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Hodge decomposition and solution formulas for some first-order time-dependent parabolic operators with non-constant coefficients

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Abstract In this paper, we present a Hodge decomposition for the L_p-space of some parabolic first-order partial differential operators with non-constant coefficients. This is done over different types of domains in Euclidean space $\mathbb{R}^n$ and on some conformally flat cylinders and the n-torus associated with different spinor bundles. Initially, we apply a regularization procedure in order to control the non-removable singularities over the hyperplane $t = 0$. Using the setting of Clifford algebras combined with a Witt basis, we introduce some specific integral and projection operators. We present an L_p-decomposition where one of the components is the kernel of the regularized parabolic Dirac operator with non-constant coefficients. After that, we study the behavior of the solutions and the validity of our results when the regularization parameter tends to zero. To round off, we give some analytic solution formulas for the special context of domains on cylinders and n-tori.

Keywords Schrödinger equation on manifolds · Regularized parabolic Dirac operator · Hypoelliptic equations · Regularization procedure · Hodge decomposition

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1 Introduction

Time evolution problems are of extreme importance in mathematical physics. Although such kind of problems are being studied by a large community of mathematicians and physicists,

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there is nevertheless still a strong need to develop further special techniques, in particular if we want to construct analytic representations for the solutions in special geometric settings.

One useful method that one frequently applies in the study of PDE's is the factorization of their associated second-order operators in terms of first-order operators. Under certain conditions, this factorization procedure allows us to obtain an orthogonal decomposition of the L_2-space. In that orthogonal decomposition, one of the components is the kernel of the corresponding first-order operator. This decomposition, when applicable, is one of the most interesting aspects of complex and hypercomplex analysis which in turn offers quite useful applications, especially to the theory of partial differential equations. In [7,12], such an orthogonal decomposition has been used in order to study elliptic boundary value problems of mathematical physics over bounded domains in scales of Hilbert spaces, like the stationary Navier–Stokes equations. The treatment of the non-stationary cases, however, carries additional difficulties due to the time-dependence. The time-dependence implies a much more complicated structure of the singularities of the corresponding fundamental solutions.

The aim of this paper is to present a Hodge decomposition for the case of the non-stationary operator $D_- M D_- - i\partial_t$ operator, where D_- is the backward parabolic Dirac operator (used for instance in [4]) and where M is a non-constant scalar L_2- homeomorphism such that $D_- M D_-$ is invertible. Then, we also extend some of the results that we are going to develop in the first part of the paper to the geometric context of some higher dimensional conformally flat cylinders and the n-torus with different conformally inequivalent spin structures. The treatment of this time-dependent operator involves an additional difficulty since the fundamental solution of D_- possesses non-removable singularities in the hyperplane $t = 0$. To overcome this problem, we use a standard regularization procedure (cf. for instance [13,14]) which gives us some degree of control over these singularities. This procedure then enables us to apply the well-known theory of hypoelliptic operators. For more details about the hypoelliptic theory, we refer the interested reader to [1].

To summarize, the paper is structured as follows: in the section "Preliminaries," we introduce some basic notions about Clifford analysis, we introduce a Witt basis and we present the regularization procedure that will be implemented. In the same section, we recall the definition of the regularized Teodorescu operator, the regularized Cauchy–Bitsadze operator and the regularized parabolic Dirac operator [4,9].

In the third section, we are going to prove a regularized Hodge decomposition for the space $L_p(\Omega)$, where Ω is a time-dependent and non-cylindrical domain. Furthermore, we will study the behavior of this decomposition for the limit case.

In Sect. 4, we revisit the particular geometric context of considering these boundary value problems for spinor sections on some conformally flat-n-dimensional cylinder and tori. As an application of the Hodge decomposition theorems proved in Sect. 3, we present some representation formulas for the solutions for the non-stationary Schrödinger-type operator $D_- M D_- - iM\partial_t$ expressed in terms of the explicitly computed fundamental solution for these manifolds. These geometric models belong to the most basic ones in modern quantum theory and quantum gravity and serve as useful models in cosmology.

2 Preliminaries

2.1 Hypercomplex and hypoelliptic analysis

We consider the n-dimensional vector space $\mathbb{R}^n$ endowed with an orthonormal basis $\{e_1, \cdots, e_n\}$. We define the universal real Clifford algebra $C\ell_{0,n}$ as the 2^n-dimensional associative algebra in which the multiplication rules $e_i e_j + e_j e_i = -2\delta_{i,j}$ hold. A vector space basis for $C\ell_{0,n}$ is given by the elements $e_0 = 1$ and $e_A = e_{h_1} \dots e_{h_k}$, where $A = \{h_1, \dots, h_k\} \subset M = \{1, \dots, n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$. Each element $x \in C\ell_{0,n}$ can be represented in the form $x = \sum_A x_A e_A,\ x_A \in \mathbb{R}$. The Clifford conjugation is defined by $\overline{e_j} = -e_j$ for all $j = 1, \dots, n$, and we have $\overline{ab} = \overline{b}\,\overline{a}$.

We introduce the complexified Clifford algebra $\mathbb{C}_n$ as the tensor product

$$\mathbb{C} \otimes C\ell_{0,n} = \left\{ w = \sum_A w_A e_A, w_A \in \mathbb{C}, A \subset M \right\}$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, that means $ie_j = e_j i$ for all $j = 1, \dots, n$. To avoid ambiguities with the Clifford conjugation, we denote the complex conjugation, which maps a complex scalar $a_A = a_{A0} + ia_{A1}$ with real components a_{A0} and a_{A1} onto $a_A = a_{A0} - ia_{A1}$, by $\sharp$. The complex conjugation leaves the elements e_j invariant, i.e., $e_j^\sharp = e_j$ for all $j = 1, \dots, n$. We also have a pseudonorm on $\mathbb{C}$ defined by $|a| := \sum_A |a_A|$ where $a = \sum_A a_A e_A$, as usual. Notice also that for $a, b \in \mathbb{C}_n$ we only have $|ab| \leq 2^n |a||b|$. The other norm criteria are fulfilled.

Next, we introduce the Euclidean Dirac operator $D = \sum_{j=1}^n e_j \partial_{x_j}$. The latter factorizes the n-dimensional Euclidean Laplacian, that is, $D^2 = -\Delta = -\sum_{j=1}^n \partial x_j^2$. A $\mathbb{C}_n$-valued function that is defined on an open set $U \subseteq \mathbb{R}^n$, $u : U \mapsto \mathbb{C}_n$, is called *left-monogenic* if it satisfies $Du = 0$ on U (resp. *right-monogenic* if it satisfies $uD = 0$ on U).

A function $u : U \mapsto \mathbb{C}_n$ has a representation $u = \sum_A u_A e_A$ with $\mathbb{C}$-valued components u_A. Such a function is continuous if each complex component is continuous in the usual way.

In order to deal with time-dependent problems, we will embed $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$ in the same way as done in [5]. For that purpose, we add two new basis elements $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ that satisfy

$$\mathfrak{f}^2 = \mathfrak{f}^{\dagger 2} = 0, \quad \mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f} = 1, \quad \mathfrak{f}e_j + e_j\mathfrak{f} = \mathfrak{f}^\dagger e_j + e_j\mathfrak{f}^\dagger = 0, \quad j = 1, \dots, n. \tag{1}$$

The extended basis is often called a Witt basis. This construction allows us to use a suitable factorization of the time evolution operators where only partial derivatives are used.

In all that follows, we shall consider $\mathbb{C}_n$-valued maps f from a time-dependent domain $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^+$, i.e., functions in the variables $(x_1, x_2, \dots, x_n; t)$ where $x_i \in \mathbb{R}$ for $i = 1, \dots, n$ and $t \in \mathbb{R}^+$ where $\mathbb{C}_n$ is the complexified Clifford algebra generated by the extended basis $e_1, \dots, e_n, \mathfrak{f}, \mathfrak{f}^\dagger$. For the sake of readability we abbreviate the space-time tuple $(x_1, x_2, \dots, x_n; t)$ simply by (x, t), assigning $x = x_1 e_1 + \cdots x_n e_n$. For additional details on Clifford analysis, we refer the interested reader for instance to [6,7].

In the sequel we will also use the short notation $L_p(\Omega)$, $C^k(\Omega)$, etc., to abbreviate for instance $L_p(\Omega, \mathbb{C}_n) := \{f : \Omega \to \mathbb{C}_n \mid \int_\Omega |f(x_1, \dots, x_n; t)|^p \mathrm{d}x_1 \dots \mathrm{d}x_n \mathrm{d}t < +\infty\}$ and $C^k(\Omega, \mathbb{C}_n)$. For convenience, we also recall the notation $W_p^a(\Omega)$ for the Sobolev space of $L_p(\Omega)$ functions that are a-times weakly differentiable in the sense of Sobolev. Furthermore, as usual, the notation $\overset{\circ}{W}{}_p^a(\Omega)$ will be used for the subspace of functions of $W_p^a(\Omega)$ that satisfy $f = 0$ at the boundary of Ω.

In the particular case $p = 2$ one can endow the Banach space $L_2(\Omega)$ with the structure of a Hilbert $\mathbb{C}_n$-module by introducing the following sort of "inner product"

$$\langle f, g \rangle_{L_2} := \int_\Omega \overline{f(x,t)}^\sharp g(x,t)\, \mathrm{d}x\mathrm{d}t, \quad f,\ g \in L_2(\Omega).$$

Actually, this inner product is $\mathbb{C}_n$-valued; however, this is a rather broadly used definition in Clifford analysis. In the real Clifford analysis setting this inner product has already used in the 1980s, see for instance [2]. In [3] this inner product has been used in the context of complexified Clifford analysis where explicit reproducing Hilbert space kernels have been computed.

Next, following for instance [1] (Section 1.1), a partial differential operator is hypoelliptic if and only if its fundamental solution is a C^∞ function in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0,0)\}$.

2.2 Regularization procedure

2.2.1 Regularized fundamental solution and regularized Teodorescu operator

Following for instance [4], we know that the fundamental solution E_- for the parabolic Dirac operator $D_\pm = D + \mathfrak{f}\partial_t \pm i\mathfrak{f}^\dagger$ has singularities at all the points of the hyperplane $t = 0$. This represents an important difference to the nature of the fundamental solution of hypoelliptic operators, involving only the classical 1-point singularity. Moreover, these singularities are not removable by standard calculation methods. This property carries an additional problem for the study of the related Teodorescu and Cauchy–Bitsadze operators. One cannot guarantee the convergence of the integrals that define those operators in the classical sense, and that appear in the construction of the Hodge decomposition for the L_p-space with general $1 \leq p < \infty$ in terms of the integral kernel of the first-order operator.

In order to overcome and to solve this problem we need to regularize the fundamental solution as well as the associated operators (cf. [4,9,13]). This process of regularization creates a sequence of operators and associated fundamental solutions E_-^ϵ, which are locally integrable in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0,0)\}$. Moreover, these families of operators (resp. families of the associated fundamental solutions) will converge to the original operators (resp. to their fundamental solutions) when the regularization parameter $\epsilon \to 0^+$.

In this sense we recall the following definitions and results.

Definition 1 (*cf.* [4]). For a function $u \in W_p^a(\Omega)$, with $1 \leq p < +\infty$ and $a \in \mathbb{N}$, we define the forward/backward regularized parabolic Dirac operator as

$$D_\pm^\epsilon u = \left(D + \mathfrak{f}\partial_t \pm \mathbf{k}\mathfrak{f}^\dagger\right)u, \tag{2}$$

where $\mathbf{k} = \frac{\epsilon+i}{\epsilon^2+1}$ and $D = \sum_{j=1}^n e_j \partial_{x_j}$ stands for the (spatial) Dirac operator.

For this regularized operator we have that $D_\pm^\epsilon : W_p^1(\Omega) \to L_p(\Omega)$. We have the following result

Theorem 1 (cf. [4]). *For the sequence of parabolic Dirac operators D_-^ϵ, with $\epsilon > 0$, we have the following convergence*

$$||D_- - D_-^\epsilon||_{L_p(\Omega)} \to 0,$$

where $D_- := D + \mathfrak{f}\partial_t - i\mathfrak{f}^\dagger$, when $\epsilon \to 0$.

Moreover, we have the family of regularized fundamental solutions for this first-order operator

Definition 2 (*cf.* [4]). A fundamental solution $E_-^\epsilon(x,t)$ of the first-order operator D_-^ϵ is defined as follows

$$E_-^\epsilon(x,t) = e_-^\epsilon(x,t)\left[\frac{-x}{2(\epsilon+i)t} + \mathfrak{f}\left(\frac{-n}{2t} + \frac{|x|^2}{4(\epsilon+i)t^2}\right) - \mathbf{k}\mathfrak{f}^\dagger\right]. \tag{3}$$

Here $e_-^\epsilon(x,t) := (\epsilon+i)e_-(x,(\epsilon+i)t)$, where

$$e_-(x,t) = i\frac{H(t)}{(4\pi i t)^{n/2}}\exp\left(-i\frac{|x|^2}{4t}\right).$$

is the fundamental solution of the usual Schrödinger operator $\Delta - i\partial_t$, $\Delta = \sum_{i=1}^n \partial x_i^2$ and $H(t)$ represents the ordinary Heaviside function.

The fundamental solution of the regularized operator D_-^ϵ now allows us to explicitly express and to introduce the corresponding Teodorescu operator and Cauchy–Bitsadze operator for D_-^ϵ. Again, following [4], we may introduce

Definition 3 Let $u \in L_p(\Omega)$. The regularized Teodorescu operator is defined as

$$T_-^\epsilon u(x,t) = \int_\Omega E_-^\epsilon(x-z,t-s)u(z,s)\mathrm{d}z\mathrm{d}s, \quad (x,t)\notin\partial\Omega. \tag{4}$$

As in the stationary elliptic cases, treated for instance in [7] (Chapter 3), also in this context here the regularized Teodorescu operator represents the right inverse of the regularized Dirac operator. To be more precise we recall

Theorem 2 (cf. [4]). *The regularized Teodorescu operator T_-^ϵ is the right inverse of the regularized parabolic Dirac operator D_-^ϵ, i.e., for a function $u \in L_p(\Omega)$ we have the following equality*

$$(D_-^\epsilon T_-^\epsilon u)(x,t) = u(x,t),$$

for every $(x,t)\in\Omega$.

The Teodorescu operator has a natural analogue for the integration over the boundary, namely

Definition 4 (c.f. [4]) Let $u \in W_p^{a-\frac{1}{p}}(\partial\Omega)$, $a\in\mathbb{N}$. We define the regularized Cauchy–Bitsadze operator as

$$F_-^\epsilon u(x,t) = \int_{\partial\Omega} E_-^\epsilon(x-z,t-r)\mathrm{d}\sigma_{z,r}u(z,r),\ (x,t)\notin\partial\Omega. \tag{5}$$

Instead of $\partial\Omega$ we often also use the abbreviated notation $\Gamma := \partial\Omega$ when no ambiguity or misunderstanding may occur.

These two integral operators will provide us with the fundamental tools to express the solution of the related boundary value problems that we want to discuss later in this paper in the particular context of cylinders and tori.

3 L_p-decomposition in a general Lipschitz domain $\Omega \subset \mathbb{R}^n$

3.1 The regularized case

In all that follows we always assume that $\Omega \subseteq \mathbb{R}^n\times\mathbb{R}^+$ is a strongly Lipschitz domain. For convenience we recall its definition.

Definition 5 (c.f. [11]) A domain $G \subset \mathbb{R}^n$ is called a strongly Lipschitz if the boundary ∂G can be covered by a finite number of open sets V_i, $(i = 1, \ldots, k)$ such that each set $G \cap V_i$, $(i = 1, \ldots, k)$ can be represented by the inequality $y_n > g(y_1, \ldots, y_{n-1})$, where g is a Lipschitz continuous function.

Moreover, the symbol $\oplus_M$ stands for a direct sum, in which all the terms are multiplied by the function M. Here, and in all that follows, M is a non-constant scalar L_2-homeomorphism such that the operator $D_- M D_-$ is invertible. The ideas presented in [5] allow us to establish the following results about the decomposition of L_p-spaces.

Theorem 3 *The space* $L_p(\Omega)$, $1 \leq p < +\infty$ *admits the following decomposition*

$$L_p(\Omega) = \left(L_p(\Omega) \cap \left(M\ker\left(D_-^{\epsilon}\right)\right)\right) \oplus_M D_-^{\epsilon}\left(\overset{\circ}{W}{}_p^1(\Omega)\right), \tag{6}$$

for all $\epsilon > 0$, *and we can define the following projectors*

$$P_M^{\epsilon} : L_p(\Omega) \to L_p(\Omega) \cap \left(M\ker\left(D_-^{\epsilon}\right)\right)$$
$$Q_M^{\epsilon} : L_p(\Omega) \to D_-^{\epsilon}\left(\overset{\circ}{W}{}_p^1(\Omega)\right),$$

where P_M^{ϵ} *and* Q_M^{ϵ} *are called Bergman projectors.*

Proof Since the operator D_-^{ϵ} is hypoelliptic, i.e., its fundamental solution (3) is a C^{∞} function in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0, 0)\}$ (for more details see [1], Section 1.1), we may immediately infer that the operator $D_-^{\epsilon} M D_-^{\epsilon}$ is hypoelliptic, too. Under these conditions and in view of [7] (Section 3.6), [14], we can guarantee the existence and uniqueness of the operator $(D_-^{\epsilon} M D_-^{\epsilon})_0^{-1}$ for the boundary value problem

$$\begin{cases} (D_-^{\epsilon} M D_-^{\epsilon})u = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases},$$

i.e., $(D_-^{\epsilon} M D_-^{\epsilon})_0^{-1}$ is such that

$$\begin{cases} u = (D_-^{\epsilon} M D_-^{\epsilon})^{-1} f & \text{in } \Omega \\ (D_-^{\epsilon} M D_-^{\epsilon})^{-1} u = 0 & \text{in } \partial\Omega \end{cases}.$$

As a first step we take a look at the intersection of the two subspaces $D_-^{\epsilon}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$ and $L_p(\Omega) \cap \left(M \ker\left(D_-^{\epsilon}\right)\right)$. Consider a function u in $\left(L_p(\Omega) \cap \left(M\ker\left(D_-^{\epsilon}\right)\right)\right) \cap D_-^{\epsilon}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$. It is immediate to see that $u = D_-^{\epsilon} v$, with $M D_-^{\epsilon} v = 0$ in Ω. There exists a function $w \in \overset{\circ}{W}{}_p^1(\Omega)$ such that $D_-^{\epsilon} w = u$ and $D_-^{\epsilon} M D_-^{\epsilon} w = 0$. This ia a consequence of $u \in D_-^{\epsilon}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$. If we apply $(D_-^{\epsilon} M D_-^{\epsilon})_0^{-1}$, then we get $w = 0$. Consequently, $u = 0$, i.e., the intersection of these subspaces only contains the zero function. Therefore, our sum is a direct sum.

Now, let us consider $u \in L_p(\Omega)$. We have

$$u_2 = M D_-^{\epsilon} (D_-^{\epsilon} M D_-^{\epsilon})_0^{-1} D_-^{\epsilon} u \in D_-^{\epsilon}\left(\overset{\circ}{W}{}_p^1(\Omega)\right).$$

Applying D_{-}^{ϵ} to $u_1^{\epsilon} = u - u_2^{\epsilon}$, we obtain

$$\begin{aligned} MD_{-}^{\epsilon}u_1 &= MD_{-}^{\epsilon}u - MD_{-}^{\epsilon}u_2 \\ &= MD_{-}^{\epsilon}u - MD_{-}^{\epsilon}MD_{-}^{\epsilon}(D_{-}^{\epsilon}MD_{-}^{\epsilon})_0^{-1}D_{-}^{\epsilon}u \\ &= MD_{-}^{\epsilon}u - M(D_{-}^{\epsilon}MD_{-}^{\epsilon})(D_{-}^{\epsilon}MD_{-}^{\epsilon})_0^{-1}D_{-}^{\epsilon}u \\ &= MD_{-}^{\epsilon}u - MD_{-}^{\epsilon}u \\ &= 0, \end{aligned}$$

i.e., $MD_{-}^{\epsilon}u_1 \in M\ker\left(D_{-}^{\epsilon}\right)$. □

Corollary 1 *For the particular case of $p = 2$, this decomposition is orthogonal.*

Proof The right linear sets $\mathcal{A} = L_2(\Omega) \cap \left(M\ker(D_{-}^{\epsilon})\right)$ and $\mathcal{B} = L_2(\Omega) \ominus \mathcal{A}$ are subspaces of $L_2(\Omega)$. For every function $u \in L_2(\Omega)$, we have that $T_{-}^{\epsilon}u \in W_2^1(\Omega)$. From this property, we may conclude that there exists a function $v \in W_2^1(\Omega)$ with $u = D_{-}^{\epsilon}v$. Let $u = D_{-}^{\epsilon}v \in \mathcal{B}$. Then, we have for all $g \in \mathcal{A}$

$$\int_{\Omega} \overline{M^{-1}D_{-}v}\, M^{-1}g \,\mathrm{d}x\,\mathrm{d}t = 0,$$

which proves the orthogonality of our decomposition in the special case $p = 2$. □

3.2 The limit case of the operator $D_{-}MD_{-} - iM\partial_t$

The aim of this section is to generalize the previous L_p-decomposition to the limit case $\epsilon \to 0$ where we deal with the original parabolic Dirac operator D_{-}; whence, we are in the framework of the generalized non-stationary Schrödinger-type operator of the form $D_{-}MD_{-} - iM\partial_t$, where M is a non-constant scalar L_2-homeomorphism such that $D_{-}MD_{-}$ is invertible.

In order to achieve this, we first recall the following two results proved in [4] concerning the convergence of the families of fundamental solutions $(E_{-}^{\epsilon})_{\epsilon>0}$ and the regularized Teodorescu operators $(T_{-}^{\epsilon})_{\epsilon>0}$.

Theorem 4 (cf. [4]) *For all $1 \leq p < +\infty$, we have the following weak convergence in $W_p^{-\frac{n}{2}-1}(\Omega)$,*

$$\left\langle E_{-}^{\epsilon}, \varphi\right\rangle \to \langle E_{-}, \varphi\rangle, \varphi \in W_p^{\frac{n}{2}+1}(\Omega), \tag{7}$$

when $\epsilon \to 0$.

Theorem 5 (cf. [4]) *The family of regularized Teodorescu operators T_{-}^{ϵ} converges weakly to T_{-} in $W_p^{-\frac{n}{2}-1}(\Omega)$ for all $1 \leq p < +\infty$.*

With these results, we are in position to study the convergence of the family of projectors Q_M^{ϵ} to the projector Q_M, with $Q_M : L_p(\Omega) \to D_{-}\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$.

Theorem 6 *The family of projectors Q_M^{ϵ} is a fundamental family in $W_p^{-\frac{n}{2}-1}(\Omega)$ for all $1 \leq p < +\infty$.*

Proof Let us start with the proof of the convergence. Consider $u \in L_p(\Omega)$ and $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$, where $1 \leq p < +\infty$. For all $\epsilon > 0$, we have $(Q_M^\epsilon)^2 = Q_M^\epsilon$ and $Q_M^\epsilon(P_M^\epsilon u) = 0$. Therefore, we have for any $\epsilon_1, \epsilon_2 > 0$

$$\begin{aligned}\left|\langle Q_M^{\epsilon_1}u - Q_M^{\epsilon_2}u,\, \varphi\rangle\right| &= \left|\langle Q_M^{\epsilon_1}(P_M^{\epsilon_1}u + Q_M^{\epsilon_1}u) - Q_M^{\epsilon_2}(P_M^{\epsilon_1}u + Q_M^{\epsilon_1}u),\, \varphi\rangle\right| \\ &= \left|\langle Q_M^{\epsilon_1}u - Q_M^{\epsilon_2}P_M^{\epsilon_2}u - Q_M^{\epsilon_2}Q_M^{\epsilon_1}u,\, \varphi\rangle\right| \\ &\leq \underbrace{\left|\langle Q_M^{\epsilon_2}P_M^{\epsilon_1}u,\, \varphi\rangle\right|}_{\textbf{(A)}} + \underbrace{\left|\langle (I - Q_M^{\epsilon_2})Q_M^{\epsilon_1}u,\, \varphi\rangle\right|}_{\textbf{(B)}}.\end{aligned}$$

For the projector P_M^ϵ and Q_M^ϵ defined in Theorem 3, taking into account the mapping properties of the regularized operators D_-^ϵ, T_-^ϵ and F_-^ϵ studied in [4], and after some calculations, we obtain for the term (**A**)

$$\begin{aligned}\left|\langle Q_M^{\epsilon_2}P_M^{\epsilon_1}u,\, \varphi\rangle\right| &= \left|\langle Q_M^{\epsilon_2}(F_-^{\epsilon_1}P_M^{\epsilon_1} - Q_M^{\epsilon_2}F_-^{\epsilon_2})P_M^{\epsilon_1}u, \varphi\rangle\right| \\ &= \left|\langle Q_M^{\epsilon_2}(I - T_-^{\epsilon_1}D_-^{\epsilon_1} - (I - T_-^{\epsilon_2}D_-^{\epsilon_2}))P_M^{\epsilon_1}u,\, \varphi\rangle\right| \\ &= \left|\langle Q_M^{\epsilon_2}(T_-^{\epsilon_1}D_-^{\epsilon_1} - T_-^{\epsilon_2}D_-^{\epsilon_2})P_M^{\epsilon_1}u,\, \varphi\rangle\right| \\ &= \left|\langle Q_M^{\epsilon_2}(T_-^{\epsilon_1}(D_-^{\epsilon_1} - D_-^{\epsilon_2}) + (T_-^{\epsilon_1} - T_-^{\epsilon_2})D_-^{\epsilon_2})P_M^{\epsilon_1}u,\, \varphi\rangle\right|\end{aligned}$$

Relying on Theorems 1 and 5, we obtain the weak convergence of (**A**), in $W_p^{-\frac{n}{2}-1}(\Omega)$ for all $1 \leq p < +\infty$ to zero. Finally, since $Q_M^{\epsilon_1}u \in D_-\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$, there exists a $g \in \overset{\circ}{W}{}_p^1(\Omega)$ such that $u = D_-^\epsilon g$. Therefore, (**B**) becomes

$$\begin{aligned}\left|\langle (I - Q_M^{\epsilon_2})Q_M^{\epsilon_1}u,\, \varphi\rangle\right| &= \left|\langle (I - Q_M^{\epsilon_2})D_-^\epsilon g,\, \varphi\rangle\right| \\ &= \left|\langle D_-^{\epsilon_1}g - Q_M^{\epsilon_2}D_-^{\epsilon_1}g + D_-^{\epsilon_2}g - D_-^{\epsilon_2}g\, \varphi\rangle\right| \\ &= \left|\langle Q_M^{\epsilon_2}(D_-^\epsilon g - D_-g) + (D_-g - D_-^\epsilon g),\, \varphi\rangle\right| \\ &= \left|\langle (D_-g - D_-^\epsilon g)(I - Q_M^{\epsilon_1}),\, \varphi\rangle\right|.\end{aligned}$$

By Theorem 1, we conclude that the preceding expression tends to zero as $\epsilon \to 0$. □

Now it remains to prove that Q_M is idempotent. Hereby, we have

$$Q_M^2 = \lim_{\epsilon \to 0}(Q_M^\epsilon)^2 = \lim_{\epsilon \to 0} Q_M^\epsilon = Q_M.$$

Theorem 7 *For a given $f \in L_p(\Omega)$, consider the solutions (u^ϵ) to the problem*

$$\begin{cases}(D_-^\epsilon M D_-^\epsilon)u^\epsilon = f \\ \qquad\quad u^\epsilon|_\Gamma = 0\end{cases}, \tag{8}$$

for each $\epsilon > 0$. Then, the family of such solutions (u^ϵ) is a fundamental family in $W_p^{-\frac{n}{p}-1}(\Omega)$, for all $1 \leq p < +\infty$. Moreover, $(D_-^\epsilon u^\epsilon)$ is a fundamental family in $W_p^{-\frac{n}{p}-1}(\Omega)$.

Proof Let us consider $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$, $f \in L_p(\Omega)$ and a family of functions (u^ϵ), such that $u^\epsilon \in D_-^\epsilon(\Omega)$ with $\epsilon > 0$, and $\epsilon_1, \epsilon_2 > 0$. Since the elements of the family are solution of the Problem (8), we have that

$$u^\epsilon = T_-^\epsilon M^{-1} Q_M^\epsilon T_-^\epsilon f, \tag{9}$$

(for more details about this assertion, we refer the reader to [4]). Then

$$\begin{aligned}
\left|\langle u^{\epsilon_1}-u^{\epsilon_2},\varphi\rangle\right| &= \left|\langle T_-^{\epsilon_1}M^{-1}Q_M^{\epsilon_1}T_-^{\epsilon_1}f - T_-^{\epsilon_2}M^{-1}Q_M^{\epsilon_2}T_-^{\epsilon_2}f,\varphi\rangle\right| \\
&= \left|\langle \left(T_-^{\epsilon_1}M^{-1}Q_M^{\epsilon_1}T_-^{\epsilon_1} - T_-^{\epsilon_2}M^{-1}Q_M^{\epsilon_2}T_-^{\epsilon_2}\right)f,\varphi\rangle\right| \\
&\leq \left|\langle \left(T_-^{\epsilon_1}M^{-1}Q_M^{\epsilon_1}\left(T_-^{\epsilon_1}-T_-^{\epsilon_2}\right)\right)f,\varphi\rangle\right| + \left|\langle \left(\left(T_-^{\epsilon_1}-T_-^{\epsilon_2}\right)M^{-1}Q_M^{\epsilon_2}T_-^{\epsilon_2}\right)f,\varphi\rangle\right| \\
&\quad + \left|\langle \left(T_-^{\epsilon_1}M^{-1}\left(Q_M^{\epsilon_1}-Q_M^{\epsilon_2}\right)T_-^{\epsilon_2}\right)f,\varphi\rangle\right|.
\end{aligned}$$

From Theorem 5 and 6, we may conclude that the right-hand side of the previous inequality tends to zero when $\epsilon_1, \epsilon_2 \to 0$. Moreover, we can guarantee that there exists a function $f \in L_p(\Omega)$ such that

$$D_-^{\epsilon_1}u^{\epsilon_1} = M^{-1}Q_M^{\epsilon_1}T_-^{\epsilon_1}f \quad \text{and} \quad D_-^{\epsilon_2}u^{\epsilon_2} = M^{-1}Q_M^{\epsilon_2}T_-^{\epsilon_2}f.$$

This in turn implies that

$$\begin{aligned}
\left|\langle \left(M^{-1}Q_M^{\epsilon_1}T_-^{\epsilon_1} - M^{-1}Q_M^{\epsilon_2}T_-^{\epsilon_2}\right)f,\varphi\rangle\right| &\leq \left|\langle \left(M^{-1}Q_M^{\epsilon_1}\left(T_-^{\epsilon_1}-T_-^{\epsilon_2}\right)\right)f,\varphi\rangle\right| \\
&\quad + \left|\langle \left(M^{-1}\left(Q_M^{\epsilon_1}-Q_M^{\epsilon_2}\right)T_-^{\epsilon_2}\right)f,\varphi\rangle\right|.
\end{aligned}$$

By Theorem 6 and 7, we conclude that the right-hand side of the previous expression converges weakly to zero when $|\epsilon_1-\epsilon_2| \to 0$, in $W_p^{-\frac{n}{2}-1}(\Omega)$, for all $1 \leq p < +\infty$. □

This result can be refined. In fact, let us denote by $u_2 \in W_p^{-\frac{n}{2}-1}(\Omega)$ the function limit of the Cauchy family that we studied. Relying on Theorem 7, we can guarantee the existence of $f \in L_p(\Omega)$ such that

$$(D_-MD_-)u_2 = f \quad \text{and} \quad (D_-MD_-)u_2^{\epsilon} = f,$$

with $u_2|_\Gamma = 0 = u_2^{\epsilon}|_\Gamma$. Since $(D_-)^{-1}$ exists and since it is unique (for more details see [4,14]), we can establish the following equality

$$u_2 - u_2^{\epsilon} = (D_-MD_-)^{-1}\left((D_-MD_-)-(D_-MD_-)\right)u_2^{\epsilon},$$

which implies that

$$||u_2-u_2^{\epsilon}||_{L_p(\Omega)} = ||(D_-MD_-)^{-1}||\;||(D_-^{\epsilon}MD_-^{\epsilon})-(D_-MD_-)||\;||u_2^{\epsilon}||_{L_p(\Omega)}.$$

Since $||(D_-^{\epsilon}MD_-^{\epsilon})-(D_-MD_-)||$ converges to zero when $\epsilon \to 0$, we may conclude that the right-hand side of the last expression also converges to zero. This fact implies that $u_2 \in L_p(\Omega)$.

Moreover, we can guarantee

(i) For any two elements $u_2^{\epsilon_1}$ and $u_2^{\epsilon_2}$ of the fundamental family studied in Theorems 6 and 7, there exist functions $g_2^{\epsilon_1}, g_2^{\epsilon_2} \in \overset{\circ}{W}{}_p^1(\Omega)$ such that

$$u_2^{\epsilon_1} = D_-^{\epsilon_1}g_2^{\epsilon_1} \quad \text{and} \quad u_2^{\epsilon_2} = D_-^{\epsilon_2}g_2^{\epsilon_1}$$

and

$$\begin{aligned}
||D_-^{\epsilon_2}(g_2^{\epsilon_1}-g_2^{\epsilon_2})||_{L_p(\Omega)} &= ||D_-^{\epsilon_2}g_2^{\epsilon_1} - D_-^{\epsilon_1}g_2^{\epsilon_1} + D_-^{\epsilon_1}g_2^{\epsilon_2} - D_-^{\epsilon_2}g_2^{\epsilon_2}||_{L_p(\Omega)} \\
&\leq ||\left(D_-^{\epsilon_2}-D_-^{\epsilon_1}\right)g_2^{\epsilon_1}||_{L_p(\Omega)} + ||u_2^{\epsilon_1}-u_2^{\epsilon_2}||_{L_p(\Omega)}.
\end{aligned}$$

By Theorem 1 and 7 and in view of the above described considerations, we conclude that the right-hand side of the previous expression converges to zero, when $|\epsilon_1 - \epsilon_2| \to 0$, i.e.,

$$||D_-^{\epsilon_2}(g_2^{\epsilon_1} - g_2^{\epsilon_2})||_{L_p(\Omega)} \to 0, \quad \text{when } |\epsilon_1 - \epsilon_2| \to 0.$$

Since $||D_-^{\epsilon}|| \to ||D_-|| < \infty$, when $\epsilon \to 0$, we conclude that $g \to g_2^{\epsilon_2} + C$, when $|\epsilon_1 - \epsilon_2| \to 0$ and $\epsilon_1, \epsilon_2 \to 0$, where $C \in \ker(D_-)$.
Under these conditions, we showed that for any function $u \in L_p(\Omega)$, there exists an function $v \in \overset{\circ}{W}{}_p^1(\Omega)$ such that $u = D_- v$.

(ii) Suppose that there are two functions $g_1, g_2 \in \overset{\circ}{W}{}_p^1(\Omega)$ that satisfy

$$u = D_- g_1 \quad \text{and} \quad u = D_- g_2,$$

for the same function $u \in L_p(\Omega)$. We have

$$(D_- M D_-) g_1 = (D_- M D_-) g_2 \Leftrightarrow g_1 = (D_- M D_-)^{-1}(D_- M D_-) g_2 \Leftrightarrow g_1 = g_2,$$

which proves our assertion.

Theorem 8 *For each $u \in L_p(\Omega)$, the family of $P_M^\epsilon u$ converges to $\hat{u}$ in $\big(M\ker(D_-^\epsilon)\big) \cap L_p(\Omega)$, for all $\epsilon > 0$ and $1 \leq p < +\infty$.*

Proof The proof is made in three steps. First consider $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$, a function $u \in L_p(\Omega)$, and a family of functions (u_1^ϵ), where $u_1^\epsilon \in \big(M\ker(D_-^\epsilon)\big) \cap L_p(\Omega)$ with $\epsilon > 0$, with $1 \leq p < +\infty$.

Let $\epsilon_1, \epsilon_2 > 0$. In view of the decomposition (6) we have for $u_1^{\epsilon_1}, u_1^{\epsilon_2}$ in $M\ker(D_-^{\epsilon_1})$, $M\ker(D_-^{\epsilon_2})$

$$\left|\langle u_1^{\epsilon_1} - u_1^{\epsilon_2}, \varphi\rangle\right| = \left|\langle (u - u_2^{\epsilon_1}) - (u - u_2^{\epsilon_2}), \varphi\rangle\right| \leq \left|\langle u_2^{\epsilon_2} - u_2^{\epsilon_1}, \varphi\rangle\right|,$$

where $u_2^{\epsilon_1}$ and $u_2^{\epsilon_1}$ are elements of the fundamental family (u_2^ϵ), where $u_2^\epsilon \in D_-^\epsilon(\overset{\circ}{W}{}_p^1(\Omega))$ for $\epsilon > 0$. From Theorem 7, we may conclude that the right-hand side of the previous expression converges weakly to zero, in $W_p^{-\frac{n}{2}-1}(\Omega)$, when $|\epsilon_1 - \epsilon_2| \to 0$. This establishes that (P_M^ϵ) is a fundamental family in $W_p^{-\frac{n}{2}-1}(\Omega)$.

Now we can refine our conclusion. On the basis of the techniques and arguments that we deduced for the family $D_-^\epsilon u^\epsilon$, with $\epsilon > 0$ and in view of Theorem 7, we can prove that the function limit belongs to $L_p(\Omega)$.

Finally, let us denote by u_1 the function limit of this fundamental family. For a given $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$, with $1 \leq p < +\infty$, we have

$$\left|\langle D_-^\epsilon u_1, \varphi\rangle\right| = \left|\langle D_-^\epsilon u_1 - D_-^\epsilon u_1^\epsilon, \varphi\rangle\right| \leq \left|\langle D_-^\epsilon (u_1 - u_1^\epsilon), \varphi\rangle\right| + \left|\langle (D_- - D_-^\epsilon) u_1^\epsilon(x,t), \varphi\rangle\right|.$$

Theorems 8 and 1 guarantee that the first and the second term of the right-hand side of the previous expression converges to 0 when $\epsilon \to 0$. $\square$

In conclusion, for each $u \in L_p(\Omega)$, we obtain $u = P_M^\epsilon u + Q_M^\epsilon u$. Also, we proved that

$$Q_M^\epsilon u \to Q_M u \quad \text{and} \quad Q_M^2 u = Q_M u,$$

which implies that Q_M is a projector and that we can define a projector P_M as

$$P_M u = u - Q_M u,$$

with $P_M u \in (M\ker(D_-)) \cap L_p(\Omega)$.

As a consequence, we may establish the following main result of this section

Theorem 9 *For* $1 \leq p < +\infty$, *the following decomposition holds,*

$$L_p(\Omega) = \big(L_p(\Omega) \cap (M\ker(D_-))\big) \oplus_M D_-(\overset{\circ}{W}{}^1_p(\Omega)).$$

This theorem allows us now to introduce the following projectors in a meaningful way

$$P_M : L_p(\Omega) \to L_p(\Omega) \cap (M\ker(D_-))$$
$$Q_M : L_p(\Omega) \to D_-\left(\overset{\circ}{W}{}^1_p(\Omega)\right).$$

P_M and Q_M are usually called Bergman projectors. Notice that $Q_M = I - P_M$, where I is the identity operator.

4 Some applications for related BVP on cylinders and tori

As an application of the results and techniques that we developed in the previous section, we are going to present in this section an analytic representation formula for the solutions of the non-stationary inhomogeneous Schrödinger problem

$$\begin{cases} (D_- M D_- - i\partial_t)u(x,t) = f(x,t) \\ u(x,t) = g(x,t) \in \partial\Omega \end{cases}, \quad (x,t) \in \Omega = U \times (0,+\infty) \subseteq \mathbb{R}^n \times \mathbb{R}^+,$$

where M is a non-constant scalar L_2-homeomorphism such that $D_- M D_-$ is invertible, and where U is a bounded strongly Lipschitz domain lying on a class of conformally flat cylinders and n-tori endowed with different spin structures. As mentioned in the introduction, these geometries serve as the most basic geometric examples in quantum theory and cosmology; therefore, we revisit them in detail.

First of all, we call from previous works that a conformally flat manifold in n real variables is a Riemannian manifold with an atlas whose transition functions are Möbius transformations. Notice that in $\mathbb{R}^n$ with $n \geq 3$, the set of Möbius transformations are the only conformal maps in the sense of Gauss. Under this viewpoint, one may regard conformally flat manifolds as higher dimensional generalizations of holomorphic Riemann surfaces.

Following for instance the classical work by Kuiper [10], a large class of conformally flat manifolds can be constructed by factoring out a simply connected domain $U \subseteq \mathbb{R}^n$ by a Kleinian group Γ that acts totally discontinuously on U. In particular, we obtain a number of higher dimensional conformally flat cylinders C_k in n real variables by taking for $U = \mathbb{R}^n$ and for Γ a k-dimensional lattice, for simplicity $\mathbb{Z}^k = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$ where k may be a positive integer from the set $\{1, \ldots, n\}$. In the case $k = n$, we obtain a flat n-torus. In another interesting subcase case represented by $n = 2, k = 1$, we re-obtain the classical infinity cylinder of radius 1 embedded in the three-dimensional Euclidean space.

Let us now fix the notation

$$C_k := \mathbb{R}^n/\mathbb{Z}^k, \quad k = 1, \ldots, n.$$

Since $\mathbb{R}^n$ is the universal covering space of all these generalized cylinders C_k, there exists a well-defined projection map $p_k : \mathbb{R}^n \to C_k, x \mapsto x \bmod \mathbb{Z}^k$. One has $p_k(x) = p_k(y)$ if and only if there exists an $\omega \in \mathbb{Z}^k$ such that $x = y + \omega$.

Next, every subset $U \subset \mathbb{R}^n$ that has the property that $x \in U$ also implies that $x + \omega \in U$ for all $\omega = \sum_{i=1}^{k} \omega_i e_i \in \mathbb{Z}^k$ gives rise to an open subset U' on C_k defined by $U' := p_k(U)$.

More generally, one can consider on C_k 2^k different spinor bundles. To construct them, we decompose the lattice $\mathbb{Z}^k$, as suggested in [8], into the direct sum of the sublattices $\mathbb{Z}^l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$ and $\mathbb{Z}^{k-l} := \mathbb{Z}e_{l+1} + \cdots + \mathbb{Z}e_k$ where l is some integer from $\{1, \ldots, k\}$. We now obtain 2^k conformally inequivalent different spinor bundles $E^{(l)}$ on C_k by making the identification $(x, X) \Longleftrightarrow (x + \omega, (-1)^{\omega_1 + \cdots + \omega_l} X)$ with $x \in \mathbb{R}^n$ and $X \in \mathbb{C}_n$.

4.1 Regularized case

We now briefly recall (for details, see [9]), how we can construct the fundamental solution to the regularized Dirac operator D_-^{ε} on the generalized cylinders C_k with values in one of the chosen spinor bundles $E^{(l)}$. This is needed, if we want to evaluate our operator equations that we developed in the preceding part of this paper.

First of all, the projection map p_k naturally induces a regularized Dirac operator on C_k associated with the chosen bundle $E^{(l)}$, viz $D_-^{\varepsilon\,\prime} = p_k(D_-^{\varepsilon})$.

Let $U \subset \mathbb{R}^n$ be an open set. Now one has to bear in mind that every function $f : U \times \mathbb{R}^+ \to \mathbb{C}_n$ of the particular quasi-periodicity behavior of the form

$$f(x + \omega, t) = (-1)^{\omega_1 + \cdots + \omega_l} f(x, t) \quad \forall \omega \in \mathbb{Z}^k \tag{10}$$

descends to a well-defined spinor section on $C_k \times \mathbb{R}^+$ with values in the spinor bundle $E^{(l)}$ such as constructed above, again by applying the projection map $p_k(f) =: f' : U' \times \mathbb{R}^+ \to \mathbb{C}_n$. If additionally f is a null solution to the regularized Dirac operator D_-^{ϵ} on Euclidean space, then its projection $f' := p_k(f)$ turns out to be a well-defined hypoelliptic spinor section on $C_k \times \mathbb{R}^+$ that is annihilated by the cylindrical regularized Dirac operator $D_-^{\varepsilon\,\prime}$.

As shown in [9], we can construct a fundamental solution for the regularized Dirac operator on $C_k \times \mathbb{R}^+$ with values in $E^{(l)}$ by periodizing the fundamental solution $E_-^{\epsilon}(x, t)$ over the period lattice $\mathbb{Z}^k$ in a way taking care of the right transformation behavior (10). This is achieved by taking the infinite multiple sum

$$\wp_{k,l}^{\epsilon}(x, t) := \sum_{m \in \mathbb{Z}^l} \sum_{n \in \mathbb{Z}^{k-l}} (-1)^{m_1 + \cdots + \cdots m_l} E_-^{\epsilon}(x + m + n, t). \tag{11}$$

The normal convergence of this series is proved in [9]. By a direct rearrangement argument, one directly verifies that

$$\wp_{k,l}^{\epsilon}(x + \omega, t) = (-1)^{m_1 + \cdots + m_l} \wp_{k,l}^{\epsilon}(x, t), \quad \forall \omega \in \mathbb{Z}^k.$$

Its projection $G'^{\epsilon}_{k,l}(x', t) := p_k(\wp_{k,l}^{\epsilon}(x, t))$ is well defined and represents the fundamental solution of the regularized cylindrical regularized Dirac operator on the manifold $C_k \times \mathbb{R}^k$ with values in the spinor bundle $E^{(l)}$.

Now suppose that $\Omega' \subset C_k \times \mathbb{R}^+$ is a bounded domain with a strongly Lipschitz boundary $\partial\Omega'$ and that $u' : U' \times \mathbb{R}^+ \to \mathbb{C}_n$ is an L_p-section with values in the chosen spinor bundle $E^{(l)}$.

The associated Teodorescu transformation and Cauchy–Bitsadze operator for the regularized Dirac operators on these cylinders can now explicitly be expressed by

$$T^{\epsilon}_{-C_{k,l}} u'(x', t) = \int_{\Omega'} G'^{\epsilon}{}_{k,l}(x' - z', t - s) u(z', s)\, \mathrm{d}z'\, \mathrm{d}s, \quad (x', t) \notin \partial\Omega'.$$

and, analogously,

$$F^{\epsilon}_{C_k,l}u'(x',t) = \int_{\partial\Omega'} G'^{\epsilon}_{k,l}E^{\epsilon}_{-}(x'-z',t-r)\mathrm{d}\sigma_{z',r}u'(z',r), \quad (x',t)\notin\partial\Omega'.$$

By means of these properly adapted operators, we can establish a similar direct decomposition of the $L_p(\Omega')$ space as presented earlier in Theorem 3.1. in the particular context of these cylinders. By completely analogous arguments, we can also establish

$$L_p(\Omega') = \left(L_p(\Omega')\cap\left(M\ker\left(D^{\epsilon}_{-}{}'\right)\right)\right)\oplus_M D^{\epsilon}_{-}{}'\left(\overset{\circ}{W}{}^1_p(\Omega')\right).$$

In the case $p=2$, this decomposition turns again out to be orthogonal.

The arising Bergman projectors will be denoted by $P^{\epsilon}_{MC_k,l}$ and $Q^{\epsilon}_{MC_k,l}$. Alternatively, the latter projectors can also be obtained by periodizing the Bergman kernels of the operators P_M resp. Q_M in a same way as we obtain from the usual Cauchy kernel E the periodized kernel G, namely by making precisely the analogous series constructions over the period lattice, but taking care of the special minus sign associated with the special spinor bundle. However, to do this, we need to know first the kernels for P_M and Q_M. This, however, is very difficult in general, because the Bergman kernel depends on the geometry of the domain. Actually, following [7] (Sections 3.6 and 4.2), the projector $Q^{\epsilon}_{M,C_{k,l}}$ can also directly be expressed in terms of the cylindrical Teodorescu and Cauchy–Bitsadze operator, namely as:

$$Q^{\epsilon}_{MC_k,l} = I - [F^{\epsilon}_{C_k,l}(\mathrm{tr}_{\Gamma}T^{\epsilon}_{-C_k,l}F^{\epsilon}_{C_k,l})^{-1}]\mathrm{tr}_{\Gamma}T^{\epsilon\epsilon}_{-C_k,l},$$

where $\Gamma=\partial\Omega$, and tr_{Γ} stands for the usual trace operator used for instance in [7] (Sections 3.6 and 4.2) in a similar context. This is an advantage. In terms of these operators, we may express the kernel function, namely $G'^{\epsilon}_{k,l}$.

The explicit knowledge of the fundamental solution $G'^{\varepsilon}_{k,l}$ thus enables us already completely to express the solutions to the boundary value problem, where now f' is a spinor valued L_p-section on a domain $\Omega'\subset C_k\times\mathbb{R}$ (with values in the bundle $E^{(l)}$)

$$\left\{\begin{array}{r} (D^{\epsilon}_{-}{}'MD^{\epsilon}_{-}{}')u'^{\epsilon} = f' \\ u'^{\epsilon}|_{\Gamma} = 0 \end{array}\right., \tag{12}$$

explicitly analytically, by

$$u'^{\epsilon}(x',t) = T^{\epsilon}_{-C_k,l}M^{-1}Q^{\epsilon}_{MC_k,l}T^{\epsilon}_{-C_k,l}f'(x',t).$$

Notice, that all these integral operators only involve the kernel functions $G'^{\varepsilon}_{k,l}$.

The other Bergman projector $P^{\epsilon}_{MC_k,l} = I - Q^{\epsilon}_{MC_k,l}$ allows us finally to express the solutions of the more general boundary value problem of the form

$$\left\{\begin{array}{r} (D^{\epsilon}_{-}{}'MD^{\epsilon}_{-}{}')u'^{\epsilon}(x',t) = f'(x',t) \\ u'^{\epsilon}|_{\Gamma}(x',t) = g'(x',t) \end{array}\right.. \tag{13}$$

In view of the validity of our decomposition theorem (Theorem 3.1.), we are allowed to apply the same calculation steps as in the stationary case performed in [7] (Section 4.2) so that we obtain the following representation for the solution:

$$\begin{aligned} u'^{\epsilon}(x',t) = {} & F^{\epsilon}_{C_k,l}g'(x',t) + T^{\epsilon}_{-C_k,l}M^{-1}P^{\epsilon}_{MC_k,l}D_{-}{}^{\epsilon}{}_{C_k,l}h'(x',t) \\ & + T^{\epsilon}_{-C_k,l}M^{-1}Q^{\epsilon}_{MC_k,l}T^{\epsilon}_{-C_k,l}f'(x',t), \end{aligned}$$

where h' represents the unique $W^2_{p+2}(\Omega')$ extension of g'.

4.2 The limit case

In view of Theorem 3.3, we can construct the fundamental solution of D_- on the manifolds $C_k \times \mathbb{R}^+$ associated with the spinor bundle $E^{(l)}$ by simply taking the limit

$$\wp_{k,l}(x,t) := \lim_{\epsilon \to 0^+} \sum_{m \in \mathbb{Z}^l} \sum_{n \in \mathbb{Z}^{k-l}} (-1)^{m_1+\cdots+\cdots m_l} E_-^{\epsilon}(x+m+n,t).$$

and by applying after that the projection map $p_k(\wp_{k,l}(x,t)) =: G'_{k,l}(x',t)$. In view of Theorem 3.8., the Hodge decomposition remains valid in the limit case (under the mentioned conditions in which convergence is guaranteed). Consequently, we obtain the same representation formulas for the solution of the boundary value problem

$$\begin{cases} (D_-{}' M D_-{}') u'^{\epsilon}(x',t) = f'(x',t) \\ u'^{\epsilon}|_{\Gamma}(x',t) = g'(x',t) \end{cases}, \tag{14}$$

just by replacing the kernel function $G'^{\epsilon}_{k,l}(x',t)$ by its limit $G'_{k,l}(x',t)$ when evaluating the integral operators in the solution formulas.

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