Hölder a priori estimates for second order tangential operators on CR manifolds

Annamaria Montanari[*]

Abstract

On a real hypersurface M in C^{n+1} of class $C^{2,\alpha}$ we consider a local CR structure by choosing n complex vector fields W_j in the complex tangent space. Their real and imaginary parts span a $2n$-dimensional subspace of the real tangent space, which has dimension $2n+1$. If the Levi matrix of M is different from zero at every point, than we can generate the missing direction. Under this assumption we prove interior a priori estimates of Schauder type for solutions of a class of second order partial differential equations with C^α coefficients, which are not elliptic because they involve second-order differentiation only in the directions of the real and imaginary part of the tangential operators W_j. In particular, our result applies to a class of fully nonlinear PDE's naturally arising in the study of domains of holomorphy in the theory of holomorphic functions of several complex variables.

Key words: Vector fields with $C^{1,\alpha}$ coefficients; Levi matrix; Control distance; Freezing metod; Hölder a priori estimates.

2000 Mathematics Subject Classification: Primary 35J70, 35H20; Secondary 32W50, 22E30.

1 Introduction

In this paper we prove a priori estimates for solutions of the linear subelliptic equation $Hv = f$ in $\mathbb{R}^{2n+1}$, where

$$H = \sum_{m,j=1}^{2n} h_{mj} Z_m Z_j - \lambda \partial_t, \tag{1}$$

the coefficients λ, h_{mj} are α-Hölder continuous and such that $h_{mj} = h_{jm}$, $m, j = 1, \dots, 2n$, and

$$\sum_{m,j=1}^{2n} h_{mj} \eta_m \eta_j \geq M \sum_{j=1}^{2n} \eta_j^2, \quad \forall \eta = (\eta_1, \dots, \eta_{2n}) \in \mathbb{R}^{2n} \tag{2}$$

---

[*] Investigation supported by University of Bologna. Funds for selected research topics.

for a suitable positive constant M. Here the first order differential operators Z_j are

$$\begin{aligned} Z_{2l} &= \frac{\partial}{\partial y_l} + \omega_{2l}\frac{\partial}{\partial t}, \\ Z_{2l-1} &= \frac{\partial}{\partial x_l} + \omega_{2l-1}\frac{\partial}{\partial t}, \\ Z &= (Z_1, Z_2, \dots, Z_{2n}), \end{aligned} \tag{3}$$

where $(x_1, y_1, \dots, x_n, y_n, t) \in \mathbb{R}^{2n+1}$ and the coefficients $\omega = (\omega_1, \dots, \omega_{2n})$ are of class $C^{1,\alpha}$.
The operator H in (1) is not elliptic at any point. In order to overcome the lack of ellipticity we make the following crucial hypothesis: we assume that the missing direction is generated by one of the commutators $[Z_l, Z_p], l \neq p$.
We explicitly remark that we can not apply to our operator H the regularity theory developed in [15], [16], [25], [3], because in those works the smoothness hypothesis on the coefficients of the vector fields is crucial.

Schauder-type estimates for sum of squares of smooth linear vector fields satisfying Hörmander condition have been proved by C. J. Xu in [31]. In that paper also operators formally of the type (1) were considered, with coefficients $\omega_j \in C^\infty$ and $h_{ij} \in C^{1,\alpha}$, but neither that result nor that technique work in our situation, because in our case the coefficients ω of Z are only $C^{1,\alpha}$. Moreover, even if the coefficients of the vector fields were smooth, operators of the type H as in (1) are studied in [31] by simple using a change of variables, which transforms the operator in a sum of squares. If the coefficients h_{ij} are only C^α, as for the linearized Levi Monge-Ampère equation (see [22]), this change of variable is not possible.

The motivation for studying operators of the type (1) in our assumptions is very strong. Indeed, the vector fields in (3) naturally arise in the study of envelopes of holomorphy in the theory of holomorphic functions in $\mathbb{C}^{n+1}$ (see [14], [18], [20], [24], [27], [28], [30] for details).

In order to clarify our motivation let us introduce some notations. Denote by $z = (z_1, \dots, z_{n+1})$ a point of $\mathbb{C}^{n+1}$ and by $M = \{z : \rho(z) = 0\}$ a real hypersurface in $\mathbb{C}^{n+1}$. Assume for example $\partial_{z_{n+1}}\rho \neq 0$ at $z_0 \in M$. Denote by $T_0^{\mathbb{C}}M$ the complex tangent hyperplane to M at z_0, and choose

$$h_l = e_l - \frac{\partial_{z_l}\rho}{\partial_{z_{n+1}}\rho} e_{n+1},$$

with $(e_p)_{p=1,\dots,n+1}$ the canonical basis of $\mathbb{C}^{n+1}$.

Since, for every $l = 1, \dots, n$

$$\langle h_l, \partial_{\bar z}\rho\rangle = \langle e_l - \frac{\partial_{z_l}\rho}{\partial_{z_{n+1}}\rho} e_{n+1}, \sum_{j=1}^{n+1} (\partial_{\bar z_j}\rho) e_j\rangle = \partial_{z_l}\rho - \frac{\partial_{z_l}\rho}{\partial_{z_{n+1}}\rho}\partial_{z_{n+1}}\rho = 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^{n+1}$, then $\mathcal{U} = \{h_l, l = 1, \dots, n\}$ is a complex basis of $T_0^{\mathbb{C}}M$.

By identifying e_p with the first order complex differential operator ∂_{z_p} for every $p = 1, \dots, n+1$ and h_l with the first order complex differential operator

$$W_l = \partial_{z_l} - \frac{\partial_{z_l}\rho}{\partial_{z_{n+1}}\rho}\partial_{z_{n+1}}, \tag{4}$$

for every $l = 1, \dots, n$, we obviously get $W_l\rho = 0$ for every $l = 1, \dots, n$.

In the sequel we will denote by

$$W_{\bar l} = \overline{W_l}, \tag{5}$$

for every $l = 1, \dots, n$.

If ρ is of class C^2 then the vector fields in (4) and (5) introduce a CR structure on M, because they are linear independent and $[W_l, W_p] = 0$ (see for example [14, p.93]). Moreover, let us define

$$T = \frac{1}{\partial_{z_{n+1}}\rho}\partial_{z_{n+1}} - \frac{1}{\partial_{\bar{z}_{n+1}}\rho}\partial_{\bar{z}_{n+1}}. \tag{6}$$

Then,

$$[W_l, W_{\bar{p}}] = A_{l\bar{p}}(\rho)T. \tag{7}$$

This defines the $n \times n$ Hermitian matrix $A_{l\bar{p}}(\rho)$, which is called the Levi matrix and we assume it is different from zero at every point.

Since we have assumed $\partial_{z_{n+1}}\rho \neq 0$ at $z_0 \in M$, it is not restrictive to take its imaginary part different from zero. With this convention, there is a neighborhood U_{z_0} of z_0 such that $M \cap U_{z_0}$ is the graph of a C^2 function $u : \Omega \to \mathbb{R}$, with Ω an open bounded subset in $\mathbb{R}^{2n+1}$. Then we can choose the defining function of M as $\rho = Im(z_{n+1}) - u(z_1, \dots, z_n, Re(z_{n+1}))$. By the coordinate change

$$\zeta_j = z_j \quad 1 \leq j \leq n, \quad t = Re(z_{n+1}), \quad r = Im(z_{n+1}) - u(z_1, \dots, z_n, Re(z_{n+1}))$$

the vector fields $W_{\bar{j}}$ become (see for example [29, p.547]) the following tangential Cauchy Riemann operators on M

$$W_{\bar{j}} = \frac{\partial}{\partial\bar{\zeta}_j} + \frac{\frac{\partial u}{\partial\bar{\zeta}_j}}{i - \frac{\partial u}{\partial t}}(\zeta_1, \dots, \zeta_n, t)\frac{\partial}{\partial t}. \tag{8}$$

Introduce real coordinates $\zeta_l = x_l + iy_l$ for every $l = 1, \dots, n$ and put

$$\begin{aligned} Z_{2l} &= 2Im\Big(\frac{\partial}{\partial\bar{\zeta}_l} + \frac{\frac{\partial u}{\partial\bar{\zeta}_l}}{i - \frac{\partial u}{\partial t}}(\zeta_1, \dots, \zeta_n, t)\frac{\partial}{\partial t}\Big), \\ Z_{2l-1} &= 2Re\Big(\frac{\partial}{\partial\bar{\zeta}_l} + \frac{\frac{\partial u}{\partial\bar{\zeta}_l}}{i - \frac{\partial u}{\partial t}}(\zeta_1, \dots, \zeta_n, t)\frac{\partial}{\partial t}\Big). \end{aligned} \tag{9}$$

Then, the vector fields Z have the same structure as those in (3) with coefficients

$$\begin{aligned} \omega_{2l} &= -\frac{u_{x_l} + u_{y_l}u_t}{1 + u_t^2}, \\ \omega_{2l-1} &= \frac{u_{y_l} - u_{x_l}u_t}{1 + u_t^2} \end{aligned} \tag{10}$$

where subscripts denote partial derivatives.

A regularity theory for sum of squares of $C^{1,\alpha}$ vector fields of the type (9) has been recently established by Citti in [4], [5], [6] and by Citti and the author in [12], [13].

In particular, by using the techniques developed in [12, Theorem 4.1.], one can prove the following.

Proposition 1.1. *Let $h_{ij}, \lambda \in C^{m-1,\alpha}_{Z,loc}(\Omega)$, $\omega \in C^{m,\alpha}_{Z,loc}(\Omega)$, $m \geq 2$ and let $v \in C^{2,\alpha}_{Z,loc}(\Omega)$ be a solution of equation $Hv = f$ with H as in (1) and $f \in C^{m-1,\alpha}_{Z,loc}(\Omega)$. Then the solution v belongs to $C^{m+1,\beta}_{Z,loc}(\Omega)$ for every $\beta \in (0, \alpha)$.*

Here $C^{m,\alpha}_Z$ denotes the class of functions whose tangent derivatives of order m are α-Hölder continuous with respect to a distance d_Z naturally associated to the vector fields Z_j (see (12) and (13) for precise definitions).

This result has been used in [12] to study regularity properties of quasilinear equations of Levi's type, but it is not useful for studying fully nonlinear equations such as the Levi Monge-Ampère equation, whose second order part is the determinant of the Levi matrix in (7)(see [21]). In that case the coefficients h_{ij} depend on the second tangential derivatives of a solution and ω depends on the first tangential derivatives of a solution. In particular, if $u \in C^{2,\alpha}_{Z,loc}(\Omega)$ is a solution of the Levi Monge-Ampère equation, then $h_{ij} \in C^{\alpha}_{Z,loc}(\Omega)$ and $\omega \in C^{1,\alpha}_{Z,loc}(\Omega)$ and it is not possible to apply to it Proposition 1.1.

In Section 2, by means of a method relying on the lifting argument first introduced by Rothshild and Stein in [25], and of a non standard freezing method already used in [12], [13], [4], [5], [6], we reduce the study of the operator H to the analysis of a family $\widetilde{H}_{\xi_0}$ of left invariant operators on a free nilpotent Lie group of dimension $N = 2n^2 + n + 1$. The fundamental solution $\widetilde{\Gamma}_{\xi_0}$ of the operator $\widetilde{H}_{\xi_0}$ is used as a parametrix of the operator H in (1) and provides an explicit representation formula for solutions of the linear equation $Hv = f$ in spaces of Hölder continuous functions $C^{2,\alpha}_Z$. Then, we twice differentiate this formula with respect to the intrinsic derivatives $Z_j, j = 1, \ldots, 2n$ and in Section 3 we estimate it at two different points.

Our main result is the following interior Schauder-type estimate for classical solutions of $Hv = f$, with $h_{ij}, \lambda, f \in C^{\alpha}$, and the coefficients ω of Z of class $C^{1,\alpha}$.

Theorem 1.1. *Let $h_{ij}, \lambda \in C^{\alpha}_Z(\Omega)$, $\omega \in C^{1,\alpha}_Z(\Omega)$ and $v \in C^{2,\alpha}_Z(\Omega)$ be a solution of equation $Hv = f \in C^{\alpha}_Z(\Omega)$. Then if $\Omega' \subset\subset \Omega$ with $d_Z(\Omega', \partial\Omega) \geq \delta > 0$, there is a positive constant c such that for every $\beta \in (0, \alpha)$*

$$\delta |Zv|^Z_{0;\Omega'} + \delta^2 |Z^2 v|^Z_{0;\Omega'} + \delta^{2+\beta} [Z^2 v]^Z_{\beta;\Omega'} \leq c(\sup_{\Omega} |v| + |f|^Z_{0,\alpha;\Omega}) \tag{11}$$

where c depends only on the constant M in (2), on $|h_{ij}|^Z_{0,\alpha;\Omega}$, $|\lambda|^Z_{0,\alpha;\Omega}$, $|\omega|^Z_{1,\alpha;\Omega}$, as well as on $n, \alpha, \delta, \Omega$.

Our method also requires interpolation inequalities between some weighted norms naturally associated to the geometry of the problem. The proof of these inequalities is inspired to a standard method for the elliptic case (see [17]), however in Appendix 1 we carry on it in details for reader convenience.

In a forthcoming paper [22] we will apply our Theorem 1.1 to prove smoothness of strictly Levi convex solutions of the fully nonlinear Levi Monge-Ampère equation.

2 Preliminaries

In this section we first introduce some classes $C^{m,\alpha}_Z$ of Hölder continuous functions naturally arising from the geometry of the problem. We then write a representation formula for $C^{2,\alpha}_Z$-solutions of $Hv = f$ with H the linear operator defined in (1).

For every $l = 1, \ldots, n$ let us define the first order vector fields Z_l as in (3) with coefficients $\omega \in C^{1,\alpha}(\Omega)$. Moreover, let us assume that the vector fields $Z_1, \ldots, Z_{2n}, [Z_1, Z_2]$ are linearly independent at every point and span $\mathbb{R}^{2n+1}$.

If the coefficients of the vector fields were smooth, then the linear operator H would satisfy Hörmander's condition of hypoellipticity. In our context the coefficients are only $C^{1,\alpha}(\Omega)$. However, for every $\xi, \xi_0 \in \Omega$ there exists an absolutely continuous mapping $\gamma : [0, 1] \to \mathbb{R}^{2n+1}$, which is a piecewise integral curve of the vector fields Z introduced in (3), which connects ξ_0 and ξ. Then there exists a Carnot-Carathéodory distance $d_Z(\xi, \xi_0)$ naturally associated to the geometry of the problem (see for example the distance ϱ_4 defined in [23, page 113]). Precisely, if $C(\delta)$ denotes the class of absolutely continuous mappings $\varphi : [0,1] \to \Omega$ which almost everywhere satisfy $\varphi'(t) = \sum_{j=1}^{2n} a_j(t) Z_j(\varphi(t))$ with $|a_j(t)| < \delta$, define

$$d_Z(\xi_0, \xi) = \inf\{\delta > 0 \,:\, \exists\, \varphi \in C(\delta) \text{ such that } \varphi(0) = \xi_0,\, \varphi(1) = \xi\} \quad . \tag{12}$$

The fact that d_Z is finite follows because the commutators of the vector fields Z span $\mathbb{R}^{2n+1}$ at every point. This was first proved by Carathéodory for smooth vector fields; for vector fields with $C^{1,\alpha}$ coefficients the proof is contained in [4].

We now define the class of Hölder continuous functions in terms of d_Z: for $0 < \alpha < 1$

$$C^{\alpha}_Z(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \text{ s.t. there exists a constant } c > 0 : \\ |v(\xi) - v(\xi_0)| \le c\, d^{\alpha}_Z(\xi, \xi_0) \text{ for all } \xi, \xi_0 \in \Omega \Big\}$$

and

$$C^{1,\alpha}_Z(\Omega) = \{ v \in C^{\alpha}_Z(\Omega) \,:\, \exists\, Z_j v \in C^{\alpha}_Z(\Omega) \quad \forall\, j = 1, \dots, 2n\}.$$

If the coefficients $\omega \in C^{m-1,\alpha}_Z(\Omega)$, $m \ge 2$, we define

$$C^{m,\alpha}_Z(\Omega) = \{ v \in C^{m-1,\alpha}_Z(\Omega) \,:\, Z_j v \in C^{m-1,\alpha}_Z(\Omega) \quad \forall\, j = 1, \dots, 2n\}. \tag{13}$$

Obviously (see [12])

$$C^{m,\alpha}(\Omega) \subset C^{m,\alpha}_Z(\Omega) \subset C^{m/2,\alpha/2}(\Omega).$$

For every $m \ge 0$ we also define spaces of locally Hölder continuous functions :

$$C^{m,\alpha}_{Z,loc}(\Omega) = \{ v : \Omega \to \mathbb{R} \,:\, v \in C^{m,\alpha}_Z(\Omega') \quad \forall\, \Omega' \subset\subset \Omega\}.$$

If $v \in C^{\alpha}_Z(\Omega)$ we define

$$[v]^Z_{\alpha;\Omega} = \sup_{\xi,\zeta \in \Omega} \frac{|v(\xi) - v(\zeta)|}{d^{\alpha}_Z(\xi, \zeta)}.$$

Denote by

$$Z^I = Z_{i_1} Z_{i_2} \cdots Z_{i_m},$$

where

$$I = (i_1, \dots, i_m) \tag{14}$$

is a multi-index of length $|I| = m$. If $v \in C^{m,\alpha}_Z(\Omega)$, with $m = 0, 1, 2, \dots$, and $0 < \alpha < 1$ we define the seminorm

$$[v]^Z_{m;\Omega} = \sup_{|I|=m} \sup_{\Omega} |Z^I v|$$
$$[v]^Z_{m,\alpha;\Omega} = \sup_{|I|=m} [Z^I v]^Z_{\alpha;\Omega},$$

and the norms

$$|v|^Z_{m;\Omega} = \sum_{j=0}^{m} \left(\sup_{|I|=j} \sup_{\Omega} |Z^I v| \right),$$
$$|v|^Z_{m,\alpha;\Omega} = |v|^Z_{m;\Omega} + [v]^Z_{m,\alpha;\Omega}.$$

We must remark that the Lie algebra generated by the vector fields Z_j is of step 2, because we need one commutator to generate the whole space. But our fields do not satisfy the minimal number of relations at every point, so that the Lie algebra is not free up to step 2. So we need to apply the technique introduced in [25] to add new variables and lift the vector until the algebra becomes free.

Denote by $\xi = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, t)$ in such a way that $\xi_{2n+1} = t$. We now proceed to lift the vector fields Z_j as follows.
We have $2n$ fields and need one relation to generate the whole space, so we must add $\binom{2n}{2} = n(2n-1)$ variables to obtain a free algebra. The total number of variables becomes

$$N = 2n + 1 + n(2n-1) = 2n^2 + n + 1.$$

If $\widetilde{\xi} = (\xi_1, \ldots, \xi_{2n+1}, \xi_{2n+2}, \ldots, \xi_N) \in \mathbb{R}^N$, we denote by $\partial_j = \frac{\partial}{\partial \xi_j}$ for every $j = 1, \ldots, N$ and define

$$\begin{aligned}
&\widetilde{Z}_1 = Z_1\\
&\widetilde{Z}_2 = Z_2\\
&\widetilde{Z}_3 = Z_3 + \xi_1 \partial_{2n+2} + \xi_2 \partial_{2n+3}\\
&\ldots\\
&\widetilde{Z}_k = Z_k + \sum_{j=1}^{k-1} \xi_j \partial_{2n + \frac{(k-2)(k-1)}{2} + j} \qquad \text{for } 3 \le k \le 2n, \qquad \text{and}\\
&\widetilde{T} = \lambda \partial_{2n+1} + \partial_N.
\end{aligned}$$

Then we introduce the *lifted* linear operator

$$\widetilde{H} = \sum_{i,j=1}^{2n} h_{ij} \widetilde{Z}_i \widetilde{Z}_j - \widetilde{T}.$$

For every $f \in C_Z^{1,\alpha}(\Omega)$ we define the first order Taylor polynomial of f at $\xi_0 \in \Omega$ in the directions of the vector fields Z_j:

$$P_{\xi_0} f(\xi) = f(\xi_0) + \sum_{j=1}^{2n} Z_j f(\xi_0)(\xi - \xi_0)_j.$$

We need the following lemma whose proof can be found in [4, Remark 2.3].

Lemma 2.1. *If $f \in C_Z^{1,\alpha}(\Omega)$ and $d_Z(\xi, \xi_0) < 1$, the following inequality holds:*

$$|P_{\xi_0} f(\xi) - f(\xi)| \le [f]^Z_{1,\alpha;\Omega} d_Z^{1+\alpha}(\xi, \xi_0), \qquad \forall \xi \in \Omega.$$

It is easy to check that for every $f \in C_Z^{1,\alpha}(\Omega)$ and $\xi, \xi_0, \zeta \in \Omega$

$$P_{\xi_0} f(\zeta) - P_\xi f(\zeta) = P_{\xi_0} f(\xi) - f(\xi) + \sum_{j=1}^{2n} (Z_j f(\xi_0) - Z_j f(\xi))(\zeta - \xi)_j$$

and from this equality, together with Lemma 2.1 we also get

Lemma 2.2. *If $f \in C_Z^{1,\alpha}(\Omega)$ and $\xi, \xi_0 \in \Omega$, $d_Z(\xi, \xi_0) < 1$, the following inequality holds:*

$$|P_{\xi_0} f(\zeta) - P_\xi f(\zeta)| \le [f]^Z_{1,\alpha;\Omega} \Big(d_Z^{1+\alpha}(\xi, \xi_0) + d_Z^{\alpha}(\xi, \xi_0) d_Z(\xi, \zeta) \Big), \qquad \forall \zeta \in \Omega.$$

For $k = 1, \dots, 2n$ we define the *frozen* vector fields

$$\widetilde{Z}_{k,\xi_0} = \partial_k + P_{\xi_0}(\omega_k)\partial_{2n+1} + (\widetilde{Z}_k - Z_k), \qquad \widetilde{T}_{\xi_0} = \lambda(\xi_0)\partial_{2n+1} + \partial_N. \tag{15}$$

We recall that, from the definition of the fields $\widetilde{Z}_k$'s, we have

$$\widetilde{Z}_k - Z_k = \sum_{j=1}^{k-1} \xi_j \partial_{2n+\frac{(k-2)(k-1)}{2}+j} \qquad .$$

We remark that the following identity holds:

$$[\widetilde{Z}_{1,\xi_0}, \widetilde{Z}_{2,\xi_0}] := g(\xi_0)\partial_{2n+1}$$

where the map $\xi_0 \mapsto g(\xi_0)$ is of class C^α and $g(\xi_0) \neq 0$, so that the $\widetilde{Z}_{k,\xi_0}$'s are nilpotent vector fields of step 2. Moreover, the Lie algebra generated by the vector fields $\widetilde{Z}_{k,\xi_0}$'s and $\widetilde{T}_{\xi_0}$ is free, by construction. Then we can define the *frozen* operator

$$\widetilde{H}_{\xi_0} = \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\widetilde{Z}_{i,\xi_0}\widetilde{Z}_{j,\xi_0} - \widetilde{T}_{\xi_0}.$$

The matrix $(h_{ij})_{i,j=1}^{2n}$ is positive definite and the functions $\xi_0 \mapsto h_{ij}(\xi_0)$ are α-Hölder continuous; then we can find an orthogonal $2n \times 2n$ matrix $\widetilde{U}$ such that

$$(h_{ij}(\xi_0))_{i,j=1}^{2n} = \widetilde{U}(\xi_0)\widetilde{U}^T(\xi_0) \qquad \widetilde{U}^T(\xi_0) = (u_{ij})_{i,j=1}^{2n}.$$

The maps $\xi_0 \mapsto u_{ij}(\xi_0)$ are of class C^α as composition of analytic functions with α-Hölder continuous functions, mainly due to the fact that the matrix $(h_{ij})_{i,j}$ is positive definite.
Put

$$\widetilde{W}_{\xi_0} = \widetilde{U}^T(\xi_0)\,\widetilde{Z}_{\xi_0}, \tag{16}$$

with $\widetilde{W}_{\xi_0} = (\widetilde{W}_{1,\xi_0}, \dots, \widetilde{W}_{2n,\xi_0})$, then for every $i = 1, \dots, 2n$,

$$\widetilde{W}_{i,\xi_0} = \sum_{j=1}^{2n} u_{ij}(\xi_0)\widetilde{Z}_{j,\xi_0}, .$$

We stress that the fields $\widetilde{W}_{i,\xi_0}$, $\widetilde{T}_{\xi_0}$ are still linearly independent and generate a free algebra of step 2 in $\mathbb{R}^N$. Moreover, the operator can be written in terms of the new fields as a sum of squares plus a potential:

$$\widetilde{H}_{\xi_0} = \sum_{j=1}^{2n} \widetilde{W}_{j,\xi_0}^2 - \widetilde{T}_{\xi_0}$$

and we call $\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}, \cdot)$ its fundamental solution with pole at $\widetilde{\xi}$.
We can introduce a pseudo-distance $\widetilde{d}_{\xi_0}$ associated to the frozen fields $\widetilde{W}_{j,\xi_0}$ and $\widetilde{T}_{\xi_0}$ in the following way: for every $\widetilde{\xi}, \widetilde{\zeta} \in \mathbb{R}^N$ let γ be the integral curve such that

$$\begin{cases} \dot{\gamma} = \displaystyle\sum_{j=1}^{2n} e_j \widetilde{W}_{j,\xi_0}\gamma + \sum_{\substack{l,j=1 \\ l<j}}^{2n} e_{lj}[\widetilde{W}_{l,\xi_0}, \widetilde{W}_{j,\xi_0}]\gamma + e_N \widetilde{T}_{\xi_0}\gamma \\ \gamma(0) = \widetilde{\xi} \\ \gamma(1) = \widetilde{\zeta} \quad . \end{cases}$$

Define

$$\widetilde{d}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) = ||(e_1,\dots,e_{2n},(e_{ij})_{i<j},e_N)||$$

where, for every $\eta = (\eta_1,\dots,\eta_N) \in \mathbb{R}^N$,

$$||\eta|| = \Big(\sum_{j=1}^{2n}(\eta_j)^4 + \sum_{j=2n+1}^{N}(\eta_j)^2\Big)^{\frac{1}{4}}. \tag{17}$$

Then the homogeneous dimension of $\mathbb{R}^N$ with respect to $||\cdot||$ is

$$\widetilde{Q} = 2n + 2n(2n-1) + 2 = 4n^2 + 2.$$

For every $\xi,\zeta \in \mathbb{R}^{2n+1}$, let $\widetilde{\xi} = (\xi,0)$, $\widetilde{\zeta} = (\zeta,0)$ and define

$$d_{\xi_0}(\xi,\zeta) := \widetilde{d}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}).$$

Precisely, it is $d_{\xi_0}(\xi,\zeta) = \Big(\sum_{j=1}^{2n}(e_j)^4 + (e_{12})^2\Big)^{\frac{1}{4}}$ and the homogeneous dimension of $\mathbb{R}^{2n+1}$ with respect to it is $Q = 2n+2$. By the results in [4] the following equivalence locally holds:

$$d_{\xi_0}(\xi_0,\zeta) \approx d_Z(\xi_0,\zeta) \tag{18}$$

where the distance d_Z was defined in (12).

Now, let $J = (j_1,\dots,j_s)$, $j_h = 1,\dots,2n$ for every $h = 1,\dots,s$, be a multi-index of length $|J| = s$; we denote by $\widetilde{W}^J_{\xi_0}$, $\widetilde{Z}^J_{\xi_0}$ the derivative operators of order s

$$\begin{aligned}\widetilde{W}^J_{\xi_0} &= \widetilde{W}_{j_1,\xi_0}\widetilde{W}_{j_2,\xi_0}\cdots\widetilde{W}_{j_s,\xi_0}\\ \widetilde{Z}^J_{\xi_0} &= \widetilde{Z}_{j_1,\xi_0}\widetilde{Z}_{j_2,\xi_0}\cdots\widetilde{Z}_{j_s,\xi_0}\quad .\end{aligned}$$

Then by [26], for every compact set $K \subset \mathbb{R}^N$ and for every multi-index J there is a positive constant c_J such that:

$$\begin{aligned}|\widetilde{W}^J_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})| &\le c_J\widetilde{d}_{\xi_0}^{\,-\widetilde{Q}+2-|J|}(\widetilde{\xi},\widetilde{\zeta})\\ |\widetilde{Z}^J_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})| &\le c_J\widetilde{d}_{\xi_0}^{\,-\widetilde{Q}+2-|J|}(\widetilde{\xi},\widetilde{\zeta})\end{aligned} \tag{19}$$

for every $\widetilde{\xi},\widetilde{\zeta} \in K$.

Remark 2.1. *If in (2) we choose* $\widetilde{\xi} = (\xi_0,0)$ *then we get the canonical coordinates of* $\widetilde{\zeta}$ *around* $(\xi_0,0)$*, see for example [25]. Moreover, the change of variable*

$$\begin{aligned}&\psi_{\xi_0} : \Omega \times \mathbb{R}^{N-(2n+1)} \to \mathbb{R}^N\\ &\psi_{\xi_0}(\widetilde{\zeta}) = (e_1,\dots,e_{2n},(e_{ij})_{i<j},e_N)\end{aligned} \tag{20}$$

is such that for every function $f \in C^1(\Omega\times\mathbb{R}^{N-(2n+1)},\mathbb{R})$

$$\begin{aligned}\widetilde{W}_{i,\xi_0}f &= \overline{W}_i(f\circ\psi_{\xi_0}), \quad \forall i = 1,\dots,2n,\\ \widetilde{T}_{\xi_0}f &= \overline{T}(f\circ\psi_{\xi_0}),\end{aligned}$$

where the first order vector fields $\overline{W}_i$*, for all* $i = 1,\dots,2n$ *and* $\overline{T}$ *are left invariant on a nilpotent Lie group and do not depend on the frozen point* $(\xi_0,0)$*.*

In the sequel we will denote by $\overline{\Gamma}$ the fundamental solution of the second order operator $\sum_{j=1}^{2n}\overline{W}_j^2 - \overline{T}$, and by $\overline{d}$ the distance defined by the norm in (17)

$$\overline{d}(0,\eta) = ||\eta||.$$

This remark has been used in [13] to prove estimates of the dependence of the fundamental solution on the frozen point in a similar situation to that considered here. Precisely we have:

Proposition 2.1. *Let $\xi, \xi_0, \zeta \in \Omega'$ and let $\widetilde{\xi}$, $\widetilde{\xi}_0$, $\widetilde{\zeta} \in \Omega' \times \mathbb{R}^{N-2n-1}$ defined as $\widetilde{\xi} = (\xi, 0)$, $\widetilde{\xi}_0 = (\xi_0, 0)$. Then, for every multi-index J, there exists a constant $c_J > 0$ which depends only on J and on the compact set Ω', such that*

$$|\widetilde{W}_{\xi_0}^J \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) - \widetilde{W}_{\xi}^J \widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta})| \leq c_J \left(\frac{\widetilde{d}_{\xi_0}^{\alpha}(\widetilde{\xi}_0, \widetilde{\xi})}{\widetilde{d}_{\xi_0}^{\widetilde{Q}-2+|J|}(\widetilde{\xi}_0, \widetilde{\zeta})} + \frac{\widetilde{d}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\xi})}{\widetilde{d}_{\xi_0}^{\widetilde{Q}-1+|J|}(\widetilde{\xi}_0, \widetilde{\zeta})} \right) \tag{21}$$

$$|\widetilde{Z}_{\xi_0}^J \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) - \widetilde{Z}_{\xi}^J \widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta})| \leq c_J \left(\frac{\widetilde{d}_{\xi_0}^{\alpha}(\widetilde{\xi}_0, \widetilde{\xi})}{\widetilde{d}_{\xi_0}^{\widetilde{Q}-2+|J|}(\widetilde{\xi}_0, \widetilde{\zeta})} + \frac{\widetilde{d}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\xi})}{\widetilde{d}_{\xi_0}^{\widetilde{Q}-1+|J|}(\widetilde{\xi}_0, \widetilde{\zeta})} \right) \tag{22}$$

for every $\widetilde{\zeta}$ such that $\widetilde{d}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) \geq 2\widetilde{d}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\xi})$.

Proof. Inequality (21) was proved in [12, Proposition 3.5]. In order to show that inequality (22) holds, we first recall that $\widetilde{Z} = \widetilde{V}\widetilde{W}$, with $\widetilde{V} = (\widetilde{U}^T)^{-1} := (v_{ij})_{i,j}$ (see (16)). By inequalities (19), (21) and the fact that the coefficients of the matrix $\widetilde{V}$ are α-Hölder continuous, we get

$$\begin{aligned} |\widetilde{Z}_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) - \widetilde{Z}_{\xi}\widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta})| &= |(\widetilde{V}_{\xi_0} - \widetilde{V}_{\xi})\widetilde{W}_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\xi_0, \zeta) + \widetilde{V}_{\xi}(\widetilde{W}_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\xi_0, \zeta) - \widetilde{W}_{\xi}\widetilde{\Gamma}_{\xi}(\xi, \zeta))| \\ &\leq |(\widetilde{V}_{\xi_0} - \widetilde{V}_{\xi})\widetilde{W}_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\xi_0, \zeta)| + |\widetilde{V}_{\xi}(\widetilde{W}_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\xi_0, \zeta) - \widetilde{W}_{\xi}\widetilde{\Gamma}_{\xi}(\xi, \zeta))| \\ &\leq \text{const} \cdot \left(\widetilde{d}_{\xi_0}^{\alpha}(\widetilde{\xi}_0, \widetilde{\xi})\, \widetilde{d}_{\xi_0}^{2-\widetilde{Q}}(\widetilde{\xi}_0, \widetilde{\zeta}) + \frac{\widetilde{d}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\xi})}{\widetilde{d}_{\xi_0}^{\widetilde{Q}-1}(\widetilde{\xi}_0, \widetilde{\zeta})} \right) \qquad . \end{aligned}$$

For multi-indexes I, J as in (14), we define:

$$\widetilde{V}_J^I = v_{i_1 j_1} \cdots v_{i_k j_k} \tag{23}$$

so that

$$\widetilde{Z}^J = \sum_{|I|=|J|} \widetilde{V}_I^J \widetilde{W}^I,$$

and inequality (22) follows by applying the same proceeding as in the case $|J| = 1$ treated above. □

In the following proposition we write a representation formula in term of $\widetilde{\Gamma}_{\xi_0}$ for the solution v of the linear equation $Hv = f$. This representation formula will be the main tool in the proof of Theorem 1.1.

Proposition 2.2. *Let $v \in C_Z^{2,\alpha}(\Omega)$. For every $K_1 \subset\subset K_2 \subset\subset \Omega$ we choose $\widetilde{K}_1 \subset\subset \widetilde{K}_2 \subset\subset \Omega \times \mathbb{R}^{N-(2n+1)}$ such that*

$$\widetilde{K}_1 \cap \{(\zeta, 0) \in \mathbb{R}^N \,:\, \zeta \in \mathbb{R}^{2n+1}\} = K_1$$

$$\widetilde{K}_2 \cap \{(\zeta, 0) \in \mathbb{R}^N \,:\, \zeta \in \mathbb{R}^{2n+1}\} = K_2$$

and fix a real valued function $\phi \in C_0^2(\widetilde{K}_2)$ *such that* $\phi|_{\widetilde{K}_1} \equiv 1$. *For every* $\xi_0 \in \Omega$ *and* $\widetilde{\xi} = (\xi, \xi') \in \widetilde{K}_1$ *we have*

$$\begin{aligned}
v(\xi) =& v(\xi)\phi(\widetilde{\xi}) = -\int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) H v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) v(\zeta) \widetilde{H}_{\xi_0}\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& + \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1} v(\zeta) \widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) Z_i v(\zeta) \widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& + \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (\lambda(\xi_0) - \lambda(\zeta))\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \sum_{i,j=1}^{2n} \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (h_{ij}(\xi_0) - h_{ij}(\zeta)) Z_i Z_j v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& + 2\sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}_{j,\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& + \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (Z_j\omega_i(\xi_0) - Z_j\omega_i(\zeta))\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (P_{\xi_0}\omega_i - \omega_i)(P_{\xi_0}\omega_j - \omega_j)\partial_{2n+1} v(\zeta)\partial_{2n+1}\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \sum_{i,j=1}^{2n} h_{ij}(\xi_0) PV_{\xi_0}\Big(\int \partial_{2n+1}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) (P_{\xi_0}\omega_i - \omega_i)(P_{\xi_0}\omega_j - \omega_j)\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta}\Big).
\end{aligned}$$

In the last integral $PV_{\xi_0}\int$ *denotes a principal value integral depending on* ξ_0 *as in [12, Definition 3.1].*

Proof. Let $v \in C^2(\Omega)$. By taking into account that v is a function of the first $2n+1$ variables, for every $\widetilde{\xi} = (\xi, \xi') \in \widetilde{K}_1$ we have:

$$\begin{aligned}
v(\xi) =& v(\xi)\phi(\widetilde{\xi}) = -\int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) \widetilde{H}_{\xi_0}(v(\zeta)\phi(\widetilde{\zeta})) d\widetilde{\zeta} \\
=& -\int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) H_{\xi_0} v(\zeta)\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) v(\zeta) \widetilde{H}_{\xi_0}\phi(\widetilde{\zeta}) d\widetilde{\zeta} \\
& - \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) Z_{i,\xi_0} v(\zeta) \widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta}) d\widetilde{\zeta}
\end{aligned}$$

$$\begin{aligned}
&= -\int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) H v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})(H_{\xi_0} - H)v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})v(\zeta)\widetilde{H}_{\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta}) Z_i v(\zeta)\widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})(Z_{i,\xi_0} - Z_i)v(\zeta)\widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta}.
\end{aligned}$$

Let us compute $H_{\xi_0} - H$.

$$\begin{aligned}
H_{\xi_0} - H &= \sum_{i,j=1}^{2n}\Big(h_{ij}(\xi_0)Z_{i,\xi_0}Z_{j,\xi_0} - h_{ij}(\zeta)Z_iZ_j\Big) - \Big(\lambda(\xi_0) - \lambda(\zeta)\Big)\partial_{2n+1} \\
&= \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\Big(Z_{i,\xi_0}Z_{j,\xi_0} - Z_iZ_j\Big) + \sum_{i,j=1}^{2n}\Big(h_{ij}(\xi_0) - h_{ij}(\zeta)\Big)Z_iZ_j - \Big(\lambda(\xi_0) - \lambda(\zeta)\Big)\partial_{2n+1}
\end{aligned}$$

where

$$\begin{aligned}
Z_{i,\xi_0}Z_{j,\xi_0} - Z_iZ_j =&(Z_{i,\xi_0} - Z_i)Z_{j,\xi_0} + Z_i(Z_{j,\xi_0} - Z_j) \\
=&(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1}Z_{j,\xi_0} + Z_i((P_{\xi_0}\omega_j - \omega_j)\partial_{2n+1}) \\
=&(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1}Z_{j,\xi_0} + (Z_i\omega_j(\xi_0) - Z_i\omega_j(\zeta))\partial_{2n+1} + (P_{\xi_0}\omega_j - \omega_j)Z_i\partial_{2n+1} \\
=&(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1}Z_{j,\xi_0} + (Z_i\omega_j(\xi_0) - Z_i\omega_j(\zeta))\partial_{2n+1} \\
&+ (P_{\xi_0}\omega_j - \omega_j)(Z_i - Z_{i,\xi_0})\partial_{2n+1} + (P_{\xi_0}\omega_j - \omega_j)Z_{i,\xi_0}\partial_{2n+1} \\
=&(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1}Z_{j,\xi_0} + (Z_i\omega_j(\xi_0) - Z_i\omega_j(\zeta))\partial_{2n+1} \\
&- (P_{\xi_0}\omega_j - \omega_j)(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1}^2 + (P_{\xi_0}\omega_j - \omega_j)Z_{i,\xi_0}\partial_{2n+1}.
\end{aligned}$$

By replacing the expression of $Z_{i,\xi_0}Z_{j,\xi_0} - Z_iZ_j$ in $H_{\xi_0} - H$ and this last in the representation formula for v we get

$$\begin{aligned}
v(\xi) =& v(\xi)\phi(\tilde{\xi}) = -\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta}) H v(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)\partial_{2n+1}Z_{j,\xi_0}v(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})(Z_i\omega_j(\xi_0)-Z_i\omega_j(\zeta))\partial_{2n+1}v(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&+ \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})(P_{\xi_0}\omega_j-\omega_j)(P_{\xi_0}\omega_i-\omega_i)\partial^2_{2n+1}v(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})(P_{\xi_0}\omega_j-\omega_j)Z_{i,\xi_0}\partial_{2n+1}v(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \sum_{i,j=1}^{2n} \int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})\Big(h_{ij}(\xi_0)-h_{ij}(\zeta)\Big)Z_iZ_jv(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&+ \int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})\Big(\lambda(\xi_0)-\lambda(\zeta)\Big)\partial_{2n+1}v(\zeta)\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})v(\zeta)\widetilde{H}_{\xi_0}\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})Z_iv(\zeta)\widetilde{Z}_{j,\xi_0}\phi(\tilde{\zeta})d\tilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{\Gamma}_{\xi_0}(\tilde{\xi},\tilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)\partial_{2n+1}v(\zeta)\widetilde{Z}_{j,\xi_0}\phi(\tilde{\zeta})d\tilde{\zeta}.
\end{aligned}$$

By remarking that $[\partial_{2n+1}, Z_{j,\xi_0}] = 0$ for every $j = 1, \dots, 2n$, and that the formal adjoint operator of $\widetilde{Z}_{j,\xi_0}$ is $-\widetilde{Z}_{j,\xi_0}$, integrate by part the second integral of the previous equality with respect to $\widetilde{Z}_{j,\xi_0}$, the fourth with respect to ∂_{2n+1}, and the fifth with respect to $\widetilde{Z}_{i,\xi_0}$. Then remark that

$$\begin{aligned}
Z_{j,\xi_0}&(P_{\xi_0}\omega_i-\omega_i)+Z_{i,\xi_0}(P_{\xi_0}\omega_j-\omega_j)-\partial_{2n+1}\Big((P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)\Big) \\
=&(Z_{j,\xi_0}-Z_j)(P_{\xi_0}\omega_i-\omega_i)+(Z_j\omega_i(\xi_0)-Z_j\omega_i(\zeta))+(Z_{i,\xi_0}-Z_i)(P_{\xi_0}\omega_j-\omega_j) \\
&+(Z_i\omega_j(\xi_0)-Z_i\omega_j(\zeta))-\partial_{2n+1}\Big((P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)\Big) \\
=&(P_{\xi_0}\omega_j-\omega_j)\partial_{2n+1}(P_{\xi_0}\omega_i-\omega_i)+(Z_j\omega_i(\xi_0)-Z_j\omega_i(\zeta))+(P_{\xi_0}\omega_i-\omega_i)\partial_{2n+1}(P_{\xi_0}\omega_j-\omega_j) \\
&+(Z_i\omega_j(\xi_0)-Z_i\omega_j(\zeta))-\partial_{2n+1}\Big((P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)\Big) \\
=&+(Z_j\omega_i(\xi_0)-Z_j\omega_i(\zeta))+(Z_i\omega_j(\xi_0)-Z_i\omega_j(\zeta)).
\end{aligned}$$

By taking into account that $h_{ij} = h_{ji}$ the thesis follows. □

We now differentiate the representation formula of Proposition 2.2 with respect to the vector fields at ξ_0.

Proposition 2.3. *Let $v \in C^{2,\alpha}_Z(\Omega)$. For every multi-index $I = (i_1, i_2)$ of length 2 and $\widetilde{\xi}_0 = (\xi_0, 0) \in \widetilde{K}_1$*

$$\begin{aligned}
Z^I v(\xi_0) = &- \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(Hv(\zeta)\phi(\widetilde{\zeta}) - Hv(\xi_0)\phi(\widetilde{\xi}_0))d\widetilde{\zeta} - Hv(\xi_0) \sum_{|J|=2} \widetilde{V}^I_J(\xi_0)\sigma^J \\
&- \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) v(\zeta) \widetilde{H}_{\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&+ \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1} v(\zeta) \widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) Z_i v(\zeta) \widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&+ \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(\lambda(\xi_0) - \lambda(\zeta))\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \sum_{i,j=1}^{2n} \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(h_{ij}(\xi_0) - h_{ij}(\zeta)) Z_i Z_j v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&+ 2\sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}^I_{\xi_0} \widetilde{Z}_{j,\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&+ \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(Z_j\omega_i(\xi_0) - Z_j\omega_i(\zeta))\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(P_{\xi_0}\omega_i - \omega_i)(P_{\xi_0}\omega_j - \omega_j)\partial_{2n+1} v(\zeta)\partial_{2n+1}\phi(\widetilde{\zeta})d\widetilde{\zeta} \\
&- \sum_{i,j=1}^{2n} h_{ij}(\xi_0) \int \widetilde{Z}^I_{\xi_0} \partial_{2n+1}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(P_{\xi_0}\omega_i - \omega_i)(P_{\xi_0}\omega_j - \omega_j)\partial_{2n+1} v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta},
\end{aligned}$$

with $\widetilde{V}^J_I$ as in (23) and by using the notations of Remark 2.1

$$\sigma^J = \int_{\{\eta \in \mathbb{R}^N : \overline{d}(0,\eta) = 1\}} \overline{W}_{j_2}\overline{\Gamma}(0,\eta) \frac{\overline{W}_{j_1}\overline{d}(0,\eta)}{|D\overline{d}(0,\eta)|} d\mathcal{H}^{N-1},$$

for $J = (j_1, j_2)$.

Proof. Let us call $v(\xi) = \sum_{l=1}^{10} v_l(\xi, \xi_0)$ with $v_l(\xi, \xi_0)$ the l-th line of the representation formula proved in Proposition 2.2. It is a standard fact that, for any multi-index I of length 2

$$Z^I v_1(\xi_0, \xi_0) = - \int \widetilde{Z}^I_{\xi_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})(Hv(\zeta)\phi(\widetilde{\zeta}) - Hv(\xi_0)\phi(\widetilde{\xi}_0))d\widetilde{\zeta} - Hv(\xi_0) \sum_{|J|=2} \widetilde{V}^I_J(\xi_0)\sigma^J$$

and

$$\begin{aligned}
Z^I v_2(\xi_0,\xi_0) &= -\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})v(\zeta)\widetilde{H}_{\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta}\\
Z^I v_3(\xi_0,\xi_0) &= +\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)\partial_{2n+1}v(\zeta)\widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta}\\
Z^I v_4(\xi_0,\xi_0) &= -\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})Z_i v(\zeta)\widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta})d\widetilde{\zeta}\\
Z^I v_5(\xi_0,\xi_0) &= +\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(\lambda(\xi_0)-\lambda(\zeta))\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta}\\
Z^I v_6(\xi_0,\xi_0) &= -\sum_{i,j=1}^{2n}\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(h_{ij}(\xi_0)-h_{ij}(\zeta))Z_iZ_j v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta}\\
Z^I v_8(\xi_0,\xi_0) &= +\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(Z_j\omega_i(\xi_0)-Z_j\omega_i(\zeta))\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta}\\
Z^I v_9(\xi_0,\xi_0) &= -\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}^I_{\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)\partial_{2n+1}v(\zeta)\partial_{2n+1}\phi(\widetilde{\zeta})d\widetilde{\zeta}.
\end{aligned}$$

Note that v_{10} is a principal value integral. However, we can define

$$w(\xi_0)=-\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}^I_{\xi_0}\partial_{2n+1}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta},$$

and the integrals are well defined, because by (19) and by Lemma 2.1

$$|\widetilde{Z}^I_{\xi_0}\partial_{2n+1}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi_0},\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)|\le c\widetilde{d}_{\xi_0}^{-\widetilde{Q}+2\alpha}(\widetilde{\xi_0},\widetilde{\zeta}).$$

Let us fix a function $\theta\in C^\infty(\mathbb{R})$ such that $0\le\theta\le 1$, $\theta(\tau)=0$ for all $\tau\le 1$ and $\theta(\tau)=1$ for all $\tau\ge 2$. For every $\varepsilon>0$ let us define

$$v_{10}^{(\varepsilon)}(\xi)=-\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \partial_{2n+1}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)(P_{\xi_0}\omega_j-\omega_j)\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})\theta\Big(\frac{\widetilde{d}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})}{\varepsilon}\Big)d\widetilde{\zeta}.$$

Arguing as in [12], we get

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2}|v_{10}^{(\varepsilon)}(\xi)-v_{10}(\xi,\xi_0)|\le c_1\varepsilon^{2+2\alpha},$$

and for any multi-index I of length 2

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2}|Z^I v_{10}^{(\varepsilon)}(\xi)-w(\xi_0)|\le c_2\varepsilon^{2\alpha},$$

with c_1, c_2 positive constants independent of ξ_0. Thus, we conclude that $Z^I v_{10}(\xi_0,\xi_0)=w(\xi_0)$.

Analogously, define

$$v_7^{(\varepsilon)}(\xi)=2\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}_{j,\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})\theta\Big(\frac{\widetilde{d}_{\xi_0}(\widetilde{\xi},\widetilde{\zeta})}{\varepsilon}\Big)d\widetilde{\zeta}$$

and

$$W(\xi) = 2\sum_{i,j=1}^{2n} h_{ij}(\xi_0)\int \widetilde{Z}^I_{\xi_0}\widetilde{Z}_{j,\xi_0}\widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0,\widetilde{\zeta})(P_{\xi_0}\omega_i-\omega_i)\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})d\widetilde{\zeta}.$$

We have

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2} |v_7^{(\varepsilon)}(\xi) - v_7(\xi,\xi_0)| \le c_1\varepsilon^{2+\alpha},$$

and for any multi-index I of length 2

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2} |Z^I v_7^{(\varepsilon)}(\xi) - W(\xi_0)| \le c_2\varepsilon^{\alpha},$$

with c_1, c_2 positive constants independent of ξ_0. Thus, we conclude that $Z^I v_7(\xi_0,\xi_0) = W(\xi_0)$.

3 Schauder-type Interior Estimates

In this section we prove Theorem 1.1 for the operator H defined in (1).
Proof of Theorem 1.1. We divide the proof in four steps.
I Step. Let $K_1, \widetilde{K}_1, K_2, \widetilde{K}_2$ as in Proposition 2.2. For every $(\xi,0)=\widetilde{\xi}\in\widetilde{K}_1$ and $|I|\le 4$ we set

$$w^I(\xi) = \int \widetilde{Z}^I_{\xi}\widetilde{\Gamma}_{\xi}(\widetilde{\xi},\widetilde{\zeta})g_{\xi}(\widetilde{\zeta})d\widetilde{\zeta}$$

with $\widetilde{\zeta}\to g_{\xi}(\widetilde{\zeta})$ a C^α function with compact support in $\widetilde{K}_2$ and such that $g_{\xi}(\widetilde{\xi})\equiv 0$.

In this section, in order to simplify notations, we shall denote by d_ξ the distance $\widetilde{d}_\xi$ introduced in Section 2, and for every $(\xi,0),(\xi_0,0)\in\widetilde{K}_1$ we set $d = d_Z(\xi,\xi_0)$.

We will prove the following statement:

If $|I|\le 4$ *and for every* $(\xi,0)=\widetilde{\xi}, (\xi_0,0)=\widetilde{\xi}_0\in\widetilde{K}_1$ *the functions*

$$\widetilde{\zeta}\to\frac{|g_{\xi_0}(\widetilde{\zeta})|}{d_{\xi_0}^{|I|-2+\alpha}(\widetilde{\xi}_0,\widetilde{\zeta})},\quad \widetilde{\zeta}\to\frac{|g_{\xi_0}(\widetilde{\zeta})-g_{\xi}(\widetilde{\zeta})|}{d_{\xi_0}^{|I|-2}(\widetilde{\xi}_0,\widetilde{\zeta})}$$

are bounded over $\widetilde{K}_2$, *then*

$$|w^I(\xi_0)-w^I(\xi)| \le c\,d^\alpha \sup_{\widetilde{\zeta}\in\widetilde{K}_2}\frac{|g_{\xi_0}(\widetilde{\zeta})|}{d_{\xi_0}^{|I|-2+\alpha}(\widetilde{\xi}_0,\widetilde{\zeta})} + c\,(\ln M - \ln 2d)\sup_{\widetilde{\zeta}\in\widetilde{K}_2}\frac{|g_{\xi_0}(\widetilde{\zeta})-g_{\xi}(\widetilde{\zeta})|}{d_{\xi_0}^{|I|-2}(\widetilde{\xi}_0,\widetilde{\zeta})}, \tag{24}$$

with $M=\sup\{d_\xi(\widetilde{\xi},\widetilde{\zeta}) : (\xi,0)=\widetilde{\xi}\in\widetilde{K}_1,\ \widetilde{\zeta}\in\widetilde{K}_2\}$.

Denote by $d_{\widetilde{Z}}$ the Carnot-Carathéodory distance associated to the vector fields $\widetilde{Z}$ as in (12). Let us set

$$M_1 = \{\widetilde{\zeta}\in\widetilde{K}_2 : d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})\le 2d\}$$
$$M_2 = \{\widetilde{\zeta}\in\widetilde{K}_2 : d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta}) > 2d\}.$$

Hence, for every $|I| \leq 4$ we get

$$
\begin{aligned}
|w^I(\xi_0) - w^I(\xi)| \leq & \int_{M_1} |\widetilde{Z}^I_{\widetilde{\xi}_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta})||g_{\xi_0}(\widetilde{\zeta})| d\widetilde{\zeta} \\
& + \int_{M_1} |\widetilde{Z}^I_{\widetilde{\xi}} \widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta})||g_{\xi}(\widetilde{\zeta})| d\widetilde{\zeta} \\
& + \int_{M_2} |\widetilde{Z}^I_{\widetilde{\xi}_0} \widetilde{\Gamma}_{\xi_0}(\widetilde{\xi}_0, \widetilde{\zeta}) - \widetilde{Z}^I_{\widetilde{\xi}} \widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta})||g_{\xi_0}(\widetilde{\zeta})| d\widetilde{\zeta} \\
& + \int_{M_2} |\widetilde{Z}^I_{\widetilde{\xi}} \widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta})||g_{\xi_0}(\widetilde{\zeta}) - g_{\xi}(\widetilde{\zeta})| d\widetilde{\zeta} \\
= & A^I_1 + A^I_2 + A^I_3 + A^I_4.
\end{aligned}
$$

We shall first show that *if* $|I| \leq 4$ *and for every* $(\xi, 0) = \widetilde{\xi} \in \widetilde{K}_1$ *the function*

$$
\widetilde{\zeta} \to \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{|I|-2+\alpha}(\widetilde{\xi}, \widetilde{\zeta})}
$$

is bounded over $\widetilde{K}_2$, *then*

$$
A^I_1 + A^I_2 + A^I_3 \leq c\, d^{\alpha} \sup_{\widetilde{\zeta} \in \widetilde{K}_2} \frac{|g_{\xi_0}(\widetilde{\zeta})|}{d_{\xi_0}^{|I|-2+\alpha}(\widetilde{\xi}_0, \widetilde{\zeta})}. \tag{25}
$$

By the triangle inequality, for every $\widetilde{\zeta} \in M_1$ we get

$$
d_{\widetilde{Z}}(\widetilde{\xi}_0, \widetilde{\zeta}) \leq d_{\widetilde{Z}}(\widetilde{\xi}_0, \widetilde{\xi}) + d_{\widetilde{Z}}(\widetilde{\xi}, \widetilde{\zeta}) < 3d. \tag{26}
$$

Hence $M_1 \subset \widetilde{M_1} = \{\widetilde{\zeta} : d_{\widetilde{Z}}(\widetilde{\xi}_0, \widetilde{\zeta}) < 3d\}$. Then we use (26) and the estimate (19) to obtain

$$
\begin{aligned}
A^I_1 + A^I_2 \leq & c \int_{\widetilde{M}_1} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+2-|I|} |g_{\xi}(\widetilde{\zeta})| d\widetilde{\zeta} \\
\leq & c \int_{\widetilde{M}_1} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+\alpha} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} d\widetilde{\zeta} \\
\leq & c \int_0^{3d} \rho^{-1+\alpha} d\rho \sup_{\widetilde{\zeta} \in \widetilde{M}_1} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} \\
\leq & c\, d^{\alpha} \sup_{\widetilde{\zeta} \in \widetilde{M}_1} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})}.
\end{aligned}
$$

By estimate (22)

$$
\begin{aligned}
A^I_3 \leq & c\, d^{\alpha} \int_{M_2} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+2-|I|} |g_{\xi}(\widetilde{\zeta})| d\widetilde{\zeta} + c\, d \int_{M_2} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+1-|I|} |g_{\xi}(\widetilde{\zeta})| d\widetilde{\zeta} \\
\leq & c\, d^{\alpha} \int_{M_2} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+\alpha} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} d\widetilde{\zeta} + d \int_{M_2} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+1-|I|+\alpha} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} d\widetilde{\zeta} \\
\leq & c \left(d^{\alpha} \int_{2d}^{M} \rho^{-1+\alpha} d\rho + d \int_{2d}^{M} \rho^{-2+\alpha} d\rho \right) \sup_{\widetilde{\zeta} \in M_2} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} \\
\leq & c\, d^{\alpha} \sup_{\widetilde{\zeta} \in M_2} \frac{|g_{\xi}(\widetilde{\zeta})|}{d_{\xi}^{\alpha+|I|-2}(\widetilde{\xi}, \widetilde{\zeta})}.
\end{aligned}
$$

We shall now prove that *if $|I| \leq 4$ and for every $(\xi, 0) = \widetilde{\xi}, (\xi_0, 0) = \widetilde{\xi}_0 \in \widetilde{K}_1$ the function*

$$\widetilde{\zeta} \to \frac{|g_\xi(\widetilde{\zeta}) - g_{\xi_0}(\widetilde{\zeta})|}{d_\xi^{|I|-2}(\widetilde{\xi}, \widetilde{\zeta})}$$

is bounded over $\widetilde{K}_2$, then

$$A_4^I \leq c\,(\ln M - \ln 2d) \sup_{\widetilde{\zeta} \in \widetilde{K}_2} \frac{|g_\xi(\widetilde{\zeta}) - g_{\xi_0}(\widetilde{\zeta})|}{d_\xi^{|I|-2}(\widetilde{\xi}_0, \widetilde{\zeta})}. \tag{27}$$

Again by estimate (19)

$$\begin{aligned}
A_4^I &\leq c \int_{M_2} d_\xi(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}+2-|I|} |g_{\xi_0}(\widetilde{\zeta}) - g_\xi(\widetilde{\zeta})| d\widetilde{\zeta} \\
&\leq c \int_{M_2} d_\xi(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}} \frac{|g_{\xi_0}(\widetilde{\zeta}) - g_\xi(\widetilde{\zeta})|}{d_\xi^{|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} d\widetilde{\zeta} \\
&\leq c \int_{2d}^{M} \rho^{-1} d\rho \sup_{\widetilde{\zeta} \in M_2} \frac{|g_{\xi_0}(\widetilde{\zeta}) - g_\xi(\widetilde{\zeta})|}{d_\xi^{|I|-2}(\widetilde{\xi}, \widetilde{\zeta})} \\
&\leq c\,(\ln M - \ln 2d) \sup_{\widetilde{\zeta} \in M_2} \frac{|g_{\xi_0}(\widetilde{\zeta}) - g_\xi(\widetilde{\zeta})|}{d_\xi^{|I|-2}(\widetilde{\xi}, \widetilde{\zeta})}.
\end{aligned}$$

By interchanging ξ with ξ_0 in (25) and (27) we get (24).

II Step. For every $\xi_0 \in \Omega$ let $\delta = d_Z(\xi_0, \partial\Omega)$ and $r = \mu\delta$ with $\mu \in]0, 1/4[$ a positive constant to be specified later.

Let $K_1 = \{\zeta \in \Omega : d_Z(\xi_0, \zeta) < r\}$ and $K_2 = \{\zeta \in \Omega : d_Z(\xi_0, \zeta) < 2r\}$. For every $\xi \in K_1, \xi \neq \xi_0$, we will apply estimates (24) to $Z^I v(\xi_0) - Z^I v(\xi)$ and, by also choosing the cut-off function ϕ in the representation formula of Proposition 2.3, we will prove that

$$r^{2+\beta} \frac{|Z^I v(\xi_0) - Z^I v(\xi)|}{d_Z^\beta(\xi, \xi_0)} \leq c\Big(r^{2+\alpha}[Hv]_{\alpha;K_2} + r^2 \sup_{K_1} |Hv| + r^2 [v]^Z_{2;K_2} + r^{1+\alpha}[v]^Z_{1,\alpha;K_2} + r[v]^Z_{1;K_2} + \sup_{K_2} |v|\Big). \tag{28}$$

By Proposition 2.3 for every $\xi \in K_1$ and $|I| = 2$

$$-Z^I v(\xi) = \sum_{i=1}^{6} w_i^I(\widetilde{\xi}) + \sum_{j=1}^{2n} w_7^{(I,j)}(\widetilde{\xi}) + w_8^{(I,1,2)}(\widetilde{\xi}) + Hv(\xi) \sum_{|J|=2} \widetilde{V}_J^I(\xi)\sigma^J,$$

where $|(I, j)| = 3$ and $|(I, 1, 2)| = 4$.

Choose the cut-off function ϕ in Proposition 2.3 as follows: $\phi(\widetilde{\zeta}) = \varphi(\rho)$ with $\rho = d_{\xi_0}(\widetilde{\zeta}, \widetilde{\xi}_0)$, and $\varphi \in C_0^\infty(]0, 2r[)$, such that $0 \leq \varphi(\rho) \leq 1$ and

$$\varphi(\rho) = \begin{cases} 1\,, & \rho < r \\ 0\,, & \rho \geq 2r \end{cases}, \quad |\varphi'(\rho)| \leq \frac{1}{r}, \quad |\varphi''(\rho)| \leq \frac{1}{r^2} \quad \forall \rho \in [0, 2r[.$$

For every $\widetilde{\zeta} \in \widetilde{K}_2 = \{\widetilde{\zeta} = (\zeta, \zeta') \in K_2 \times \mathbb{R}^{N-(2n+1)} : d_{\widetilde{Z}}(\widetilde{\zeta}, \widetilde{\xi}_0) < 2r\}$ set $g_\xi^{(1)}(\widetilde{\zeta}) = Hv(\zeta)\phi(\widetilde{\zeta}) - Hv(\xi)\phi(\widetilde{\xi})$. Then,

$$\begin{aligned}|g_\xi^{(1)}(\widetilde{\zeta})| =& |Hv(\zeta)\phi(\widetilde{\zeta}) - Hv(\xi)\phi(\widetilde{\xi})| \leq |Hv(\zeta) - Hv(\xi)||\phi(\widetilde{\zeta})| + |Hv(\xi)||\phi(\widetilde{\zeta}) - \phi(\widetilde{\xi})|\\ \leq& [Hv]^Z_{\alpha;K_2} d^\alpha_Z(\xi, \zeta) + r^{-1}|Hv(\xi)| d_{\widetilde{Z}}(\widetilde{\xi}, \widetilde{\zeta})\\ \leq& c d^\alpha_{\widetilde{Z}}(\widetilde{\xi}, \widetilde{\zeta})([Hv]^Z_{\alpha;K_2} + r^{-\alpha} \sup_{K_1} |Hv|).\end{aligned}$$

Remark that, as in (18), the equivalence $d_{\widetilde{Z}}(\widetilde{\xi}, \widetilde{\zeta}) \approx d_\xi(\widetilde{\xi}, \widetilde{\zeta})$ locally holds. Hence,

$$\frac{|g_\xi^{(1)}(\widetilde{\zeta})|}{d_\xi^{|I|-2+\alpha}(\widetilde{\xi}, \widetilde{\zeta})} = \frac{|g_\xi^{(1)}(\widetilde{\zeta})|}{d_\xi^{\alpha}(\widetilde{\xi}, \widetilde{\zeta})} \leq c([Hv]^Z_{\alpha;K_2} + r^{-\alpha} \sup_{K_1} |Hv|).$$

Moreover, since $d_{\widetilde{Z}}(\widetilde{\xi}, \widetilde{\xi}_0) = d_Z(\xi, \xi_0) < r$

$$\begin{aligned}|g_\xi^{(1)}(\widetilde{\zeta}) - g_{\xi_0}^{(1)}(\widetilde{\zeta})| =& |Hv(\xi)\phi(\widetilde{\xi}) - Hv(\xi_0)\phi(\widetilde{\xi}_0)|\\ \leq& |(Hv(\xi) - Hv(\xi_0))\phi(\widetilde{\xi})| + |Hv(\xi_0)(\phi(\widetilde{\xi}) - \phi(\widetilde{\xi}_0))|\\ \leq& [Hv]^Z_{\alpha,K_1} d^\alpha_Z(\xi, \xi_0) + r^{-1} \sup_{K_1} |Hv| d_{\widetilde{Z}}(\widetilde{\xi}, \widetilde{\xi}_0)\\ \leq& d^\alpha_Z(\xi, \xi_0)([Hv]^Z_{\alpha,K_1} + r^{-\alpha} \sup_{K_1} |Hv|).\end{aligned}$$

For every multi-index I of length 2,

$$w_1^I(\xi) = \int \widetilde{Z}^I_\xi \widetilde{\Gamma}_\xi(\widetilde{\xi}, \widetilde{\zeta}) g_\xi^{(1)}(\widetilde{\zeta}) d\widetilde{\zeta}.$$

Hence, by (24)

$$\begin{aligned}r^\beta \frac{|w_1^I(\xi) - w_1^I(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq& c\Big(\frac{d_Z(\xi, \xi_0)}{r}\Big)^{\alpha-\beta}(r^\alpha [Hv]^Z_{\alpha;K_2} + \sup_{K_1} |Hv|)\\ &+ c\Big(\frac{d_Z(\xi, \xi_0)}{r}\Big)^{\alpha-\beta} \ln\Big(\frac{r}{d_Z(\xi, \xi_0}\Big)(r^\alpha [Hv]^Z_{\alpha,K_1} + \sup_{K_1} |Hv|).\end{aligned} \tag{29}$$

For every $\widetilde{\zeta} \in \widetilde{K}_2$ set $g_\xi^{(2)}(\widetilde{\zeta}) = v(\zeta)\widetilde{H}_\xi \phi(\widetilde{\zeta})$. Then,

$$\begin{aligned}|g_\xi^{(2)}(\widetilde{\zeta})| =& |v(\zeta)\widetilde{H}_\xi \phi(\widetilde{\zeta})|\\ \leq& r^{-2} \sup_{K_2} |v|.\end{aligned}$$

Moreover,

$$
\begin{aligned}
|g_\xi^{(2)}(\widetilde{\zeta}) - g_{\xi_0}^{(2)}(\widetilde{\zeta})| =& |v(\zeta)(\widetilde{H}_\xi \phi(\widetilde{\zeta}) - \widetilde{H}_{\xi_0}\phi(\widetilde{\zeta}))| = |v(\zeta)||(H_\xi - \widetilde{H}_{\xi_0})\phi(\widetilde{\zeta})| \\
=& |v(\zeta)|\Big|\Big(\sum_{i,j=1}^{2n} h_{ij}(\xi)\Big(Z_{i,\xi}Z_{j,\xi} - Z_{i,\xi_0}Z_{j,\xi_0}\Big) + \sum_{i,j=1}^{2n}\Big(h_{ij}(\xi) - h_{ij}(\xi_0)\Big)Z_{i,\xi_0}Z_{j,\xi_0} \\
&- \Big(\lambda(\xi) - \lambda(\xi_0)\Big)\partial_{2n+1}\Big)\phi(\widetilde{\zeta})\Big| \\
\leq& |v(\zeta)|\Big|\Big(\sum_{i,j=1}^{2n} h_{ij}(\xi)\Big((P_\xi\omega_i - P_{\xi_0}\omega_i)\partial_{2n+1}Z_{j,\xi_0} + (Z_i\omega_j(\xi) - Z_i\omega_j(\xi_0))\partial_{2n+1} \\
&+ (P_\xi\omega_j - P_{\xi_0}\omega_j)(P_\xi\omega_i - P_{\xi_0}\omega_i)\partial_{2n+1}^2 + (P_\xi\omega_j - P_{\xi_0}\omega_j)Z_{i,\xi_0}\partial_{2n+1}\Big)\phi(\widetilde{\zeta})\Big| \\
&+ |v(\zeta)|\Big|\Big(\sum_{i,j=1}^{2n}\Big(h_{ij}(\xi) - h_{ij}(\xi_0)\Big)Z_{i,\xi_0}Z_{j,\xi_0} - \Big(\lambda(\xi) - \lambda(\xi_0)\Big)\partial_{2n+1}\Big)\phi(\widetilde{\zeta})\Big|
\end{aligned}
$$

by Lemma 2.2

$$
\begin{aligned}
&\leq \sup_{K_2}|v|r^{-2}d_Z^\alpha(\xi,\xi_0)([\lambda]_{\alpha;K_1} + \sum_{i,j=1}^{2n}[h_{ij}]_{\alpha;K_1}) \\
&\quad + \sup_{K_2}|v|\Big(\sum_{i,j=1}^{2n}\sup_{K_1}|h_{ij}|\Big(r^{-2}[\omega_j]_{1,\alpha;K_1}d_Z^\alpha(\xi,\xi_0) \\
&\qquad + r^{-3}[\omega_i]_{1,\alpha;K_2}(d_Z^{1+\alpha}(\xi,\xi_0) + d_Z^\alpha(\xi,\xi_0)d_Z(\zeta,\xi_0)) \\
&\qquad + r^{-4}[\omega_i]_{1,\alpha;K_2}[\omega_j]_{1,\alpha;K_2}(d_Z^{1+\alpha}(\xi,\xi_0) + d_Z^\alpha(\xi,\xi_0)d_Z(\zeta,\xi_0))^2\Big)\Big) \\
&\quad \leq c\,d_Z^\alpha(\xi,\xi_0)r^{-2}\sup_{K_2}|v|.
\end{aligned}
$$

For every $(\xi,0) = \widetilde{\xi} \in \widetilde{K}_1$ and for every multi-index I of length 2,

$$
w_2^I(\xi) = \int \widetilde{Z}_\xi^I\widetilde{\Gamma}_\xi(\widetilde{\xi},\widetilde{\zeta})g_\xi^{(2)}(\widetilde{\zeta})d\widetilde{\zeta}.
$$

Remark that we can not directly apply (24) to it because $\widetilde{\zeta} \to \frac{|g_\xi^{(2)}(\widetilde{\zeta})|}{d_{\widetilde{Z}}^\alpha(\widetilde{\xi},\widetilde{\zeta})}$ is not bounded in $\widetilde{K}_2$. However $g_\xi^{(2)}(\widetilde{\zeta}) \equiv 0$ in $\widetilde{K}_1$, so if $2d_Z(\xi,\xi_0) \leq r$ then $A_1^I + A_2^I = 0$, while if $2d_Z(\xi,\xi_0) > r$ we can estimate $A_1^I + A_2^I$ as follows

$$
\begin{aligned}
A_1^I + A_2^I \leq& c\int_{M_1\cap(\widetilde{K}_2\setminus\widetilde{K}_1)} d_\xi(\widetilde{\xi},\widetilde{\zeta})^{-\widetilde{Q}}d\widetilde{\zeta}\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})| \\
\leq& c\int_r^{2d_Z(\xi,\xi_0)}\rho^{-1}d\rho\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})| \\
\leq& c\ln\Big(\frac{2d_Z(\xi,\xi_0)}{r}\Big)\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})|.
\end{aligned}
$$

Moreover,

$$\begin{aligned}
A_3^I \leq & c\Big(d_Z^\alpha(\xi,\xi_0)\int_{M_2\cap(\widetilde{K}_2\setminus\widetilde{K}_1)} d_\xi(\widetilde{\xi},\widetilde{\zeta})^{-\widetilde{Q}}d\widetilde{\zeta} + d_Z(\xi,\xi_0)\int_{M_2\cap(\widetilde{K}_2\setminus\widetilde{K}_1)} d_\xi(\widetilde{\xi},\widetilde{\zeta})^{-\widetilde{Q}-1}d\widetilde{\zeta}\Big)\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})| \\
\leq & c\Big(d_Z^\alpha(\xi,\xi_0)\int_{\max\{r,2d_Z(\xi,\xi_0)\}}^{2r}\rho^{-1}d\rho + d_Z(\xi,\xi_0)\int_{\max\{r,2d_Z(\xi,\xi_0)\}}^{2r}\rho^{-2}d\rho\Big)\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})| \\
\leq & c\Big(d_Z^\alpha(\xi,\xi_0)\ln\Big(\frac{2r}{\max\{r,2d_Z(\xi,\xi_0)\}}\Big) + \Big(\frac{d_Z(\xi,\xi_0)}{\max\{r,2d_Z(\xi,\xi_0)\}} - \frac{d_Z(\xi,\xi_0)}{2r}\Big)\Big)\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})| \\
\leq & c\Big(d_Z^\alpha(\xi,\xi_0)\ln\Big(\frac{r}{\max\{d_Z(\xi,\xi_0)\}}\Big) + \frac{d_Z(\xi,\xi_0)}{2r}\Big)\Big)\sup_{\widetilde{\zeta}\in\widetilde{K}_2}|g_\xi^{(2)}(\widetilde{\zeta})|.
\end{aligned}$$

To estimate A_4^I we use (27)

$$A_4^I \leq c(\ln M - \ln 2d)d_Z(\xi,\xi_0)r^{-2}\sup_{K_2}|v|.$$

Hence, by the previous estimates

$$\begin{aligned}
r^\beta\frac{|w_2^I(\xi)-w_2^I(\xi_0)|}{d_Z^\beta(\xi,\xi_0)} \leq & c\Big(\max\Big\{0,\Big(\frac{r}{2d_Z(\xi,\xi_0)}\Big)^\beta\ln\Big(\frac{2d_Z(\xi,\xi_0)}{r}\Big)\Big\} + \Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{1-\beta}\Big)(r^{-2}\sup_{K_2}|v|) \\
& + c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}\ln\Big(\frac{r}{d_Z(\xi,\xi_0)}\Big)(r^{\alpha-2}\sup_{K_2}|v|) \\
\leq & c(r^{\alpha-2}\sup_{K_2}|v|).
\end{aligned} \tag{30}$$

For every $\widetilde{\zeta}\in\widetilde{K}_2$ set $g_\xi^{(3)}(\widetilde{\zeta}) = h_{ij}(\xi)(P_\xi\omega_i - \omega_i)\partial_{2n+1}v(\zeta)\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})$. Then,

$$\begin{aligned}
|g_\xi^{(3)}(\widetilde{\zeta})| = & |h_{ij}(\xi)(P_\xi\omega_i - \omega_i)\partial_{2n+1}v(\zeta)\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})| \\
\leq & r^{-1}|h_{ij}(\xi)|[\omega_i]^Z_{1,\alpha;K_2}d_Z^{1+\alpha}(\xi,\zeta)\sup_{\zeta\in K_2}|\partial_{2n+1}v(\zeta)| \\
\leq & cd_Z^\alpha(\xi,\zeta)[v]^Z_{2;K_2}.
\end{aligned}$$

Moreover, by Lemma 2.1 and Lemma 2.2

$$\begin{aligned}
|g_\xi^{(3)}(\widetilde{\zeta}) - g_{\xi_0}^{(3)}(\widetilde{\zeta})| \leq & |(h_{ij}(\xi)-h_{ij}(\xi_0))(P_\xi\omega_i - \omega_i)\partial_{2n+1}v(\zeta)\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})| \\
& + |h_{ij}(\xi_0)(P_\xi\omega_i - P_{\xi_0}\omega_i)\partial_{2n+1}v(\zeta)\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})| \\
& + |h_{ij}(\xi_0)(P_{\xi_0}\omega_i - \omega_i)\partial_{2n+1}v(\zeta)(\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta}) - \widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta}))| \\
\leq & r^{-1}[h_{ij}]_{\alpha;K_1}d_Z^\alpha(\xi,\xi_0)[\omega_i]^Z_{1,\alpha;K_2}d_Z^{1+\alpha}(\xi,\zeta)\sup_{K_2}|\partial_{2n+1}v| \\
& + r^{-1}|h_{ij}(\xi_0)|(d_Z^{\alpha+1}(\xi,\xi_0) + d_Z^\alpha(\xi,\xi_0)d_Z(\xi,\zeta))[\omega_i]^Z_{1,\alpha;K_1}\sup_{K_2}|\partial_{2n+1}v| \\
& + r^{-2}|h_{ij}(\xi_0)||[\omega_i]^Z_{1,\alpha;K_2}d_Z^{1+\alpha}(\xi,\zeta)\sup_{K_2}|\partial_{2n+1}v|(d_Z^{\alpha+1}(\xi,\xi_0) + d_Z^\alpha(\xi,\xi_0)d_Z(\xi,\zeta))[\omega_i]^Z_{1,\alpha;K_1} \\
\leq & cd_Z^\alpha(\xi,\xi_0)[v]^Z_{2;K_2}
\end{aligned}$$

For every $(\xi,0)=\widetilde{\xi}\in\widetilde{K}_1$ and for every multi-index I of length 2,

$$w_3^I(\xi) = \int\widetilde{Z}_\xi^I\widetilde{\Gamma}_\xi(\widetilde{\xi},\widetilde{\zeta})g_\xi^{(3)}(\widetilde{\zeta})d\widetilde{\zeta}.$$

Hence, by (24)

$$
\begin{aligned}
r^{\beta}\frac{|w_3^I(\xi)-w_3^I(\xi_0)|}{d_Z^{\beta}(\xi,\xi_0)} \leq & c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}(r^{\alpha}[v]^Z_{2;K_2}) \\
& + c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}\ln\Big(\frac{r}{d_Z(\xi,\xi_0)}\Big)(r^{\alpha}[v]^Z_{2;K_2}).
\end{aligned}
\tag{31}
$$

For every $\widetilde{\zeta}\in\widetilde{K}_2$ set $g_\xi^{(4)}(\widetilde{\zeta})=h_{ij}(\xi)(Z_iv(\zeta)\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})-Z_iv(\xi)\widetilde{Z}_{j,\xi}\phi(\widetilde{\xi}))$. Then,

$$
\begin{aligned}
|g_\xi^{(4)}(\widetilde{\zeta})| \leq & |h_{ij}(\xi)(Z_iv(\zeta)-Z_iv(\xi))\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})|+|h_{ij}(\xi)Z_iv(\xi)(\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})-\widetilde{Z}_{j,\xi}\phi(\widetilde{\xi})| \\
\leq & r^{-1}|h_{ij}(\xi)|[v]^Z_{1,\alpha;K_2}d_Z^{\alpha}(\xi,\zeta)+r^{-2}|h_{ij}(\xi)|[v]^Z_{1;K_2}d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta}) \\
\leq & c d^{\alpha}_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})\Big(r^{-1}[v]^Z_{1,\alpha;K_2}+r^{-1-\alpha}[v]^Z_{1;K_2}\Big)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|g_\xi^{(4)}(\widetilde{\zeta})-g_{\xi_0}^{(4)}(\widetilde{\zeta})| \leq & |(h_{ij}(\xi)-h_{ij}(\xi_0))(Z_iv(\zeta)\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})-Z_iv(\xi)\widetilde{Z}_{j,\xi}\phi(\widetilde{\xi})| \\
& + |h_{ij}(\xi_0)Z_iv(\zeta)(\widetilde{Z}_{j,\xi}\phi(\widetilde{\zeta})-\widetilde{Z}_{j,\xi_0}\phi(\widetilde{\zeta}))| \\
& + |h_{ij}(\xi_0)||(Z_iv(\xi)-Z_iv(\xi_0))\widetilde{Z}_{j,\xi}\phi(\widetilde{\xi})|+|h_{ij}(\xi_0)||Z_iv(\xi_0)||\widetilde{Z}_{j,\xi}\phi(\widetilde{\xi})-\widetilde{Z}_{j,\xi_0}\phi(\widetilde{\xi}_0)|
\end{aligned}
$$

by Lemma 2.2

$$
\begin{aligned}
\leq & r^{-1}[h_{ij}]_{\alpha;K_1}d_Z^{\alpha}(\xi,\xi_0)[v]^Z_{1;K_2} \\
& + r^{-2}|h_{ij}(\xi_0)|(d_Z^{\alpha+1}(\xi,\xi_0)+d_Z^{\alpha}(\xi,\xi_0)d_Z(\xi,\zeta))[\omega_i]^Z_{1,\alpha;K_1}[v]^Z_{1;K_2} \\
& + r^{-1}|h_{ij}(\xi_0)|[v]^Z_{1,\alpha;K_2}d_Z^{\alpha}(\xi,\xi_0)+r^{-2}|h_{ij}(\xi_0)||Z_iv(\xi_0)|d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\xi}_0) \\
\leq & c d_Z^{\alpha}(\xi,\xi_0)\Big(r^{-1}[v]^Z_{1;K_2}+r^{-1}[v]^Z_{1,\alpha;K_2}+r^{-1-\alpha}[v]^Z_{1;K_1}\Big).
\end{aligned}
$$

For every $(\xi,0)=\widetilde{\xi}\in\widetilde{K}_1$ and for every multi-index I of length 2,

$$
w_4^I(\xi)=\int\widetilde{Z}^I_\xi\widetilde{\Gamma}_\xi(\widetilde{\xi},\widetilde{\zeta})g_\xi^{(4)}(\widetilde{\zeta})d\widetilde{\zeta}.
$$

Hence, by (24)

$$
\begin{aligned}
r^{\beta}\frac{|w_4^I(\xi)-w_4^I(\xi_0)|}{d_Z^{\beta}(\xi,\xi_0)} \leq & c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}\Big(r^{-1+\alpha}[v]^Z_{1,\alpha;K_2}+r^{-1}[v]^Z_{1;K_2}\Big) \\
& + c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}\ln\Big(\frac{r}{d_Z(\xi,\xi_0)}\Big)\Big(r^{-1+\alpha}[v]^Z_{1;K_2}+r^{-1+\alpha}[v]^Z_{1,\alpha;K_2}+r^{-1}[v]^Z_{1;K_1}\Big).
\end{aligned}
\tag{32}
$$

For every $\widetilde{\zeta}\in\widetilde{K}_2$ and $|J|=2$ set $g_\xi^{(5)}(\widetilde{\zeta})=(g(\xi)-g(\zeta))Z^Jv(\zeta)\phi(\widetilde{\zeta})$, with $g(\xi)=\lambda(\xi)+h_{ij}(\xi)+Z_j\omega_i(\xi)$. Then,

$$
\begin{aligned}
|g_\xi^{(5)}(\widetilde{\zeta})| = & |(g(\xi)-g(\zeta))||Z^Jv(\zeta)||\phi(\widetilde{\zeta})| \\
\leq & [g]^Z_{\alpha;K_2}d_Z^{\alpha}(\xi,\zeta)[v]^Z_{2;K_2} \\
\leq & c d^{\alpha}_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})[v]^Z_{2;K_2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|g_\xi^{(5)}(\widetilde{\zeta})-g_{\xi_0}^{(5)}(\widetilde{\zeta})| = & |g(\xi)-g(\xi_0)||Z^Jv(\zeta)||\phi(\widetilde{\zeta})| \leq [g]^Z_{\alpha;K_1}d_Z^{\alpha}(\xi,\xi_0)[v]^Z_{2;K_2} \\
\leq & c d_Z^{\alpha}(\xi,\xi_0)[v]^Z_{2;K_2}.
\end{aligned}
$$

For every $(\xi, 0) = \widetilde{\xi} \in \widetilde{K}_1$ and for every multi-index I of length 2,

$$w_5^I(\xi) = \int \widetilde{Z}_\xi^I \widetilde{\Gamma}_\xi(\widetilde{\xi}, \widetilde{\zeta}) g_\xi^{(5)}(\widetilde{\zeta}) d\widetilde{\zeta}.$$

Hence, by (24)

$$\begin{aligned} r^\beta \frac{|w_5^I(\xi) - w_5^I(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq & c\Big(\frac{d_Z(\xi, \xi_0)}{r}\Big)^{\alpha-\beta} \Big(r^\alpha [v]_{2;K_2}^Z\Big) \\ & + c\Big(\frac{d_Z(\xi, \xi_0)}{r}\Big)^{\alpha-\beta} \ln\Big(\frac{r}{d_Z(\xi, \xi_0)}\Big) \Big(r^\alpha [v]_{2;K_2}^Z\Big). \end{aligned} \tag{33}$$

For every $\widetilde{\zeta} \in \widetilde{K}_2$ set $g_\xi^{(6)}(\widetilde{\zeta}) = h_{ij}(\xi)(P_\xi \omega_i - \omega_i)(P_\xi \omega_j - \omega_j)\partial_{2n+1} v(\zeta) \partial_{2n+1}\phi(\widetilde{\zeta})$. Then

$$\begin{aligned} |g_\xi^{(6)}(\widetilde{\zeta})| = & |h_{ij}(\xi)||(P_\xi \omega_i - \omega_i)||(P_\xi \omega_j - \omega_j)||\partial_{2n+1} v(\zeta)||\partial_{2n+1}\phi(\widetilde{\zeta})| \\ \leq & r^{-2}|h_{ij}(\xi)|[\omega_i]_{1,\alpha;K_2}^Z [\omega_j]_{1,\alpha;K_2}^Z d_Z^{2+2\alpha}(\xi, \zeta) \sup_{\zeta \in K_2} |\partial_{2n+1} v(\zeta)| \\ \leq & c d_{\widetilde{Z}}^\alpha(\widetilde{\xi}, \widetilde{\zeta}) r^\alpha [v]_{2;K_2}^Z. \end{aligned}$$

Moreover, by Lemma 2.1 and Lemma 2.2

$$\begin{aligned} |g_\xi^{(6)}(\widetilde{\zeta}) - g_{\xi_0}^{(6)}(\widetilde{\zeta})| \leq & |(h_{ij}(\xi) - h_{ij}(\xi_0))(P_\xi \omega_i - \omega_i)(P_\xi \omega_j - \omega_j)\partial_{2n+1} v(\zeta) \partial_{2n+1}\phi(\widetilde{\zeta})| \\ & + |h_{ij}(\xi_0)(P_\xi \omega_i - P_{\xi_0}\omega_i)(P_\xi \omega_j - \omega_j)\partial_{2n+1} v(\zeta) \partial_{2n+1}\phi(\widetilde{\zeta})| \\ & + |h_{ij}(\xi_0)(P_{\xi_0} \omega_i - \omega_i)(P_\xi \omega_j - P_{\xi_0}\omega_j)\partial_{2n+1} v(\zeta) \partial_{2n+1}\phi(\widetilde{\zeta})| \\ \leq & r^{-2}[h_{ij}]_{\alpha;K_1} d_Z^\alpha(\xi, \xi_0)[\omega_i]_{1,\alpha;K_2}^Z [\omega_j]_{1,\alpha;K_2}^Z d_Z^{2+2\alpha}(\xi, \zeta) \sup_{K_2} |\partial_{2n+1} v| \\ & + r^{-2}|h_{ij}(\xi_0)|(d_Z^{\alpha+1}(\xi, \xi_0) + d_Z^\alpha(\xi, \xi_0) d_Z(\xi, \zeta))(d_Z^{1+\alpha}(\xi, \zeta) \\ & \quad + d_Z^{1+\alpha}(\xi_0, \zeta))[\omega_i]_{1,\alpha;K_2}^Z [\omega_i]_{1,\alpha;K_2}^Z \sup_{K_2} |\partial_{2n+1} v| \\ \leq & c d_Z^\alpha(\xi, \xi_0) r^\alpha [v]_{2;K_2}^Z. \end{aligned}$$

For every $(\xi, 0) = \widetilde{\xi} \in \widetilde{K}_1$ and for every multi-index I of length 2,

$$w_6^I(\xi) = \int \widetilde{Z}_\xi^I \widetilde{\Gamma}_\xi(\widetilde{\xi}, \widetilde{\zeta}) g_\xi^{(6)}(\widetilde{\zeta}) d\widetilde{\zeta}.$$

Hence, by (24)

$$\begin{aligned} r^\beta \frac{|w_6^I(\xi) - w_6^I(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq & c\Big(\frac{d_Z(\xi, \xi_0)}{r}\Big)^{\alpha-\beta} \Big(r^{2\alpha} [v]_{2;K_2}^Z\Big) \\ & + c\Big(\frac{d_Z(\xi, \xi_0)}{r}\Big)^{\alpha-\beta} \ln\Big(\frac{r}{d_Z(\xi, \xi_0)}\Big) \Big(r^{2\alpha} [v]_{2;K_2}^Z\Big). \end{aligned} \tag{34}$$

For every $\widetilde{\zeta} \in \widetilde{K}_2$ let $g_\xi^{(7)}(\widetilde{\zeta}) = h_{ij}(\xi)(P_\xi \omega_i - \omega_i)(\zeta)\partial_{2n+1} v(\zeta) \phi(\widetilde{\zeta})$. Then,

$$\begin{aligned} |g_\xi^{(7)}(\widetilde{\zeta})| = & |h_{ij}(\xi)||(P_\xi \omega_i - \omega_i)(\zeta)||\partial_{2n+1} v(\zeta)||\phi(\widetilde{\zeta})| \\ \leq & \sup_{K_1} |h_{ij}| [\omega_i]_{1,\alpha;K_2}^Z d_Z^{1+\alpha}(\xi, \zeta) [v]_{2;K_2}^Z \\ \leq & c d_{\widetilde{Z}}^{1+\alpha}(\widetilde{\xi}, \widetilde{\zeta}) [v]_{2;K_2}^Z. \end{aligned}$$

Moreover,

$$\begin{aligned}|g^{(7)}_{\xi}(\widetilde{\zeta})-g^{(7)}_{\xi_0}(\widetilde{\zeta})| \leq &|h_{ij}(\xi)-h_{ij}(\xi_0)||(P_\xi\omega_i-\omega_i)(\zeta)||\partial_{2n+1}v(\zeta)||\phi(\widetilde{\zeta})| \\ &+|h_{ij}(\xi_0)||(P_\xi\omega_i-P_{\xi_0}\omega_i)(\zeta)||\partial_{2n+1}v(\zeta)||\phi(\widetilde{\zeta})| \\ \leq &[h_{ij}]^Z_{\alpha;K_1}d^\alpha_Z(\xi,\xi_0)[\omega_i]^Z_{1,\alpha;K_2}d^{1+\alpha}_Z(\xi,\zeta)[v]^Z_{2;K_2} \\ &+|h_{ij}(\xi_0)|(d^{\alpha+1}_Z(\xi,\xi_0)+d^\alpha_Z(\xi,\xi_0)d_Z(\xi,\zeta))[v]^Z_{2;K_2} \\ \leq &c\Big(d^\alpha_Z(\xi,\xi_0)d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})+d^{1+\alpha}_Z(\xi,\xi_0)\Big)[v]^Z_{2;K_2}.\end{aligned}$$

For every $(\xi,0)=\widetilde{\xi}\in\widetilde{K}_1$ and for every multi-index I of length 3,

$$w^I_7(\xi)=\int \widetilde{Z}^I_\xi\widetilde{\Gamma}_\xi(\widetilde{\xi},\widetilde{\zeta})g^{(7)}_\xi(\widetilde{\zeta})d\widetilde{\zeta}.$$

By (25) for every multi-index I of length 3 we get

$$A^I_1+A^I_2+A^I_3\leq cd_Z(\xi,\xi_0)[v]^Z_{2;K_2}.$$

Let us remark that the function $\widetilde{\zeta}\to\frac{|g^{(7)}_\xi(\widetilde{\zeta})-g^{(7)}_{\xi_0}(\widetilde{\zeta})|}{d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})}$ is not bounded over $\widetilde{K}_2$. However, in this case

$$\begin{aligned}A^I_4 \leq &c\int_{M_2\cap\widetilde{K}_2} d_\xi(\widetilde{\xi},\widetilde{\zeta})^{-\widetilde{Q}-1}|g^{(7)}_\xi(\widetilde{\zeta})-g^{(7)}_{\xi_0}(\widetilde{\zeta})|d\widetilde{\zeta} \\ \leq &c\int_{M_2\cap\widetilde{K}_2} d_\xi(\widetilde{\xi},\widetilde{\zeta})^{-\widetilde{Q}-1}\Big(d^\alpha_Z(\xi,\xi_0)d_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})+d^{1+\alpha}_Z(\xi,\xi_0)\Big)[v]^Z_{2;K_2}d\widetilde{\zeta} \\ \leq &c\Big(d^\alpha_Z(\xi,\xi_0)\int_{2d_Z(\xi,\xi_0)}^{2r}\rho^{-1}d\rho+d^{1+\alpha}_Z(\xi,\xi_0)\int_{2d_Z(\xi,\xi_0)}^{2r}\rho^{-2}d\rho\Big)[v]^Z_{2;K_2} \\ \leq &c\Big(d^\alpha_Z(\xi,\xi_0)\ln\big(\frac{r}{d_Z(\xi,\xi_0)}\big)+d^{1+\alpha}_Z(\xi,\xi_0)\big(\frac{1}{2d_Z(\xi,\xi_0)}-\frac{1}{2r}\big)\Big)[v]^Z_{2;K_2} \\ \leq &cd^\alpha_Z(\xi,\xi_0)\Big(\ln\big(\frac{r}{d_Z(\xi,\xi_0)}\big)+\frac{1}{2}\big(1-\frac{d^\alpha_Z(\xi,\xi_0)}{r}\big)\Big)[v]^Z_{2;K_2}\end{aligned}$$

Hence, by the previous estimates

$$\begin{aligned}r^\beta\frac{|w^I_7(\xi)-w^I_7(\xi_0)|}{d^\beta_Z(\xi,\xi_0)} \leq &c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}\Big(r^\alpha[v]^Z_{2;K_2}\Big) \\ &+c\Big(\frac{d_Z(\xi,\xi_0)}{r}\Big)^{\alpha-\beta}\Big(\ln\Big(\frac{r}{d_Z(\xi,\xi_0)}\Big)+1\Big)r^\alpha[v]^Z_{2;K_2}.\end{aligned} \tag{35}$$

For every $\widetilde{\zeta}\in\widetilde{K}_2$ let $g^{(8)}_\xi(\widetilde{\zeta})=h_{ij}(\xi)(P_\xi\omega_i-\omega_i)(\zeta)(P_\xi\omega_j-\omega_j)(\zeta)\partial_{2n+1}v(\zeta)\phi(\widetilde{\zeta})$. Then, by arguing as for $g^{(6)}_\xi$

$$\begin{aligned}|g^{(8)}_\xi(\widetilde{\zeta})| \leq &|h_{ij}(\xi)|[\omega_i]^Z_{1,\alpha;K_2}[\omega_j]^Z_{1,\alpha;K_2}d^{2+2\alpha}_Z(\xi,\zeta)\sup_{\zeta\in K_2}|\partial_{2n+1}v(\zeta)| \\ \leq &cd^{2+\alpha}_{\widetilde{Z}}(\widetilde{\xi},\widetilde{\zeta})r^\alpha[v]^Z_{2;K_2}.\end{aligned}$$

Moreover,

$$\begin{aligned}|g^{(8)}_\xi(\widetilde{\zeta})-g^{(8)}_{\xi_0}(\widetilde{\zeta})| \leq &[h_{ij}]_{\alpha;K_1}d^\alpha_Z(\xi,\xi_0)[\omega_i]^Z_{1,\alpha;K_2}[\omega_j]^Z_{1,\alpha;K_2}d^{2+2\alpha}_Z(\xi,\zeta)\sup_{K_2}|\partial_{2n+1}v| \\ &+|h_{ij}(\xi_0)|(d^{\alpha+1}_Z(\xi,\xi_0)+d^\alpha_Z(\xi,\xi_0)d_Z(\xi,\zeta))(d^{1+\alpha}_Z(\xi,\zeta) \\ &\quad+d^{1+\alpha}_Z(\xi_0,\zeta))[\omega_i]^Z_{1,\alpha;K_2}[\omega_i]^Z_{1,\alpha;K_2}\sup_{K_2}|\partial_{2n+1}v| \\ \leq &c\Big(d^\alpha_Z(\xi,\xi_0)d^{2+2\alpha}_Z(\xi,\zeta)+d^{1+\alpha}_Z(\xi,\xi_0)d^{1+\alpha}_Z(\xi,\zeta)+d^\alpha_Z(\xi,\xi_0)d^{2+\alpha}_Z(\xi,\zeta)\Big)[v]^Z_{2;K_2}.\end{aligned}$$

For every $(\xi, 0) = \widetilde{\xi} \in \widetilde{K}_1$ and for every multi-index I of length 4,

$$w_8^I(\xi) = \int \widetilde{Z}_{\xi}^I \widetilde{\Gamma}_{\xi}(\widetilde{\xi}, \widetilde{\zeta}) g_{\xi}^{(8)}(\widetilde{\zeta}) d\widetilde{\zeta}.$$

By (25) we get

$$A_1^I + A_2^I + A_3^I \leq c d_Z(\xi, \xi_0) r^{\alpha} [v]_{2;K_2}^Z$$

for every multi-index I of length 4.

The function $\widetilde{\zeta} \to \frac{|g_{\xi}^{(8)}(\widetilde{\zeta}) - g_{\xi_0}^{(8)}(\widetilde{\zeta})|}{d_{\widetilde{Z}}^2(\widetilde{\xi}, \widetilde{\zeta})}$ is not bounded over $\widetilde{K}_2$. However, in this case

$$\begin{aligned}
A_4^I \leq & c \int_{M_2 \cap \widetilde{K}_2} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}-1} |g_{\xi}^{(8)}(\widetilde{\zeta}) - g_{\xi_0}^{(8)}(\widetilde{\zeta})| d\widetilde{\zeta} \\
\leq & c \int_{M_2 \cap \widetilde{K}_2} d_{\xi}(\widetilde{\xi}, \widetilde{\zeta})^{-\widetilde{Q}-2} \Big(d_Z^{\alpha}(\xi, \xi_0) d_{\widetilde{Z}}^{2+\alpha}(\widetilde{\xi}, \widetilde{\zeta}) + d_Z^{1+\alpha}(\xi, \xi_0) d_{\widetilde{Z}}^{1+\alpha}(\widetilde{\xi}, \widetilde{\zeta}) \Big) [v]_{2;K_2}^Z d\widetilde{\zeta} \\
\leq & c \Big(d_Z^{\alpha}(\xi, \xi_0) \int_{2d_Z(\xi,\xi_0)}^{2r} \rho^{-1+\alpha} d\rho + d_Z^{1+\alpha}(\xi, \xi_0) \int_{2d_Z(\xi,\xi_0)}^{2r} \rho^{-2+\alpha} d\rho \Big) [v]_{2;K_2}^Z \\
\leq & c \Big(d_Z^{\alpha}(\xi, \xi_0) \big((2r)^{\alpha} - (2d_Z(\xi, \xi_0))^{\alpha} \big) + d_Z^{1+\alpha}(\xi, \xi_0) \big((2d_Z(\xi, \xi_0))^{-1+\alpha} - (2r)^{-1+\alpha} \big) \Big) [v]_{2;K_2}^Z \\
\leq & c d_Z^{\alpha}(\xi, \xi_0) r^{\alpha} [v]_{2;K_2}^Z.
\end{aligned}$$

Hence, by the previous estimates

$$r^{\beta} \frac{|w_7^I(\xi) - w_7^I(\xi_0)|}{d_Z^{\beta}(\xi, \xi_0)} \leq c \Big(\frac{d_Z(\xi, \xi_0)}{r} \Big)^{\alpha-\beta} \Big(r^{2\alpha} [v]_{2;K_2}^Z \Big). \tag{36}$$

Obviously

$$r^{\beta} \frac{|Hv(\xi) V_J^I(\xi) - Hv(\xi_0) V_J^I(\xi_0)|}{d_Z^{\beta}(\xi, \xi_0)} \leq \Big(\frac{d_Z(\xi, \xi_0)}{r} \Big)^{\alpha-\beta} r^{\alpha} [Hv]_{\alpha;K_1}^Z \leq r^{\alpha} [Hv]_{\alpha;K_1}^Z,$$

and estimate (28) follows by estimates (29)–(36).

III Step. For every $\xi, \xi_0 \in \Omega$, $\xi \neq \xi_0$, assume $\delta = d_Z(\xi_0, \partial\Omega) \leq d_Z(\xi, \partial\Omega)$ and $r = \mu\delta$. For every multi-index I, $|I| = 2$, if $d_Z(\xi, \xi_0) \geq r$ then

$$r^{2+\beta} \frac{|Z^I v(\xi) - Z^I v(\xi_0)|}{d_Z^{\beta}(\xi, \xi_0)} \leq r^2 |Z^I v(\xi)| + |Z^I v(\xi_0)| \leq 2r^2 [v]_{2;\Omega}^Z.$$

Since $r = \mu\delta$ then

$$\delta^{2+\beta} \frac{|Z^I v(\xi) - Z^I v(\xi_0)|}{d_Z^{\beta}(\xi, \xi_0)} \leq 2\mu^{-\beta} \delta^2 [v]_{2;\Omega}^Z.$$

If $d_Z(\xi, \xi_0) < r$ by (28) we get

$$\begin{aligned}
\delta^{2+\beta} \frac{|Z^I v(\xi_0) - Z^I v(\xi)|}{d_Z^{\beta}(\xi, \xi_0)} \leq & c \Big(\mu^{\alpha-\beta} \delta^{2+\alpha} [Hv]_{\alpha;K_2} + \mu^{-\beta} \delta^2 \sup_{K_1} |Hv| + \mu^{-\beta} \delta^2 [v]_{2;K_2}^Z \\
& + \mu^{\alpha-\beta-1} \delta^{1+\alpha} [v]_{1,\alpha;K_2}^Z + \mu^{-1-\beta} \delta [v]_{1;K_2}^Z + \mu^{-2-\beta} \sup_{K_2} |v| \Big),
\end{aligned}$$

so that, by combing these two inequalities, we obtain

$$
\begin{aligned}
\delta^{2+\beta}\frac{|Z^I v(\xi_0) - Z^I v(\xi)|}{d_Z^{\beta}(\xi,\xi_0)} \leq & c\Big(\mu^{\alpha-\beta}\delta^{2+\alpha}[Hv]_{\alpha;\Omega} + \mu^{-\beta}\delta^2 \sup_{\Omega}|Hv| \\
& + \mu^{-\beta}\delta^2[v]^Z_{2;\Omega} + \mu^{\alpha-\beta-1}\delta^{1+\alpha}[v]^Z_{1,\alpha;\Omega} + \mu^{-1-\beta}\delta[v]^Z_{1;\Omega} + \mu^{-2-\beta}\sup_{\Omega}|v|\Big) \\
& + 2\mu^{-\alpha}\delta^2[v]^Z_{2;\Omega}.
\end{aligned} \tag{37}
$$

IV Step. We now need the following interpolation inequality, whose proof is contained in Appendix 1.

Proposition 3.1. *Let* $v \in C_Z^{2,\alpha}(\Omega)$, *where* Ω *is an open subset of* $\mathbb{R}^{2n+1}$. *Define*

$$
[v]^{*Z}_{j,0;\Omega} = \sup_{\substack{\xi\in\Omega \\ |J|=j}} \delta_\xi^j |Z^J v(\xi)|, \qquad [v]^{*Z}_{j,\alpha;\Omega} = \sup_{\substack{\xi,\xi_0\in\Omega \\ |J|=j}} \delta_{\xi,\xi_0}^{j+\alpha}\frac{|Z^J v(\xi) - Z^J v(\xi_0)|}{d_Z^{\alpha}(\xi,\xi_0)},
$$

where $\delta_\xi = d_Z(\xi,\partial\Omega)$, $\delta_{\xi,\xi_0} = \min\{\delta_\xi,\delta_{\xi_0}\}$. *Then for any* $\varepsilon > 0$ *there is a positive constant* $C = C(\varepsilon)$ *such that*

$$
[v]^{*Z}_{j,\beta;\Omega} \leq C\sup_{\Omega}|v| + \varepsilon[v]^{*Z}_{2,\alpha;\Omega}
$$

for every $j = 0,1,2$; $0 \leq \alpha,\beta \leq 1$, $j+\beta < 2+\alpha$.

By Proposition 3.1 and estimate (37) we get

$$
\begin{aligned}
[v]^{*Z}_{2,\beta;\Omega} \leq & c\Big(\mu^{\alpha-\beta}\delta^{2+\alpha}[Hv]^Z_{\alpha;\Omega} + \mu^{-\beta}\delta^2\sup_{\Omega}|Hv| \\
& + \mu^{-\beta}[v]^{*Z}_{2;\Omega} + \mu^{\alpha-\beta-1}[v]^{*Z}_{1,\alpha;\Omega} + \mu^{-1-\beta}[v]^{*Z}_{1;\Omega} + \mu^{-2-\beta}\sup_{\Omega}|v|\Big) + 2\mu^{-\alpha}[v]^{*Z}_{2;\Omega} \\
\leq & C\Big(|Hv|^Z_{\alpha;\Omega} + \sup_{\Omega}|v|\Big) + c\varepsilon\Big(\mu^{-\beta} + \mu^{\alpha-\beta-1} + \mu^{-1-\beta} + 2\mu^{-\alpha}\Big)[v]^{*Z}_{2,\beta;\Omega}
\end{aligned}
$$

Choosing $\varepsilon = \mu^{1+\alpha}$ we obtain

$$
[v]^{*Z}_{2,\beta;\Omega} \leq C(\mu)\Big(|Hv|^Z_{\alpha;\Omega} + \sup_{\Omega}|v|\Big) + c\mu^{\alpha-\beta}[v]^{*Z}_{2,\beta;\Omega}.
$$

We now choose $\mu \in]0,1/4[$ such that $c\mu^{\alpha-\beta} \leq 1/2$ and use again Proposition 3.1 to arrive at the estimate

$$
[v]^{*Z}_{j,\beta;\Omega} \leq C\Big(|Hv|^Z_{\alpha;\Omega} + \sup_{\Omega}|v|\Big),
$$

for every $j = 0,1,2$; $0 \leq \alpha,\beta \leq 1$, $j+\beta < 2+\alpha$. In particular, for every $\Omega' \subset\subset \Omega$, if $\delta = d_Z(\Omega',\partial\Omega)$, then

$$
\delta^{j+\beta}[v]^Z_{j,\beta;\Omega'} \leq [v]^{*Z}_{j,\beta;\Omega} \leq C\Big(|Hv|^Z_{\alpha;\Omega} + \sup_{\Omega}|v|\Big),
$$

for every $j = 0,1,2$; $0 \leq \alpha,\beta \leq 1$, $j+\beta < 2+\alpha$. □

Appendix 1

In this appendix we give a proof of the interpolation inequality stated in Proposition 3.1. For smooth vector fields an analogous result was proved in [31] (see also [17] for the elliptic case).

Proof of Proposition 3.1.

I Case. Assume $j = 1; \alpha = \beta = 0$. For every $\xi_0 \in \Omega$ let $\delta_{\xi_0} = d_Z(\xi, \xi_0)$. We set $r = \mu\delta_{\xi_0}$ with $\mu \leq 1/2$ a positive constant to be specified later and $D = D_Z(\xi_0, r) = \{\xi \in \mathbb{R}^{2n+1} : d_Z(\xi, \xi_0) < r\}$. For every $i = 1, \dots, 2n$ let $\gamma_i : [0, 2r] \to \Omega$ be the integral curve of the vector field Z_i such that $\gamma_i(r) = \xi_0$. Precisely γ_i is the solution of the Cauchy problem:

$$\begin{cases} \gamma_i'(\tau) = & Z_i\gamma_i(t), \\ \gamma_i(r) = & \xi_0. \end{cases}$$

Let us set $\xi' = \gamma_i(0), \xi'' = \gamma_i(2r)$. For some $\bar{r} \in]0, 2r[$ we get

$$\begin{aligned} v(\xi'') - v(\xi') = & \int_0^{2r} \frac{d}{dt}(v \circ \gamma_i)(t)dt \\ = & \int_0^{2r} Z_i v(\gamma_i(t))dt \\ = & 2r Z_i v(\gamma_i(\bar{r})). \end{aligned}$$

Let $\bar{\xi} = \gamma_i(\bar{r})$. Then

$$|Z_i v(\bar{\xi})| = \frac{|v(\xi'') - v(\xi')|}{2r} \leq \frac{1}{r} \sup_{\overline{D}} |v|. \tag{38}$$

Moreover,

$$\begin{aligned} Z_i v(\xi_0) = & Z_i v(\gamma_i(r)) = Z_i v(\gamma_i(\bar{r})) + \int_{\bar{r}}^{r} \frac{d}{dt}(Z_i v \circ \gamma_i)(t)dt \\ = & Z_i v(\bar{\xi}) + \int_{\bar{r}}^{r} (Z_i Z_i v)(\gamma_i(t))dt \end{aligned}$$

and by (38)

$$\begin{aligned} |Z_i v(\xi_0)| \leq & |Z_i v(\bar{\xi})| + |r - \bar{r}| \sup_{\overline{D}} |Z_i Z_i v| \\ \leq & \frac{1}{r} \sup_{\overline{D}} |v| + r \sup_{\overline{D}} |Z_i Z_i v|. \end{aligned} \tag{39}$$

For every $\xi \in \overline{D}$ $\delta_\xi \geq \delta_{\xi_0} - r = (1 - \mu)\delta_{\xi_0} \geq \delta_{\xi_0}/2$. By (39) we get

$$\delta_{\xi_0} |Z_i v(\xi_0)| \leq \mu^{-1} \sup_{\overline{D}} |v| + 4\mu [v]^{*Z}_{2;\overline{D}}.$$

Now, for every $\varepsilon > 0$, choose $\mu \leq \varepsilon/4$ and $C = \mu^{-1}$ to obtain

$$[v]^{*Z}_{1;\Omega} \leq C \sup_{\Omega} |v| + \varepsilon [v]^{*Z}_{2;\Omega}. \tag{40}$$

II Case. We assume $j = 2; \beta = 0, \alpha > 0$. With notations of the first case we have

$$|Z_i Z_l v(\bar{\xi})| = \frac{|Z_l v(\xi') - Z_l v(\xi'')|}{2r} \leq \frac{1}{r} \sup_{\overline{D}} |Z_l v|$$

and

$$|Z_i Z_l v(\xi_0)| \leq |Z_i Z_l v(\bar{\xi})| + |Z_i Z_l v(\xi_0) - Z_i Z_l v(\bar{\xi})|.$$

For every $\xi \in \overline{D}$ we have $\delta_\xi, \delta_{\xi,\xi_0} \geq \delta_{\xi_0}/2$ and

$$\delta_{\xi_0}^2 |Z_i Z_l v(\xi_0)| \leq 2\mu^{-1} \sup_{\xi \in \overline{D}} (\delta_\xi |Z_l v(\xi)|) + 2^{2+\alpha} \mu^\alpha [v]^{*Z}_{2,\alpha;\overline{D}}.$$

Hence, by also using estimate (40), for every $\varepsilon > 0$ there is a positive constant $C = C(\varepsilon)$ such that

$$[v]^{*Z}_{j;\Omega} \leq C \sup_\Omega |v| + \varepsilon [v]^{*Z}_{2,\alpha;\Omega} \tag{41}$$

for every $j = 0, 1, 2$.

III Case. Assume $j < 2; \beta > 0, \alpha \geq 0$. Let $\xi, \xi_0 \in \Omega$ with $\delta_{\xi_0} < \delta_\xi$ so that $\delta_{\xi,\xi_0} = \delta_{\xi_0}$. Choose $r = \mu \delta_{\xi_0}$ with $\mu \leq 1/2$ and $D = D_Z(\xi_0, r)$.

If $\xi \in D$ then there exists an absolutely continuous mapping $\gamma : [0,1] \to D$ such that $\gamma(0) = \xi_0, \gamma(1) = \xi$ and almost everywhere $\gamma'(t) = \sum_{i=1}^{2n} u_i(t) Z_i \gamma(t)$ with $|u_i(t)| \leq d_Z(\xi, \xi_0)$ for every $i = 1, \dots, 2n$. Hence,

$$\begin{aligned} v(\xi) - v(\xi_0) =& v(\gamma(1)) - v(\gamma(0)) = \int_0^1 \frac{d}{dt}(v \circ \gamma)(t) dt \\ =& \int_0^1 \Big(\sum_{i=1}^{2n} u_i(t) (Z_i v)(\gamma(t) \Big) dt, \end{aligned}$$

and

$$|v(\xi) - v(\xi_0)| \leq d_Z(\xi, \xi_0) \sum_{i=1}^{2n} \sup_D |Z_i v|.$$

In particular

$$\delta_{\xi_0}^\beta \frac{|v(\xi) - v(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq \mu^{1-\beta} \delta_{\xi_0} \sum_{i=1}^{2n} \sup_D |Z_i v| \tag{42}$$

for every $\xi \in D$.

If $\xi \notin D$ then

$$\delta_{\xi_0}^\beta \frac{|v(\xi) - v(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq 2\mu^{-\beta} \sup_\Omega |v|. \tag{43}$$

Combining inequalities (42), (43) and using (40), (41), we obtain for $0 < \beta \leq 1$ and for every $\varepsilon > 0$ there is $C > 0$ such that

$$[v]^{*Z}_{0,\beta;\Omega} \leq C \sup_\Omega |v| + \varepsilon [v]^{*Z}_{2,\alpha;\Omega}. \tag{44}$$

The proof for $j = 1$ proceeds in the same way after replacing v with $Z_i v$. In place of (42) we now have for every $\xi \in D$

$$\delta_{\xi_0}^{1+\beta} \frac{|Z_i v(\xi) - Z_i v(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq \mu^{1-\beta} \delta_{\xi_0}^2 \sum_{j=1}^{2n} \sup_D |Z_j Z_i v|$$

and for $\xi \notin D$ in place of (43) we now have

$$\delta_{\xi_0}^{1+\beta} \frac{|Z_i v(\xi) - Z_i v(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq 2\mu^{-\beta} \delta_{\xi_0} \sup_\Omega |Z_i v|.$$

IV Case. Assume $j = 2; \beta < \alpha$. With the same notations as above, if $\xi \in D$

$$\delta_{\xi_0}^{2+\beta} \frac{|Z_i Z_j v(\xi) - Z_i Z_j v(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq \mu^{\alpha-\beta} \delta_{\xi_0}^{2+\alpha} \frac{|Z_i Z_j v(\xi) - Z_i Z_j v(\xi_0)|}{d_Z^\alpha(\xi, \xi_0)},$$

while if $\xi \notin D$

$$\delta_{\xi_0}^{2+\beta} \frac{|Z_i Z_j v(\xi) - Z_i Z_j v(\xi_0)|}{d_Z^\beta(\xi, \xi_0)} \leq 2\mu^{-\beta} [v]^{*Z}_{2;\Omega}.$$

Combining these inequalities and taking the supremum over $\xi, \xi_0 \in \Omega$ we get the desired estimate. □

References

[1] E. Bedford, B. Gaveau *Hypersurfaces with Bounded Levi Form,* Indiana University Journal **27** No. 5 (1978), 867-873.

[2] L. Caffarelli, J.J. Kohn, L. Niremberg, J. Spruck, *The Dirichlet problem for non-linear second order elliptic equations II: Complex Monge-Ampère and uniformly elliptic equations,* Comm. Pure Appl. Math. **38** (1985), 209-252.

[3] L. Capogna, D. Danielli, N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations.* Am. J. Math. **118**, No.6, (1996), 1153-1196.

[4] G. Citti, *C^∞ regularity of solutions of a quasilinear equation related to the Levi operator* Ann. Scuola Norm. Sup. di Pisa Cl. Sci., Serie **4** Vol. XXIII (1996), 483-529.

[5] G. Citti, *C^∞ regularity of solutions of the Levi equation*, Ann. Inst. H. Poincare, Anal. non Linéaire, 15, 4 (1998) 517-534.

[6] G. Citti, *Regularity of solutions of a nonlinear Hörmander type equation*, Nonlinear Analysis, **47** (2001) 479-489.

[7] G. Citti, E. Lanconelli, A. Montanari, *On the smoothness of viscosity solutions of the prescribed Levi-curvature equation*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.,**10** (1999) 61-68.

[8] G. Citti, E. Lanconelli, A. Montanari, *Smoothness of Lipschitz continuous graphs with non vanishing Levi curvature*, Acta Mathematica **188** (2002) 87-128.

[9] G. Citti, A. Montanari, *Strong solutions for the Levi curvature equation*, Adv. in Diff. Eq. Vol **5** (1-3) (2000), 323-342.

[10] G. Citti, A. Montanari, *Regularity properties of Levi flat graphs* C.R. Acad. Sci. Paris, t. 329, Série 1 (1999) 1049-1054.

[11] G. Citti, A. Montanari, *Analytic estimates for solutions of the Levi equation*, J. Differential Equations **173**, (2001) 356-389.

[12] G. Citti, A. Montanari, *C^∞ regularity of solutions of an equation of Levi's type in $\mathbb{R}^{2n+1}$*, Ann. Mat. Pura Appl. **180** (2001), 27-58.

[13] G. Citti, A. Montanari, *Regularity properties of solutions of a class of elliptic-parabolic nonlinear Levi type equations*, Trans. Amer. Math. Soc. **354** (2002), 2819-2848.

[14] J.P. D'Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Florida, 1993.

[15] G.B. Folland, *Subelliptic estimates and functions spaces on nilpotent Lie groups*, Ark. Mat. **13**, (1975), 161-207.

[16] G.B. Folland, E.M. Stein, *Estimates for the* $\bar\partial_b$ *complex and analysis on the Heisenberg group* Comm. Pure and Appl. Math. **20**,(1974) , 429-522.

[17] D. Gilgarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehrer der Math. Wiss. vol. 224, Springer-Verlag, New York (1977).

[18] L. Hörmander, *An Introduction to Complex Analysis in Several Variables* Von Nostrand, Princeton, NJ, 1966.

[19] L. Hörmander, *Hypoelliptic second order differential equations,* Acta Math. **119**,(1967) , 147-171.

[20] S. Krantz, *Function Theory of Several Complex Variables,* Wiley, New York, 1982.

[21] F. Lascialfari, A. Montanari, *Smooth regularity for solutions of the Levi Monge-Ampère equation*, to appear on Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **12** (2001) pagg. 115-123.

[22] A. Montanari, F. Lascialfari, *The Levi Monge-Ampère equation: smooth regularity of strictly Levi convex solutions*, preprint.

[23] A. Nagel, E.M. Stein, S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985) 103-147.

[24] R.M. Range, *Holomorphic Functions and Integral Representation Formulas in Several Complex Variables* Springer-Verlag, New York, 1986.

[25] L.P. Rothschild, E.M. Stein, *Hypoelliptic differential operators on nilpotent groups* Acta Math., **137** (1977), 247-320.

[26] A. Sánchez-Calle, *Fundamental solutions and geometry of the sum of squares of vector fields*, Invent. Math., **78** (1984), 143-160.

[27] Z. Slodkowski, G. Tomassini, *The Levi equation in higher dimension and relationships to the envelope of holomorphy*, Amer. J. of Math. **116** (1994), 479-499.

[28] Z. Slodkowski, G. Tomassini, *Weak solutions for the Levi equation and Envelope of Holomorphy*, J. Funct. Anal. **101**, no. **4** (1991), 392-407.

[29] E. M. Stein *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton university Press, Princeton, New Jersey 1993.

[30] G. Tomassini, *Geometric Properties of Solutions of the Levi equation*, Ann. Mat. Pura Appl. (4), **152** (1988), 331-344.

[31] C.J. Xu, *Regularity for Quasilinear Second-Order Subelliptic Equations*, Comm. Pure and Appl. Math., **45** (1992), 77-96.

Dipartimento di Matematica, Università di Bologna
Piazza Porta San Donato 5, 40126 Bologna, ITALY.
E-mail: montanar@dm.unibo.it