Hitting Time and Hitting Places for Non-Lattice Recurrent Random Walks

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1. Introduction. An interesting chapter of modern probability theory began when Spitzer [7] established the existence of a recurrent potential operator for arbitrary recurrent lattice random walks, and used this operator to prove the existence of a limit at infinity for the hitting distribution and Green's function for a finite set. This potential theory was also invaluable in establishing the asymptotic behavior for large times of the hitting time for a finite set. It has been an unsolved problem to establish the proper analogues of these results in the case of a recurrent non-lattice random walk. Great progress was made in this direction when Ornstein [5] showed that the hitting distribution for bounded intervals had a limit at infinity for an arbitrary non-lattice 1-dimensional random walk, and that for a non-singular random walk on the line there was a recurrent potential operator which acted on null-functions (*i.e.*, functions with 0 integral) having compact support. (The methods used by Ornstein will also work in the case of a two dimensional recurrent random walk.) Counter-examples show that in the general case there is no potential operator acting on any non-trivial class of null functions with compact support. One of the main results of this paper (see Theorem 3.1) is that there is a potential operator acting on a subclass of the integrable functions whose *Fourier transforms* have compact support which yields useful results. In particular, we will give a complete analogue of the results on the limits of the hitting distribution and Green's function at infinity for an arbitrary recurrent random walk. (see §3). In the asymptotically stable case we obtain the asymptotic behavior of the hitting times. Incomplete results of this nature are obtained in the general case.

The introduction of a potential operator acting on functions whose Fourier transform has compact support opens new possibilities of attacking the general problem. It is readily shown (see §3) that the existence of such a potential operator can be reduced to a certain problem in Fourier analysis. This problem is easily solved in the case of a two dimensional random walk, but even for a lattice random walk on the line, it seems impossible to solve this problem

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directly by Fourier analytic methods. However, to solve the problem it suffices to approximate the original random walk to a sufficient accuracy by another random walk for which the potential operator does exist. Initially we tried to do this using a lattice random walk, but unfortunately the approximation can only be carried out when $\log (1 + |x|)$ is integrable with respect to the probability measure which generates the original random walk. When we learned of Ornstein's result we naturally tried the approximation using a non-singular walk, and this works.

In summary then we do the following. In section 2 we introduce the notation to be used throughout the paper, and the real work begins in section 3. There the general results for an arbitrary recurrent random walk are stated together with some of the simpler proofs, the major technicalities being deferred to section 6. The arguments here were suggested by the corresponding facts for stable processes used by Port [6]. In section 4 improved versions of the results in section 3 are given for a non-singular random walk, and again the more difficult proofs are postponed till section 7. In section 5 we give our results on the time dependent behavior of random walks, the proofs of which will be given in section 8.

Work related to that of this paper can also be found in [1], [2] and [3].

2. Notation. Let μ be a probability measure on d-dimensional Euclidean space R^d, and let $\mu^{(n)}$ be the n-fold convolution of μ with itself. Let X denote the smallest closed subgroup of R^d such that $\mu(X) = 1$. We assume throughout this paper that μ generates a non-degenerate recurrent random walk on R^d, *i.e.*, that X has dimension d, and that $\sum_{n=1}^{\infty} \mu^{(n)}$ assigns infinite measure to some compact subset of X. Then it must be that $d = 1$ or 2 and if $d = 1$ then

$$\sigma^2 = \int x^2 \mu(dx) > 0.$$

(Henceforth, all such unspecified integrals on x will be assumed over X.)

Let dx denote Haar measure on X, and for an integrable function f on X let $\hat{f}(\theta)$ denote its Fourier transform, *i.e.*,

$$\hat{f}(\theta) = \int e^{i\theta \cdot x} f(x)\, dx.$$

If γ is a finite measure on X let $\hat{\gamma}(\theta)$ be its Fourier-Stieltjes transform,

$$\hat{\gamma}(\theta) = \int e^{i\theta \cdot x} \gamma(dx).$$

In particular, the characteristic function of μ is just $\hat{\mu}(\theta)$. There is a closed subset Φ of R^d, essentially the dual of X, and a measure $d\theta$ (a multiple of Lebesgue measure) such that $\hat{\mu}(\theta) \neq 1$ for all $\theta \in \Phi - \{0\}$, and such that if f is continuous and integrable, and $\hat{f}(\theta)$ is integrable, then

$$f(x) = (2\pi)^{-d} \int_{\Phi} e^{-i\theta \cdot x} \hat{f}(\theta)\, d\theta.$$

(From now on integrals involving θ are always understood to range over $\theta \in \Phi$). Let Z denote the integers. If $X = R^d$ then $\Phi = R^d$. If $X = Z^d$, then $\Phi = [-\pi, \pi]$ or $[-\pi, \pi] \times [-\pi, \pi]$, according as $d = 1$ or 2. If $X = R \oplus Z$ then $\Phi = R \times [-\pi, \pi]$. In all these cases $d\theta$ is Lebesgue measure. By a suitable non-singular linear transformation on R^d the general case can be reduced to one of these special cases.

For a function f on X let $f_y(x) = f(x - y)$, and for an integrable function f, let

$$J(f) = \int f(x)\, dx.$$

Let ξ_n, $n \geqq 1$ be independent random variables having the common distribution μ, and having partial sums S_n'. Then the random walk generated by μ is the Markov process $S_n = S_n' + S_0$, where S_0 is independent of the S_n, $n \geqq 1$. If B is a Borel subset of X then

$$T_B = \min \{n > 0: S_n \in B\}$$

is the hitting time after time 0 of B. If $1_B(x)$ is the indicator function of B, then the occupation time in B by time n is the random variable

$$N_n(B) = \sum_{k=1}^{n} 1_B(S_k).$$

For $n \geq 1$, let

$$P^n(x, dy) = \mu^{(n)}(dy - x)$$

be the n-fold transition function of the random walk. The "taboo" transition function ${}_BP^n(x, dy)$ is defined as

$$P_x(T_B \geqq n, S_n \in dy) = {}_BP^n(x, dy).$$

When convenient, we may think of these quantities as operators on functions in the usual way.

Let I_B be the "identity restricted to B", *i.e.*

$$I_B f(x) = 1_B(x) f(x).$$

For $0 < \lambda < 1$ define the measure

$$\mu^\lambda = \sum_{n=1}^{\infty} \lambda^n \mu^{(n)}$$

and define operators on functions as follows:

$$U^\lambda = \sum_{n=1}^{\infty} \lambda^n P^n, \qquad G_B^\lambda = \sum_{n=1}^{\infty} \lambda^n {}_BP^n, \qquad \Pi_B^\lambda = G_B^\lambda I_B\,.$$

Let c^λ be constants (to be chosen later) and define

$$A^\lambda f = c^\lambda J(f) - U^\lambda f.$$

Set

$$Q_B^\lambda(x) = \sum_{n=0}^{\infty} P_x(T_B > n)\lambda^n,$$

and set

$$L_B^\lambda(x) = (1 - \lambda)c^\lambda Q_B^\lambda(x).$$

Note that if $P_x(T_B < \infty) = 1$ for all x, then

$$Q_B^\lambda(x) = [1 - \Pi_B^\lambda 1_B(x)](1 - \lambda)^{-1}.$$

When any of the above functions of λ have a finite limit as $\lambda \uparrow 1$, we will denote this limit by the same symbol without the λ, *e.g.*

$$\lim_{\lambda \uparrow 1} \Pi_B^\lambda = \Pi_B .$$

Let F denote the set of all integrable functions f on X such that $\hat{f}(\theta)$ vanishes off of a compact set and $x^2 f(x)$ is integrable.

Let B denote the class of all relatively compact Borel subsets of X.

We introduce the following notational convention. If f is a function on X and $d = 1$ define

$$\operatorname*{Lim}_{|x|\to\infty} f(x) = \tfrac{1}{2}[\lim_{x\to+\infty} f(x) + \lim_{x\to-\infty} f(x)]$$

whenever both limits on the right exist and are finite. If $d = 2$ set Lim = lim.

3. The general case. Let g be a fixed element of F such that $J(g) = 1$. Set $c^\lambda = U^\lambda g(0)$, $0 < \lambda < 1$. For our purposes the most convenient form of the result that a "recurrent potential operator exists" is

Theorem 3.1. *For $f \varepsilon$ F and $x \varepsilon X$*

$$\lim_{\lambda \uparrow 1} A^\lambda f(x) = Af(x)$$

exists and is finite. Also

$$\operatorname*{Lim}_{|y|\to\infty} (Af_y(x) - Af_y(0)) = 0$$

and, if $d = 1$,

$$\lim_{y\to\pm\infty} (Af_y(x) - Af_y(0)) = \mp x\sigma^{-2}J(f).$$

The convergence in these limits is uniform on compacts.

An alternative form of this result is

Theorem 3.1′. *For $f \varepsilon$ F with $J(f) = 0$ and $x \varepsilon X$*

$$\lim_{\lambda \uparrow 1} U^\lambda f(x)$$

exists and is finite. Also

$$\operatorname*{Lim}_{|y|\to\infty} \lim_{\lambda\uparrow 1} U^\lambda f_y(x) = 0$$

and, if $d = 1$,

$$\lim_{y\to\pm\infty} \lim_{\lambda\uparrow 1} U^\lambda f_y(x) = \mp\sigma^{-2} \int z f(z)\, dz.$$

The convergence in these limits is uniform on compacts.

These results can be reformulated in terms of Fourier analysis. We observe that for $f \in F$ and $x \in X$

$$U^\lambda f(x) = \frac{\lambda}{(2\pi)^d} \int \frac{e^{ix\cdot\theta} \hat{f}(-\theta)}{1 - \lambda\hat{\mu}(\theta)}\, \hat{\mu}(\theta)\, d\theta,$$

$$A^\lambda f(x) = \frac{\lambda}{(2\pi)^d} \int \frac{\hat{g}(-\theta) J(f) - e^{ix\cdot\theta} \hat{f}(-\theta)}{1 - \lambda\hat{\mu}(\theta)}\, \hat{\mu}(\theta)\, d\theta,$$

and

$$A^\lambda f_y(x) - A^\lambda f_y(0) = \frac{\lambda}{(2\pi)^d} \int \frac{e^{-iy\cdot\theta}(1 - e^{ix\cdot\theta}) \hat{f}(-\theta)}{1 - \lambda\hat{\mu}(\theta)}\, \hat{\mu}(\theta)\, d\theta.$$

We note that $|1 - \lambda\hat{\mu}(\theta)| \geqq \lambda\, |1 - \hat{\mu}(\theta)| \geqq \lambda c\, |\theta|^2$ for some constant $c > 0$. If $d = 2$, then $|\theta|^{-1}$ is integrable on compact neighborhoods of $\theta = 0$ and Theorems 3.1 and 3.1′ follow immediately. If $d = 1$ Theorems 3.1 and 3.1′ are each equivalent to

Theorem 3.1″. $(d = 1)$ *Let C be a compact neighborhood of $\theta = 0$. Then*

$$\lim_{\lambda\uparrow 1} \int_C \frac{\theta}{1 - \lambda\hat{\mu}(\theta)}\, d\theta$$

exists and is finite. Also

$$\lim_{y\to\pm\infty} \lim_{\lambda\uparrow 1} \frac{i}{2\pi} \int_C \frac{\theta e^{iy\theta}}{1 - \lambda\hat{\mu}(\theta)}\, d\theta = \mp\sigma^{-2}.$$

We will prove Theorem 3.1″ in section 6, reducing it to the non-singular case where we can apply Ornstein's result.

It is easily seen that

$$P^n = \sum_{k=1}^{n-1} {}_BP^k I_B P^{n-k} + {}_BP^n, \qquad n \geqq 1,$$

and consequently that

$$U^\lambda = G_B^\lambda + \Pi_B^\lambda U^\lambda, \qquad 0 < \lambda < 1.$$

Thus

$$A^\lambda f(x) - \Pi_B^\lambda A^\lambda f(x) = -G_B^\lambda f(x) + c^\lambda(1 - \Pi_B^\lambda 1_X(x))J(f).$$

We restate this fundamental identity as (if $P_x(T_B < \infty) = 1$)

$$A^\lambda f(x) - \Pi_B^\lambda A^\lambda f(x) = -G_B^\lambda f(x) + L_B^\lambda(x)J(f). \tag{3.1}$$

As our first application of Theorem 3.1 and equation (3.1) we will get

Theorem 3.2. *Let $B \in \mathrm{B}$ have a non-empty interior. Then*

$$\lim_{\lambda \uparrow 1} L_B^\lambda(x) = L_B(x)$$

exists and is finite. For $f \in \mathrm{F}$

$$\lim_{\lambda \uparrow 1} G_B^\lambda f(x) = G_B f(x).$$

The convergence in these limits is uniform on compacts. For $f \in \mathrm{F}$

$$Af(x) - \Pi_B Af(x) = -G_B f(x) + L_B(x)J(f), \qquad x \in X. \tag{3.2}$$

The proof of Theorem 3.2 will be given in section 6. From Theorem 3.1 and equation (3.2) we get that for $f \in \mathrm{F}$

$$\operatorname*{Lim}_{|y| \to \infty} G_B f_y(x) = L_B(x)J(f)$$

and, if $d = 1$,

$$\lim_{y \to \pm\infty} G_B f_y(x) = \left[L_B(x) \pm \sigma^{-2} \int (x - z)\Pi_B(x, dz)\right]J(f).$$

The convergence in these limits is uniform on compacts. In section 6 we will "unsmooth" these results and obtain

Theorem 3.3. *Let $A, B \in \mathrm{B}$ be such that* int B *is non-empty and $|\partial A| = 0$. Then*

$$\operatorname*{Lim}_{|y| \to \infty} G_B(x, y + A) = L_B(x)\, |A|$$

and, if $d = 1$,

$$\lim_{y \to \pm\infty} G_B(x, y + A) = \left[L_B(x) \pm \sigma^{-2} \int (x - z)\Pi_B(x, dz)\right] |A|.$$

The convergence in these limits is uniform on compacts.

Let $\tilde{\mu}$ denote the dual measure to μ defined by $\tilde{\mu}(dx) = \mu(-dx)$, so that for a non-negative Borel function f

$$\int f(-x)\tilde{\mu}(dx) = \int f(x)\mu(dx).$$

It follows easily that for any Borel sets A and C

$$\int_C \mu(A - x)\, dx = \int_A \tilde{\mu}(C - x)\, dx.$$

Associated with $\tilde{\mu}$ we have a non-degenerate recurrent random walk. We use $\sim$ to denote quantities corresponding to this random walk. Then

$$\int_C P(x, A)\, dx = \int_A \tilde{P}(x, C)\, dx.$$

We recall that ${}_BP = P$ and, for $n \geqq 2$,

$${}_BP^n = ({}_BP^{n-1}I_{B^c})P = (PI_{B^c})_BP^{n-1}.$$

Given non-negative Borel functions f and g, set

$$[f, g] = \int f(x)g(x)\, dx.$$

Then $[Pf, g] = [f, \tilde{P}g]$. A simple induction argument shows that $[{}_BP^nf, g] = [f, {}_B\tilde{P}^ng]$ and hence $[G_Bf, g] = [f, \tilde{G}_Bg]$. In particular

$$\int_C G_B(x, A)\, dx = \int_A \tilde{G}_B(x, C)\, dx.$$

Applying Theorem 3.3 to the dual process we obtain

Theorem 3.4. *Let $A, B, C \in \mathrm{B}$ be such that* int B *is non-empty, and* $|\partial C| = 0$. *Then*

$$\operatorname*{Lim}_{|x|\to\infty} \int_C G_B(x + z, A)\, dz = |C| \int_A \tilde{L}_B(x)\, dx$$

and, if $d = 1$,

$$\lim_{x\to\pm\infty} \int_C G_B(x + z, A)\, dz = |C| \int_A \left[\tilde{L}_B(x) \pm \sigma^{-2} \int (x - z)\tilde{\Pi}_B(x, dz)\right] dx.$$

Letting $A = B$ we obtain the following

Corollary. *If $B \in \mathrm{B}$ and* int B *is non-empty, then*

$$\int_B L_B(x)\, dx = 1$$

and, in particular, $L_B(x)$ is not identically zero.

In the next theorem we "unsmooth" the result of Theorem 3.4. Theorem 3.5 generalizes a result of Ornstein [5] referred to in the introduction. The proof will be given in Section 6.

Theorem 3.5. *Let $A, B \in \mathrm{B}$ be such that* int B *is non-empty and* $|\partial A| = |\partial B| = 0$. *Then*

$$\operatorname*{Lim}_{|x|\to\infty} G_B(x, A) = \int_A \tilde{L}_B(x)\, dx$$

and, if $d = 1$,

$$\lim_{x \to \pm\infty} G_B(x, A) = \int_A \left[\tilde{L}_B(x) \pm \sigma^{-2} \int (x - z)\tilde{\Pi}_B(x, dz) \right] dx.$$

4. The non-singular case. We assume throughout this section that *some* $\mu^{(n)}$ *is non-singular*. Ornstein's result [5] on the recurrent potential operator in the non-singular case will be sharpened in the next theorem, the proof of which will be given in section 7.

Theorem 4.1. *The constants* c^λ, $0 < \lambda < 1$, *can be chosen so that*

$$A^\lambda(x, dy) = a^\lambda(y - x)\, dy - U_2^\lambda(x, dy),$$

where $U_2^\lambda(x, dy)$ *is a finite positive measure that increases as* $\lambda \uparrow 1$ *to a finite measure* $U_2(x, dy)$, $a^\lambda(x)$ *is continuous in* x,

$$\lim_{\lambda \uparrow 1} a^\lambda(x) = a(x)$$

exists and is finite,

$$\operatorname*{Lim}_{|y| \to \infty} (a(y - x) - a(y)) = 0,$$

and, if $d = 1$,

$$\lim_{y \to \pm\infty} (a(y - x) - a(y)) = \mp\sigma^{-2}x;$$

the convergence in these limits being uniform on compacts.

Let $B \in \mathrm{B}$ have positive measure. Then, as in section 3, we have the identity

$$A^\lambda(x, dy) - \int \Pi_B^\lambda(x, dz) A^\lambda(z, dy) = -G_B^\lambda(x, dz) + L_B^\lambda(x)\, dy.$$

Consequently

$$G_B^\lambda(x, dy) = g_B^\lambda(x, y)\, dy + {}_2G_B^\lambda(x, dy),$$

where

$${}_2G_B^\lambda(x, dy) = U_2^\lambda(x, dy) - \int \Pi_B^\lambda(x, dz) U_2^\lambda(z, dy)$$

and

$$a^\lambda(y - x) - \int \Pi_B^\lambda(x, dz) a^\lambda(y - z) = -g_B^\lambda(x, y) + L_B^\lambda(x).$$

Then, as $\lambda \uparrow 1$, ${}_2G_B^\lambda(x, dy)$ converges to the finite measure

$${}_2G_B(x, dy) = U_2(x, dy) - \int \Pi_B(x, dz) U_2(z, dy).$$

Also, as before

$$\lim_{\lambda \uparrow 1} L_B^\lambda(x) = L_B(x)$$

and the convergence is uniform on compacts. Therefore, as $\lambda \uparrow 1$, $g_B^\lambda(x, y)$ converges to a finite value $g_B(x, y)$ and

$$a(y - z) - \int \Pi_B(x, dz)a(y - z) = -g_B(x, y) + L_B(x).$$

Thus

$$(a(y - x) - a(y)) - \int \Pi_B(x, dz)(a(y - z) - a(y)) = -g_B(x, y) + L_B(x).$$

Consequently

$$\operatorname{Lim}_{|y| \to \infty} g_B(x, y) = L_B(x)$$

and, if $d = 1$,

$$\lim_{y \to \pm\infty} g_B(x, y) = L_B(x) \pm \sigma^{-2} \int (x - z)\Pi_B(x, dz).$$

The convergence in these limits is uniform on compacts. Thus we have

Theorem 4.2. *If A, B ε* B *and $|B| > 0$, then the conclusion of Theorem* 3.3 *holds for these sets.*

We will prove in section 7 that Theorem 3.5 can also be sharpened in the non-singular case.

Theorem 4.3. *If A, B, ε* B *and $|B| > 0$, then the conclusion of Theorem* 3.5 *holds for these sets.*

5. Hitting times. In this section we will investigate the asymptotic behavior, for large n, of the quantities $P_x(N_n(B) = k)$, $k \geq 0$. The only place we have really satisfactory results is in the asymptotically stable case. Complete results for the lattice case were obtained by Kesten and Spitzer [4].

Theorem 5.1. *Assume S_n'/b_n converges in law to a stable distribution having exponent α and density p. Let B ε* B *have a non-empty interior. Then*

$$\lim_{n \to \infty} \frac{P_x(T_B > n)\left(\frac{d}{\alpha}\right)\Gamma\left(2 - \frac{d}{\alpha}\right)\Gamma\left(1 + \frac{d}{\alpha}\right)p(0)}{\left(\sum_{k=1}^{n} b_k^{-d}\right)^{-1}} = L_B(x),$$

and the convergence is uniform in x on compacts. Moreover, if the random walk is also non-singular, then the above holds for all B ε B *of positive measure.*

Corollary 5.1. *Assume μ has mean zero and finite covariance matrix Σ. Let $B \in \mathcal{B}$ have non-empty interior. Then*

$$\lim_{n\to\infty} n^{1/2} P_x(T_B > n) = (2/\pi)^{1/2} \sigma L_B(x), \quad \text{if} \quad d = 1$$

and

$$\lim_{n\to\infty} \log n P_x(T_B > n) = 2\pi \, |\Sigma|^{1/2} \, L_B(x) \quad \text{if} \quad d = 2.$$

Moreover, if the random walk is also non-singular, then we need only assume that $B \in \mathcal{B}$ has positive measure.

Theorem 5.2. *Let $m \geqq 1$, and assume the conditions of Theorem 5.1 hold. Then*

$$\lim_{n\to\infty} \frac{P_x(N_n(B) = m)\left(\frac{d}{\alpha}\right)\Gamma\left(2 - \frac{d}{\alpha}\right)\Gamma\left(1 + \frac{d}{\alpha}\right)p(0)}{\left(\sum_{k=1}^{n} b_k^{-d}\right)^{-1}} = \int_B \Pi_B^m(x, dz) L_B(z)$$

where $\Pi_B^1 = \Pi_B$, and $\Pi_B^{m+1} = \Pi_B \Pi_B^m$, $m \geqq 1$.

In the general case our results are much more meager. We first show that if we start the random walk at "random" over R^d, then the times to hit sets are proportional. More precisely,

Theorem 5.3. *Assume A, B, are two sets in $\mathcal{B}$ both having non-empty interior. Then, as $n \to \infty$*

$$\int P_x(T_B \leqq n) \, dx \sim \int P_x(T_A \leqq n).$$

Moreover, if the random walk is non-singular then we need only assume A, B have positive measure.

Now observe that

$$\int \tilde{P}_x(T_B \leqq n) \, dx = \sum_{j=1}^{n} [1_X \, , {}_B\tilde{P}^j 1_B] = \sum_{j=1}^{n} [{}_B P^j 1_X \, , 1_B] = \sum_{j=1}^{n} \int_B P_x(T_B \geqq j).$$

The above theorem now yields the following

Corollary 5.2. *Assume $A, B \in \mathcal{B}$ and have non-empty interior. Then*

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} \int_B P_x(T_B > j) \, dx}{\sum_{j=0}^{n} \int_A P_x(T_A > j) \, dx} = 1.$$

If the random walk is also non-singular, then we need only assume A, B have positive measure.

In intuitive terms the result of the corollary is that the mean return times to sets are proportional. Our final results are on linking passage times to B with the return times to B.

Theorem 5.4. *Let $B \in \mathbf{B}$ have non-empty interior and let $A \in \mathbf{B}$ be such that $|\partial A| = 0$. Then*

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} \int_A P_x(T_B > j)\,dx}{\sum_{j=0}^{n} \int_B P_x(T_B > j)\,dx} = \int_A L_B(x)\,dx.$$

In the non-singular case one can improve upon this result.

Theorem 5.5. *Assume the random walk is non-singular. Then for $B \in \mathbf{B}$ having positive measure,*

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} P_x(T_B > j)}{\sum_{j=0}^{n} \int_B P_x(T_B > j)\,dx} = L_B(x),$$

and the convergence is uniform on compacts.

6. Proof of results of section 3. We give here the proofs of Theorems 3.1″, 3.2, 3.3 and 3.5.

Proof of Theorem 3.1″. Suppose first that some $\mu^{(n)}$ is non-singular. Let f be a continuous function with compact support such that $\hat{f}$ is integrable, $J(f) = 0$, and

$$\int x f(x)\,dx \neq 0.$$

Then

$$U^{\lambda} f(x) = \frac{\lambda}{2\pi} \int \frac{e^{ix\theta} \hat{f}(-\theta)\hat{\mu}(\theta)}{1 - \lambda\hat{\mu}(\theta)}\,d\theta.$$

Ornstein [5] has shown that in this case $\lim_{\lambda\uparrow 1} U^{\lambda} f(x)$ exists and is finite. This implies that

$$\lim_{\lambda\uparrow 1} \int_{-1}^{1} \frac{\theta}{1 - \lambda\hat{\mu}(\theta)}\,d\theta$$

exists and is finite. Ornstein [5] has also shown that in this case

$$\lim_{x\to\pm\infty} \lim_{\lambda\uparrow 1} U^{\lambda} f(x)$$

exists and is finite. This implies that

$$\lim_{x\to\pm\infty} \lim_{\lambda\uparrow 1} \frac{i}{2\pi} \int_{-1}^{1} \frac{\theta e^{ix\theta}}{1-\lambda\hat{\mu}(\theta)}\, d\theta = c_\pm$$

exists and is finite. We also have that

$$\lim_{x\to\pm\infty} \lim_{\lambda\uparrow 1} U^\lambda f(x) = -c_\pm \int x f(x)\, dx.$$

We proceed next to identify $c_\pm$. Suppose first that $\sigma^2 < \infty$. Then, as is well known,

$$\int \left| \frac{\mathfrak{I}\hat{\mu}(\theta)}{\theta^3} \right| d\theta < \infty.$$

It follows easily that

$$\lim_{\lambda\uparrow 1} \frac{i}{2\pi} \int_{-1}^{1} \frac{\theta}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \frac{1}{2\pi} \int_{-1}^{1} \Re\left(\frac{i\theta}{1-\hat{\mu}(\theta)} \right) d\theta$$

and

$$\lim_{\lambda\uparrow 1} \frac{i}{2\pi} \int_{-1}^{1} \frac{\theta e^{ix\theta}}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \frac{1}{2\pi} \int_{-1}^{1} \Re\left(\frac{i\theta e^{ix\theta}}{1-\hat{\mu}(\theta)} \right) d\theta.$$

Thus

$$\begin{aligned} c_+ &= \lim_{x\to\infty} \frac{1}{x} \int_0^x dy\, \frac{1}{2\pi} \int_{-1}^{1} \Re\left(\frac{i\theta e^{ix\theta}}{1-\hat{\mu}(\theta)} \right) d\theta \\ &= \lim_{x\to\infty} \frac{1}{2\pi x} \int_{-1}^{1} \Re\left(\frac{e^{ix\theta}-1}{1-\hat{\mu}(\theta)} \right) d\theta. \end{aligned}$$

It is readily seen that

$$\lim_{x\to\infty} \frac{1}{2\pi x} \int_{-1}^{1} \frac{\sin x\theta\, \mathfrak{I}\hat{\mu}(\theta)}{|1-\hat{\mu}(\theta)|^2}\, d\theta = 0$$

and

$$\lim_{x\to\infty} \frac{1}{2\pi x} \int_{-1}^{1} \frac{(\cos x\theta - 1)\Re(1-\hat{\mu}(\theta))}{|1-\hat{\mu}(\theta)|^2}\, d\theta = \lim_{x\to\infty} \frac{1}{2\pi x} \int_{-1}^{1} \frac{\cos x\theta - 1}{\sigma^2\theta^2}\, d\theta = -\sigma^{-2}.$$

Thus $c_+ = -\sigma^{-2}$. A similar argument shows that $c_- = \sigma^{-2}$.

Suppose next that $\sigma^2 = \infty$. Then

$$\lim_{\lambda\uparrow 1} \frac{i}{2\pi} \int_{-1}^{1} \frac{\theta e^{ix\theta}}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \lim_{\lambda\uparrow 1} \frac{i}{2\pi} \int_{-1}^{1} \frac{\theta e^{-ix\theta}}{1-\lambda\hat{\mu}(\theta)}\, d\theta = -\frac{1}{\pi} \int_{-1}^{1} \frac{\theta \sin x\theta}{1-\hat{\mu}(\theta)}\, d\theta.$$

Thus

$$\begin{aligned} c_+ - c_- &= -\lim_{x\to\infty} \frac{1}{x} \int_0^x dy\, \frac{1}{\pi} \int_{-1}^{1} \frac{\theta \sin y\theta}{1-\hat{\mu}(\theta)}\, d\theta \\ &= \lim_{x\to\infty} \frac{1}{\pi x} \int_{-1}^{1} \frac{\cos\theta x - 1}{1-\hat{\mu}(\theta)}\, d\theta = 0 \end{aligned}$$

(the proof is that of Spitzer [7, pp. 345-6]). Thus $c_+ = c_-$.

Choose $y > 0$ and let g be a continuous function with compact support such that $\hat{g}$ is integrable and $J(g) \neq 0$. Let f be defined by $f(x) = g(x) - g(x + y)$. Then $J(f) = 0$ and

$$\lim_{x \to \pm\infty} \lim_{\lambda \uparrow 1} U^\lambda f(x) = -c_\pm y J(g).$$

Also $U^\lambda f(x) = U^\lambda g(x) - U^\lambda g(x + y)$. Thus

$$\lim_{n \to \pm\infty} \lim_{\lambda \uparrow 1} U^\lambda g(ny) - U^\lambda g((n + 1)y) = -c_\pm y J(g).$$

Consequently

$$-c_+ y J(g) = \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} \lim_{\lambda \uparrow 1} (U^\lambda g(my) - U^\lambda g((m + 1)y))$$

$$= \lim_{n \to \infty} \frac{1}{n} \lim_{\lambda \uparrow 1} U^\lambda g(0) - U^\lambda g(ny).$$

Note that

$$U^\lambda g(0) - U^\lambda g(x) = \frac{\lambda}{2\pi} \int \frac{(1 - e^{ix\theta})\hat{g}(-\theta)\hat{\mu}(\theta)\, d\theta}{1 - \lambda\hat{\mu}(\theta)}.$$

It is now easily seen that $U^\lambda g(0) - U^\lambda g(x)$ is bounded in λ uniformly for x in compact sets. Let g be non-negative as well. It follows that $U^\lambda g(0) - U^\lambda g(x)$ is bounded from below uniformly in λ and x. Thus $c_+ \leqq 0$. A similar argument shows that $c_- \geqq 0$. Since $c_+ = c_-$ we have that $c_+ = c_- = 0$.

This completes the proof of Theorem 3.1″ under the added assumption that μ be non-singular.

In general, suppose we can find a probability measure γ on R which defines a recurrent random walk, has variance σ^2, is such that some $\gamma^{(n)}$ is non-singular, and is close enough to μ so that

$$\int \left| \frac{\hat{\mu}(\theta) - \hat{\gamma}(\theta)}{\theta^3} \right| d\theta < \infty.$$

Since Theorem 3.1″ holds for γ it holds for μ as well.

To construct such a γ we first find non-negative functions f and g having compact support such that $0 \leqq g \leqq 1$,

$$\int g(x)\mu(dx) = \int f(x)\, dx > 0,$$

$$\int x g(x)\mu(dx) = \int x f(x)\, dx,$$

and

$$\int x^2 g(x)\mu(dx) = \int x^2 f(x)\, dx.$$

We then define γ by $\gamma(dx) = (1 - g(x))\mu(dx) + f(x)dx$. This is the desired measure.

This completes the proof of Theorem 3.1″ and hence Theorems 3.1 and 3.1′ as well.

Proof of Theorem 3.2. Let A, $B \in \mathrm{B}$ and let B have non-empty interior. Since the random walk is recurrent there is an n_0 such that $\sum_1^{n_0} P^n(x, B) \geqq 1$ for $x \in A$. Consequently $P_x(T_B \leqq n_0) \geqq n_0^{-1}$ for $x \in A$. Using an argument similar to that of Lemma 5.1 of Port [6], we get that

$$\lim_{\lambda \uparrow 1} G_B^\lambda(x, A) = G_B(x, A) < \infty$$

and that $G_B(x, A)$ is bounded in x. Choose $f \in \mathrm{F}$. Then

$$\lim_{\lambda \uparrow 1} (A^\lambda f(x) - \Pi_B^\lambda A^\lambda f(x)) = Af(x) - \Pi_B Af(x)$$

and the convergence is uniform on compacts.

We will show next that $L_B^\lambda(x)$ is bounded in x and λ for x in compacts. Suppose otherwise. Then we can find λ_k and $x_k \in X$ such that the x_k lie in a compact set and $L_B^{\lambda_k}(x_k) \to \infty$ as $k \to \infty$. It follows from equation (3.1) that if $f \in \mathrm{F}$ and $J(f) \neq 0$, then

$$G_B^{\lambda_k} f(x_k) \sim L_B^{\lambda_k}(x_k) J(f) \quad \text{as} \quad k \to \infty.$$

Also for any compact set A

$$G_B^{\lambda_k} f(x_k) \sim \int_{A^c} G_B^{\lambda_k}(x_k, dy) f(y) \quad \text{as} \quad k \to \infty.$$

Suppose we can find two functions f and g in F such that $J(f) = J(g) \neq 0$ and for some positive constants a and b

$$f(x) \geqq a\,|x|^{-4} \quad \text{and} \quad g(x) \leqq b\,|x|^{-6}$$

for x sufficiently large. Then we can find a compact set A such that $f(x) \geqq 2g(x) \geqq 0$ for $x \in A^c$. But now we have two contradictory results, namely

$$\int_{A^c} G_B^{\lambda_k}(x_k, dy) f(y) \sim \int_{A^c} G_B^{\lambda_k}(x, dy) g(y)$$

and

$$\int_{A^c} G_B^{\lambda_k}(x_k, dy) f(y) \geqq 2 \int_{A^c} G_B^{\lambda_k}(x, dy) g(y).$$

As examples of two such functions, when $X = R$, we can take

$$f(x) = \left(\frac{\sin x}{x}\right)^4 + \left(\frac{\sin (x + \pi/2)}{x + \pi/2}\right)^4$$

and

$$g(x) = c\left[\left(\frac{\sin x}{x}\right)^6 + \left(\frac{\sin (x + \pi/2)}{x + \pi/2}\right)^6\right],$$

where $c > 0$ is chosen so that $J(f) = J(g)$. It is easily seen that f and g are the desired functions. Similar constructions work if $X = R^2$ or $X \cong Z \oplus R$. If $X \cong Z$ or Z^2 is trivial to construct such functions, since in these cases Φ is compact.

This completes the proof of the fact that $L_B^\lambda(x)$ is bounded in x and λ for x in compacts. It follows from equation (3.1) that, for $f \in \mathrm{F}$, $G_B^\lambda f(x)$ is bounded in x and λ for x in compacts. Chose $f \in \mathrm{F}$ such that f is non-negative and $J(f) > 0$. Then $G_B^\lambda f(x)$ is monotonic in λ so that $\lim_{\lambda \uparrow 1} G_B^\lambda f(x) = G_B f(x)$ exists and is finite. This proves that $\lim_{\lambda \uparrow 1} L_B^\lambda(x) = L_B(x)$ exists and is finite, and the rest of Theorem 3.2 follows immediately from Theorem 3.1.

Proof of Theorem 3.3. We give the proof for the case $X = R^d$, only obvious modifications being required in the general case. Let k be a fixed non-negative element of F such that $J(k) = 1$. For x and y in X and A a relatively compact Borel set, define $V(x, A, a) = G_B f(x)$, where

$$f(x) = \int 1_A(x - z)a^{-d}k(a^{-1}z)\, dz.$$

Then $f \in \mathrm{F}$, $J(f) = |A|$, and

$$\begin{aligned} V(x, y + A, a) &= G_B f_y(x) \\ &= \int a^{-d}k(a^{-1}z)\, dz\, G_B(x, y + z + A). \end{aligned}$$

Let C be a compact subset of X. There is a constant $c < \infty$ such that $L_B(x) \leqq c$ for $x \in C$ and, if $d = 1$,

$$L_B(x) \pm \sigma^{-2} \int (z - x)\Pi_B(x, dz) \leqq c.$$

Let A be as in the statement of Theorem 3 and choose $\epsilon > 0$. We can find relatively compact Borel subsets A_1 and A_2 of X such that $\bar{A}_1 \leqq \operatorname{int} A$, $\bar{A} \leqq \operatorname{int} A_2$, $|A - A_1| \leqq \epsilon c^{-1}$ and $|A_2 - A| \leqq \epsilon c^{-1}$. We can find an open neighborhood S of the origin such that $A_1 + S \leqq A$ and $A - S \leqq A_2$. Then $z + A_2 \geqq A$ for $z \in S$ and hence

$$V(x, y + A_2, a) \geqq G_B(x, y + A) \int_S a^{-d}k(a^{-1}z)\, dz.$$

Thus $G_B(x, y + A)$ has a finite upper bound N for $x \in C$ and $y \in X$. Since $z + A_1 \leqq A$ for $z \in S$, we see that

$$V(x, y + A_1, a) \leqq G_B(x, y + A) + N \int_{S^c} a^{-d}k(a^{-1}z)\, dz.$$

Choose $a_0 > 0$ such that

$$N \int_{S^c} a_0^{-d}k(a_0^{-1}z)\, dz \leqq \epsilon.$$

Then for $x \varepsilon C$ and $y \varepsilon X$

$$V(x, y + A_1, a_0) - \epsilon \leqq G_B(x, y + A) \leqq V(x, y + A_2, a_0) + \epsilon.$$

We now complete the proof of Theorem 3 for the case $d = 2$ or $d = 1$ and $\sigma^2 = \infty$, the case $d = 1$ and $\sigma^2 < \infty$ requiring only slight modifications. We can find a compact set C_1 such that for $x \varepsilon C$ and $y \not\varepsilon C_1$

$$V(x, y + A_1, a_0) \geqq L_B(x)\, |A_1| - \epsilon \geqq L_B(x)\, |A| - 2\epsilon$$

and

$$V(x, y + A_2, a_0) \leqq L_B(x)\, |A_2| + \epsilon \leqq L_B(x)\, |A| + 2\epsilon.$$

Then for $x \varepsilon C$ and $y \not\varepsilon C_1$

$$|G_B(x, y + A) - L_B(x)\, |A|| \leqq 3\epsilon;$$

that is

$$\lim_{y \to \infty} G_B(x, y + A) = L_B(x)\, |A|,$$

and the convergence is uniform for x in compact sets.

This completes the proof of Theorem 3.3.

Proof of Theorem 3.5. We first prove the following

Lemma. *If A and B are as in Theorem* 3.5, *then $G_B(x, A)$ is continuous almost everywhere.*

Proof of Lemma. The function $G_B(x, A)$ is discontinuous at x only if $P_x(T_{\partial A \cup \partial B} < \infty) > 0$, *i.e.*, only if for some $n > 0$, $P_x(S_n \varepsilon \partial A \cup \partial B) > 0$. But

$$\int dx\, P_x(S_n \varepsilon \partial A \cup \partial B) = |\partial A \cup \partial B| = 0,$$

so that $P_x(S_n \varepsilon \partial A \cup \partial B) = 0$ almost everywhere, from which the lemma follows immediately.

In proving Theorem 3.5 we assume $X = R^d = R^2$, only obvious modifications being required in the other cases.

Suppose we know that Theorem 3.5 holds whenever B is a square and $A \subset B$. Then it holds in general. For let B_1 be a square containing A and B. Then

$$G_B(x, A) = G_{B_1}(x, A) + \int \Pi_{B_1}(x, dz) 1_{B^c}(z) G_B(z, A).$$

By the lemma $1_{B^c}(z)G_B(z, A)$ is continuous almost everywhere, and hence

$$\lim_{|x| \to \infty} G_B(x, A) = \left(\int_A L_{B_1}(x)\, dx\right)\left(1 + \int_{B_1 \cap B^c} G_B(z, A)\, dz\right)$$

exists as desired.

To complete the proof of Theorem 3.5, we will show that the theorem holds if B is a square and $A \subset B$. Suppose this is false. We will derive a contradiction. We can take

$$B = \{(x_1, x_2) | -N \leqq x_1, x_2 \leqq N\}.$$

We can find an $\epsilon > 0$ such that if D is any open set containing ∂B, there exist $x_n \varepsilon X$ with $|x_n| \to \infty$ and

$$\Pi_{B \cup D}(x_n, D) \geqq \epsilon.$$

(This follows from Theorem 3.4 and our supposition that Theorem 3.5 is false for A and B.) By changing x_n slightly if necessary we have the following result: we can find $\epsilon > 0$ such that if D is any open set containing ∂B, there exist $x_n \varepsilon X$ with $|x_n| \to \infty$ and

$$\Pi_B(x_n, B \cap D) \geqq \epsilon.$$

For $\delta > 0$ set

$$C^\delta = \{(x_1, x_2) | -\delta \leqq x_1, x_2 \leqq \delta\}.$$

Then $C^\delta = C_1^\delta \cup C_2^\delta \cup C_3^\delta \cup C_4^\delta$, where

$$C_1^\delta = \{(x_1, x_2) | -\delta \leqq x_1, x_2 \leqq 0\},$$

$$C_2^\delta = \{(x_1, x_2) | 0 \leqq x_1, x_2 \leqq \delta\},$$

$$C_3^\delta = \{(x_1, x_2) | 0 \leqq x_1 \leqq \delta, -\delta \leqq x_2 \leqq 0\},$$

and

$$C_4^\delta = \{(x_1, x_2) | -\delta \leqq x_1 \leqq 0, 0 \leqq x_2 \leqq \delta\}.$$

Then we can find an $\epsilon > 0$ such that if D is any open set containing ∂B, there exist $x_n \varepsilon X$ with $|x_n| \to \infty$ such that if $\delta > 0$ is sufficiently small then for all n there is a j, $1 \leqq j \leqq 4$, such that

$$\Pi_B(z, B \cap D) \geqq \epsilon/4, \qquad z \varepsilon x_n + C_j^\delta.$$

Thus by Theorem 3.4

$$|C_\delta| \int_{B \cap D} \tilde{L}_B(x)\, dx = \lim_{n \to \infty} \int_{x_n + C_\delta} dz\, \Pi_B(z, B \cap D) \geqq \epsilon\, |C_\delta|/16.$$

Consequently

$$\int_{B \cap D} \tilde{L}_B(x)\, dx \geqq \epsilon/16$$

for any open set D containing ∂B. But this is clearly impossible since $|\partial B| = 0$.

7. Proof of results of section 4.

Proof of Theorem 4.1. We make a decomposition similar to that used by Stone [9]. We can find an n_0 and probability measures φ and χ such that $\mu^{(n_0)} = (\varphi + \chi)/2$, φ has compact support, is absolutely continuous and has a twice continuously differentiable density. Note that

$$\lambda\hat{\mu}(1 - \lambda\hat{\mu})^{-1} = (1 - \lambda^{n_0}\hat{\mu}^{n_0})^{-1} \sum_{n=1}^{n_0} \lambda^n \mu^n$$

and

$$(1 - \lambda^{n_0}\hat{\mu}^{n_0})^{-1} = \left(1 - \frac{\lambda^{n_0}\hat{\chi}}{2}\right)^{-1}\left(1 + \frac{\lambda^{n_0}\hat{\varphi}}{2}(1 - \lambda^{n_0}\hat{\mu}^{n_0})^{-1}\right).$$

Set

$$\mu_3^\lambda = \sum_{n=0}^{\infty} \lambda^{nn_0} \mu^{(nn_0)}$$

$$\mu_2^\lambda = \left(\sum_{n=1}^{n_0} \lambda^n \mu^{(n)}\right) * \sum_{n=0}^{\infty} 2^{-n} \lambda^{nn_0} \chi^{(n)},$$

and

$$\mu_1^\lambda = (\lambda/2)\varphi * \mu_2^\lambda * \mu_3^\lambda .$$

Then $\mu^\lambda = \mu_1^\lambda + \mu_2^\lambda$. It is clear that, as $\lambda \uparrow 1$, μ_2^λ increases to a finite measure

$$\mu_2 = \left(\sum_{n=1}^{n_0} \mu^{(n)}\right) * \sum_{n=0}^{\infty} 2^{-n} \chi^{(n)}.$$

It is also clear that μ_1^λ, $0 < \lambda < 1$, is absolutely continuous and has a continuous density p^λ, and that

$$p^\lambda(y) = \frac{\lambda}{2(2\pi)^d} \int \frac{e^{-iy\cdot\theta}\hat{\varphi}(\theta)\hat{\mu}_2^\lambda(\theta)}{1 - \lambda^{n_0}\hat{\mu}^{n_0}(\theta)}\, d\theta.$$

Set $c^\lambda = p^\lambda(0)$ and $a^\lambda(y) = c^\lambda - p^\lambda(y) = p^\lambda(0) - p^\lambda(y)$. Then

$$a^\lambda(y) = \frac{\lambda}{2(2\pi)^d} \int \frac{(1 - e^{-iy\cdot\theta})\hat{\varphi}(\theta)\hat{\mu}_2^\lambda(\theta)}{1 - \lambda^{n_0}\hat{\mu}^{n_0}(\theta)}\, d\theta$$

and

$$a^\lambda(y - x) - a^\lambda(y) = \frac{\lambda}{2(2\pi)^d} \int \frac{e^{-iy\cdot\theta}(e^{ix\cdot\theta} - 1)\hat{\varphi}(\theta)\hat{\mu}_2^\lambda(\theta)}{1_s\, \lambda^{n_0}\hat{\mu}^{n_0}(\theta)}\, d\theta.$$

Let D denote a compact neighborhood of $\theta = 0$. It is easily seen that as $\lambda \uparrow 1$ the contribution $a_{D^c}^\lambda(y)$ to $a^\lambda(y)$ of the integral over D^c converges to $a_{D^c}(y)$ uniformly for y in compact sets, and that, by the Riemann–Lebesgue Lemma,

$$\lim_{|y|\to\infty} a_{D^c}(y - x) - a_{D^c}(y) = 0,$$

and the limit is uniform for x in compact sets. Set

$$ {}_1a_D^\lambda(y) = \frac{1}{(2\pi)^d} \int_D \frac{(1 - e^{iy\cdot\theta})((\lambda/2)\hat{\varphi}(\theta)\hat{\mu}_2^\lambda(\theta) - n_0)}{1 - \lambda^{n_0}\hat{\mu}^{n_0}(\theta)}\, d\theta. $$

It is easily seen that, as $\lambda \uparrow 1$, ${}_1a_D^\lambda(y) \to {}_1a_D(y)$ uniformly for y in compact sets and that

$$ \lim_{|y|\to\infty} ({}_1a_D(y - x) - {}_1a_D(y)) = 0, $$

the convergence being uniform for x in compact sets. Set

$$ {}_2a_D^\lambda(y) = \frac{n_0}{(2\pi)^d} \int_D \frac{1 - e^{-iy\cdot\theta}}{1 - \lambda^{n_0}\hat{\mu}^{n_0}(\theta)}\, d\theta. $$

Then, as $\lambda \uparrow 1$, ${}_2a_D^\lambda(y)$ converges to ${}_2a_D(y)$ uniformly for y in compact sets. This result is immediate if $d = 2$; it follows from Theorem 3.1″ if $d = 1$. Similarly if $d = 2$ (or $d = 1$ and $\sigma^2 = \infty$)

$$ \lim_{|y|\to\infty} ({}_2a_D(y - x) - {}_2a_D(y)) = 0, $$

the convergence being uniform for x in compact sets. If $d = 1$, then by Theorem 3.1″

$$ \lim_{y\to\pm\infty} {}_2a_D(y - x) - {}_2a_D(y) = \mp\sigma^{-2}x, $$

the convergence being uniform for x in compact sets. Theorem 4.1 now follows immediately.

Proof of Theorem 4.3. We recall first the known fact that if some iterate of μ is non-singular, then

$$ \lim_{n\to\infty} \int |P^n(y, dz) - P^n(0, dz)| = 0 $$

and the convergence is uniform on compacts.

Let A and B be as in the theorem. Then there is a finite upper bound N to $G_B(x, A)$, $x \in X$. Let C be a compact set such that $|\partial C| = 0 < |C|$. Choose $\epsilon > 0$ and choose $n > 0$ such that

$$ \int |P^n(y, dz) - P^n(0, dz)| \leq \epsilon N^{-1}, \qquad y \in C. $$

Now

$$ G_B(x + z, A) = \sum_{k=1}^{n} {}_BP^k(x + z, A) + \int_{B^c} P^n(x + z, dy) G_B(y, A). $$

Thus for $z \in C$

$$ |G_B(x + z, A) - G_B(x, A)| \leq \epsilon N^{-1} + \sum_{k=1}^{n} ({}_BP^k(x, A) + {}_BP^k(x + z, A)). $$

Now

$$\lim_{|x|\to\infty} \sup_{z \in C} \sum_{k=1}^{n} ({}_BP^k(x, A) + {}_BP^k(x + z, A)) = 0.$$

Thus

$$\limsup_{|x|\to\infty} \sup_{z \in C} |G_B(x + z, A) - G_B(x, A)|$$

$$\leqq \limsup_{|x|\to\infty} \sup_{z \in C} \int_{B^c} |P^n(x + z, dy) - P^n(x, dy)| \, G_B(z, A) \leqq \epsilon$$

and hence

$$\lim_{|x|\to\infty} \sup_{z \in C} |G_B(x + z, A) - G_B(x, A)| = 0.$$

Theorem 4.3 now follows immediately from Theorem 3.4.

8. Proofs of results in section 5.

Proof of Theorem 5.1. The following result may be found in Stone [8]. If $g \in \mathrm{F}$ and S_n/b_n is as in Theorem 5.1 then

$$P^n g(0) \sim p(0) J(g) b_n^{-d}.$$

Added in Proof: This is true if $|\hat{\mu}(\theta)| \neq 1$ for $\theta \in \Phi$ and $\theta \neq 0$. Otherwise some minor modifications in the proof are necessary.

Consider the case when $d = 1$ and $1 < \alpha \leqq 2$. Then for some slowly varying function $h(n)$,

$$b_n = n^{1/\alpha} h(n)$$

and thus by a standard Abelian theorem

$$c^\lambda = \sum_{n=1}^{\infty} \lambda^n P^n g(0) \sim p(0)\Gamma(2 - 1/\alpha)(1 - 1/\alpha)^{-1}(1 - \lambda)^{-1+1/\alpha} h\left(\frac{1}{1 - \lambda}\right)^{-1}.$$

On the other hand if $d = \alpha$, then $b_n = nh'(n)$ for some slowly varying function $h'(n)$ and thus for some other slowly varying function $h(n)$,

$$c^\lambda \sim p(0) h\left(\frac{1}{1 - \lambda}\right)^{-1}.$$

By Theorem 3.2 we know that, uniformly in x on compacts,

$$\lim_{\lambda \uparrow 1} (1 - \lambda) c^\lambda Q_B^\lambda(x) = \lim_{\lambda \uparrow 1} L_B^\lambda(x) = L_B(x)$$

and thus by standard Tauberian theorems, for $1 < \alpha \leqq 2$,

$$P_x(T_B > n) \sim L_B(x) p(0) \frac{n^{1/\alpha - 1} h(n)\left(1 - \dfrac{1}{\alpha}\right)\left(\dfrac{1}{\alpha}\right)}{\Gamma\left(2 - \dfrac{1}{\alpha}\right)\Gamma\left(1 + \dfrac{1}{\alpha}\right)}.$$

This, together with the fact that

$$\left[\sum_{k=1}^{n} b_k^{-1}\right]^{-1} \sim (1 - 1/\alpha)n^{1/\alpha-1}h(n),$$

establishes Theorem 5.1 in the case $1 < \alpha \leqq 2$. The result for the case $\alpha = d$ follows by essentially the same argument. The stronger assertions made in the non-singular case follow from the results in section 4.

Proof of Theorem 5.2. Let

$$\psi_k^\lambda(x) = \sum_{n=1}^{\infty} \lambda^n P_x(N_n(B) = k).$$

Then it is easy to see that for $k \geqq 1$,

$$\psi_k^\lambda(x) = \int_B \Pi_B^\lambda(x, dz)\psi_{k-1}^\lambda(z).$$

An easy induction argument shows that

$$\lim_{\lambda \uparrow 1} (1 - \lambda)c^\lambda \psi_k^\lambda(x) = \int_B \Pi_B^k(x, dz)L_B(z),$$

and thus as $\lambda \uparrow 1$

$$\sum_{n=0}^{\infty} P_x(N_n \leqq k)\lambda^n \sim (I + \Pi_B + \cdots \Pi_B^k)L_B(z)[(1 - \lambda)c^\lambda]^{-1}.$$

Since for each fixed k, $P_x(N_n(B) \leqq k)$ is monotone in n, we may conclude that as $n \to \infty$,

$$P_x(N_n(B) \leqq k) \sim \frac{(I + \Pi_B + \cdots \Pi_B^k)L_B(z)}{\left[\sum_{j=1}^{n} b_j^{-d}\right]^{-1}},$$

from which the theorem follows.

Proof of Theorem 5.3. It suffices to establish the theorem for sets A, B such that $A \subset B$, and B has non-empty interior. Set

$$E_B(n) = \int P_x(T_B \leqq n)\, dx,$$

and set $\varphi_n(y) = P_y(T_A > n)$. Then since

$$P_x(T_A \leqq n) = P_x(T_B \leqq n) - \sum_{k=1}^{n} \int_{B-A} {}_BP^k(x, dz)\varphi_{n-k}(z),$$

we see that

$$E_A(n) = E_B(n) - \int \sum_{k=1}^{n} {}_BP^k I_{B-A}\varphi_{n-k}(x)\, dx$$

$$= E_B(n) - \sum_{k=1}^{n} [{}_BP^k I_{B-A}\varphi_{n-k}, 1_X].$$

Thus to establish the result it is enough to show that

$$\lim_{n\to\infty} \frac{1}{E_B(n)} \sum_{k=1}^{n} [{}_BP^k I_{B-A}\varphi_{n-k}\, ,\, 1_X] = 0. \tag{5.1}$$

Now

$$\begin{aligned}\sum_{k=1}^{n} [{}_BP^k I_{B-A}\varphi_{n-k}\, ,\, 1_X] &\leqq \sum_{k=1}^{n} [1_B\varphi_{n-k}\, ,\, {}_B\tilde{P}^k 1_X] \\ &\leqq \sum_{k=1}^{n} \sup_x 1_B(x)\varphi_{n-k}(x) \int_B \tilde{P}_x(T_B \geqq k)\, dx.\end{aligned} \tag{5.2}$$

But, in general,

$$\lim_{n\to\infty} \sup_x 1_B(x)\varphi_{n-k}(x) = 0$$

whenever $B \in \mathrm{B}$ has non-empty interior. Since

$$E_B(n) = \sum_{k=1}^{n} \int_B \tilde{P}_x(T_B \geqq k)\, dx$$

we see that (5.1) follows from (5.2) by a simple summability argument.

In the proofs of Theorems 5.4 and 5.5 we will exclude the case $d = 1$, $\sigma^2 < \infty$. The result for this case is an easy consequence of Corollary 5.1.

Proof of Theorem 5.4. Observe that for $n \geqq 1$

$$P_x(T_B > n) = \int_{B^c} G_B(x, dz) P_z(T_B = n) \tag{5.3}$$

and thus

$$\int_A P_x(T_B > n)\, dx = \int_{B^c} \tilde{G}_B(z, A) P_z(T_B = n)\, dz.$$

Consequently

$$\begin{aligned}\sum_{j=1}^{n} \int_A P_x(T_B > j)\, dx &= \int_{B^c} \tilde{G}_B(z, A) P_z(T_B \leqq n)\, dz \\ &= \int_{B^c} \left[\tilde{G}_B(z, A) - \int_A L_B(x)\, dx\right] P_z(T_B \leqq n) + \int_A L_B(x)\, dx \int_{B^c} P_z(T_B \leqq n)\, dz.\end{aligned}$$

Also, as $n \to \infty$,

$$\int_A L_B(x)\, dx \int_{B^c} P_z(T_B \leqq n)\, dz \sim \int_A L_B(x) E_B(n)\, dx.$$

Thus to establish the theorem we need only show that

$$\lim_{n\to\infty} \frac{\int_{B^c} \left[\tilde{G}_B(z, A) - \int_A L_B(x)\, dx\right] P_z(T_B \leqq n)\, dz}{E_B(n)} = 0.$$

However this easily follows from the fact that

$$\operatorname*{Lim}_{|z|\to\infty} \tilde{G}_B(z, A) = \int_A L_B(x)\, dx.$$

Proof of Theorem 5.5. From (5.3) we see that

$$\left(\frac{1}{E_B(n)}\right) \sum_{j=1}^{n} P_x(T_B > j) = \left(\frac{1}{E_B(n)}\right) \int_{B^c} G_B(x, dz) P_z(T_B \leqq n).$$

From the results of section 4 we know that

$$G_B(x, dz) = g_B(x, z)\, dz + {}_2G_B(x, dz)$$

where

$$\operatorname*{Lim}_{|z|\to\infty} g_B(x, z) = L_B(x)$$

uniformly in x on compacts and ${}_2G_B(x, X)$ is bounded on compacts. The theorem now easily follows from these three facts.

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