting to conjecture that $h(i, j)$ converges to $\mu^{-1}(1-p^j)/j$ as $i \to \infty$. The following corollary shows that this is not the case for the Greenwood model. In fact (see the remark after the proof of Corollary 1.3), $h(i, j)$ slightly oscillates around the value $\mu^{-1}(1-p^j)/j$ as $i$ varies. The Greenwood model therefore serves as a counterexample, where $h(i, j)$ does not converge as $i \to \infty$, and the proof of this oscillating behavior is one of the main contributions of this article. For more information on (near constancy) oscillation related to truncation of continuous random variables, we refer the reader to Janson [15] and the references therein. Similar oscillatory effects are known for the (expected) duration of the leader election algorithm (see, for example, Prodinger [20] or Fill *et al*. [5], Theorem 2).

**Corollary 1.3 (Asymptotic behavior of the hitting probabilities).** *Fix $p \in (0, 1)$ and $j \in \mathbb{N}$. For every $\alpha \in [0, 1]$ there exists a subsequence $(i_l)_{l\in\mathbb{N}}$ (depending on $p$, $j$ and $\alpha$) such that*

$$\lim_{l\to\infty} h(i_l, j) = \frac{1}{j!}\sum_{n\in\mathbb{Z}}\left(e^{-p^{n-\alpha}} - e^{-p^{n-1-\alpha}}\right)p^{j(n-\alpha)} = \frac{1}{j!}\mathbb{E}\left(p^{j(Z-\alpha)}\right),$$

*where $Z = Z(p, \alpha)$ is an integer valued random variable with distribution function $\mathbb{P}(Z \le n) = e^{-p^{n-\alpha}}$, $n \in \mathbb{Z}$. In particular, the sequence $(h(i, j))_{i\ge j}$ has infinitely many accumulation points and, hence, does not converge as $i \to \infty$.*

***Remark (Discrete Gumbel distribution).*** Define $\mu := -\log p \in (0, \infty)$. If $G$ is standard Gumbel distributed with distribution function $x \mapsto e^{-e^{-x}}$, $x \in \mathbb{R}$, then the truncated random variable $Z := \lfloor G/\mu + \alpha\rfloor + 1$ has distribution function $\mathbb{P}(Z \le n) = \mathbb{P}(G < \mu(n-\alpha)) = e^{-e^{-\mu(n-\alpha)}} = e^{-p^{n-\alpha}}$, $n \in \mathbb{Z}$, so the random variable $Z$ in Corollary 1.3 has a kind of discrete Gumbel distribution. Note that $Z - \alpha = \lfloor G/\mu + \alpha\rfloor - \alpha + 1$ coincides with the random variable $X_\alpha$ studied in Example 2.7 of [15] with $c := 1/\mu$. If $U$ is uniformly distributed on $[0, 1]$ and independent of $X := G/\mu$, then $X_U \stackrel{d}{=} X + U$.

**Proposition 1.4 (Resolvent).** *Let $\alpha \in (0, 1)$. The resolvent $R_\alpha := (I - \alpha P)^{-1}$ of the Greenwood model is a lower left triangular matrix having entries*

$$r_\alpha(i, j) = \binom{i}{j}\sum_{k=j}^{i}\binom{i-j}{k-j}\frac{(-1)^{k-j}}{1-\alpha p^k}, \qquad i, j \in \mathbb{N}_0, i \ge j. \tag{6}$$

We now turn to the question of the existence of a Markov chain $Y$ which is dual to the chain $X$. Dual Markov processes often arise naturally when considering some random phenomenon forwards and backwards in time. Typical examples are for instance known from the physics literature on interacting particle systems and from mathematical population genetics. Duality has been proven to be a powerful tool in order to analyze the underlying processes. For some general information on duality of Markov processes, we refer the reader to the book of Liggett [16], Chapter II, Section 3.

Recall that $K_i := \sum_{k=1}^{i} \nu_k$, where $\nu_1, \nu_2, \dots$ are i.i.d. Bernoulli random variables with $\mathbb{P}(\nu_1 = 1) = p$ and $\mathbb{P}(\nu_1 = 0) = q$. For every $j \in \mathbb{N}_0$ the map $i \mapsto \mathbb{P}(X_{k+1} \le j | X_k = i) = \mathbb{P}(K_i \le j)$ is non-increasing in $i \in \mathbb{N}_0$, that is, $X$ is stochastically monotone. Moreover, the latter expression tends to 0 as $i \to \infty$. Via Siegmund duality [21] it follows that there exists a Markov chain $Y = (Y_k)_{k\in\mathbb{N}_0}$ with state space $\mathbb{N}_0$ which is dual to $X$ with respect to the duality kernel $H : \mathbb{N}_0^2 \to \{0, 1\}$ defined via $H(i, j) := 1$ if $i \le j$ and $H(i, j) := 0$ otherwise. We have

$$\mathbb{P}(Y_{k+1} \ge i | Y_k = j) = \mathbb{P}(X_{k+1} \le j | X_k = i) = \mathbb{P}(K_i \le j), \qquad i, j \in \mathbb{N}_0.$$

Thus, the chain $Y$ has transition probabilities

$$\begin{aligned} \pi_{ij} &:= \mathbb{P}(Y_{k+1} = j | Y_k = i) = \mathbb{P}(Y_{k+1} \ge j | Y_k = i) - \mathbb{P}(Y_{k+1} \ge j + 1 | Y_k = i) \\ &= \mathbb{P}(K_j \le i) - \mathbb{P}(K_{j+1} \le i) = \mathbb{P}(K_j \le i, K_{j+1} > i) \\ &= \mathbb{P}(K_j = i, K_{j+1} = i + 1) \\ &= \mathbb{P}(K_{j+1} = i + 1 | K_j = i)\mathbb{P}(K_j = i) = p p_{ji} = \binom{j}{i} p^{i+1} q^{j-i}, \qquad i, j \in \mathbb{N}_0. \end{aligned} \tag{7}$$

Note that similar computations are also valid for more general stochastic processes $(K_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{N}_0$ satisfying $K_{n+1} - K_n \in \{0, 1\}$ for all $n \in \mathbb{N}_0$. For more details, we refer the reader to the Appendix. Conditional on $Y_k = i$, $Y_{k+1} + 1$ is Pascal distributed with parameters $i + 1$ and $p$, i.e. $Y_{k+1} + 1$ counts the total number of trials in a Bernoulli sequence with success parameter $p$ until you have obtained $i + 1$ successes. The chain $Y = (Y_k)_{k\in\mathbb{N}_0}$ can be pathwise constructed (see, for instance, [2], Example 9) from $Y_0$ and a family $\{\nu_j^{(k)} : j \in \mathbb{N}, k \in \mathbb{N}_0\}$ of independent and identically distributed (i.i.d.) Bernoulli variables with success parameter $p$ via the recursion $Y_{k+1} := \inf\{j \in \mathbb{N}_0 : \nu_1^{(k)} + \cdots + \nu_{j+1}^{(k)} > Y_k\}$, $k \in \mathbb{N}_0$. The transition matrix $\Pi := (\pi_{ij})_{i,j\in\mathbb{N}_0}$ of $Y$ is upper right triangular. Note that $\mathbb{E}(Y_{k+1} | Y_k = i) = (i + 1)/p - 1 \ge i$ for all $i \in \mathbb{N}_0$. Thus, $Y$ is a submartingale. Since the process $Y$ does not jump downwards it returns to the state $i \in \mathbb{N}_0$ in $k$ steps with probability $\pi_{ii}^{(k)} = \pi_{ii}^k = (p^{i+1})^k = p^{(i+1)k}$. In particular, $\sum_{k=0}^{\infty} \pi_{ii}^{(k)} = 1/(1 - p^{i+1}) < \infty$. Thus, $Y$ is transient. Again, we are interested in the hitting probabilities

$$\tilde{h}(i, j) := \mathbb{P}(Y_k = j \text{ for some } k \in \mathbb{N}_0 | Y_0 = i), \qquad i, j \in \mathbb{N}_0.$$

We call $\tilde{h}(i, j)$ the dual hitting probabilities. Clearly, $\tilde{h}(i, i) = 1$ for all $i$ and $\tilde{h}(i, j) = 0$ for all $i > j$, since $\Pi$ is upper right triangular. We can therefore assume without loss of generality that $i < j$. The following proposition provides formulas for $\tilde{h}(i, j)$ for $i < j$.

**Theorem 1.5 (Hitting probabilities of the dual chain Y).** *For the Greenwood model with parameter $p \in (0,1)$, the Siegmund dual Markov chain $Y$ has hitting probabilities*

$$\tilde{h}(i,j) = \big(1-p^{j+1}\big)\binom{j}{i}\sum_{k=i}^{j}\binom{j-i}{k-i}\frac{(-1)^{k-i}}{1-p^{k+1}} \tag{8}$$

$$= \big(1-p^{j+1}\big)\binom{j}{i}\sum_{k=1}^{\infty} p^{(i+1)k}\big(1-p^k\big)^{j-i} \tag{9}$$

$$= \binom{j}{i}\sum_{k=1}^{\infty} p^{(i+1)k}\Big(\big(1-p^k\big)^{j-i} - \big(p-p^k\big)^{j-i}\Big), \qquad i,j \in \mathbb{N}_0, i<j. \tag{10}$$

***Remark.*** A formula for the generating function $\psi_i(z) := \sum_{j=i+1}^{\infty} \tilde{h}(i,j) z^j/j!$, $z \in \mathbb{C}$, is provided in (15).

In order to understand the behavior of $\tilde{h}(i,j)$ as $j$ varies let us start with the following heuristics. The sum on the right-hand side in (9) is approximately equal to $\int_0^\infty p^{(i+1)t}(1-p^t)^{j-i}\,\mathrm{d}t = \mu^{-1}\int_0^1 x^i(1-x)^{j-i}\,\mathrm{d}x = \mu^{-1}B(i+1, j-i+1)$. Thus, $\tilde{h}(i,j)$ can be roughly estimated by $(1-p^{j+1})\binom{j}{i}\mu^{-1}B(i+1,j-i+1) = \mu^{-1}(1-p^{j+1})/(j+1)$. For large $j$ a reasonable approximation for $\tilde{h}(i,j)$ is $\mu^{-1}/j$. A similar argument (see the remark after the proof of Corollary 1.6) as provided for $h(i,j)$ shows however that $j\tilde{h}(i,j)$, seen as a function of $j$, oscillates around $\mu^{-1}$. A precise statement is provided in the following corollary.

**Corollary 1.6 (Asymptotics of the dual hitting probabilities).** *Fix $p \in (0,1)$ and $i \in \mathbb{N}_0$. For every $\beta \in [0,1]$ there exists a subsequence $(j_l)_{l\in\mathbb{N}}$ (depending on $p$, $i$ and $\beta$) such that*

$$\lim_{l\to\infty} j_l \tilde{h}(i, j_l) = \frac{1}{i!}\sum_{n\in\mathbb{Z}} e^{-p^{n-\beta}} p^{(i+1)(n-\beta)}.$$

*In particular, the sequence $(j\tilde{h}(i,j))_{j\ge i}$ has infinitely many accumulation points and, hence, does not converge as $j \to \infty$.*

We finish the results with the following proposition.

**Proposition 1.7 (Resolvent of the dual chain Y).** *Let $\alpha \in (0,1)$. The resolvent $\tilde{R}_\alpha := (I - \alpha\Pi)^{-1}$ of the dual chain $Y$ is an upper right triangular matrix with entries*

$$\tilde{r}_\alpha(i,j) = \binom{j}{i}\sum_{k=i}^{j}\binom{j-i}{k-i}\frac{(-1)^{k-i}}{1-\alpha p^{k+1}}, \qquad i,j \in \mathbb{N}_0, i \le j.$$

## 2. Proofs concerning the Markov chain X

**Proof of Proposition 1.1.** It is readily checked that the transition matrix $P$ with entries (1) has spectral decomposition $P = RDL$, where $D$ is the diagonal matrix with entries $d_{ii} := p^i$ and $R = (r_{ij})_{i,j\in\mathbb{N}_0}$ and $L = (l_{ij})_{i,j\in\mathbb{N}_0}$ have entries $r_{ij} := \binom{i}{j}$ and $l_{ij} := (-1)^{i-j}\binom{i}{j}$.

The Markov chain $X$ has Green matrix (see, for example, Norris [18], page 145) $G = \sum_{n=0}^{\infty} P^n = \sum_{n=0}^{\infty}(RDL)^n = R(\sum_{n=0}^{\infty} D^n)L$. Thus, for $i, j \in \mathbb{N}$ with $i \geq j$, the $(i, j)$-entry of $G$ is

$$\begin{aligned} g(i,j) &= \sum_{k=j}^{i} r_{ik} \sum_{n=0}^{\infty} d_{kk}^n l_{kj} = \sum_{k=j}^{i} r_{ik} \frac{1}{1-p^k} l_{kj} \\ &= \sum_{k=j}^{i} \binom{i}{k} \frac{1}{1-p^k} (-1)^{k-j} \binom{k}{j} = \binom{i}{j} \sum_{k=j}^{i} \binom{i-j}{k-j} \frac{(-1)^{k-j}}{1-p^k}. \end{aligned}$$

Multiplying $g(i, j)$ with $1 - f_j$, where $f_j = p_{jj} = p^j$ is the return probability for $j$, we obtain (see, for example, Norris [18], page 145) the formula for $h(i, j)$ for $i, j \in \mathbb{N}$ with $i \geq j$. Clearly, $h(i, 0) = 1$ for all $i \in \mathbb{N}_0$, since the state 0 is absorbing. □

***Remark (Absorption time).*** The spectral decomposition $P = RDL$ used in the previous proof is helpful to derive the distribution of the absorption time $\tau_n := \inf\{m \in \mathbb{N}_0 : X_m = 0, X_0 = n\}$ of the Markov chain $X$. Since $P^m = (RDL)^m = RD^mL$ it follows that $\tau_n$ has distribution function $\mathbb{P}(\tau_n \leq m) = \mathbb{P}(X_m = 0|X_0 = n) = (P^m)_{n0} = (RD^mL)_{n0} = \sum_{k=0}^{n} r_{nk} d_{kk}^m l_{k0} = \sum_{k=0}^{n} \binom{n}{k} (p^k)^m (-1)^k = (1 - p^m)^n$, $m \in \mathbb{N}_0$. Thus, $\tau_n$ has the same distribution as the maximum of $n$ i.i.d. random variables $G_1, \ldots, G_n$ with distribution $\mathbb{P}(G_1 = k) = qp^{k-1}$, $k \in \mathbb{N}$, which is also clear from the model, because $G_i$ can be interpreted as the time of death of the $i$th individual, $1 \leq i \leq n$. Note that $\tau_n$ has mean $\mathbb{E}(\tau_n) = \sum_{k=0}^{\infty} \mathbb{P}(\tau_n > k) = \sum_{k=0}^{\infty}(1 - (1 - p^k)^n) \sim \int_0^{\infty}(1 - (1 - p^u)^n)\,\mathrm{d}u = \mu^{-1} \int_0^1 (1 - (1 - x)^n)/x\,\mathrm{d}x = \mu^{-1} h_n \sim \mu^{-1}\log n$ as $n \to \infty$, where $\mu := -\log p \in (0, \infty)$ and $h_n := \sum_{i=1}^{n} 1/i$ denotes the $n$th harmonic number, $n \in \mathbb{N}$. For more information on the (mean of the) maximum of $n$ i.i.d. geometric random variables, we refer the reader to [4].

More generally, for $k \in \{0, \ldots, n\}$, let $\tau_{nk} := \inf\{m \in \mathbb{N}_0 : X_m \leq k, X_0 = n\}$ denote the first time the chain $X$, started in state $n$, jumps to a state smaller than or equal to $k$. Note that $\tau_{n0} = \tau_n$. For all $m \in \mathbb{N}_0$,

$$\begin{aligned} \mathbb{P}(\tau_{nk} \leq m) = \mathbb{P}(X_m \leq k|X_0 = n) &= \sum_{j=0}^{k} (P^m)_{nj} \\ &= \sum_{j=0}^{k} \sum_{i=j}^{n} r_{ni} d_{ii}^m l_{ij} = \sum_{j=0}^{k} \sum_{i=j}^{n} \binom{n}{i} (p^i)^m (-1)^{i-j} \binom{i}{j} \qquad (11) \\ &= \sum_{j=0}^{k} \binom{n}{j} (p^m)^j \sum_{i=j}^{n} \binom{n-j}{i-j} (-p^m)^{i-j} = \sum_{j=0}^{k} \binom{n}{j} (p^m)^j (1 - p^m)^{n-j}. \end{aligned}$$

Thus, $\tau_{nk}$ has distribution function $\mathbb{P}(\tau_{nk} \le m) = \mathbb{P}(K_{nm} \le k)$, where $K_{nm}$ is binomially distributed with parameters $n$ and $p^m$. Clearly, $\lim_{n\to\infty} \mathbb{P}(\tau_{nk} \le m) = 0$ for all $k, m \in \mathbb{N}_0$. By the central limit theorem, for large $n$ the probability $\mathbb{P}(\tau_{nk} \le m)$ can be approximated by $\Phi((k - np^m)/\sqrt{np^m(1-p^m)})$, where $\Phi$ denotes the distribution function of the standard normal distribution.

**Proof of Theorem 1.2.** Two proofs are presented. The first proof exploits the formula (2) for the hitting probability $h(i, j)$ whereas the second proof is based on generating functions.

*Alternative* 1. Without loss of generality, assume that $1 \le j < i$. Rewriting the sum on the right-hand side of (2) as

$$\sum_{k=j}^{i} \binom{i-j}{k-j} \frac{(-1)^{k-j}}{1-p^k} = \sum_{l=0}^{i-j} \binom{i-j}{l} \frac{(-1)^l}{1-p^{l+j}} = \sum_{l=0}^{i-j} \binom{i-j}{l} (-1)^l \sum_{k=0}^{\infty} p^{(l+j)k}$$
$$= \sum_{k=0}^{\infty} p^{jk} \sum_{l=0}^{i-j} \binom{i-j}{l} (-p^k)^l = \sum_{k=1}^{\infty} p^{jk} (1-p^k)^{i-j}$$

it follows from (2) that

$$h(i, j) = (1-p^j) \binom{i}{j} \sum_{k=1}^{\infty} (1-p^k)^{i-j} p^{jk}$$
$$= \binom{i}{j} \sum_{k=1}^{\infty} (1-p^k)^{i-j} p^{jk} - \binom{i}{j} \sum_{k=1}^{\infty} (1-p^k)^{i-j} p^{j(k+1)}$$
$$= \binom{i}{j} \sum_{k=1}^{\infty} (1-p^k)^{i-j} p^{jk} - \binom{i}{j} \sum_{k=2}^{\infty} (1-p^{k-1})^{i-j} p^{jk}$$
$$= \binom{i}{j} \sum_{k=1}^{\infty} \big((1-p^k)^{i-j} - (1-p^{k-1})^{i-j}\big) p^{jk},$$

which is (4).

*Alternative* 2. The following proof is based on generating functions and does not make use of (2). It exploits similar ideas as in [17]. For $j \in \mathbb{N}_0$ define the generating function $\phi_j : \mathbb{C} \to \mathbb{C}$ via $\phi_j(z) := \sum_{i=j+1}^{\infty} h(i, j) z^i / i!$. Note that $|\phi_j(z)| \le \sum_{i=j+1}^{\infty} |z|^i / i! \le e^{|z|} < \infty$. We have (see Norris [18], page 13, Theorem 1.3.2)

$$\phi_j(z) = \sum_{i=j+1}^{\infty} h(i, j) \frac{z^i}{i!} = \sum_{i=j+1}^{\infty} \sum_{k=j}^{i} p_{ik} h(k, j) \frac{z^i}{i!} = \sum_{i=j+1}^{\infty} p_{ij} \frac{z^i}{i!} + \sum_{i=j+1}^{\infty} \sum_{k=j+1}^{i} p_{ik} h(k, j) \frac{z^i}{i!},$$

since $h(j, j) = 1$. The first sum on the right-hand side above reduces to

$$\sum_{i=j+1}^{\infty} p_{ij} \frac{z^i}{i!} = \sum_{i=j+1}^{\infty} \binom{i}{j} q^{i-j} p^j \frac{z^i}{i!} = \frac{(pz)^j}{j!} \sum_{i=j+1}^{\infty} \frac{(qz)^{i-j}}{(i-j)!} = \frac{(pz)^j}{j!} \big(e^{qz} - 1\big).$$

Concerning the remaining part one may exchange the two sums and conclude that the remaining part equals

$$\sum_{k=j+1}^{\infty} h(k, j) \sum_{i=k}^{\infty} p_{ik} \frac{z^i}{i!} = \sum_{k=j+1}^{\infty} h(k, j) \frac{(pz)^k}{k!} \sum_{i=k}^{\infty} \frac{(qz)^{i-k}}{(i-k)!} = \phi_j(pz) e^{qz}.$$

Thus, for each $j \in \mathbb{N}_0$, the generating function $\phi_j$ is a solution to the equation

$$\phi_j(z) = \frac{(pz)^j}{j!} \big(e^{qz} - 1\big) + e^{qz} \phi_j(pz), \qquad |z| < 1. \tag{12}$$

This equation is of the form $\phi_j(z) = a_j(z) + b(z)\phi_j(pz)$ with $a_j(z) := (pz)^j (e^{qz} - 1)/j!$ and $b(z) := e^{qz}$. Iteration leads to

$$\phi_j(z) = \sum_{k=0}^{K} a_j\big(p^k z\big) \prod_{l=0}^{k-1} b\big(p^l z\big) + \phi_j\big(p^{K+1} z\big) \prod_{l=0}^{K} b\big(p^l z\big), \qquad |z| < 1,\ K \in \mathbb{N}_0. \tag{13}$$

For $K = 0$ equation (13) coincides with (12) and for arbitrary $K \in \mathbb{N}_0$ equation (13) is obtained by induction on $K$. Letting $K \to \infty$ in (13) and noting that $\phi_j(p^{K+1} z) \to \phi_j(0) = 0$ as $K \to \infty$ and that

$$\prod_{l=0}^{K} b\big(p^l z\big) = \prod_{l=0}^{K} e^{qp^l z} = e^{\sum_{l=0}^{K} qp^l z} = e^{(1-p^{K+1})z} \to e^z$$

as $K \to \infty$ leads to the solution

$$\begin{aligned}
\phi_j(z) &= \sum_{k=0}^{\infty} a_j\big(p^k z\big) \prod_{i=0}^{k-1} b\big(p^i z\big) = \sum_{k=0}^{\infty} \frac{(p^{k+1} z)^j}{j!} \big(e^{qp^k z} - 1\big) \prod_{i=0}^{k-1} e^{qp^i z} \\
&= \frac{z^j}{j!} \sum_{k=0}^{\infty} p^{j(k+1)} \big(e^{qp^k z} - 1\big) e^{(1-p^k)z} \\
&= \frac{z^j}{j!} \sum_{k=0}^{\infty} p^{j(k+1)} \big(e^{(1-p^{k+1})z} - e^{(1-p^k)z}\big) \\
&= \frac{z^j}{j!} \sum_{k=1}^{\infty} p^{jk} \big(e^{(1-p^k)z} - e^{(1-p^{k-1})z}\big).
\end{aligned} \tag{14}$$

Replacing the exponentials by their power series expansions we obtain

$$\phi_j(z) = \frac{z^j}{j!}\sum_{k=1}^{\infty} p^{jk} \sum_{l=1}^{\infty}\big((1-p^k)^l - (1-p^{k-1})^l\big)\frac{z^l}{l!}.$$

The index transformation $i = j + l$ leads to

$$\phi_j(z) = \frac{1}{j!}\sum_{i=j+1}^{\infty} \frac{z^i}{(i-j)!}\sum_{k=1}^{\infty} p^{jk}\big((1-p^k)^{i-j} - (1-p^{k-1})^{i-j}\big).$$

Thus, the coefficient $h(i, j)$ in front of $z^i/i!$ is given by (4). □

**Proof of Corollary 1.3.** Fix $p \in (0, 1)$ and $j \in \mathbb{N}$ and let $\mu := -\log p \in (0, \infty)$. For $i \in \mathbb{N}$ with $i > j$ define $a_i := \mu^{-1}\log(i - j)$ and let $\alpha_i := a_i - \lfloor a_i \rfloor$ denote the fractional (non-integer) part of $a_i$. Since $a_i \to \infty$ as $i \to \infty$ and $a_{i+1} - a_i = \mu^{-1}\log(1 + 1/(i - j)) \to 0$ as $i \to \infty$ it follows that $\{\alpha_i : i = j + 1, j + 2, \dots\}$ is dense in [0, 1]. Thus, for every $\alpha \in [0, 1]$ there exists a subsequence $(i_l)_{l\in\mathbb{N}}$ (depending on $p$, $j$ and $\alpha$) such that $i_1 < i_2 < \cdots$ and $\lim_{l\to\infty} \alpha_{i_l} = \alpha$. By Theorem 1.2,

$$h(i_l, j) = \binom{i_l}{j}\sum_{k=1}^{\infty}\big((1-p^k)^{i_l-j} - (1-p^{k-1})^{i_l-j}\big)p^{jk}.$$

Using the index transformation $n := k - \lfloor a_{i_l}\rfloor = k + \alpha_{i_l} - a_{i_l}$ and noting that $p^{a_{i_l}} = (e^{-\mu})^{\mu^{-1}\log(i_l-j)} = 1/(i_l - j)$, we obtain

$$h(i_l, j) = \binom{i_l}{j}\sum_{n=1-\lfloor a_{i_l}\rfloor}^{\infty}\left(\left(1 - \frac{p^{n-\alpha_{i_l}}}{i_l - j}\right)^{i_l-j} - \left(1 - \frac{p^{n-1-\alpha_{i_l}}}{i_l - j}\right)^{i_l-j}\right)\frac{p^{j(n-\alpha_{i_l})}}{(i_l - j)^j}.$$

For $l \to \infty$ this expression converges by dominated convergence to

$$\frac{1}{j!}\sum_{n=-\infty}^{\infty}\big(e^{-p^{n-\alpha}} - e^{-p^{n-1-\alpha}}\big)p^{j(n-\alpha)} = \frac{1}{j!}\mathbb{E}\big(p^{j(Z-\alpha)}\big) =: f(\alpha).$$

In order to verify that the function $f$ is non-constant, we proceed as follows. Let $G$ be standard Gumbel distributed with distribution function $x \mapsto e^{-e^{-x}}$, $x \in \mathbb{R}$. Define $X := G/\mu$ and $X_\alpha := \lfloor X + \alpha\rfloor - \alpha + 1$ for all $\alpha \in [0, 1]$. It is readily checked that $Z - \alpha \overset{d}{=} X_\alpha$ for all $\alpha \in [0, 1]$. By Janson [15], page 1810, Example 2.7 (formula for $\mathbb{E}(e^{tX_\alpha})$ applied with $c := 1/\mu$ and $t := -j\mu$) the function $f$ satisfies

$$f(\alpha) = \frac{1}{j!}\mathbb{E}\big(p^{jX_\alpha}\big) = \frac{1}{j!}\mathbb{E}\big(e^{-j\mu X_\alpha}\big) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k\alpha}, \qquad \alpha \in [0, 1],$$

with Fourier coefficients

$$c_k := \frac{1}{j!}\frac{1-p^j}{j\mu - 2\pi \mathrm{i}k}\Gamma(1+j-2\pi\mathrm{i}k/\mu) = \frac{1-p^j}{j!\mu}\Gamma(j-2\pi\mathrm{i}k/\mu) \neq 0, \qquad k \in \mathbb{Z}.$$

Therefore, $f$ cannot be constant because otherwise all Fourier coefficients $c_k$, $k \in \mathbb{Z} \setminus \{0\}$, would vanish, which is not the case. From the non-constancy and the continuity of $f$, we conclude that the image $f([0, 1])$ of $f$ is a non-empty (closed) interval. It follows that the sequence $(h(i, j))_{i\geq j}$ has infinitely many accumulation points. In particular, $h(i, j)$ does not converge as $i \to \infty$. □

***Example.*** For $j = 1$, $p = 1/2$ and $\alpha = 0$ a possible subsequence is $i_l = j + 2^l = 1 + 2^l$, and we have $h(i_l, 1) \to \mathbb{E}((1/2)^Z)$ as $l \to \infty$, where $Z$ is an integer valued random variable with distribution function $\mathbb{P}(Z \leq n) = e^{-(1/2)^n}$, $n \in \mathbb{Z}$.

***Remark.*** The function $f$ occurring at the end of the previous proof takes in average the value $\int_0^1 f(\alpha)\,\mathrm{d}\alpha = c_0 = (1 - p^j)/(j\mu)$ showing that, for large $i$, the hitting probabilities $h(i, j)$ oscillate around $(1 - p^j)/(j\mu)$ as $i$ varies. Since $|\Gamma(-\mathrm{i}t)| = |\Gamma(\mathrm{i}t)| = \sqrt{\pi/(t\sinh(\pi t))} \sim \sqrt{2\pi/t}e^{-\pi t/2}$ as $t \to \infty$, the Fourier coefficients $c_k$ decrease rapidly with increasing $k$. In fact all Fourier coefficients $c_k$, $k \in \mathbb{Z} \setminus \{0\}$, are quite small in comparison to the leading Fourier coefficient $c_0$. The amplitude of the oscillation is hence rather small.

***Remark.*** For a sequence $(a_n)_{n\in\mathbb{N}}$ and $0 \leq a < b \leq 1$ define $I_n(a, b) := \{k \in \mathbb{N} : n - 1 + a < a_k \leq n - 1 + b\}$ and $I_n := I_n(0, 1) = \{k \in \mathbb{N} : n - 1 < a_k \leq n\}$ for all $n \in \mathbb{N}$. The sequence $(a_n)_{n\in\mathbb{N}}$ is said to have density $f : [0, 1] \to \mathbb{R}$, if $\lim_{n\to\infty} |I_n(a, b)|/|I_n| = \int_a^b f(x)\,\mathrm{d}x$ for all $0 \leq a < b \leq 1$.

Fix $j \in \mathbb{N}$ and $\mu \in (0, \infty)$. For the particular sequence $(a_n)_{n\in\mathbb{N}}$ used in the proof of Corollary 1.3 defined via $a_n := 0$ for $n \leq j$ and $a_n := \mu^{-1}\log(n - j)$ for $n > j$ we have $I_n(a, b) = \{k > j : \mu(n - 1 + a) < \log(k - j) \leq \mu(n - 1 + b)\} = \{k > j : j + e^{\mu(n-1+a)} < k \leq j + e^{\mu(n-1+b)}\}$ for all sufficiently large $n$. Thus, $|I_n(a, b)| = e^{\mu(n-1+a)} - e^{\mu(n-1+b)} + O(1) = e^{\mu(n-1)}(e^{\mu b} - e^{\mu a}) + O(1)$ from which we conclude that

$$\frac{|I_n(a, b)|}{|I_n|} \to \frac{e^{\mu b} - e^{\mu a}}{e^{\mu\cdot 1} - e^{\mu\cdot 0}} = \frac{e^{\mu b} - e^{\mu a}}{e^{\mu} - 1}$$

as $n \to \infty$. Thus, the sequence $(a_n)_{n\in\mathbb{N}}$ has density $f(x) = \mu e^{\mu x}/(e^{\mu} - 1)$, $x \in [0, 1]$, and distribution function $F(x) = (e^{\mu x} - 1)/(e^{\mu} - 1)$, $x \in [0, 1]$.

The behavior is quite different for the sequence $a_n := n^{\alpha}$ for some $0 < \alpha < 1$. In this case, we still have $a_{n+1} - a_n \sim \alpha n^{\alpha-1} \to 0$ as $n \to \infty$, so the fractional parts of $(a_n)_n$ are still dense in $[0, 1]$. But $|I_n(a, b)| = (n - 1 + b)^{1/\alpha} - (n - 1 + a)^{1/\alpha} + O(1) = (b - a)(1/\alpha)n^{1/\alpha-1} + o(n^{1/\alpha-1})$, from which we conclude that $(a_n)_n$ has density $f(x) = 1$, $x \in [0, 1]$, so $(a_n)_n$ is uniformly distributed on $[0, 1]$.

**Proof of Proposition 1.4.** From the spectral decomposition $P = RDL$ used in the proof of Proposition 1.1 it follows that $R_\alpha = (I - \alpha P)^{-1} = \sum_{k=0}^{\infty} \alpha^k P^k = R(\sum_{k=0}^{\infty} \alpha^k D^k)L = RD(\alpha)L$, where $D(\alpha)$ is the diagonal matrix with diagonal entries $d_{ii}(\alpha) := \sum_{k=0}^\infty (\alpha d_{ii})^k = 1/(1-\alpha d_{ii}) = 1/(1-\alpha p^i)$, $i \in \mathbb{N}_0$. Thus, for all $i, j \in \mathbb{N}_0$ with $i \ge j$,

$$r_\alpha(i,j) = \sum_{k=j}^{i} r_{ik} d_{kk}(\alpha) l_{kj} = \sum_{k=j}^{i} \binom{i}{k} \frac{1}{1-\alpha p^k} (-1)^{k-j} \binom{k}{j} = \binom{i}{j} \sum_{k=j}^{i} \binom{i-j}{k-j} \frac{(-1)^{k-j}}{1-\alpha p^k},$$

which is (6). □

## 3. Proofs concerning the dual Markov chain Y

**Proof of Theorem 1.5.** A straightforward computation shows that the transition matrix $\Pi$ with entries (7) has spectral decomposition $\Pi = \tilde{R}\tilde{D}\tilde{L}$, where $\tilde{D}$ is the diagonal matrix with entries $\tilde{d}_{ii} := p^{i+1}$, $i \in \mathbb{N}_0$, and $\tilde{R} := (\tilde{r}_{ij})_{i,j \in \mathbb{N}_0}$ and $\tilde{L} := (\tilde{l}_{ij})_{i,j \in \mathbb{N}_0}$ have entries $\tilde{r}_{ij} := (-1)^{j-i}\binom{j}{i}$ and $\tilde{l}_{ij} := \binom{j}{i}$. Now proceed as in the proof of Proposition 1.1, which yields the formula (8) for $\tilde{h}(i, j)$.

Two proofs of (9) and (10) are now provided. The first proof uses (8). The second proof is based on generating functions and does not make use of (8). Since both proofs are almost the same as those provided for Theorem 1.2, only the key ideas and main steps are provided.

*Alternative* 1. Performing in the sum on the right-hand side of (8) the index transformation $l = k - i$, making use of $1/(1-p^{l+i+1}) = \sum_{k=0}^\infty p^{(l+i+1)k}$ and interchanging afterwards the two sums $\sum_{l=0}^{j-i}$ and $\sum_{k=0}^\infty$ shows that (8) equals (9). Moreover, (10) is equal to (9), since

$$\begin{aligned}
&\left(1-p^{j+1}\right) \sum_{k=1}^\infty p^{(i+1)k} \left(1-p^k\right)^{j-i} \\
&= \sum_{k=1}^\infty p^{(i+1)k} \left(1-p^k\right)^{j-i} - \sum_{k=1}^\infty p^{(i+1)(k+1)} p^{j-i} \left(1-p^k\right)^{j-i} \\
&= \sum_{k=1}^\infty \left(\left(1-p^k\right)^{j-i} - \left(p-p^k\right)^{j-i}\right) p^{(i+1)k}.
\end{aligned}$$

*Alternative* 2. For $i \in \mathbb{N}_0$ and $z \in \mathbb{C}$ consider

$$\psi_i(z) := \sum_{j=i+1}^\infty \tilde{h}(i,j) \frac{z^j}{j!} = \sum_{j=i+1}^\infty \sum_{k=i}^{j} \pi_{ik} \tilde{h}(k,j) \frac{z^j}{j!} = \sum_{j=i+1}^\infty \pi_{ij} \frac{z^j}{j!} + \sum_{j=i+1}^\infty \sum_{k=i}^{j-1} \pi_{ik} \tilde{h}(k,j) \frac{z^j}{j!}.$$

Using $\pi_{ij} = \binom{j}{i} p^{i+1} q^{j-i}$ it is readily checked that $\sum_{j=i+1}^\infty \pi_{ij} z^j/j! = p^{i+1} z^i (e^{qz}-1)/i! =: a_i(z)$. Interchanging the two sums the other part reduces to $\sum_{k=i}^\infty \pi_{ik} \sum_{j=k+1}^\infty \tilde{h}(k,j) z^j/j! =$

$\sum_{k=i}^{\infty} \pi_{ik}\psi_k(z)$. Thus, $\psi_i(z) = a_i(z) + \sum_{k=i}^{\infty} \pi_{ik}\psi_k(z)$. Therefore, for $z, u \in \mathbb{C}$,

$$\psi(z,u) := \sum_{i=0}^{\infty} \psi_i(z)u^i = \sum_{i=0}^{\infty}\left(a_i(z) + \sum_{k=i}^{\infty} \pi_{ik}\psi_k(z)\right)u^i = \sum_{i=0}^{\infty} a_i(z)u^i + \sum_{k=0}^{\infty} \psi_k(z) \sum_{i=0}^{k} \pi_{ik}u^i.$$

Since $\sum_{i=0}^{\infty} a_i(z)u^i = (e^{qz} - 1)pe^{pzu} =: a(z,u)$ and $\sum_{i=0}^{k} \pi_{ik}u^i = \sum_{i=0}^{k} \binom{k}{i} p^{i+1}q^{k-i}u^i = p(pu+q)^k$ it follows that $\psi(z,u) = (e^{qz} - 1)pe^{pzu} + \sum_{k=0}^{\infty} \psi_k(z)p(pu+q)^k = a(z,u) + p\psi(z, pu+q)$. Iteration gives $\psi(z,u) = \sum_{k=0}^{K-1} p^k a(z, p^k u + 1 - p^k) + p^K \psi(z, p^K u + 1 - p^K)$, $K \in \mathbb{N}$. Letting $K \to \infty$ and noting that $|\psi(z,1)| < \infty$ yields $\psi(z,u) = \sum_{k=0}^{\infty} p^k a(z, p^k u + 1 - p^k)$. Plugging in $a(z, p^k u + 1 - p^k) = p(e^{qz} - 1)e^{pz(p^k u + 1 - p^k)} = p(e^{qz} - 1)e^{p^{k+1}zu}e^{pz}e^{-p^{k+1}z}$ leads to the solution

$$\begin{aligned}
\psi(z,u) &= \big(e^{qz} - 1\big)e^{pz} \sum_{k=0}^{\infty} p^{k+1} e^{p^{k+1}zu} e^{-p^{k+1}z} \\
&= \big(e^{z} - e^{pz}\big) \sum_{k=0}^{\infty} p^{k+1} e^{p^{k+1}zu} e^{-p^{k+1}z} \\
&= \sum_{k=0}^{\infty} p^{k+1} e^{p^{k+1}zu} \big(e^{(1-p^{k+1})z} - e^{(p-p^{k+1})z}\big) \\
&= \sum_{k=1}^{\infty} p^{k} e^{p^{k}zu} \big(e^{(1-p^{k})z} - e^{(p-p^{k})z}\big).
\end{aligned}$$

Series expansion $e^{p^k zu} = \sum_{i=0}^{\infty} p^{ik}(zu)^i/i!$ yields

$$\psi(z,u) = \sum_{i=0}^{\infty} (zu)^i/i! \sum_{k=1}^{\infty} p^{(i+1)k}\big(e^{(1-p^k)z} - e^{(p-p^k)z}\big),$$

from which it follows that the coefficient $\psi_i(z) = [u^i]\psi(z,u)$ in front of $u^i$ is

$$\psi_i(z) = \frac{z^i}{i!} \sum_{k=1}^{\infty} p^{(i+1)k}\big(e^{(1-p^k)z} - e^{(p-p^k)z}\big). \tag{15}$$

Further series expansion $e^{(1-p^k)z} - e^{(p-p^k)z} = \sum_{j=i+1}^{\infty} ((1-p^k)^{j-i} - (p-p^k)^{j-i})z^{j-i}/(j-i)!$ shows that the coefficient $\tilde{h}(i,j) = j![z^j]\psi_i(z)$ is given by (10). □

***Remark (Distributional equality of waiting times).*** For $k, n \in \mathbb{N}_0$ with $k \le n$ let $\sigma_{kn} := \inf\{m \in \mathbb{N}_0 : Y_m \ge n, Y_0 = k\}$ denote the first time the chain $Y$, started in state $k$, jumps to a state larger than or equal to $n$. By Siegmund duality, for all $m \in \mathbb{N}_0$, $\mathbb{P}(\sigma_{kn} \le m) = \mathbb{P}(Y_m \ge n | Y_0 = k) = \mathbb{P}(X_m \le k | X_0 = n) = \mathbb{P}(\tau_{nk} \le m)$. Thus $\sigma_{kn}$ has the same distribution as $\tau_{nk}$ (see the remark after the proof of Proposition 1.1). In particular, $\sigma_n := \sigma_{0n}$ has the same distribution as the absorption time $\tau_n := \tau_{n0}$ of the Markov chain $X$ started at $X_0 = n$. The spectral decomposition $\Pi = \tilde{R}\tilde{D}\tilde{L}$ used at the beginning of the proof of Theorem 1.5 shows that $\sigma_{kn}$ has distribution function

$$\begin{aligned}
\mathbb{P}(\sigma_{kn} \le m) = \mathbb{P}(Y_m \ge n | Y_0 = k) &= \sum_{j=n}^{\infty} \big(\Pi^m\big)_{kj} = \sum_{j=n}^{\infty} \big(\tilde{R}\tilde{D}^m\tilde{L}\big)_{kj} \\
&= \sum_{j=n}^{\infty}\sum_{i=k}^{j} \tilde{r}_{ki}\tilde{d}_{ii}^m \tilde{l}_{ij} = \sum_{j=n}^{\infty}\sum_{i=k}^{j} (-1)^{i-k}\binom{i}{k}\big(p^{i+1}\big)^m\binom{j}{i} \\
&= \big(p^m\big)^{k+1}\sum_{j=n}^{\infty}\binom{j}{k}\sum_{i=k}^{j}\binom{j-k}{i-k}\big(-p^m\big)^{i-k} \\
&= \big(p^m\big)^{k+1}\sum_{j=n}^{\infty}\binom{j}{k}\big(1-p^m\big)^{j-k}, \qquad m \in \mathbb{N}_0.
\end{aligned} \tag{16}$$

That (16) coincides with (11) follows either from the Siegmund duality stated at the beginning of this remark or alternatively from equation (18) in the Appendix with $p$ replaced by $p^m$.

**Proof of Corollary 1.6.** We proceed as in the proof of Corollary 1.3. Fix $p \in (0, 1)$ and $i \in \mathbb{N}_0$ and let $\mu := -\log p \in (0, \infty)$. For $j > i$ define $b_j := \mu^{-1}\log(j-i)$ and put $\beta_j := b_j - \lfloor b_j \rfloor$. Since $b_j \to \infty$ as $j \to \infty$ and $b_{j+1} - b_j = \mu^{-1}\log(1 + 1/(j-i)) \to 0$ as $j \to \infty$ it follows that $\{\beta_j : j = i+1, i+2, \ldots\}$ is dense in $[0, 1]$. Thus, for every $\beta \in [0, 1]$ there exist integers $j_1 < j_2 < \cdots$ such that $\lim_{l\to\infty}\beta_{j_l} = \beta$. By Theorem 1.5, $\tilde{h}(i, j_l) = (1 - p^{j_l+1})\binom{j_l}{i}\sum_{k=1}^{\infty}(1 - p^k)^{j_l-i}p^{(i+1)k}$. Using the index transformation $n := k - \lfloor b_{j_l}\rfloor = k + \beta_{j_l} - b_{j_l}$ and noting that $p^{b_{j_l}} = (e^{-\mu})^{\mu^{-1}\log(j_l-i)} = 1/(j_l - i)$ we obtain

$$\tilde{h}(i, j_l) = \big(1 - p^{j_l+1}\big)\binom{j_l}{i}\sum_{n=1-\lfloor b_{j_l}\rfloor}^{\infty}\left(1 - \frac{p^{n-\beta_{j_l}}}{j_l - i}\right)^{j_l-i}\frac{p^{(i+1)(n-\beta_{j_l})}}{(j_l-i)^{i+1}}.$$

This expression, multiplied with $j_l$, tends by dominated convergence to $\lim_{l\to\infty} j_l\tilde{h}(i, j_l) = (1/i!)\sum_{n\in\mathbb{Z}} e^{-p^{n-\beta}}p^{(i+1)(n-\beta)} =: g(\beta)$. Note that $g(\beta)$ is defined for all $\beta \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is 1-periodic. It is easy to see that $g$ is continuously differentiable. Clearly, $g$ has Fourier coefficients

$$c_k := \int_0^1 g(\beta)e^{-2\pi \mathrm{i}k\beta}\,\mathrm{d}\beta = \frac{1}{i!}\sum_{n\in\mathbb{Z}}\int_0^1 e^{-p^{n-\beta}}p^{(i+1)(n-\beta)}e^{-2\pi \mathrm{i}k\beta}\,\mathrm{d}\beta, \qquad k \in \mathbb{Z}.$$

Using the substitution $x := p^{n-\beta}$ and noting that $\mathrm{d}\beta/\mathrm{d}x = c/x$ with $c := 1/\mu$ and that $e^{-2\pi \mathrm{i} k\beta} = e^{-2\pi \mathrm{i} k(n+c\log x)} = x^{-2\pi \mathrm{i} kc}$ it follows that

$$c_k = \frac{c}{i!}\sum_{n\in\mathbb{Z}}\int_{p^n}^{p^{n-1}} e^{-x}x^{i-2\pi \mathrm{i} kc}\,\mathrm{d}x = \frac{c}{i!}\int_0^\infty e^{-x}x^{i-2\pi \mathrm{i} kc}\,\mathrm{d}x$$
$$= \frac{c}{i!}\Gamma(i+1-2\pi \mathrm{i} kc) \neq 0, \qquad k \in \mathbb{Z}.$$

Since $g$ is continuously differentiable the Fourier series $\beta \mapsto \sum_{k\in\mathbb{Z}} c_k e^{2\pi \mathrm{i} k\beta}$ coincides with $g$. As in the proof of Corollary 1.3 it follows that the image $g([0, 1])$ is a non-empty interval. Thus, $(j\tilde{h}(i, j))_{j\geq i}$ has infinitely many accumulation points and $j\tilde{h}(i, j)$ does not converge as $j \to \infty$. □

***Remark.*** The function $g$ occurring at the end of the previous proof takes in average the value $\int_0^1 g(\beta)\,\mathrm{d}\beta = c_0 = 1/\mu$ showing that, for large $j$, the values $j\tilde{h}(i, j)$ oscillate around $1/\mu$ as $j$ varies. The same argument as already used in the first remark after the proof of Corollary 1.3 shows that the Fourier coefficients $c_k$ decrease rapidly with increasing $k$ and that the amplitude of the oscillation is rather small.

**Proof of Proposition 1.7.** Proceeding in the same way as in the proof of Proposition 1.4, but with the spectral decomposition $\Pi = \tilde{R}\tilde{D}\tilde{L}$, we obtain for all $i, j \in \mathbb{N}_0$ with $i \leq j$

$$\tilde{r}_\alpha(i, j) = \sum_{k=i}^{j} \tilde{r}_{ik}\tilde{d}_{kk}(\alpha)\tilde{l}_{kj} = \sum_{k=i}^{j}(-1)^{k-i}\binom{k}{i}\frac{1}{1-\alpha p^{k+1}}\binom{j}{k}$$
$$= \binom{j}{i}\sum_{k=i}^{j}\binom{j-i}{k-i}\frac{(-1)^{k-i}}{1-\alpha p^{k+1}}.$$

□

# Appendix

The following lemma provides a formula for a particular class of stochastic processes $K = (K_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{N}_0$ satisfying $K_{n+1} - K_n \in \{0, 1\}$ for all $n \in \mathbb{N}_0$. Such processes arise for example, in the Chinese restaurant process (see, for example, [19]), where $K_n$ counts the number of occupied tables after the $n$th customer has entered the restaurant. Other applications stem from population genetics, where $K_n$ is the number of types in a sample of size $n$ taken from a population consisting of individuals of different types. The lemma is useful to derive combinatorial identities, as indicated in the examples afterwards.

**Lemma A.1.** *Suppose that $K = (K_n)_{n\in\mathbb{N}_0}$ is a stochastic process with state space $\mathbb{N}_0$ satisfying $K_{n+1} - K_n \in \{0, 1\}$ for all $n \in \mathbb{N}_0$. Then, for all $n, k \in \mathbb{N}_0$,*

$$\mathbb{P}(K_n \leq k) - \lim_{N\to\infty}\mathbb{P}(K_N \leq k) = \sum_{j=n}^{\infty}\mathbb{P}(K_j = k, K_{j+1} = k+1). \tag{17}$$

**Proof.** From $K_{j+1} - K_j \in \{0, 1\}$ for all $j \in \mathbb{N}_0$ we conclude that $\mathbb{P}(K_j \le k) - \mathbb{P}(K_{j+1} \le k) = \mathbb{P}(K_j \le k, K_{j+1} > k) = \mathbb{P}(K_j = k, K_{j+1} = k+1)$ for all $j, k \in \mathbb{N}_0$. For all $n, k \in \mathbb{N}_0$ it follows by summation over all $j \ge n$ that $\sum_{j=n}^{\infty} \mathbb{P}(K_j = k, K_{j+1} = k+1) = \sum_{j=n}^{\infty} (\mathbb{P}(K_j \le k) - \mathbb{P}(K_{j+1} \le k)) = \mathbb{P}(K_n \le k) - \lim_{N\to\infty} \mathbb{P}(K_N \le k)$. □

In the following three examples are provided. The first example is related to the Greenwood model, the other examples are related to (generalized) Stirling numbers and Chinese restaurant processes and are hence of independent interest. In the first and second example, the chain $K$ is time-homogeneous whereas in the third example $K$ is in general time-inhomogeneous.

***Examples.***

1. Let $p \in (0, 1]$. If $K = (K_n)_{n\in\mathbb{N}_0}$ is the Markov chain with initial state $K_0 := 0$ moving one step to the right with probability $p$ and staying where it is with probability $q := 1 - p$, then $K_n$ is binomially distributed with parameters $n$ and $p$. The chain $K$ is transient, since $p > 0$. In particular, $\lim_{N\to\infty} \mathbb{P}(K_N \le k) = 0$ for all $k \in \mathbb{N}_0$. Thus, (17) reduces to

$$\sum_{j=0}^{k} \binom{n}{j} p^j q^{n-j} = p^{k+1} \sum_{j=n}^{\infty} \binom{j}{k} q^{j-k}, \qquad n \in \mathbb{N}, k \in \mathbb{N}_0. \tag{18}$$

Note that (18) is essentially a combinatorial reformulation of the Siegmund duality of the chains $X$ and $Y$ studied in this article, as pointed out in the remark around (16).

2. Fix $N \in \mathbb{N}$. Suppose that $K = (K_n)_{n\in\mathbb{N}_0}$ is a Markov chain with initial state $K_0 := 0$ which moves from state $k \in \{0, \dots, N\}$ to state $k + 1$ with probability $1 - k/N$ and stays in state $k$ with probability $k/N$. In this case, $K_n$ counts the number of non-empty boxes when $n$ balls are allocated randomly and independently to $N$ boxes. One may also interpret $K_n$ as follows. Imagine a restaurant having $N$ tables each of infinite capacity. Customers successively enter the restaurant. Each customer chooses randomly one of the $N$ tables and takes place at this table. Then $K_n$ is the number of occupied tables after the $n$th customer has been seated. The states $0, \dots, N - 1$ are transient, the state $N$ is absorbing and the chain $K$ never moves to a state above $N$. It is known and easily checked by induction on $n \in \mathbb{N}_0$ that $K_n$ has distribution $\mathbb{P}(K_n = k) = N^{-n}(N)_k S(n, k)$, $k \in \mathbb{N}_0$, where $(N)_0 := 1$, $(N)_k := N(N - 1)\cdots(N - k + 1)$ for $k \in \mathbb{N}$, and $S(\cdot, \cdot)$ denote the Stirling numbers of the second kind. Since all states $k < N$ are transient we have $\lim_{N\to\infty} \mathbb{P}(K_N \le k) = 0$ for $k < N$ and from (17) we conclude that

$$\frac{1}{N^n} \sum_{j=0}^{k} (N)_j S(n, j) = (N)_k \left(1 - \frac{k}{N}\right) \sum_{j=n}^{\infty} \frac{S(j, k)}{N^j}, \qquad n \in \mathbb{N}_0, 0 \le k < N. \tag{19}$$

3. Fix $0 \le a \le b \le 1$ and $\theta_1, \theta_2 \in \mathbb{R}$ with $0 < \theta_1 \le \theta_2$. Define $r := \theta_2 - \theta_1$. Let $K = (K_n)_{n\in\mathbb{N}_0}$ be a (in general time-inhomogeneous) Markov chain with $K_0 := 0$ and the following transition probabilities. Given the chain is at time $n \in \mathbb{N}_0$ in state $k \in \{0, \dots, n\}$ it moves to state $k+1$ with probability $p_k(n) := (\theta_1 + ak)/(\theta_2 + bn)$ and stays in state $k$ with probability $1 - p_k(n)$. Note that $0 < p_k(n) \le 1$ for $0 \le k \le n$. Exploiting the recursion $S(n + 1, k) = S(n, k - 1) + (bn -$

$ak + r)S(n,k)$ for the generalized Stirling numbers $S(n,k) := S(n,k;-b,-a,r)$ as defined in Hsu and Shiue [13] it follows by induction on $n \in \mathbb{N}_0$ that $K_n$ has distribution

$$\mathbb{P}(K_n = k) = \frac{[\theta_1|a]_k}{[\theta_2|b]_n} S(n,k;-b,-a,r), \qquad k \in \mathbb{N}_0, \tag{20}$$

where $[x|y]_0 := 1$ and $[x|y]_n := \prod_{j=0}^{n-1}(x+jy)$ for $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Clearly, (20) holds for $n = 0$, since $K_0 = 0$. The induction step from $n \in \mathbb{N}_0$ to $n+1$ reads

$$\begin{aligned}\mathbb{P}(K_{n+1} = k) &= p_{k-1}(n)\mathbb{P}(K_n = k-1) + \big(1 - p_k(n)\big)\mathbb{P}(K_n = k)\\ &= \frac{\theta_1 + a(k-1)}{\theta_2 + bn}\frac{[\theta_1|a]_{k-1}}{[\theta_2|b]_n} S(n,k-1) + \frac{bn - ak + r}{\theta_2 + bn}\frac{[\theta_1|a]_k}{[\theta_2|b]_n} S(n,k)\\ &= \frac{[\theta_1|a]_k}{[\theta_2|b]_{n+1}}\big(S(n,k-1) + (bn - ak + r)S(n,k)\big) = \frac{[\theta_1|a]_k}{[\theta_2|b]_{n+1}} S(n+1,k).\end{aligned}$$

From (17), we obtain the duality relation

$$\begin{aligned}&\frac{1}{[\theta_2|b]_n}\sum_{j=0}^{k}[\theta_1|a]_j S(n,j;-b,-a,r)\\ &\quad = [\theta_1|a]_{k+1}\sum_{j=n}^{\infty}\frac{S(j,k;-b,-a,r)}{[\theta_2|b]_{j+1}}, \qquad n,k \in \mathbb{N}_0.\end{aligned} \tag{21}$$

For $a = 0$, $b = 1$ and $\theta_1 = \theta_2 =: \theta$ the Stirling numbers $S(n,k;-1,0,0)$ coincide (see, for example, [13], page 368) with the usual unsigned Stirling numbers $|s(n,k)|$ of the first kind and (21) implies that

$$\frac{1}{\Gamma(\theta+n)}\sum_{j=0}^{k}\theta^j\big|s(n,j)\big| = \theta^{k+1}\sum_{j=n}^{\infty}\frac{1}{\Gamma(\theta+j+1)}\big|s(j,k)\big|, \qquad n,k \in \mathbb{N}_0, \theta > 0. \tag{22}$$

The combinatorial identity (22) is equivalent (see [6], Section 4) to the Siegmund duality of the block counting process and the fixation line of the Poisson–Dirichlet coalescent.

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