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Research Article

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Hodge-Deligne polynomials of character varieties of free abelian groups

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Abstract: Let F be a finite group and X be a complex quasi-projective F-variety. For $r \in \mathbb{N}$, we consider the mixed Hodge-Deligne polynomials of quotients X^r/F, where F acts diagonally, and compute them for certain classes of varieties X with simple mixed Hodge structures (MHSs). A particularly interesting case is when X is the maximal torus of an affine reductive group G, and F is its Weyl group. As an application, we obtain explicit formulas for the Hodge-Deligne and E-polynomials of (the distinguished component of) G-character varieties of free abelian groups. In the cases $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$, we get even more concrete expressions for these polynomials, using the combinatorics of partitions.

Keywords: free abelian group, character variety, mixed Hodge structures, Hodge-Deligne polynomials, equivariant E-polynomials, finite quotients

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1 Introduction

The study of the geometry, topology and arithmetic of character varieties is an important topic of contemporary research. Given a reductive complex algebraic group G, and a finitely presented group Γ, the G-character variety of Γ is the (affine) geometric invariant theory (GIT) quotient

$$\mathcal{M}_\Gamma G := \mathsf{Hom}(\Gamma, G)/\!/G.$$

When the group Γ is the fundamental group of a Riemann surface (or more generally, a Kähler group), these spaces are homeomorphic to moduli spaces of G-Higgs bundles via the non-abelian Hodge correspondence (see, e.g. [1,2]) and have found interesting connections to important problems in Mathematical Physics in the context of mirror symmetry and the geometric Langlands correspondence.

Recently, some interesting formulas were obtained by Hausel, Letellier and Rodriguez-Villegas for the so-called E-polynomial of smooth $GL(n, \mathbb{C})$-character varieties of surface groups, by applying arithmetic harmonic analysis to their $\mathbb{Z}$-models and proving these are polynomial count [3,4]. By computing indecomposable bundles on algebraic curves over finite fields, Schiffmann determined the Poincaré polynomial of the moduli spaces of stable Higgs bundles, hence of the corresponding $GL(n, \mathbb{C})$-character varieties of surface groups [5]. Other methods based on point counting were employed by Mereb [6] (the $SL(n, \mathbb{C})$ case) and Baraglia-Hekmati [7] (the singular, small n case).

Moreover, geometric tools were developed by Lawton, Logares, Muñoz and Newstead to calculate the E-polynomials using stratifications of character varieties (over $\mathbb{C}$) of surface groups, exploring directly

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the additivity of these polynomials [8,9]. This led to the development of a Topological Quantum Field Theory for character varieties by González-Prieto et al. [10,11].

In the present article, we deal instead with G-character varieties of *free abelian groups*, and with the determination of their *mixed Hodge structures* (MHSs) for a general complex reductive G. In particular, we explicitly compute the mixed Hodge polynomials of these varieties. The mixed Hodge polynomial μ_X is a three variable polynomial $\mu_X = \mu_X(t, u, v)$ defined for any (complex) quasi-projective variety X and encodes all numerical information about the MHS on the cohomology of X, generalizing both the Poincaré and the E-polynomials.

To present our main results, denote the G-character variety of the free abelian group $\Gamma \cong \mathbb{Z}^r$, $r \in \mathbb{N}$, by:

$$\mathcal{M}_r G := \mathcal{M}_{\mathbb{Z}^r} G = \mathsf{Hom}(\mathbb{Z}^r, G)/\!/G,$$

where // stands for the (affine) GIT quotient (see, e.g., [12,13]) for the natural G-action, by conjugation, on the space of representations $\mathsf{Hom}(\mathbb{Z}^r, G)$. This later space consists of pairwise commuting r-tuples of elements of G and is of relevance in Mathematical Physics, namely, in the context of supersymmetric Yang-Mills theory [14]. When r is even, $\mathbb{Z}^r$ is also a Kähler group (the fundamental group of a Kähler manifold) and the smooth locus of $\mathcal{M}_{\mathbb{Z}^{2m}}(G)$ is diffeomorphic to a certain moduli space of G-Higgs bundles over a m-dimensional abelian variety (see, for instance [15]).

The topology and geometry of character varieties of free abelian groups have been studied by Florentino-Lawton, Sikora, Ramras-Stafa, among others (see, e.g., [16–19]). It is known that the affine algebraic variety $\mathcal{M}_r G$ is not in general irreducible, but the irreducible component of the trivial $\mathbb{Z}^r$-representation, denoted $\mathcal{M}_r^0 G$, has a normalization $\mathcal{M}_r^* G$ isomorphic to T^r/W ([17, Theorem 2.1]), where $T \subset G$ is a maximal torus and W is the Weyl group, acting diagonally on T^r (hence also on its cohomology). Thus, the varieties $\mathcal{M}_r^* G$ are singular orbifolds of dimension $r \dim T$ with a special kind of MHSs, called *balanced* or of Hodge-Tate type and they satisfy the analogue of Poincaré duality for MHS. When $r = 2$, Thaddeus proved that $\mathcal{M}_2^* G$ are of crucial importance in mirror symmetry and Langlands duality and computed their *orbifold* E-polynomials [20]. Here, we obtain the following explicit formula for *mixed Hodge polynomials* of $\mathcal{M}_r^* G$.

Theorem 1.1. *Let $r \geq 1$, G be a complex reductive group with maximal torus T and Weyl group W. Then,*

$$\mu_{\mathcal{M}_r^* G}(t, u, v) = \frac{1}{|W|} \sum_{g \in W} \det(I + tuvA_g)^r, \tag{1}$$

where A_g is the automorphism induced on $H^1(T, \mathbb{C})$ by $g \in W$, and I is the identity automorphism.

One consequence of this result is a formula for the (compactly supported) E-polynomial of the irreducible component $\mathcal{M}_r^0 G \subset \mathcal{M}_r G$, for every such G (Theorem 5.4).

Our approach to Theorem 1.1 is based on working with *equivariant* MHSs and their corresponding *equivariant polynomials*, defined for varieties with an action of a finite group, and focusing on certain classes of balanced varieties. In particular, we generalize to the context of the equivariant E-polynomial, some of the techniques introduced in [21] for dealing with equivariant weight polynomials.

For the groups $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$, we have that $\mathcal{M}_r G$ is an irreducible normal variety, and the formula in Theorem 1.1 can be made even more concrete, in terms of partitions of n, and allows explicit computations of the Hodge-Deligne, E- and Poincaré polynomials of the corresponding character varieties $\mathcal{M}_r G = \mathcal{M}_r^* G$. We state the main results below in the compactly supported version, the one which is relevant in arithmetic geometry (see [3], Appendix).

Let $\mathcal{P}_n$ denote the set of partitions of $n \in \mathbb{N}$. By $\underline{n} = [1^{a_1} 2^{a_2} \cdots n^{a_n}] \in \mathcal{P}_n$ we denote the partition of n with $a_j \geq 0$ parts of size $j = 1, \ldots, n$, so that $n = \sum_j j a_j$.

Theorem 1.2. *Let $G = SL(n, \mathbb{C})$ and $r \geq 1$. The compactly supported E-polynomial of $\mathcal{M}_r G$ is*

$$E^c_{\mathcal{M}_r G}(x) = \frac{1}{(x-1)^r} \sum_{\underline{n} \in \mathcal{P}_n} \prod_{j=1}^n \frac{(x^j - 1)^{a_j r}}{a_j! j^{a_j}},$$

where $\underline{n} = [1^{a_1} 2^{a_2} \cdots n^{a_n}] \in \mathcal{P}_n$.

Theorem 1.2 generalizes, to every $r, n \geq 1$, some formulas recently obtained in [7,9] (the cases $n = 2$ and $n = 3$) by different methods, which are only tractable for low values of n: the approach in [8,9] uses stratifications and fibrations to compute E-polynomials of character varieties of free groups respectively, surface groups; the computations in [7] apply representation theory of finite groups and point counting of varieties over finite fields.

By substituting $x = 1$ in $E^c_{\mathcal{M}_r G}$, we obtain the Euler characteristics of these moduli spaces. Moreover, by showing that $\mathcal{M}_r G$ have very special MHSs (that we call *round*, see Definition 3.7), Theorems 1.1 and 1.2 immediately provide explicit formulas for their mixed and Poincaré polynomials (Theorem 5.13).

The $GL(n, \mathbb{C})$ case is particularly symmetric, as the generating function of mixed Hodge polynomials gives precisely the formula of J. Cheah [22] for the mixed Hodge numbers of symmetric products. On the other hand, by examining the action of W on the cohomology of a maximal torus, our methods allow for the computation of $\mu_{\mathcal{M}_r G}$ for all the classical complex semisimple groups G. These will be addressed in upcoming work.

We now outline the contents of the article. In Section 2, we review necessary background on MHS, quasi-projective varieties, etc., and define the relevant polynomials, providing examples and focusing on balanced varieties. In Section 3, we study properties of special MHS, related to notions defined in [21], and pay special attention to round varieties, for which the knowledge of either the Poincaré polynomial or the E-polynomial allows the determination of μ. Section 4 is devoted to equivariant MHS, character formulas and the cohomology of finite quotients. Finally, in Section 5 we prove our main theorem and provide explicit calculations of Hodge-Deligne and E-polynomials (and Euler characteristics) of character varieties of $\mathbb{Z}^r$, in particular for $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$; in the $GL(n, \mathbb{C})$ case, the computations are related to MHS on symmetric products, thereby obtaining a curious combinatorial identity. In the Appendix, we present a proof, based on [21], of the equivariant version of a theorem in [8,9] on the multiplicative property of the E-polynomial for fibrations.

A preliminary version of the main results has been announced in [23].

2 Preliminaries on character varieties and on MHSs

We start by recalling the relevant definitions and properties of character varieties and of mixed Hodge structures (MHSs) on quasi-projective varieties, which serves to fix terminology and notation.

2.1 Character varieties

Given a finitely generated group Γ and a complex affine reductive group G, the G-character variety of Γ is defined to be the (affine) GIT quotient (see [12,13]; [24] for topological aspects):

$$\mathcal{M}_\Gamma(G) = \mathsf{Hom}(\Gamma, G)/\!/G.$$

Note that $\mathsf{Hom}(\Gamma, G)$, the space of homomorphisms $\rho : \Gamma \to G$, is an affine variety, as Γ is defined by algebraic relations, and it is also a G-variety when considering the action of G by conjugation on $\mathsf{Hom}(\Gamma, G)$.

The aforementioned GIT quotient is the maximal spectrum of the ring $\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G$ of G-invariant regular functions on $\mathsf{Hom}(\Gamma, G)$:

$$\mathsf{Hom}(\Gamma, G)/\!/G := \mathrm{Specmax}(\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G).$$

The GIT quotient does not parametrize all orbits, since some of them may not be distinguishable by invariant functions. In fact, it can be shown (see, e.g., [13]) that the conjugation orbits of two representations $\rho, \rho' : \Gamma \to G$ define the same point in $\mathsf{Hom}(\Gamma, G)/\!/G$ if and only if their closures intersect: $\overline{G \cdot \rho} \cap \overline{G \cdot \rho'} \neq \emptyset$ (in either the Zariski or the complex topology coming from an embedding $\mathsf{Hom}(\Gamma, G) \hookrightarrow \mathbb{C}^N$). For detailed definitions and properties of general character varieties, we refer to [16,25].

In this article, we will be mostly concerned with the case when Γ is a finitely generated free abelian group, $\Gamma = \mathbb{Z}^r$ for some natural number r, the rank of Γ. The corresponding G-character varieties:

$$\mathcal{M}_{\mathbb{Z}^r} G = \mathsf{Hom}(\mathbb{Z}^r, G)/\!\!/G,$$

have many interesting properties, as representations in $\mathsf{Hom}(\mathbb{Z}^r, G)$ can be naturally identified with r-tuples of group elements $(A_1, \dots, A_r) \in G^r$ that pairwise commute: $A_i A_j = A_j A_i$, for all $i, j = 1, \dots, n$.

When K is a compact Lie group, the analogous space of representations $\mathsf{Hom}(\mathbb{Z}^r, K)$ is of central importance in determining the so-called moduli space of vacua of supersymmetric gauge theories on a r-dimensional torus, as studied in [14,26] and others.

2.2 MHSs

On a compact Kähler manifold X the complex cohomology satisfies the Hodge decomposition $H^k(X, \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X)$, which verifies $H^{q,p}(X) \cong \overline{H^{p,q}(X)}$. This decomposition of $H^k(X, \mathbb{C})$, a *pure Hodge structure of weight* k, can be described, equivalently, by a decreasing filtration:

$$H^k(X, \mathbb{C}) = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{k+1} = 0,$$

satisfying $F_p \cap \overline{F_q} = 0$ and $F_p \oplus \overline{F_q} = H^k(X, \mathbb{C})$ for all $p + q = k + 1$.

This notion can be generalized to quasi-projective algebraic varieties X over $\mathbb{C}$, possibly non-smooth and/or non-compact. Namely, the complex cohomology of any such variety is also endowed with a natural filtration, the *Hodge filtration* F, and moreover, there is a special second increasing filtration on the *rational* cohomology:

$$0 = W^{-1} \subseteq \cdots \subseteq W^{2k} = H^k(X, \mathbb{Q}),$$

the *weight filtration* W, satisfying a compatibility condition with respect to the Hodge filtration: the latter induces a filtration on the weighted graded pieces of the former that needs to be a pure Hodge structure. The vector space $H^k(X, \mathbb{Q})$, together with the filtrations F and W, is the prototype of an MHS. We denote the *graded pieces* of the associated decomposition by

$$H^{k,p,q}(X) := Gr_F^p Gr_{p+q}^{W_{\mathbb{C}}} H^k(X, \mathbb{C}),$$

where $W_{\mathbb{C}}$ stands for the complexified weight filtration. Note that, even though different filtrations may lead to isomorphic graded pieces, for convenience, we sometimes refer to the collection of these $H^{k,p,q}(X)$ as the MHS of X. For background and proofs we refer to [27] and the original articles by Deligne [28,29].

The above constructions can be reproduced for the *compactly supported* cohomology groups $H_c^k(X, \mathbb{C})$, yielding an analogous decomposition:

$$H_c^{k,p,q}(X) := Gr_F^p Gr_{p+q}^{W_{\mathbb{C}}} H_c^k(X, \mathbb{C}).$$

MHSs in the compactly supported context have interesting connections to number theory as illustrated, for example, in the Appendix of [3] by N. Katz.

MHSs satisfy some nice properties, as follows.

Proposition 2.1. *Let X and Y be complex quasi-projective varieties. Then,*
1. *For all k, p, q, we have $H^{k,q,p}(X) \cong \overline{H^{k,p,q}(X)}$;*
2. *The weight and Hodge filtrations are preserved by algebraic maps. Therefore, so are MHSs;*
3. *The Hodge and weight filtrations are preserved by the Künneth isomorphism. Therefore, so are MHSs;*
4. *The MHSs are compatible with the cup product*:

$$H^{k,p,q}(X) \smile H^{k',p',q'}(X) \hookrightarrow H^{k+k',p+p',q+q'}(X);$$

5. *If X is smooth of complex dimension n, MHSs are compatible with Poincaré duality*:

$$H^{k,p,q}(X) \cong (H_c^{2n-k,n-p,n-q}(X))^*.$$

Proof. All of these statements are standard. For convenience, we point to appropriate references. The proof of (1) follows from the purity of the Hodge structure on the graded pieces of the weight filtration.

The other proofs are found in chapters 4 to 6 of the book [27]. Specifically, for (2) see [27, Proposition 4.18]; (3) and (4) appear in [27, Theorem 5.44, Corollary 5.45] and (5) in [27, Proposition 6.19]. □

2.3 Hodge polynomials and balanced varieties

The spaces $H^{k,p,q}(X)$ are holomorphic invariants and encode important geometric information (diffeomorphic complex manifolds may have non-isomorphic MHSs, as in Example 2.6).

The *mixed Hodge numbers of* X are the complex dimensions of the MHS pieces

$$h^{k,p,q}(X) := \dim_{\mathbb{C}} H^{k,p,q}(X),$$

and are typically assembled in a polynomial. By definition, for a pure Hodge structure, $h^{k,p,q}(X) \neq 0$ unless $k = p + q$.

Definition 2.2. Let X be a complex quasi-projective variety of complex dimension d. The *mixed Hodge polynomial* of X (also called *Hodge-Deligne polynomial*) is the three-variable polynomial of degree $\leq 2d$

$$\mu_X(t,u,v) := \sum_{k.p,q\geq 0} h^{k,p,q}(X) t^k u^p v^q.$$

Its specialization for $t = -1$

$$E_X(u,v) := \sum_{k.p,q\geq 0} h^{k,p,q}(X)(-1)^k u^p v^q$$

is called the *E-polynomial* of X.

Remark 2.3.

(1) The specialization of μ_X for $u = v = 1$ gives the *Poincaré polynomial* of X:

$$P_X(t) := \sum_{k\geq 0} b_k(X) t^k,$$

with $b_k(X) := \dim_{\mathbb{C}} H^k(X,\mathbb{C})$ being the Betti numbers of X. Note that the coefficients of μ_X and of P_X are *non-negative* integers, whereas E_X lives in the ring $\mathbb{Z}[u,v]$.

(2) As mentioned earlier, there is an entirely parallel theory for the compactly supported cohomology. Here, the associated Hodge numbers are denoted by $h_c^{k,p,q} := \dim_{\mathbb{C}} H_c^{k,p,q}(X)$. If $\mathcal{P}_X$ stands for one of the polynomials in the aforementioned definition, we will distinguish its compactly supported version by writing $\mathcal{P}_X^c$.

(3) *Comment on terminology*: there are inconsistencies in the literature on the terminology used for these polynomials. Since $h^{k,p,q}(X)$ are generally called Hodge-Deligne (or mixed Hodge) numbers, we refer to μ_X as *Hodge-Deligne* or *mixed Hodge polynomial*. To emphasize the distinction, the *compactly supported E-polynomial* E_X^c will also be called the *Serre polynomial* of X, since its crucial behavior, as a generalized Euler characteristic, was first used by Serre in connection with the Weil conjectures (see [30]).

(4) Many specializations of the E-polynomial have been studied in the literature. There is, for example, the *weight polynomial* $W_X(y) := \sum_{k,p}(-1)^k w^{k,p}(X) y^p$, using the graded pieces of the weight filtration $w^{k,p}(X) := \dim_{\mathbb{C}} Gr_p^{W_{\mathbb{C}}} H^k(X,\mathbb{C})$ (see [21]). This is a specialization of the E-polynomial since $W_X(y) = E_X(y,y)$. Also, Hirzebruch's χ_y-*genus* and the *signature* σ of a complex manifold X are given, in terms of $E_X(u,v)$, as: $\chi_y(X) = E_X(-y,1)$ and $\sigma(X) = E_X(-1,1)$, respectively (see Hirzebruch [31]).

We now collect some well-known important properties of these polynomials, for later use.

Proposition 2.4. *For a quasi-projective variety X, we have:*

(1) *The polynomials μ_X and E_X are symmetric in the variables u and v; in particular, if $h^{k,p,q}(X) \neq 0$ then $h^{k,q,p}(X) \neq 0$.*

(2) *Let $h^{k,p,q}(X) \neq 0$. Then $p, q \leq k$. Moreover, if X is smooth, then $p + q \geq k$; if X is projective, then $p + q \leq k$. In particular, if X is a compact Kähler manifold $p + q = k$.*

(3) *The (topological) Euler characteristic $\chi(X)$ is given by $\chi(X) = E_X(1, 1)$.*

(4) *The Serre polynomial (compactly supported E-polynomial) E_X^c is additive for stratifications of X by locally closed subsets, and its degree is equal to $2\dim_{\mathbb{C}} X$.*

(5) *All polynomials μ_X, P_X and E_X are multiplicative under Cartesian products.*

Proof. (1) Follows from item (1) of Proposition 2.1. Item (2) is proved in [27, Proposition 4.20] and [27, Theorem 5.39]. Item (3) is immediate from the definition. The proof of (4) can be found in [27, Corollary 5.57] and (5) follows directly from 2.1(4). ☐

A common feature of the varieties in this paper is that their MHS is "diagonal:" for each k, the only non-zero mixed Hodge numbers are $h^{k,p,q}$ with $p = q$.

Definition 2.5. A quasi-projective variety X is said to be *balanced* or of *Hodge-Tate type* if for every non-negative integer $k \in \mathbb{N}_0$, and all $p \neq q$, $h^{k,p,q}(X) = 0$. In other words, if $h^{k,p,q}(X) \neq 0$, then $q = p$. We call $p + q$ the *total weight* of $H^{k,p,q}(X)$.

Example 2.6.

(1) If X is connected, $H^0(X, \mathbb{C}) \cong \mathbb{C}$ has always a pure Hodge structure, with trivial decomposition $H^0(X, \mathbb{C}) = H^{0,0,0}(X)$. Dually, when X is also smooth, the compactly supported cohomology is also a trivial decomposition $H_c^{2n}(X, \mathbb{C}) = H_c^{2n,n,n}(X)$.

(2) $X = \mathbb{C}^n$ is a (non-compact) Kähler manifold with cohomology only in degree zero. By the above, it has trivial pure Hodge structure: $H^*(X, \mathbb{C}) = H^0(X, \mathbb{C}) = H^{0,0,0}(X)$ and so

$$\mu_{\mathbb{C}^n}(t, u, v) = 1, \quad \mu^c_{\mathbb{C}^n}(t, u, v) = t^{2n}u^n v^n,$$

where the compactly supported version follows from Poincaré duality.

(3) Let $X = \mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Although Kähler, its cohomology has no pure Hodge structure, since $\dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 1$. Being smooth, using Proposition 2.1(1)–(2), the only non-zero $h^{1,p,q}$ is $h^{1,1,1}$, so $h^{0,0,0} = h^{1,1,1} = 1$ and all other Hodge numbers vanish. We then get

$$\mu_{\mathbb{C}^*}(t, u, v) = 1 + tuv,$$

and the only non-zero $h_c^{k,p,q}$ are $h_c^{2,1,1} = h_c^{1,0,0} = 1$, by Poincaré duality. Thus, $\mu^c_{\mathbb{C}^*}(t, u, v) = t^2uv + t$ and $E^c_{\mathbb{C}^*}(u, v) = uv - 1$. Observe that this is compatible with the decomposition into locally closed subsets $\mathbb{C}^1 = \mathbb{C}^* \sqcup \mathbb{C}^0$ as in Proposition 2.4(4). Hence, $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ are examples of balanced varieties. For a simple example of a non-balanced variety we can take an elliptic curve or any compact Riemann surface of positive genus.

(4) Consider the total space X of the trivial line bundle over an elliptic curve $X \cong (\mathbb{C}/\Lambda) \times \mathbb{C}$, where Λ is a rank two lattice in $(\mathbb{C}, +)$. It is easy to see that X is *real analytically* isomorphic to $(\mathbb{C}^*)^2$ (but not complex analytically or algebraically isomorphic). From the Künneth isomorphism and considerations analogous to Example 2.6, we get:

$$\mu_{(\mathbb{C}^*)^2}(t, u, v) = (1 + tuv)^2 = 1 + 2tuv + t^2u^2v^2,$$
$$\mu_X(t, u, v) = (1 + tu)(1 + tv) = 1 + t(u + v) + t^2uv.$$

Indeed, $(\mathbb{C}^*)^2$ is balanced, whereas the cohomology of X is pure.

Remark 2.7.

(1) The last example is a very special case (the genus 1, rank 1 case) of the non-abelian Hodge correspondence mentioned in Section 1, which produces diffeomorphisms between (Zariski open subsets of) moduli spaces of flat connections and certain moduli spaces of Higgs bundles over a given Riemann surface. The fact that one diffeomorphism type is balanced (the flat connection side of the correspondence) and the other is pure is a general feature (see [3,4]).

(2) If X is balanced, its E-polynomial depends only on the product uv, so it is common to adopt the change of variables $x \equiv uv$. When written in this variable, the degree of $E^c_X(x)$ is now equal to $\dim_{\mathbb{C}} X$, instead of $2\dim_{\mathbb{C}} X$.

3 Separably pure, elementary and round varieties

In this section, we collect many properties of MHS that are necessary later on. We also describe the types of Hodge structures that allow the recovery of the mixed Hodge polynomial given the E- or the Poincaré polynomial (Theorem 3.6), and concentrate on the case of round varieties, which are the Hodge types of our character varieties. We tried to be self-contained for the benefit of researchers in the field of character varieties or Higgs bundles that may not be familiar with MHS.

3.1 Elementary and separably pure varieties

The MHSs on the cohomology of a given quasi-projective variety X may be trivial, i.e., the decomposition of every $H^k(X, \mathbb{C})$ is the trivial one, and many such examples are considered here. When this happens, the only non-zero $h^{k,p,q}(X)$ satisfy $q = p$ (by Proposition 2.4(1)) and much of what can be said about the cohomology can be transported to MHSs. Adapting some notions from [21] (who worked with the weight polynomial), we introduce the following terminology.

Definition 3.1. Let X be a quasi-projective variety. X (or its cohomology) is called *elementary* if its MHSs are trivial decompositions of the cohomology, so that for every $k \in \mathbb{N}$ there is only one $p \in \mathbb{N}$ such that $h^{k,p,p}(X) \neq 0$ (and $h^{k,p,q}(X) \neq 0$ for $q \neq p$).

X is said to be *separably pure* if the MHS on each $H^k(X, \mathbb{C})$ is in fact pure of total weight w_k, and such that $w_j \neq w_k$ for every $j \neq k$.

Remark 3.2.

(1) Note that X is elementary if it is *balanced* and there is a *weight function* $k \mapsto p_k$ (defined only for those $k \in \mathbb{N}_0$ with $H^k(X, \mathbb{C}) \neq 0$) such that $h^{k,p,q} = 0$ for every pair (p, q) not equal to (p_k, p_k). In this case,

$$Gr^{W_{\mathbb{C}}}_{2p_k} H^k(X, \mathbb{C}) = H^{k,p_k,p_k}(X) = H^k(X, \mathbb{C}). \tag{2}$$

A general weight function is not enough to recover μ_X from the weight or the E-polynomials (different degrees of cohomology may have equal total weights). However, this can be done (see Theorem 3.6) if the weight function $k \mapsto p_k$ is injective, in which case the equality (2) takes the stronger form: $\oplus_m Gr^{W_{\mathbb{C}}}_{2p_k} H^m(X, \mathbb{C}) = H^{k,p_k,p_k}(X) = H^k(X, \mathbb{C})$.

(2) In a pure Hodge structure of total weight k on $H^k(X, \mathbb{Q})$ the only non-zero weight summand is $Gr^W_{2k} H^k(X, \mathbb{Q})$. So, a pure total cohomology is separably pure, but not conversely, as the case $\mathbb{C}^*$ shows (Example 2.6).

(3) When X is separably pure, instead of the weight function, one can define a *degree function* $(p,q) \mapsto k = k(p,q)$ (defined *only* on pairs (p,q) such that $h^{k,p,q}(X) \neq 0$). Noting that, in fact, the degree k only depends on the total weight $p+q$ (being separably pure) we can write this as $(p,q) \mapsto k_{p+q}$.

In this article, most varieties are both separably pure and balanced, and an alternative characterization follows.

Lemma 3.3. *A quasi-projective variety X is separably pure and balanced if and only if it is elementary and its weight function $k \mapsto p_k$ is injective.*

Proof. If X is separably pure, the total weight in each $H^k(X,\mathbb{C})$ has to be constant. But if X is also balanced, given k, all $h^{k,p,q}(X)$ vanish except for a unique pair $(p,q) = (p_k, p_k)$, so we have an assignment $k \mapsto p_k$ proving that X is elementary. Moreover, since the total weights are different for distinct k, the weight function is injective. The converse statement is easy since an elementary variety is Hodge-Tate and an injective weight function implies injectivity for total weights. □

Example 3.4. A family of balanced and pure varieties are the smooth projective toric varieties (hence these are elementary). Indeed, every such toric variety X, where the $\mathbb{C}^*$ action has d_j orbits in (complex) dimension $j = 0,\dots,\dim_{\mathbb{C}} X$, has (see [27, Example 5.58]):

$$\mu_X(t,u,v) = \sum_{j=0}^{n} d_j (t^2uv - 1)^j,$$

and the weight function is $2j \mapsto j$ (the weights being (j,j)). For example, the Hodge-Deligne polynomials of projective spaces are $\mu_{\mathbb{P}^n_{\mathbb{C}}}(t,u,v) = \sum_{j=0}^{n} t^{2j}u^jv^j$.

As in the case of the complex affine multiplicative group $\mathbb{C}^*$, more general complex affine algebraic groups are balanced, but not necessarily pure or separably pure.

Example 3.5. The Poincaré polynomial of $GL(n,\mathbb{C})$ is well known, given by $P_{GL(n,\mathbb{C})}(t) = \prod_{j=1}^{n}(1+t^{2j-1})$. Also, by [29, Theorem 9.1.5], we have:

$$\mu_{GL(n,\mathbb{C})}(t,u,v) = \prod_{j=1}^{n}(1 + t^{2j-1}u^jv^j).$$

For example, $\mu_{GL(3,\mathbb{C})}(t,x) = 1 + tx + t^3x^2 + t^4x^3 + t^5x^3 + t^6x^4 + t^9x^6$ (writing $x = uv$, see Remark 2.7(2)). So, $GL(3,\mathbb{C})$ is elementary (hence balanced) but *not separably pure*: both degrees 4 and 5 have associated total weight 6 (the terms with x^3), so $GL(n,\mathbb{C})$ is not separably pure, for $n \geq 3$. Moreover, the same argument readily shows that $GL(n,\mathbb{C})$ is *not elementary* for $n \geq 5$.

The aforementioned examples show that this "yoga of weights," as alluded by Grothendieck, is very useful in understanding general properties of certain classes of varieties. When we know that a particular variety X has a degree or a weight function as above, we can determine the full collection of triples (k,p,q), such that $h^{k,p,q}(X) \neq 0$.

In Figure 1, the shaded area illustrates Lemma 3.3; for the definition of round, see Section 3.2. The next result shows that elementary and separably pure are indeed the correct notions to be able to determine the mixed Hodge polynomial from the Poincaré or the E-polynomial, respectively.

Theorem 3.6. *Let X be a quasi-projective variety of dimension n. Then:*
(1) *If X is elementary, with known weight function, its Poincaré polynomial determines its Hodge-Deligne polynomial.*
(2) *If X is separably pure, with known degree function, its E-polynomial determines its Hodge-Deligne polynomial.*

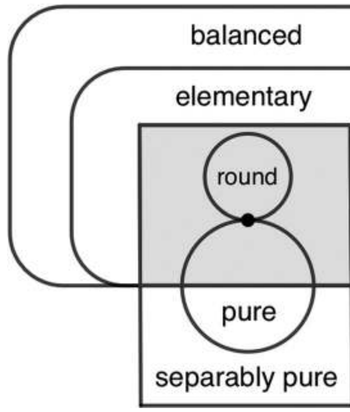

Figure 1: Venn diagram with several classes of MHS.

Proof.
(1) Suppose the Poincaré polynomial of X is $P_X(t) = \sum_k b_k t^k$ and the weight function is $k \mapsto (p_k, p_k)$. Then, since the only non-trivial mixed Hodge pieces are $H^{k,p_k,p_k}(X)$, we get $\mu_X(t,u,v) = \sum_k b_k t^k u^{p_k} v^{p_k}$.
(2) Similarly, writing $E_X(u,v) = \sum_{p,q} a_{p,q} u^p v^q$, and the degree function as $(p,q) \mapsto k_{p+q}$, since the total weights are in one-to-one correspondence with the degrees of cohomology, we obtain $\mu_X(t,u,v) = \sum_{k,p,q} a_{p,q} (-t)^{k_{p+q}} u^p v^q$. □

3.2 Round varieties

From Theorem 3.6, if a variety X is both balanced *and* separably pure, then μ_X can be recovered from either E_X or P_X, knowing their degree/weight functions. A specially interesting case is the following.

Definition 3.7. Let X be a quasi-projective variety. If the only non-zero Hodge numbers are of type $h^{k,k,k}(X)$, $k \in \{0, \dots, 2\dim_{\mathbb{C}} X\}$, we say that X is *round*.

In other words, a round variety is both elementary and separably pure and its only k-weights have the form (k,k). Round varieties are referred to as "minimally pure" balanced varieties in Dimca-Lehrer (see [21, Definition 3.1(iii)]).

Remark 3.8. In general, Cartesian products of elementary varieties are not elementary, and similarly for separably pure varieties. For instance, using Example 3.5 with $n = 2$, we see that $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$ is not separably pure. On the other hand, the following result holds for round varieties.

Proposition 3.9. *Let X and Y be round varieties. Then:*
(1) *The Hodge-Deligne polynomial of X reduces to a one-variable polynomial, and can be reconstructed from either the E or the Poincaré polynomial:*
$$\mu_X(t,u,v) = P_X(tuv) = E_X(-tu, v).$$
(2) *The Cartesian product $X \times Y$ is round.*

Proof.

(1) By definition, if X is round, we can write:

$$\mu_X(t,u,v) = \sum_{k\geq 0} h^{k,k,k}(X)t^k u^k v^k,$$

so that μ_X is a polynomial in tuv. Moreover, its Betti numbers are $b_k(X) = h^{k,k,k}(X)$ giving the first equality. The second follows from $E_X(x) = \sum_{k\geq 0}(-1)^k h^{k,k,k}(X)x^k$.

(2) This follows at once from (1) and from Proposition 2.4(5). □

Remark 3.10.

(1) If X satisfies Poincaré duality on MHS, and $\dim_{\mathbb{C}} X = n$, one has

$$\mu_X^c(t,u,v) = (t^2uv)^n \mu_X\left(\frac{1}{t},\frac{1}{u},\frac{1}{v}\right), \quad P_X^c(t) = t^{2n}P_X\left(\frac{1}{t}\right), \quad E_X^c(u,v) = (uv)^n E_X\left(\frac{1}{u},\frac{1}{v}\right).$$

In particular, $\chi(X) = E_X^c(1,1)$. If X is additionally round, analogously to Proposition 3.9, μ_X^c can be reconstructed from P_X^c and E_X^c as:

$$\mu_X^c(t,u,v) = (uv)^{-n}P_X^c(tuv) = (-t)^n E_X^c(-tu,v).$$

(2) A sufficient condition for roundness is the following: if X is balanced and separably pure and its cohomology *has no gaps*, in the sense that for every $k \in \mathbb{N}$, the condition $H^k(X,\mathbb{C}) \neq 0$ implies $H^{k-1}(X,\mathbb{C}) \neq 0$, then X is round. This is easy to see from Lemma 3.3 and the restrictions on weights (Proposition 2.4(2)).

4 Cohomology and MHSs for finite quotients

Let F be a finite group and X a complex quasi-projective F-variety. In this section, we outline some results on the cohomology and MHSs of quotients of the form X^r/F, where F acts diagonally on the Cartesian product X^r, for general $r \geq 1$. Of special relevance is a formula, in Corollary 4.8, for the Hodge-Deligne polynomial of X^r/F for an elementary variety X whose cohomology is a simple exterior algebra.

4.1 Equivariant MHSs

The MHS of the ordinary quotient X/F is related to the one of X and its F-action, as follows. Since F acts algebraically on X, it induces an action on its cohomology ring preserving the degrees, and, by Proposition 2.1(1), the MHSs. Therefore, $H^{k,p,q}(X)$ and $Gr_{p+q}^{W_{\mathbb{C}}}H^k(X,\mathbb{C})$ are also F-modules. Denoting these by $[Gr_{p+q}^{W}H^k(X,\mathbb{C})]$ and $[H^{k,p,q}(X)]$, and calling them *equivariant MHS*, one may codify this information in polynomials with coefficients belonging to the representation ring of F, $R(F)$ (cf. [21]).

Definition 4.1. The *equivariant mixed Hodge polynomial* is defined as:

$$\mu_X^F(t,u,v) = \sum_{k,p,q}[H^{k,p,q}(X)]\,t^k u^p v^q \in R(F)[t,u,v].$$

Evaluating at $t=-1$, gives us the *equivariant E-polynomial*:

$$E_X^F(u,v) = \sum_{k,p,q}(-1)^k[H^{k,p,q}(X)]\,u^p v^q \in R(F)[u,v].$$

As in the non-equivariant case, we adopt the change of variable $x = uv$ when X is balanced. As in Proposition 2.1, several simple properties can be deduced.

Proposition 4.2. *Let X be a quasi-projective F-variety, for a finite group F, and let $\mathcal{P}_X^F$ be one of the polynomials in Definition* 4.1. *Then*
(1) *$\mathcal{P}_X$ is obtained by replacing each representation in $\mathcal{P}_X^F$ by its dimension*;
(2) *The Künneth formula and Poincaré duality, for X smooth, are compatible with equivariant MHS*:

$$\mathcal{P}_{X\times Y}^F = \mathcal{P}_X^F \otimes \mathcal{P}_Y^F,$$
$$[Gr_{p+q}^{W_{\mathbb{C}}} H^k(X,\mathbb{C})] = [(Gr_{2n-(p+q)}^{W_{\mathbb{C}}} H_c^{2n-k}(X,\mathbb{C}))^*],$$
$$[H^{k,p,q}(X)] = [(H_c^{2n-k,n-p,n-q}(X))^*],$$

where $\otimes$ means that we take tensor products of graded F-representations.

Proof. (1) This follows immediately from the definition of dimension of representation. For (2), it suffices to see that the Künneth and Poincaré maps are also morphisms in the category of F-modules, which is easily checked. □

4.2 Cohomology of finite quotients

We recall some known facts concerning the usual and the compactly supported cohomology of the quotient X/F. Consider its equivariant cohomology, defined on rational cohomology by

$$H_F^*(X,\mathbb{Q}) := H^*(EF \times_F X, \mathbb{Q}),$$

where EF is the universal principal bundle over BF, the classifying space of F, and $EF \times_F X$ is the quotient under the natural action, which admits an algebraic map $EF \times_F X \xrightarrow{\pi} X/F$. Since F is finite, so is the stabilizer of any point for the F action, and the Vietoris-Begle theorem (see e.g. [32, page 344]) implies that the pullback $\pi^* : H^*(X/F,\mathbb{Q}) \longrightarrow H^*(EF \times_F X,\mathbb{Q})$ provides an isomorphism

$$H^*(X/F,\mathbb{Q}) \simeq H_F^*(X,\mathbb{Q}). \tag{3}$$

Moreover, the fibration

$$X \longrightarrow EF \times_F X \longrightarrow BF$$

has an induced Serre spectral sequence satisfying (see [33], for example)

$$E_2^{p,q} \cong H^p(BF, H^q(X)) \cong H^p(F, H^q(X)) \Rightarrow H_F^{p+q}(X,\mathbb{Q}).$$

Since F is finite, one can deduce $H^p(F, H^q(X)) = 0$ for all $p > 0$ and all q, since sheaf cohomology vanishes in degrees higher than $\dim F$. Then the Serre spectral sequence converges at the second step, and this gives ([34])[1]

$$H_F^*(X,\mathbb{Q}) \simeq H^*(X,\mathbb{Q})^F. \tag{4}$$

Combining equations (3) and (4), one gets an isomorphism of graded vector spaces:[2]

$$H^*(X/F,\mathbb{Q}) \cong H^*(X,\mathbb{Q})^F. \tag{5}$$

Proposition 4.3. *Let F be a finite group and X a smooth quasi-projective F-variety. Then, the pullback of the quotient map $\pi^* : H^*(X/F,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ is injective and has $H^*(X,\mathbb{Q})^F$ as its image.*

---

1 For polynomials, the superscript F means we are taking the equivariant version, for vector spaces with an F action, the superscript denotes the fixed subspace.

2 We thank Donu Arapura for a suggestion leading to a shorter proof of Proposition 4.3.

Proof. Assume first that F acts freely on X. Then, X/F has a well-defined manifold structure, and one can realize the pullback in cohomology by the pullback in differential forms. In particular, this shows that the image of the pullback $\pi^* : H^*(X/F, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is given by $H^*(X, \mathbb{Q})^F$. Using (5), this means that the pullback map is bijective onto $H^*(X, \mathbb{Q})^F$.

If F does not act freely, the same argument can be reproduced for the de Rham orbifold cohomology, in which representatives of orbifold cohomology classes are sections of exterior powers of the orbifold cotangent bundle (see [35]). The result then follows because, for manifolds such as X, the de Rham orbifold cohomology reduces to the usual de Rham cohomology. □

The isomorphism of (5) can be obtained as the pullback of the algebraic map $\pi : X \to X/F$. Given that pullbacks of algebraic maps preserve MHSs, we see that this isomorphism respects MHS (see also [8])

$$H^{*,*,*}(X, \mathbb{Q})^F \cong H^{*,*,*}(X/F, \mathbb{Q}). \tag{6}$$

Moreover, since orbifolds satisfy Poincaré duality (see Satake [36], where these are called V-manifolds), this isomorphism is also valid for the compactly supported cohomology.

Corollary 4.4. *Let X be a smooth complex quasi-projective F-variety, for F a finite group, and $\mathcal{P}_X(t, u, v)$ denote either the Poincaré, Hodge-Deligne or E-polynomials, for the usual or the compactly supported cohomologies. Then, $\mathcal{P}_{X/F}(t, u, v)$ equals the coefficient of the trivial representation in $\mathcal{P}^F_X(t, u, v)$.*

Proof. Given equation (6), the cohomology of X/F coincides with the invariant part of the induced action $F \curvearrowright H^*(X, \mathbb{C})$, and this is precisely the coefficient of the trivial representation in each equivariant polynomial $\mathcal{P}^F_X$. □

Corollary 4.5. *Let X be a smooth complex quasi-projective F-variety, as above. If $h^{k,p,q}(X/F) \neq 0$, then $h^{k,p,q}(X) \neq 0$. Consequently, if X is balanced, separably pure or round, then the same is true for X/F.*

Proof. By Corollary 4.4, $H^{k,p,q}(X/F) \cong H^{k,p,q}(X)^F \subset H^{k,p,q}(X)$, so the first sentence follows. Since all the properties of being balanced, etc, are relations between the coefficients of t and u, v, they survive to the quotient. □

4.3 Character formulas

For a F-variety X is useful to consider the characters of the representations $H^{k,p,q}(X)$, when this space is viewed as a F-module.

For this, let $A_g \in \mathrm{Aut}(H^{k,p,q}(X))$ be the induced automorphism of $H^{k,p,q}(X)$ given by the action of an element $g \in F$. Given k-weights (p, q), denote by

$$\begin{aligned} \chi_{k,p,q} &: F \to \mathbb{C} \\ g &\mapsto \mathrm{tr}(A_g) \end{aligned}$$

the character of $H^{k,p,q}(X)$. In general, if we denote the character of an F-module V by χ_V, because of the properties of these with respect to direct sums, we have:

$$\chi_{\mu^F_X(t,u,v)}(g) = \sum_{k,p,q} \chi_{k,p,q}(g) t^k u^p v^q, \tag{7}$$

where $\mu^F_X(t, u, v)$ is viewed as an F-module, and equivalently as a direct sum of modules graded according to the triples (k, p, q). Let $|F|$ be the cardinality of F.

Theorem 4.6. *Let X be a quasi-projective F-variety. Then*

$$\mu_{X/F}(t,u,v)=\frac{1}{|F|}\sum_{g\in F}\sum_{k,p,q}\chi_{k,p,q}(g)t^k u^p v^q.$$

Proof. If V is an F-module, and $V=\oplus_i V_i$ is a decomposition of V into irreducible sub-representations, then by the Schur orthogonality relations, the coefficient of the trivial one-dimensional representation $\mathbf{1}$ is given by:

$$\langle\chi_V,\chi_{\mathbf{1}}\rangle=\frac{1}{|F|}\sum_{g\in F}\chi_V(g)\overline{\chi_{\mathbf{1}}(g)}=\frac{1}{|F|}\sum_{g\in F}\chi_V(g).$$

Applying this to $V=\mu_X^F(t,u,v)$ gives, in view of Corollary 4.4:

$$\mu_{X/F}(t,u,v)=\langle\chi_{\mu_X^F(t,u,v)},\chi_{\mathbf{1}}\rangle=\frac{1}{|F|}\sum_{g\in F}\chi_{\mu_X^F(t,u,v)}(g),$$

and the wanted formula follows from equation (7). □

Example 4.7. Let $X=\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ and consider the natural permutation action of $S_2\cong\mathbb{Z}_2$. If $\mathbf{S}$ denotes the one-dimensional sign representation, by describing the induced action on cohomology, it is not difficult to show that $[H^{0,0,0}(X)]\cong[H^{4,2,2}(X)]\cong\mathbf{1}$ (the trivial one-dimensional representation) and that $[H^{2,1,1}(X)]\cong\mathbf{1}\oplus\mathbf{S}$, giving:

$$\mu_X^{S_2}(t,u,v)=[H^{0,0,0}(X)]\oplus[H^{2,1,1}(X)]t^2uv\oplus[H^{4,2,2}(X)]t^4u^2v^2=\mathbf{1}\oplus(\mathbf{1}\oplus\mathbf{S})t^2u\oplus\mathbf{1}t^4u^2v^2. \tag{8}$$

Alternatively, writing $S_2=\{\pm1\}$, and taking the trivial characters $\chi_{0,0,0}(g)=\chi_{4,2,2}(g)\equiv1$, for $g\in S_2$, and $\chi_{2,1,1}(1)=2$, $\chi_{2,1,1}(-1)=0$, we can use Theorem 4.6 to get:

$$\mu_{X/S_2}(t,u,v)=\frac{1}{2}(1+2t^2uv+t^4u^2v^2)+\frac{1}{2}(1+t^4u^2v^2)=1+t^2uv+t^4u^2v^2,$$

which coincides with the coefficient of $\mathbf{1}$ in equation (8). Naturally, this is the expected polynomial, because $X/S_2=\mathsf{Sym}^2(\mathbb{P}^1_{\mathbb{C}})\simeq\mathbb{P}^2_{\mathbb{C}}$.

An interesting application of Theorem 4.6 is when the cohomology of X is an exterior algebra. To be precise, we say that $H^*(X,\mathbb{C})$ is an *exterior algebra of odd degree* k_0 if:

$$H^{k_0 l}(X,\mathbb{C})\cong\bigwedge^l H^{k_0}(X,\mathbb{C}),\quad\forall l\geq0,$$

and all other cohomology groups are zero.

Corollary 4.8. *Let X be an elementary F-variety whose cohomology is an exterior algebra of odd degree k_0, and let $H^{k_0}(X,\mathbb{C})=H^{k_0,p_0,p_0}(X)$ be its (trivial) mixed Hodge decomposition, for some $p_0\leq k_0$. Then, for $r>0$ and the diagonal action of F on X^r:*

$$\mu_{X^r/F}(t,x)=\frac{1}{|F|}\sum_{g\in F}\det\left(I+t^{k_0}x^{p_0}A_g\right)^r,$$

with $x=uv$, where A_g is the automorphism of $H^k(X,\mathbb{C})$ corresponding to $g\in F$, and I is the identity automorphism. In particular, if X is round:

$$\mu_{X^r/F}(t,x)=\frac{1}{|F|}\sum_{g\in F}\det\left(I+(tx)^{k_0}A_g\right)^r.$$

Proof. First, let $r=1$. Since X is elementary and tensor and exterior products preserve MHSs, we get for all $l\geq0$,

$$H^{lk_0}(X,\mathbb{C})=\bigwedge^l H^{k_0}(X,\mathbb{C})=\bigwedge^l H^{k_0,p_0,p_0}(X)=H^{lk_0,lp_0,lp_0}(X).$$

Applying Theorem 4.6 to this case, using $x = uv$, we get

$$\mu_{X/F}(t, x) = \frac{1}{|F|}\sum_{g\in F}\sum_{l\geq 0}\chi_{lk_0,lp_0,lp_0}(g)\,t^{lk_0}x^{lp_0}. \tag{9}$$

Now, for a general F-module V, with $g \in F$ acting as $V_g \in Aut(V)$, we have:

$$\sum_{l\geq 0}\chi_{\wedge^l V}(g)s^l = \det(I + sV_g).$$

This can be seen by expanding the characteristic polynomial of V_g in terms of traces of $\wedge^l V_g$ (see e.g., [37, p. 69]). Substituting the last equality into (9), we get the result with $s = t^{k_0}x^{p_0}$. Now, for a general $r \geq 1$, it follows from Proposition 4.2(2) that for the diagonal action $\mu^F_{X^r} = (\mu^F_X)^r$, so

$$\mu_{X^r/F}(t, x) = \frac{1}{|F|}\sum_{g\in F}\chi_{(\mu^F_X(t,x))^r}(g) = \frac{1}{|F|}\sum_{g\in F}\left(\chi_{\mu^F_X(t,x)}(g)\right)^r = \frac{1}{|F|}\sum_{g\in F}\det\left(I + t^{k_0}x^{p_0}A_g\right)^r.$$

Finally, the round case follows by setting $p_0 = k_0$. □

5 Abelian character varieties and their Hodge-Deligne polynomials

In this section, we apply the previous formulas to the computation of the Hodge-Deligne, Poincaré and E-polynomials, of the distinguished irreducible component of some families of character varieties. The important case of $GL(n, \mathbb{C})$-character varieties leads to the action of the symmetric group on a torus and is naturally related to work of I. G. Macdonald [38] and of J. Cheah [22] on symmetric products.

5.1 Mixed Hodge polynomials of abelian character varieties

As in Section 2.1, let G be a connected complex affine reductive group. For simplicity, the G-character variety of $\Gamma = \mathbb{Z}^r$, a rank r free abelian group, will be denoted by

$$\mathcal{M}_r G := \mathcal{M}_{\mathbb{Z}^r} G = \mathsf{Hom}(\mathbb{Z}^r, G)/\!/G.$$

In general, the varieties $\mathcal{M}_r G$ (as well as $\mathsf{Hom}(\mathbb{Z}^r, G)$) are not irreducible. But there is a unique irreducible subvariety containing the identity representation that we call the distinguished component and denote by $\mathcal{M}^0_r G$, which is constructed as the image under the composition

$$T^r = \mathsf{Hom}(\mathbb{Z}^r, T) \overset{\iota}{\hookrightarrow} \mathsf{Hom}(\mathbb{Z}^r, G) \overset{\pi}{\to} \mathcal{M}_r G, \tag{10}$$

where π is the GIT projection, and T is a fixed maximal torus of G. This image,

$$\mathcal{M}^0_r G := (\pi \circ \iota)(\mathsf{Hom}(\mathbb{Z}^r, T)),$$

is then a closed subvariety of $\mathcal{M}_r G$ (see [20]) that we call the *distinguished component*. Let W be the Weyl group of G, acting by conjugation on T. We quote the following result from [17]. As in Section 1, denote by $\mathcal{M}^*_r G$ the normalization of $\mathcal{M}^0_r G$ as an algebraic variety, so that there is a birational map $\nu : \mathcal{M}^*_r G \to \mathcal{M}^0_r G$.

Proposition 5.1. [17, Theorem 2.1] *Let G be a complex reductive group and $r \geq 1$. Then, $\mathcal{M}^0_r G$ is an irreducible component of $\mathcal{M}_r G$, and there is an isomorphism $\mathcal{M}^*_r G \cong T^r/W$.*

We now prove Theorem 1.1.

Theorem 5.2. *Let G be a complex reductive algebraic group. Then, $\mathcal{M}_r^*G$ is round and*

$$\mu_{\mathcal{M}_r^*G}(t,u,v) = \frac{1}{|W|}\sum_{g\in W}[\det(I + tuvA_g)]^r\,, \tag{11}$$

where A_g is the automorphism of $H^1(T,\mathbb{C})$ given by $g \in W$, and I is the identity.

Proof. Since Cartesian products of round varieties are round, and the maximal torus of G is isomorphic to $(\mathbb{C}^*)^n$ for some n, T is a round variety and has an algebraic action of W. Then W also acts diagonally on $T^r = (\mathbb{C}^*)^{nr}$, so T^r/W is also round by Corollary 4.5. Moreover, the cohomology of T is an exterior algebra of degree $k_0 = 1$, so Corollary 4.8 immediately gives the desired formula for T^r/W. The theorem follows from the isomorphism $\mathcal{M}_r^*G \cong T^r/W$ of Proposition 5.1. □

Remark 5.3. By Remark 3.10(1), we obtain, in the compactly supported case:

$$\mu^c_{\mathcal{M}_r^*G}(t,u,v) = \frac{t^{r\cdot\dim T}}{|W|}\sum_{g\in W}[\det(tuvI + A_g)]^r, \tag{12}$$

where $\dim T$ is the rank of G. We also obtain a formula for the Poincaré and for the Serre polynomial $E^c_{\mathcal{M}_r^0G}$ of the distinguished component $\mathcal{M}_r^0G$.

Theorem 5.4. *For every complex reductive algebraic group G and $r \geq 1$, the Poincaré polynomial* (*respectively, the Serre polynomial*) *of $\mathcal{M}_r^0G$ is given by substituting $u = v = 1$ in equation* (11) (*respectively*, $t = -1$ *in* (12)).

Proof. From [16, Corollary 4.9], there is a strong deformation retraction from $\mathcal{M}_r^0G$ to $\mathrm{Hom}^0(\mathbb{Z}^r, K)/K$, the path component of the space of commuting r-tuples of elements in K, containing the trivial r-tuple, up to conjugation, where K is a maximal compact subgroup of G. Hence, these spaces have the same Poincaré polynomials. On the other hand, the formula of [19, Theorem. 1.4] is the same as equation (11) with $u = v = 1$. Indeed, given the identification of $H^1(T,\mathbb{C})$ with $\mathfrak{t} \cong \mathbb{C}^n$, the Lie algebra of T:

$$H^1(T,\mathbb{C}) \cong H^1((\mathbb{C}^*)^n,\mathbb{C}) = H^1(\mathbb{C}^*,\mathbb{C})^n \cong \mathfrak{t},$$

and the fact that every cohomology class has a left invariant representative, the action of the Weyl group $W = S_n$ on $H^1(T,\mathbb{C})$ coincides with the one used in [19], in the context of compact Lie groups.[3]

As indicated in Proposition 2.4(4), the Serre polynomial (E^c-polynomial) is additive for disjoint unions of locally closed subvarieties. Therefore, for every bijective normalization morphism between algebraic varieties $f : X \to Y$ the E^c-polynomials of X and of Y coincide. In particular, the E^c-polynomials of $\mathcal{M}_r^*G$ and $\mathcal{M}_r^0G$ coincide. □

5.2 Normality of the distinguished component

Given the equalities of both Poincaré and Serre polynomials of $\mathcal{M}_r^0G$ and $\mathcal{M}_r^*G$, it is interesting to check where there is also an equality $\mu_{\mathcal{M}_r^*G} = \mu_{\mathcal{M}_r^0G}$. To handle this question, we start by considering some sufficient conditions for normality of $\mathcal{M}_r^0G$.

3 We thank S. Lawton for calling our attention to the recent preprint [19], where the Poincaré polynomial of analogous spaces for *compact* Lie groups is computed by quite different methods.

Lemma 5.5. *Let $F \subset G$ be a finite subgroup of the center of G, and $H = G/F$. If $\mathcal{M}_r^0 G$ is normal, then $\mathcal{M}_r^0 H$ is normal.*

Proof. Let $\mathrm{Hom}^0(\mathbb{Z}^r, G) := \pi^{-1}(\mathcal{M}_r^0 G)$. By definition of π in equation (10) this is the variety of homomorphisms that can be conjugated, in G, to some representation inside the maximal torus $T \subset G$ (and similarly for $\pi_H : \mathrm{Hom}^0(\mathbb{Z}^r, H) \to \mathcal{M}_r^0 H$). The fibration of algebraic groups $F \to G \to H$ induces the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathbb{Z}^r, F) & \hookrightarrow & \mathrm{Hom}(\mathbb{Z}^r, T) & \twoheadrightarrow & \mathrm{Hom}(\mathbb{Z}^r, T') \\
\| & & \downarrow & & \downarrow \\
\mathrm{Hom}(\mathbb{Z}^r, F) & \hookrightarrow & \mathrm{Hom}^0(\mathbb{Z}^r, G) & \overset{\phi}{\twoheadrightarrow} & \mathrm{Hom}^0(\mathbb{Z}^r, H) \\
 & & \downarrow \pi & & \downarrow \pi_H \\
 & & \mathcal{M}_r^0 G & \twoheadrightarrow & \mathcal{M}_r^0 H,
\end{array}$$

where $T' := T/F$ is a maximal torus of H, and the surjections on the two top rows are discrete fibrations (and finite étale morphisms). Since F is central, conjugating by G or by H are equivalent, so that $\mathcal{M}_r^0 G \cong \mathrm{Hom}^0(\mathbb{Z}^r, G)/H$, and the map ϕ is H-equivariant, we obtain isomorphisms:

$$\mathcal{M}_r^0 H \cong \mathrm{Hom}^0(\mathbb{Z}^r, H)/H \cong (\mathrm{Hom}^0(\mathbb{Z}^r, G)/F^r)/H \cong \mathcal{M}_r^0 G/F^r$$

because the actions of $F^r \cong \mathrm{Hom}(\mathbb{Z}^r, F)$ and conjugation by H on commute. Hence, as an algebraic quotient by a finite group, if $\mathcal{M}_r^0 G$ is normal, then so is $\mathcal{M}_r^0 F$. □

Remark 5.6. If the action of $F^r \cong \mathrm{Hom}(\mathbb{Z}^r, F)$ on $\mathcal{M}_r^0 G$ was free, the quotient $\mathcal{M}_r^0 G \to \mathcal{M}_r^0 H$ would also be étale, and the converse of Lemma (5.5) would be valid (see e.g. [39, Theorem 4.4(i)]). However, this is not the case for the isogeny $\mathrm{Spin}(n, \mathbb{C}) \to SO(n, \mathbb{C})$ ($\mathrm{Spin}(n, \mathbb{C})$ is the complexification of the universal cover $\mathrm{Spin}(n) \to SO(n)$, of the compact group $SO(n)$). The normality of $\mathcal{M}_r^0 \mathrm{Spin}(n, \mathbb{C})$ and of $\mathcal{M}_r^0 G$ for exceptional groups G is known to be a difficult problem (see Sikora [17, Problem 2.3]).

For a complex reductive group G, let $DG = [G, G]$ denote its derived group, which is a semisimple group. Let us call *classical semisimple group* to a group G that is a direct product of groups of the three classical families: $SL(n, \mathbb{C})$, $Sp(n, \mathbb{C})$ and $SO(n, \mathbb{C})$, $n \in \mathbb{N}$.

Lemma 5.7. *If DG is a classical semisimple group, then $\mathcal{M}_r^0 G$ is a normal variety.*

Proof. Sikora proved that, when $G = SL(n, \mathbb{C})$, $Sp(n, \mathbb{C})$ or $SO(n, \mathbb{C})$, there are algebraic isomorphisms $\mathcal{M}_r^0 G \cong T^r/W$ [17]. So for G in these three families, $\mathcal{M}_r^0 G$ is normal. It is clear that if $G = G_1 \times G_2$, then the maximal torus is also a product, and that $\mathcal{M}_r^0 G = \mathcal{M}_r^0 G_1 \times \mathcal{M}_r^0 G_2$. Thus, $\mathcal{M}_r^0 G$ is normal for any classical semisimple group G.

Finally, the result follows from Lemma 5.5, by taking finite quotients. Indeed, by the central isogeny theorem, any reductive group G is a finite central quotient of a product of its derived group DG with a torus T (and clearly $\mathcal{M}_r T = \mathcal{M}_r^0 T \cong T^r$). □

Since the hypothesis of Lemma 5.7 implies that $\mathcal{M}^* G = \mathcal{M}_r^0 G$, we have proved the following.

Theorem 5.8. *Let $r \geq 1$, and let G be a reductive group whose derived group is a classical group. Then, the mixed Hodge polynomial of $\mathcal{M}_r^0 G$ is given by formula* (11).

This motivates the following conjecture.

Conjecture 5.9. For every $r \geq 1$ and complex reductive G, formula (11) holds for $\mathcal{M}_r^0 G$.

5.3 $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$ cases

The case of $G = GL(n,\mathbb{C})$ is instructive, where the Weyl group is just the symmetric group, denoted by S_n. If X is a variety, we denote its n-fold symmetric product by $X^{(n)}$ or by $\mathsf{Sym}^n(X) = X^n/S_n$. As a set, $\mathsf{Sym}^n(X)$ is the set of unordered n-tuples of (not necessarily distinct) elements of X.

Proposition 5.10. *Let $G = GL(n,\mathbb{C})$, and let $T \cong (\mathbb{C}^*)^n$ denote a maximal torus of G. Then $\mathcal{M}_r G = \mathcal{M}_r^0 G = \mathcal{M}_r^* G$ and we have isomorphisms of affine algebraic varieties*

$$\mathcal{M}_r G \cong T^r/S_n \cong \mathsf{Sym}^n(\mathbb{C}^*)^r.$$

Proof. In [16, Corollary 5.14], it was shown that $\mathcal{M}_r G = \mathsf{Hom}(\mathbb{Z}^r, GL(n,\mathbb{C}))/\!/GL(n,\mathbb{C})$ is an irreducible variety (it is also path connected, given the strong deformation retraction from $\mathcal{M}_r G$ to the path connected compact space $\mathsf{Hom}(\mathbb{Z}^r, U(n))/U(n)$, see [14,16]). Since $\mathcal{M}_r^0 G$ is irreducible of the same dimension (by [17, Theorem 2.1(1)]), $\mathcal{M}_r G = \mathcal{M}_r^0 G$. Moreover, $\mathcal{M}_r^0 G$ is normal, hence isomorphic to $\mathcal{M}_r^* G$, and so $\mathcal{M}_r G \cong \mathcal{M}_r^* G \cong T^r/W$, by Sikora's results in [17, Theorem 2.1(2)–(3)]. Since $W \cong S_n$ acts diagonally, we have finally $T^r/W = ((\mathbb{C}^*)^n)^r/S_n \cong ((\mathbb{C}^*)^r)^n/S_n = \mathsf{Sym}^n(\mathbb{C}^*)^r$. □

We now turn to the proof of Theorem 1.2, on the $SL(n,\mathbb{C})$-character variety of $\Gamma = \mathbb{Z}^r$, and start with the case $G = GL(n,\mathbb{C})$. Let M_σ denote a $n \times n$ permutation matrix (in some basis) corresponding to $\sigma \in S_n$ and let I_n be the $n \times n$ identity matrix.

Proposition 5.11. *Let $G = GL(n,\mathbb{C})$. Then, $\mathcal{M}_r G$ is round and its mixed Hodge polynomial is given by*

$$\mu_{\mathcal{M}_r G}(t,u,v) = \frac{1}{n!}\sum_{\sigma\in S_n}[\det(I_n + tuvM_\sigma)]^r\,.$$

Proof. This formula is a direct application of Proposition 5.2, since the maximal torus is $T \cong (\mathbb{C}^*)^n$, $W \cong S_n$ acts by permutation and $|S_n| = n!$. So, the automorphism A_σ on $H^1(T,\mathbb{C}) \cong \mathbb{C}^n$ acts by permutation, given by the matrix M_σ. □

Remark 5.12. There is a strong deformation retract from $\mathcal{M}_r G \cong \mathsf{Sym}^n(\mathbb{C}^*)^r$ to $(S^1)^r/S_n \cong \mathsf{Sym}^n(S^1)^r$ (see [16]), which is the space of n (unordered) points on the compact r-torus $(S^1)^r$. So our results relate also to the study of cohomology of the so-called configuration spaces on compact Lie groups.

We now provide an even more concrete formula, and better adapted to computer calculations, using the relation between conjugacy classes of permutations and partitions of a natural number n, to compute the aforementioned determinants.

For this, we set up some notations. Let $n \in \mathbb{N}$ and $\mathcal{P}_n$ be the set of partitions of n. We denote by $\underline{n}$ a general partition in $\mathcal{P}_n$ and write it as

$$\underline{n} = [1^{a_1}2^{a_2}\cdots n^{a_n}],$$

where $a_j \geq 0$ denotes the number of parts of $\underline{n}$ of size $j = 1,\ldots,n$; then, of course $n = \sum_{j=1}^n ja_j$.

Theorem 5.13. *Let $G = GL(n,\mathbb{C})$, $x = uv$ and $\underline{n} = [1^{a_1}2^{a_2}\cdots n^{a_n}] \in \mathcal{P}_n$. The mixed Hodge polynomial of $\mathcal{M}_r G$ is given by*

$$\mu_{\mathcal{M}_r G}(t,x) = \sum_{\underline{n}\in\mathcal{P}_n}\prod_{j=1}^n \frac{(1-(-tx)^j)^{a_j r}}{a_j!\,j^{a_j}}.$$

Proof. To compute the determinant in Proposition 5.11, recall that any permutation $\sigma \in S_n$ can be written as a product of disjoint cycles (including cycles of length 1), whose lengths provide a partition of n, say

$\underline{n}(\sigma) = [1^{a_1}2^{a_2}\cdots n^{a_n}]$. Moreover, any two permutations are conjugated if and only if they give rise to the same partition, so the conjugation class of σ uniquely determines the non-negative integers $a_1, \dots, a_n$. If σ is a full cycle $\sigma = (1\cdots n) \in S_n$, and M_σ a corresponding matrix, by computing in a standard basis, we easily obtain the conjugation invariant expression $\det(I_n - \lambda M_\sigma) = 1 - \lambda^n$. So, for a general permutation $\sigma \in S_n$ with cycles given by the partition $\underline{n}(\sigma)$ we have

$$\det(I_n - \lambda M_\sigma) = \prod_{j=1}^{n}(1-\lambda^j)^{a_j}.$$

Now, let $c_{\underline{n}(\sigma)}$ be the size of the conjugacy class of the permutation σ, as a subset of S_n. Then the formula of Proposition 5.11, with $\lambda = -tuv = -tx$, becomes:

$$\mu_{\mathcal{M}_r G}(t,x) = \frac{1}{n!}\sum_{\sigma\in S_n}[\det(I_n - (-tx)M_\sigma)]^r = \frac{1}{n!}\sum_{\underline{n}\in\mathcal{P}_n} c_{\underline{n}}\prod_{j=1}^{n}(1-(-tx)^j)^{a_j r},$$

where we replaced the sum over permutations by the sum over partitions $\underline{n}$ (each repeated $c_{\underline{n}}$ times). The result then follows from the well-known formula $\frac{c_{\underline{n}}}{n!} = \prod_{j=1}^{n}\frac{1}{a_j!\,j^{a_j}}$. ☐

Remark 5.14. Since $\mathcal{M}_r G$ is an orbifold of dimension nr, it satisfies the Poincaré duality for MHSs (4.4(1)), and we compute:

$$\mu^c_{\mathcal{M}_r G}(t,x) = (-t)^{nr}\sum_{\underline{n}\in\mathcal{P}_n}\prod_{j=1}^{n}\frac{((-tx)^j-1)^{a_j r}}{a_j!\,j^{a_j}}. \tag{13}$$

We now obtain the mixed Hodge polynomial for $\mathcal{M}_r SL(n,\mathbb{C})$, by relating it to $\mathcal{M}_r GL(n,\mathbb{C})$.

Theorem 5.15. *The mixed Hodge polynomials of the free abelian character varieties of $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$ are related by $\mu_{\mathcal{M}_r GL(n,\mathbb{C})}(t,x) = (1+tx)^r\mu_{\mathcal{M}_r SL(n,\mathbb{C})}(t,x)$, giving:*

$$\mu_{\mathcal{M}_r SL(n,\mathbb{C})}(t,x) = \sum_{\underline{n}\in\mathcal{P}_n}\frac{1}{(1+tx)^r}\prod_{j=1}^{n}\frac{(1-(-tx)^j)^{a_j r}}{a_j!\,j^{a_j}}.$$

Proof. In contrast to the projection $GL(n,\mathbb{C}) \to PGL(n,\mathbb{C}) := GL(n,\mathbb{C})/\mathbb{C}^*$, there is no algebraic map $GL(n,\mathbb{C}) \to SL(n,\mathbb{C})$ that commutes with the Weyl group action. So, we need to resort to the equivariant framework. We will actually prove a stronger equality:

$$\mu^{S_n}_{T^r}(t,x) = (1+tx)^r\mu^{S_n}_{\widetilde{T}^r}(t,x),$$

where $T \cong (\mathbb{C}^*)^n$ and $\widetilde{T} = \{z \in T \mid z_1\cdots z_n = 1\}$ are the maximal torus of $GL(n,\mathbb{C})$, and of $SL(n,\mathbb{C})$, respectively, and the S_n action is the natural permutation action on the coordinates. From Corollary 4.4(2), Proposition 5.1, and from the irreducibility of these character varieties, the theorem will follow. Using the multiplicativity of the equivariant polynomials (Proposition 4.2(2)) it suffices to show this for $r = 1$. Consider the fibration of quasi-projective varieties

$$\widetilde{T} \longrightarrow T \xrightarrow{\pi} \mathbb{C}^*,$$

where $\pi(z_1,\dots,z_n) = z_1\cdots z_n$. By considering the trivial action on $\mathbb{C}^*$, this is a fibration of S_n-varieties with trivial monodromy, since it is in fact a $\widetilde{T}$-principal bundle (and $\widetilde{T}$ is a connected Lie group). Then Theorem A.1 gives us an equality of the equivariant E-polynomials:

$$E^{S_n}(T) = E^{S_n}(\widetilde{T})E(\mathbb{C}^*) = E^{S_n}(\widetilde{T})(1-x).$$

Finally, the desired formula comes from the relations in Proposition 3.9, since all varieties in consideration are round. ☐

Now, we turn to the computation of some E^c-polynomials, which relate to some formulas obtained in [9].

Corollary 5.16. *For $r \in \mathbb{N}$ and $G = GL(n, \mathbb{C})$, we have*

$$E^c_{\mathcal{M}_r G}(x) = \sum_{\underline{n} \in \mathcal{P}_n} \prod_{j=1}^{n} \frac{(x^j - 1)^{a_j r}}{a_j! j^{a_j}},$$

with $\underline{n} = [1^{a_1} \cdots n^{a_n}] \in \mathcal{P}_n$, and $\chi(\mathcal{M}_r G) = 0$.

Proof. The formula for $E^c_{\mathcal{M}_r G}(x)$ follows directly from equation (13) with $t = -1$. Given the previous theorem, the vanishing of the Euler characteristic is clear, since all factors $x^j - 1$ in $E^c_{\mathcal{M}_r G}$ vanish when $x = 1$, and $\chi(\mathcal{M}_r G) = E^c_{\mathcal{M}_r G}(1)$ by Remark 3.10(1). □

We now prove Theorem 1.2.

Theorem 5.17. *Let $G = SL(n, \mathbb{C})$. Then, we have $\chi(\mathcal{M}_r G) = n^{r-1}$ and*

$$E^c_{\mathcal{M}_r G}(x) = \sum_{\underline{n} \in \mathcal{P}_n} \frac{1}{\delta(\underline{n})} p_{\underline{n}}(x)^r,$$

where $\delta(\underline{n}) := \prod_{j=1}^n a_j! j^{a_j}$, and $p_{\underline{n}}(x) = \frac{1}{x-1} \prod_{j=1}^n (x^j - 1)^{a_j}$.

Proof. The formula for $E^c_{\mathcal{M}_r G}(x)$ follows immediately from Theorem 5.15 and Corollary 5.16. For the Euler characteristic, we need to compute $E^c_{\mathcal{M}_r G}(1)$. First note that $p_{\underline{n}}(x)$ is a polynomial and can be factorized as $p_{\underline{n}}(x) = (x-1)^m h(x)$ with $h \in \mathbb{Z}[x]$ and $m = \left(\sum_{j=1}^n a_j\right) - 1$. So, $p_{\underline{n}}(1) = 0$ unless $\sum_{j=1}^n a_j = 1$. The only partition with $\sum_{j=1}^n a_j = 1$ is $\underline{n} = [n^1]$ (just one part, of length n), which corresponds to a cyclic permutation such as $(1 \cdots n) \in S_n$. The size of its conjugacy class is $c_{\underline{n}} = (n-1)!$ and we get

$$\chi(\mathcal{M}_r G) = E^c_{\mathcal{M}_r G}(1) = \frac{1}{n!}(n-1)! \lim_{x \to 1} \left(\frac{x^n - 1}{x - 1} \right)^r = \frac{1}{n} n^r = n^{r-1},$$

as we wanted to show. □

Remark 5.18. The E^c-polynomials of $\mathcal{M}_r G$ for $SL(2, \mathbb{C})$ and $SL(3, \mathbb{C})$ are already present in [9]. For $n \geq 4$ these formulas are new and can also be upgraded to mixed Hodge polynomials by using Remark 3.10(1) and Poincaré duality.

Example 5.19. The following table gives the explicit values of $\delta(\underline{n})$ and $p_{\underline{n}}(x)$ up to $n = 5$ (in each row, the ordering is preserved). All the formulas can be easily implemented in the available computer software packages (in this paper, most of our calculations were performed with GAP). For simplicity, the notation [12] refers to a partition of $n = 3$ with two cycles: one of length 1, another of length 2 (not a cycle of length 12).

n	$\lvert\mathcal{P}_n\rvert$	$\underline{n}$	$\delta(\underline{n})$	$p_{\underline{n}}(x)$
2	2	$[2]$; $[1^2]$	2; 2	$x + 1$; $x - 1$
3	3	$[3]$;	3;	$x^2 + x + 1$;
		$[12]$; $[1^3]$	2; 6	$x^2 - 1$; $(x - 1)^2$
4	5	$[4]$;	4;	$x^3 + x^2 + x + 1$;
		$[13]$; $[2^2]$;	3; 8;	$x^3 - 1$; $(x^2 - 1)(x + 1)$;
		$[1^2 2]$; $[1^4]$	4; 24	$(x - 1)^2(x + 1)$; $(x - 1)^3$
5	7	$[5]$; $[14]$;	5; 4;	$\frac{x^5 - 1}{x - 1}$; $x^4 - 1$;
		$[1^2 3]$; $[23]$;	6; 6;	$(x^3 - 1)(x - 1)$; $(x^3 - 1)(x + 1)$;
		$[12^2]$; $[1^3 2]$; $[1^5]$	8; 12; 120	$(x^2 - 1)^2$; $(x - 1)^3(x + 1)$; $(x - 1)^4$

For example, with $n = 4$, the table immediately gives

$$E^c(x) = \frac{1}{4}(x^3 + x^2 + x + 1)^r + \frac{1}{3}(x^3 - 1)^r + \frac{1}{8}(x^2 - 1)^r(x + 1)^r + \frac{1}{4}(x - 1)^{2r}(x + 1)^r + \frac{1}{24}(x - 1)^{3r},$$

for any $r \geq 1$.

5.4 Symmetric products and Cheah's formula

So far, our approach to Hodge numbers for the character varieties $\mathcal{M}_r G$, for $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$, is well adapted to fixing $n \in \mathbb{N}$, and let r be arbitrary, as we can see from Theorems 5.13 and 5.16. On the other hand, since the $GL(n, \mathbb{C})$-character varieties of $\mathbb{Z}^r$ are symmetric products

$$\mathcal{M}_r G = \mathcal{M}_r GL(n, \mathbb{C}) = \mathsf{Sym}^n(\mathbb{C}^*)^r,$$

as in Proposition 5.10, we can apply a formula of J. Cheah [22] for the mixed Hodge numbers of symmetric products. Indeed, this will lead to an "orthogonal" approach: by fixing small values for $r \in \mathbb{N}$, we obtain simple formulas valid for all $n \in \mathbb{N}$.

Let X be a quasi-projective variety with given compactly supported Hodge numbers $h_c^{k,p,q}(X)$. Cheah's formula gives the generating function of the mixed Hodge polynomials of all symmetric products $X^{(n)} = \mathsf{Sym}^n X$ is (see [22]):

$$\sum_{n\geq 0} \mu^c_{X^{(n)}}(t, u, v) z^n = \prod_{p,q,k} (1 - (-1)^k u^p v^q t^k z)^{(-1)^{k+1} h_c^{k,p,q}(X)}. \tag{14}$$

We start by observing that, for varieties satisfying Poincaré duality, Cheah's formula stays unaffected when passing from μ^c to μ and from $h_c^{k,p,q}$ to $h^{k,p,q}$.

Proposition 5.20. *Let X satisfy Poincaré duality. Then*

$$\sum_{n\geq 0} \mu_{X^{(n)}}(t, u, v) z^n = \prod_{p,q,k} (1 - (-1)^k u^p v^q t^k z)^{(-1)^{k+1} h^{k,p,q}(X)}.$$

Proof. This is a simple calculation. Let X have complex dimension d. From the relation between μ_X and μ_X^c, in Remark 3.10(1), Cheah's formula (14) is equivalent to:

$$\sum_{n\geq 0} \mu_{X^{(n)}}(t^{-1}, u^{-1}, v^{-1})(t^2 uv)^{nd} z^n = \prod_{p,q,k} (1 - (-1)^k u^p v^q t^k z)^{(-1)^{k+1} h^{2d-k,d-p,d-q}(X)}.$$

Now, changing the indices (k, p, q) to $(k', p', q') = (2d - k, d - p, d - q)$, which preserves the parity of k, we obtain

$$\sum_{n\geq 0} \mu_{X^{(n)}}\left(\frac{1}{t}, \frac{1}{u}, \frac{1}{v}\right)((t^2 uv)^d z)^n = \prod_{p',q',k'} (1 - (-1)^{k'} u^{-p'} v^{-q'} t^{-k'} (t^{2d} u^d v^d z))^{(-1)^{k'+1} h^{k',p',q'}(X)},$$

which is clearly equivalent to the desired formula, under the substitution: $(t^{-1}, u^{-1}, v^{-1}, t^{2d} u^d v^d z) \mapsto (t, u, v, z)$. ☐

For round varieties, as before, all products reduce to a single index.

Proposition 5.21. *Let X be a round variety of dimension d satisfying Poincaré duality. Then*

$$\sum_{n\geq 0} \mu_{X^{(n)}}(t, u, v) z^n = \prod_k (1 - (-tuv)^k z)^{(-1)^{k+1} h^{k,k,k}(X)}.$$

Proof. This is immediate from Proposition (5.20), since the only non-zero Hodge numbers of a round variety X are $h^{k,k,k}(X)$, for some values of k. ☐

We are now ready to apply this formula to $\mathcal{M}_r GL(n,\mathbb{C})$. Since this space is $\mathrm{Sym}^n(\mathbb{C}^*)^r$, we should consider $X = (\mathbb{C}^*)^r$.

Corollary 5.22. *Let $G = GL(n,\mathbb{C})$. Then*:

$$\sum_{n\geq 0}\mu_{\mathcal{M}_r G}(t,u,v)z^n = \prod_{k\geq 0}(1-(-tuv)^k z)^{(-1)^{k+1}\binom{r}{k}} = \frac{\prod_{k\ \mathrm{odd}}(1+(tuv)^k z)^{\binom{r}{k}}}{\prod_{k\ \mathrm{even}}(1-(tuv)^k z)^{\binom{r}{k}}}.$$

Proof. Letting $X = (\mathbb{C}^*)^r$, $d = r = \dim X$ and $h^{k,k,k}(X) = \binom{r}{k}$, $0 \leq k \leq r$, the proof is immediate from Proposition (5.21). □

Example 5.23. The simplest example of this formula is when $r = 1$ (for $r = 0$, $\mathrm{Sym}^n(\mathbb{C}^*)^0$ is a single point). In this case, $X = \mathbb{C}^*$ and $\binom{1}{0} = \binom{1}{1} = 1$, so we expand the right hand-side as a power series in z, as:

$$\frac{1+tuvz}{1-z} = 1 + \sum_{n\geq 1}(1+uvt)z^n.$$

In particular, $\mu_{\mathcal{M}_1 GL(n,\mathbb{C})}(t,u,v) = 1 + tuv$, for $n \geq 1$, which agrees with the fact that $\mathcal{M}_1 GL(n,\mathbb{C}) = \mathrm{Sym}^n(\mathbb{C}^*) \cong \mathbb{C}^{n-1}\times\mathbb{C}^*$ has the same Hodge structure as $\mathbb{C}^*$.

The case with $r = 2$ is an interesting result in itself.

Proposition 5.24. *Let $G = GL(n,\mathbb{C})$, $r = 2$ and $n \geq 1$. Then*:

$$\mu_{\mathcal{M}_2 G}(t,u,v) = (1+tuv)^2(1+(tuv)^2+\cdots+(tuv)^{2n-2}),$$

$P_{\mathcal{M}_2 G}(t) = (1+t)^2(1+t^2+\cdots+t^{2n-2})$ *and* $\chi(\mathcal{M}_2 G) = 0$.

Proof. We now have $\binom{2}{2} = \binom{2}{0} = 1$ and $\binom{2}{1} = 2$, so we expand the right hand-side of Corollary 5.22 as a power series in z, writing $\lambda = tuv$ for simplicity:

$$\frac{(1+\lambda z)^2}{(1-z)(1-\lambda^2 z)} = 1 + \frac{(1+\lambda)^2}{1-\lambda^2}\left(\frac{1}{1-z} - \frac{1}{1-\lambda^2 z}\right) = 1 + (1+\lambda)^2\sum_{n\geq 0}(1+\lambda^2+\cdots+\lambda^{2n-2})z^n.$$

This gives the desired formulas for $\mathcal{M}_2 GL(n,\mathbb{C})$, with $\lambda = tuv$, with $uv = 1$ for $P_{\mathcal{M}_2 G}$ and $\lambda = -1$ for the Euler characteristic. □

The next corollary follows immediately from Theorem 5.15.

Corollary 5.25. *Let $G = SL(n,\mathbb{C})$, $r = 2$ and $n \geq 1$. Then*:

$$\mu_{\mathcal{M}_2 G}(t,x) = 1 + (tx)^2 + \cdots + (tx)^{2n-2},$$

$P_{\mathcal{M}_2 G}(t) = (1+t^2+\cdots+t^{2n-2})$ *and* $\chi(\mathcal{M}_2 G) = n$.

Remark 5.26. The equality of Poincaré polynomials $P_{\mathcal{M}_2 SL(n,\mathbb{C})} = P_{\mathbb{P}^{n-1}_{\mathbb{C}}}$ is not a coincidence. In fact, by non-abelian Hodge correspondence, $\mathcal{M}_2 SL(n,\mathbb{C})$ is diffeomorphic to the cotangent bundle of the projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$ parametrizing semistable bundles over an elliptic curve of rank n and trivial determinant (see [15, 40]).

5.5 A combinatorial identity

We finish the article with an interesting purely combinatorial identity. We could not find out whether this identity was noted before. Recall that I_n is the identity $n \times n$ matrix and M_σ a permutation matrix associated with $\sigma \in S_n$.

Theorem 5.27. *Fix $r \in \mathbb{N}_0$. Then, for formal variables x, z (or considering $x, z \in \mathbb{C}$ in a small disc around the origin) we have:*

$$\prod_{k\geq 0}(1 - x^k z)^{(-1)^{k+1}\binom{r}{k}} = \sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{z^n}{n!}\det(I_n - xM_\sigma)^r.$$

Proof. Putting together Corollary 5.22 and the formula for $\mu_{\mathcal{M}_r GL(n,\mathbb{C})}$ in Proposition 5.11, we obtain

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{\sigma\in S_n}[\det(I_n + tuvM_\sigma)]^r\, z^n = \prod_{k\geq 0}(1 - (-tuv)^k z)^{(-1)^{k+1}\binom{r}{k}},$$

which becomes the desired identity, by setting $x = -tuv$. Note that for $r = 0$ the formula is still valid and reduces to the geometric series. The formula holds also for $z, x \in \mathbb{C}$ where the series converges. We readily check that convergence holds whenever $|x| < 1$ and $|z| < 2^{-r}$, using the bound, valid for $|x| < 1$,

$$\sum_{\sigma\in S_n}\prod_{j=1}^{n}(1 - x^j)^{a_j r} < \sum_{\sigma\in S_n} 2^{|\sigma| r} < n!2^{nr},$$

where $|\sigma| = \sum_j a_j$ denotes the number of cycles of σ, with a_j parts of size j, as in the proof of Theorem 5.13. □

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Appendix A Multiplicativity of the E-polynomial under fibrations

In this appendix, we prove a multiplicative property of the E-polynomial under fibrations, used in Theorem 5.15. This is a consequence of the fact that the Leray-Serre spectral sequence is a spectral sequence of mixed Hodge structures.

A.1 E-polynomials of fibrations

It is well known that for an algebraic fibration of algebraic varieties

$$Z \longrightarrow E \xrightarrow{\pi} B$$

the Poincaré polynomials do not behave multiplicatively, in general. Then, *a fortiori*, a multiplicative property is not expected for the Hodge-Deligne polynomial. But when all involved varieties are smooth and the associated monodromy is trivial, this property turns out to be valid for the E-polynomials. Moreover, if there is a finite group F acting on the three varieties, and the involved maps respect the action, an equivariant fibration of F-varieties, one gets a multiplicative formula for their equivariant polynomials, under the assumption that the higher direct images sheaves $R^j\pi_*\mathbb{C}_E$ (associated with the presheaf $U \mapsto H^j(\pi^{-1}U, \mathbb{C}_E)$ for $U \subset B$) are constant, where $\mathbb{C}_E$ is the constant sheaf on E.

Theorem A.1. *Let F be a finite group and consider an algebraic fibration between smooth complex algebraic quasi-projective varieties*

$$Z \longrightarrow E \xrightarrow{\pi} B$$

(*not necessarily locally trivial in the Zariski topology*). *Suppose also that this is a fibration of F-varieties* (*all spaces are F-varieties and the maps are F-equivariant*). *If Z is connected and $R^j\pi_*\mathbb{C}_E$ are constant for every j, then*

$$E_E^F(u, v) = E_Z^F(u, v) \otimes E_B^F(u, v).$$

Proof. The non-equivariant version of this result is the content of Proposition 2.4 in [8], where it is used to calculate the Serre polynomials of certain twisted character varieties. We detail the argument here, for the reader's convenience. First, assume that the F-action is trivial on the three spaces. The Leray-Serre spectral sequence of the fibration is a sequence of mixed Hodge structures ([27, Theorem 6.5]), and it is proved in [21, Theorem 6.1] that under the given assumptions, its second page $E_2^{a,b}$ admits an isomorphism

$$E_2^{a,b} \simeq H^a(B) \otimes H^b(Z),$$

which is actually an isomorphism of mixed Hodge structures. In particular, we get an equality between their respective graded pieces:

$$\begin{aligned} Gr_F^p Gr_{p+q}^W E_2^{a,b} &= \bigoplus_{p'+p''=p} \bigoplus_{q'+q''=q} Gr_F^{p'} Gr_{p'+q'}^W H^a(B) \otimes Gr_F^{p''} Gr_{p''+q''}^W H^b(Z) \\ &= \bigoplus_{p'+p''=p} \bigoplus_{q'+q''=q} H^{a,p',q'}(B) \otimes H^{b,p'',q''}(Z). \end{aligned} \tag{15}$$

Using once more the fact that this is a spectral sequence of mixed Hodge structures, we get one spectral sequence of vector spaces for each pair (p, q):

$$E(p, q)_2^{a,b} := Gr_F^p Gr_{p+q}^W E_2^{a,b} \Rightarrow Gr_F^p Gr_{p+q}^W H^{a+b}(E).$$

Now set

$$P_{(p,q)}(t) := \sum_{k\geq 0} \dim \left(\bigoplus_{a+b=k} E(p, q)_2^{a,b} \right) t^k.$$

Given that $\oplus_{a+b=k}E(p,q)_2^{a,b} \Rightarrow \oplus_{a+b=k}Gr_F^p Gr_{p+q}^W H^{a+b}(E)$, one has

$$P_{(p,q)}(-1) = \sum_k (-1)^k \dim Gr_F^p Gr_{p+q}^W H^k(E) = \sum_k (-1)^k h^{k,p,q}(E).$$

So, by definition, $E_E(u,v) = \sum_{p,q} P_{(p,q)}(-1)u^p v^q$. On the other hand, using (15)

$$P_{(p,q)}(-1) = \sum_k \sum_{a+b=k} \sum_{p'+p''=p} \sum_{q'+q''=q} (-1)^a h^{a,p',q'}(B)(-1)^b h^{b,p'',q''}(Z),$$

substituting into $E_E(u,v) = \sum_{p,q} P_{(p,q)}(-1)u^p v^q$, and switching summation order, one gets $E_E(u,v) = E_B(u,v)$ $E_Z(u,v)$, as wanted. Succinctly, the argument follows from the fact that the spectral sequence above can be seen as an equality between derived functors and, thinking in terms of K-theory, passing to cohomology for obtaining the next sheet in the sequence does not change an alternating sum.

Finally, to prove the equivariant version, suppose that all the cohomologies are F-modules. Then, since we have a fibration of F-varieties the associated Leray-Serre spectral sequence is a spectral sequence of F-modules. The associated sequences $E(k,m)_2^{p,q}$ are also spectral sequences of F-modules, since the graded pieces for the Hodge and weight filtration are so. To get the desired equality, it suffices to proceed as before: in each step we substitute the dimension $h^{k,p,q}(\cdot)$ by the corresponding F-module $[Gr_F^p Gr_{p+q}^W H^k(\cdot)]_F$, and the operations performed in $(\mathbb{Z}, +, \times)$ are replaced by those in the ring $(R(F)[u,v], \oplus, \otimes)$. □

Remark A.2. The above proof follows the proof of Theorem 6.1(ii) in [21], where the version for the equivariant weight polynomial is obtained (which is implied by Theorem A.1, since the weight polynomial is a specialization of the E-polynomial). We also remark that the study of multiplicative invariants under fibrations goes back at least to work of Chern, Hirzebruch and Serre in the mid-fifties [41], on the signature theorem.