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Hölder Continuity of Solutions for Higher Order Degenerate Nonlinear Parabolic Equations (*).

F. NICOLOSI - I. V. SKRYPNIK

Summary. – *We study a regularity of bounded solutions for some degenerate nonlinear parabolic equations of higher order. It is established the Hölder Continuity of solutions by condition that the weighted function belongs to the class* $A_{1+q/n}$.

1. – Introduction.

Let us suppose that Ω is an open bounded subset of R^n, T is an arbitrary positive number and consider on $Q_T = \Omega \times (0, T]$ nonlinear differential equation

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u, \ldots, D^m u) = 0 \tag{1.1}$$

where $x = (x_1, \ldots, x_n) \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i is nonnegative entier number, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$, $D^k u = \{D^\alpha u \colon |\alpha| = k\}$.

We shall assume that $m \geq 2$ and the functions $A_\alpha(x, t, \xi)$ in (1.1) are Carathéodory's functions, i.e they are measurable functions with respect to $(x, t) \in Q_T$ for all $\xi = \{\xi_\alpha \colon |\alpha| \leq m\}$, $\xi_\alpha \in R^1$ and continuous functions with respect to ξ for almost all $(x, t) \in Q_T$. We suppose that following inequalities

$$\sum_{|\alpha| \leq m} A_\alpha(x, t, \xi)\, \xi_\alpha \geq C' v(x) \left\{ \sum_{|\alpha| = m} |\xi_\alpha|^p + \sum_{|\alpha| = 1} |\xi_\alpha|^q \right\} - C'' v(x) \sum_{|\alpha| = 2}^{m-1} |\xi_\alpha|^{p_\alpha} - f(x, t), \tag{1.2}$$

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Indirizzo degli AA.: F. NICOLOSI: Dipartimento di Matematica, Università di Catania, Viale A. Doria 6, 95125 Catania e-mail: FNICOLOSI@dipmat.unict.it; I. V. SKRYPNIK: Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine, str. Roza Luxemburg 74, 340114 Donetsk.

$$(1.3)\qquad \sum_{1\leq|\alpha|\leq m}[v(x)]^{-1/(p_\alpha-1)}\,|A_\alpha(x,t,\xi)|^{p_\alpha/(p_\alpha-1)}+|A_0(x,t,\xi)|\leq$$

$$\leq C''v(x)\sum_{1\leq|\beta|\leq m}|\xi_\beta|^{p_\beta}+f(x,t)$$

are satisfied where $v(x)$, $f(x,t)$ are non-negative functions with properties precised later.

The number p_α in (1.2), (1.3) are defined by equalities

$$(1.4)\qquad \begin{cases} \dfrac{1}{p_\alpha}=\dfrac{|\alpha|-1}{m-1}\cdot\dfrac{1}{p}+\dfrac{m-|\alpha|}{m-1}\cdot\dfrac{1}{q_1} & \text{for } 1<|\alpha|\leq m\,,\\ p_\alpha=q \quad \text{for } |\alpha|=1 \end{cases}$$

and the numbers p, q, q_1 satisfy the inequalities

$$(1.5)\qquad p\geq 2\,,\qquad mp<q_1<q\,.$$

The conditions (1.2)-(1.5) determine the special structure of the equation (1.1). Main characteristic of this class of the equations is more strong condition of parabolicity (1.2) then usually. We assume positiveness and prescribed growth of the left-hand side of (1.2) not only with respect to $\{\xi_\alpha\colon |\alpha|=m\}$ (as usually) but also with respect to $\{\xi_\alpha\colon |\alpha|=1\}$. As model equation of this class we can consider the next equation

$$(1.6)\qquad \frac{\partial u}{\partial t}+\sum_{|\alpha|=m}(-1)^m D^\alpha(v(x)\,|D^m u|^{p-2}D^\alpha u)-$$

$$-\sum_{|\alpha|=1}D^\alpha(v(x)\,|D^1 u|^{q-2}D^\alpha u)=F(x,t)$$

by condition $q>mp$.

We remark that results of this paper about smoothness of solutions of the equation (1.1) it is impossible to prove by the less restrictive condition that the inequality $q>mp$. This follows from the counterexample from our paper [7].

We shall suppose that the weighted function $v(x)$ belongs to class $A_{1+q/n}$, has a first derivative and a function

$$(1.7)\qquad \tilde v(x)=v(x)\left\{1+\frac{1}{v(x)}\left|\frac{\partial v(x)}{\partial x}\right|\right\}^{\varrho},\qquad \varrho=\frac{(m-1)\,pq_1}{q_1-p}$$

is such that the imbedding

$$\overset{\circ}{W}{}^1_q(\Omega,v)\subset L_{\bar q}(\Omega,\tilde v)$$

is valid with some $\bar q>q$.

By conditions (1.1)-(1.5), corresponding condition on $f(x,t)$ we prove main result of this paper about Hölder continuity of an arbitrary bounded solution of the equation (1.1). Boundedness of solutions by analogous assumptions was established in our preceding paper [7].

We introduce in Section 2 a special class of functions $B_{q,s}(Q_T, v, w)$. This class generalizes corresponding classes of O. A. LADYZHENSKAYA and N. N. URAL'TSEVA [5] for $q=2$, $s=1$ and E. Di Benedetto [1] for $q>2$, $s=1$ which were introduced in non weighted cases. Class $B_{q,s}$ for $v(x)=w(x)\equiv 1$ was studied in [11].

We prove in weighted case Hölder continuity for an arbitrary function from class $B_{q,s}(Q_T, v, w)$ and we prove that an arbitrary solution of the equation (1.1) by structural assumptions (1.2)-(1.5) belongs to the class $B_{q,s}(Q_T, v, \tilde{v})$ with the function $\tilde{v}(x)$ defined by (1.7).
In this paper we study for simplicity only interior regularity of an arbitrary solution. But it is possible to establish also the regularity near boundary for solutions of initial-boundary value problems in case of Dirichlet or Neumann boundary conditions by using some constructions from paper [8].

This paper is organized as follows. In section 2 we introduce the class $B_{q,s}(Q_T, v, w)$ and formulate main results. We prove in section 3 that an arbitrary solution of the equation (1.1) belongs to the class $B_{q,s}(Q_T, v, \tilde{v})$. The imbedding of class $B_{q,s}(Q_T, v, w)$ in the space of Hölder functions is proved in sections 4-6. In section 7 we present example in order to illustrate our general result.

2. - Formulation of assumptions and main results.

We shall denote for an arbitrary measurable set $E\subset R^n$ and for an arbitrary nonnegative integrable function $\omega(x)$ on R^n

$$\omega(E)=\int_E \omega(x)\,dx\,, \tag{2.1}$$

by $|E|$ we shall denote Lebesgue measure of set E.
We assume that the weighted function $v(x)$ satisfies next properties:

v_1) $v(x)$ is the nonnegative function on R^n which belongs to class $A_{1+q/n}$.

v_2) the function $v(x)$ has a derivative of the first order on R^n; and the function $\tilde{v}(x)$ defined by (1.7) belongs to class A_∞ and satisfies the inequality

$$\frac{R_1}{R_2}\left[\frac{\tilde{v}(B(x_0, R_1))}{\tilde{v}(B(x_0, R_2))}\right]^{1/q\kappa_1}\leq C\left[\frac{v(B(x_0, R_1))}{v(B(x_0, R_2))}\right]^{1/q} \tag{2.2}$$

for an arbitrary point $x_0\in\Omega$ and arbitrary positive numbers R_1, R_2 such that $R_1<R_2$ with constants κ_1, C independent from x_0, R_1, R_2, $\kappa_1>1$. Here $B(x_0, R)$ is a ball of radius R with a center x_0.

Definitions of classes A_p, A_∞ and properties of functions from these classes it is possible to find in [6, 3]. In particular, from [6] it is followed that by condition v_1) the weighted function $v(x)$ belongs to the class $A_{1/\kappa_0+q/n}$ with some $\kappa_0 > 1$.

We will assume that the function $f(x, t)$ from (1.2), (1.3) satisfies the condition

$$f(x, t) \in L_{\varrho_0, r_0}(Q_T) \tag{2.3}$$

with $\varrho_0 > 1$, $r_0 > 1$ and such that the equality

$$\frac{1}{r_0} + \frac{\kappa_0}{(\kappa_0 - 1)\varrho_0} = 1 - \kappa' \tag{2.4}$$

is valid with $\kappa' \in (0, 1)$.

Under conditions (1.2)-(1.5), (2.3), (2.4), v_1), v_2) we shall study the behaviour of generalized solution of the equation (1.1). We shall tell that $u(x, t) \in E^{(0)}_{\text{loc}}(Q_T, v)$ if for an arbitrary infinitely differentiable function $\varphi(x, t)$ with compact support in Q_T

$$\varphi(x, t)\, u(x, t) \in E^{(0)}(Q_T, v) = C(0, T; L_2(\Omega)) \cap L_p(0, T; \overset{\circ}{W}{}^m_p(\Omega)) \cap L_q(0, T, \overset{\circ}{W}W^1_q(\Omega)).$$

By a generalized solution of (1.1) we mean a function $u(x, t) \in E^{(0)}_{\text{loc}}(Q_T, v)$ which satisfies the identity

$$\int_\Omega u(x, t)\, \psi(x, t)\, dx \Big|_{t_1}^{t_2} + \tag{2.5}$$

$$+ \int_{t_1}^{t_2} \int_\Omega \left\{ -u(x, t)\, \frac{\partial \psi(x, t)}{\partial t} + \sum_{|\alpha| \le m} A_\alpha(x, t, u(x, t), \dots, D^m u(x, t))\, D^\alpha \psi(x, t) \right\} dx\, dt = 0$$

for all $\psi(x, t)$ with compact support in Q_T such that $\psi(x, t) \in E^{(0)}(Q_T, v)$, $(\partial \psi(x, t))/\partial t \in L_2(Q_T)$ and for arbitrary numbers t_1, t_2 such that $0 \le t_1 < t_2 \le T$.

We suppose that considered solution $u(x, t)$ satisfies the inequality

$$\text{ess sup}\, \{ |u(x, t)| : (x, t) \in Q_T \} \le M \tag{2.6}$$

with some constant M. Local boundedness of solution follows from [7].

As known parameters by study of properties of solutions of the equation (1.1) we understand $n, m, p, q, q_1, C', C'', \varrho_0, r_0, \kappa', T, M$, the norm of $f(x, t)$ in $L_{\varrho_0, r_0}(Q_T)$ and parameters connected with weighted functions $v(t)$, $\tilde{v}(x)$.

We shall prove the next main result.

THEOREM 2.1. – *Assume that conditions* (1.2)-(1.5), (2.3), (2.4), v$_1$), v$_2$) *are satisfied and let* $u(x, t)$ *be a generalized solution of the equation* (1.1), *which satisfies the inequality* (2.6). *Then there exist positive constants* A *and* a *such that the inequality*

$$\text{ess osc}\, \{ u(x, t) : (x, t) \in Q_R(x_0, t_0) \} \le A R^\alpha \tag{2.7}$$

holds for an arbitrary cylinder $Q_R(x_0, t_0) = B(x_0, R) \times (t_0 - R^{n+q}, t_0)$ *if* $Q_{2R}(x_0, t_0) \subset \subset Q_T$. *The constant* a *belongs to the interval* (0,1) *and depends only on known parameters, the constant* A *depends only on known parameters and on the distance from* $Q_R(x_0, t_0)$ *to*

$$\Gamma_T = [\partial\Omega \times (0, T)] \cup [\Omega \times \{0\}].$$

The assertion of the Theorem 2.1 follows from the fact that the solution belongs to a special class of functions denoted by $B_{q,s}(Q_T, v, w)$.

Let $v(x)$, $w(x)$ be arbitrary nonnegative functions such that $v(x)$ satisfies to the condition $v_1)$, $w(x) \in A_\infty$ and the inequality

$$\frac{R_1}{R_2}\left[\frac{w(B(x_0, R_1))}{w(B(x_0, R_2))}\right]^{1/q\kappa_1} \leqslant C\left[\frac{v(B(x_0, R_1))}{v(B(x_0, R_2))}\right]^{1/q} \tag{2.8}$$

holds with the same $x_0, R_1, R_2, q, \kappa_1, C$ as in (2.2). We shall tell that a measurable function $u(x, t)$ belongs to the class $B_{q,s}(Q_T, v, w)$ if the inequality (2.6) is valid and for an arbitrary cylinder

$$Q(R, \theta) \equiv Q(x_0, t_0; R, \theta) = B(x_0, R) \times (t_0 - \theta, t_0)$$

such that $\overline{Q(R, \theta)} \subset Q_T$ and for an arbitrary infinitely differentiable nondecreasing function $\eta(t)$ on R^1 the following inequalities hold

$$\begin{aligned} &\sup_{t_0 - \theta \leqslant t \leqslant t_0} \int\limits_{B(x_0, R - \sigma R)} [u(x, t) - k]_\pm^{s+1} \eta^q(t)\, dx + \\ &+ \int\limits_{t_0-\theta}^{t_0} \int\limits_{B(x_0, R - \sigma R)} [u(x, t) - k]_\pm^{s-1} \left|\frac{\partial u(x, t)}{\partial x}\right|^q \eta^q(t)\, v(x)\, dx\, dt \leqslant \\ &\leqslant \int\limits_{B(x_0, R)} [u(x, t_0 - \theta) - k]_\pm^{s+1} \eta^q(t_0 - \theta)\, dx + \\ &+ \gamma\left\{\frac{1}{(\sigma R)^q} \iint\limits_{Q(R, \theta)} [u(x, t) - k]_\pm^{s+q-1} \eta^q(t)\, v(x)\, dx\, dt + \right. \\ &+ \iint\limits_{Q(R, \theta)} [u(x, t) - k]_\pm^{s+1} \eta^{q-1}(t) \frac{d\eta(t)}{dt}\, dx\, dt + \\ &\left. + \int\limits_{t_0-\theta}^{t_0} w(A_{k,R}^\pm(t))\, dt + \left[\int\limits_{t_0-\theta}^{t_0} |A_{k,R}^\pm(t)|^{r/\varrho}\, dt\right]^{q(1+\kappa)/r}\right\}, \end{aligned} \tag{2.9}$$

$$(2.10)\qquad \sup_{t_0-\theta\leq t\leq t_0}\int_{B(x_0,\,R-\sigma R)}\left[\ln\frac{H^{\pm}}{H^{\pm}\mp(u(x,t)-k)+\nu}\right]_+^{s+1}dx\leq$$

$$\leq\int_{B(x_0,\,R)}\left[\ln\frac{H^{\pm}}{H^{\pm}\mp(u(x,t_0-\theta)-k)+\nu}\right]_+^{s+1}dx+$$

$$+\gamma\left\{\frac{1}{(\sigma R)^q}\int\int_{Q(R,\,\theta)}\left[\ln\frac{H^{\pm}}{H^{\pm}\mp(u(x,t)-k)+\nu}\right]_+^{s}\cdot\frac{v(x)\,dx\,dt}{[H^{\pm}\mp(u(x,t)-k)+\nu]^{2-q}}+\right.$$

$$\left.+\frac{1}{\nu^b}\left[\ln\frac{H^{\pm}}{\nu}+1\right]^s\left[\int_{t_0-\theta}^{t_0}w(A^{\pm}_{k,R}(t))\,dt+\left(\int_{t_0-\theta}^{t_0}|A^{\pm}_{k,R}(t)|^{r/\varrho}\,dt\right)^{q/r(1+\kappa)}\right]\right\}.$$

Here $q\geq 2$ and s, γ, r, ϱ, δ, b, κ are given positive numbers which independent on x_0, t_0, R, θ and satisfy restrictions $\delta<M$, $\kappa\in(0.1)$, $b\leq s$, r, $\varrho>1$ and

$$(2.11)\qquad \frac{\kappa_0-1}{\kappa_0 r}+\frac{1}{\varrho}=\frac{1}{q}$$

with the same κ_0 as in (2.4) and possible values of r are limited by the condition $r\in\in(q,\infty)$. In $(2.9)_\pm$, $(2.10)_\pm$ σ is an arbitrary number from the interval (0,1).

The following notation is also used in $(2.9)_\pm$, $(2.10)_\pm$:

$$(2.12)\qquad [u(x,t)-k]_\pm=\max\{\pm[u(x,t)-k],0\},$$

$$(2.13)\qquad A^{\pm}_{k,R}(t)=\{x\in B(x_0,R):\ \pm[u(x,t)-k]>0\}$$

and $[\ln H^{\pm}\{H^{\pm}\mp(u(x,t)-k)+\nu\}^{-1}]_+$ is understood by analogy with (2.12). In $(2.9)_\pm$, $(2.10)_\pm$ k is an arbitrary real number satisfying the condition

$$(2.14)\qquad \operatorname{ess\,sup}\{[u(x,t)-k]_\pm:(x,t)\in Q(R,\theta)\}\leq\delta$$

and $H^{\pm}$, ν are positive numbers such that

$$(2.15)\qquad \operatorname{ess\,sup}\{[u(x,t)-k]_\pm:(x,t)\in Q(R,\theta)\}\leq H^{\pm}\leq\delta,\qquad \nu\leq\min\{H^{\pm},1\}.$$

By study of properties of functions from the class $B_{q,s}(Q_T,v,w)$ we understand n, q, s, γ, r, ϱ, δ b, κ, σ, M, T, parameters connected with weighted functions $v(x)$, $w(x)$ as known parameters.

THEOREM 2.2. – *Assume that the function $v(x)$ satisfies the condition v_1), nonnegative function $w(x)$ belongs to the class A_∞ and satisfies the inequality* (2.8). *Then for an arbitrary function $u(x,t)\in B_{q,s}(Q_T,v,w)$ the inequality*

$$(2.16)\qquad \operatorname{ess\,osc}\{u(x,t):(x,t)\in Q_R(x_0,t_0)\}\leq BR^{\beta}$$

holds for an arbitrary cylinder $Q_R(x_0,t_0)$ such as in Theorem 2.1. *In* (2.16) *constant β*

belongs to the interval (0,1) *and depends only on known parameters, positive constant* B *depends only on known parameters and on the distance from* $Q_R(x_0, t_0)$ *to* Γ_T.

Theorem 2.1 follows from Theorem 2.2 and next result.

THEOREM 2.3. – *Let the conditions of the Theorem* 2.1 *be satisfied. Then* $u(x, t)$ *belongs to the class* $B_{q,s}(Q_T, v, \tilde{v})$ *with constants* s, γ, r, ϱ, δ, b, κ *dependent only on known parameters of the equation* (1.1) *and with the function* $\tilde{v}(x)$ *defined by* (1.7).

Remark that for the equation of second order in weighted case Hölder continuity of the solutions was proved in [9].

3. – Proof of the Theorem 2.3.

Let us introduce the average over t for an arbitrary function $g(x, t)$ locally integrable on Q_T:

$$g_h(x, t) = [g(x, t)]_h = \frac{1}{h} \int_t^{t+h} g(x, s)\, ds \qquad \text{for } 0 < t < T - h .$$

It is simple to verify that for the generalized solution $u(x, t)$ of the equation (1.1) the following integral identity

$$\text{(3.1)} \qquad \int_{t_1}^{t_2} \int_\Omega \left\{ \frac{\partial [u(x, \tau)]_h}{\partial \tau} \varphi(x, \tau) + \sum_{|\alpha| \leq m} [A_\alpha(x, \tau, \ldots, D^m u(x, \tau))]_h D^\alpha \varphi(x, \tau) \right\} dx\, d\tau = 0$$

holds for an arbitrary function $\varphi(x, t) \in E^{(0)}(Q_T)$ with compact support in Q_T if $h < t_1 < t_2 < T - h$.

Let us substitute $\varphi(x, t)$ in (3.1) by the test function

$$\text{(3.2)} \qquad \varphi_1(x, t) = \pm [[u(x, t]_h - k]_\pm^s \cdot \xi^{r_1}(x)\, \eta^q(t)$$

where s, r_1 are sufficiently large positive numbers to be chosen later, $\xi(x)$ is an infinitely differentiable function such that $\xi(x) \equiv 1$ for $|x - x_0| \leq (1 - \sigma) R$, $\xi(x) \equiv 0$ for $|x - x_0| \geq R$ and $|D^\alpha \xi(x)| \leq C/(\sigma R)^{|\alpha|}$ for $|\alpha| \leq m$, $\eta(t)$ is an arbitrary infinitely differentiable nondecreasing function on R^1. Here $x_0 \in \Omega$ and R is so small that $B(x_0, R) \subset \Omega$. In (3.1) let us choose sufficiently small numbers $t_2 = t \leq t_0 \in (0, T)$, $t_1 = t_0 - \theta > 0$ and h such that $h < T - t_0$. The number k obeys the restriction (2.14) with $\delta = \min\{M, (1/4)(C'/C'')\}$.

We can prove that $\varphi_1(x, t) \in E^{(0)}(Q)$ for $s \geqslant m$, this permits us to use the substitution (3.2). Transforming the term which results by substituting the function $\varphi_1(x, t)$ and includes $(\partial/\partial t)[u(x, t)]_h$ we can pass to the limit as $h \to 0$ in identity (3.1) with $\varphi(x, t) = \varphi_1(x, t)$. The resultant equality reads

$$
\begin{aligned}
(3.3)\qquad & \frac{1}{s+1} \int_{B(x_0, R)} [u(x, \tau) - k]_{\pm}^{s+1} \, \xi^r(x) \, \eta^q(\tau) \, d\tau \Big|_{\tau = t_0 - \theta}^{\tau = t} - \\
& - \frac{q}{s+1} \int_{t_0 - \theta}^{t} \int_{B(x_0, R)} [u(x, \tau) - k]_{\pm}^{s+1} \, \xi^r(x) \, \eta^{q-1}(\tau) \, \frac{d\eta(\tau)}{d\tau} \, dx \, d\tau \pm \\
& \pm \int_{t_0 - \theta}^{t} \int_{B(x_0, R)} \sum_{|\alpha| \leqslant m} A_\alpha(x, \tau, u, \ldots, D^m u) \, D^\alpha \{ [u(x, \tau) - k]_{\pm}^{s} \, \xi^r(x) \} \, \eta^q(\tau) \, dx \, d\tau = 0 \, .
\end{aligned}
$$

Using conditions (1.2), (1.3), the estimate (2.6), Young inequality and the choice of δ we can evaluate the last integral. We obtain the inequality

$$
\begin{aligned}
(3.4)\qquad & \sup_{t_0 - \theta < t < t_0} \int_{B(x_0, R)} [u(x, t) - k]_{\pm}^{s+1} \, \xi^r(x) \, \eta^q(t) \, dx + \frac{C'(s+1)s}{2} (I_m + I_1) \leqslant \\
& \leqslant \int_{B(x_0, R)} [u(x, t_0 - \theta) - k]_{\pm}^{s+1} \, \xi^r(x) \, \eta^q(t_0 - \theta) \, dx + \\
& + q \int_{t_0 - \theta}^{t_0} \int_{B(x_0, R)} [u(x, \tau) - k]_{\pm}^{s+1} \, \xi^r(x) \, \eta^{q-1}(\tau) \, \frac{d\eta(\tau)}{d\tau} \, dx \, d\tau + \\
& + C_1 \Bigg\{ \sum_{j=2}^{m-1} I_j + \int_{t_0 - \theta}^{t_0} \int_{B(x_0, R)} \big([u(x, \tau) - k]_{\pm}^{s-1} f(x, t) \, \xi^{r_1}(x) + \\
& + [u(x, \tau) - k]_{\pm}^{s+q-1} \left(\frac{1}{\sigma R} \right)^q \xi^{r_1 - q}(x) \, v(x) + \\
& + [u(x, \tau) - k]_{\pm}^{s - a_1(m+q)} \, \xi^{r_1 - a_1 m}(x) \, v(x) \big) \, \eta^q(\tau) \, dx \, d\tau \Bigg\} .
\end{aligned}
$$

Here

$$(3.5)\quad \begin{cases} I_j = \sum_{|\alpha| = j} I_\alpha , \\ I_\alpha = \int_{t_0-\theta}^{t_0} \int_{B(x_0,R)} [u(x,\tau)-k]_\pm^{(s-1)} \, |D^\alpha u(x,\tau)|^{p_\alpha} v(x)\, \xi^{r_1}(x)\, \eta^q(\tau)\, dx\, d\tau , \\ a_1 = \max\left\{ \left[\dfrac{1}{p_\alpha} - \dfrac{|\alpha|}{|\beta| p_\beta} \right]^{-1} : 2 \le |\alpha| \le m, \quad 1 \le |\beta| \le |\alpha| - 1 \right\}. \end{cases}$$

In (3.4) and in the subsequent formulas we denote by C_j, $j = 1, 2, \ldots$ constants which depend only on known parameters.

The following Lemma is used to tranform the right-hand side of (3.4).

LEMMA 3.1. – *For* $2 \le j \le m-1$ *and an arbitrary number* ε *from the interval* (0,1) *the inequality*

$$(3.6)\quad I_j \le \varepsilon (I_{j+1} + I_1) + C_2 \cdot \varepsilon^{-b_j} \int_{t_0-\theta}^{t_0} \int_{B(x_0,R)} \left\{ [u(x,\tau)-k]_\pm^{s-b_j} \, \xi^{r_1}(x) + \right.$$

$$+ \frac{1}{(\sigma R)^q} [u(x,\tau)-k]_\pm^{s+q-1} \, \xi^{r_1 - q}(x) +$$

$$\left. + [u(x,\tau)-k]_\pm^{s-1} \left[\frac{1}{v(x)} \left| \frac{\partial v(x)}{\partial x} \right| \right]^\varrho \xi^{r_1}(x) \right\} \cdot v(x)\, \eta^q(\tau)\, dx\, d\tau$$

holds with some constants b_j *dependent only on known parameters and with* $\varrho = ((m-1)\, pq_1)/pq_1$.

PROOF. – Let $|\alpha| = j$ and let $\alpha = \beta + \gamma$ where $|\beta| = j-1$, $|\gamma| = 1$. Integrating by parts we obtain

$$(3.7)\quad I_\alpha = -\int_{t_0-\theta}^{t_0} \int_{B(x_0,R)} [u(x,\tau)-k]_\pm^{s-1} \left\{ \pm (s-1)[u(x,\tau)-k]_\pm^{-1} D^\gamma u D^\alpha u |D^\alpha u|^{p_\alpha - 2} + \right.$$

$$+ (p_\alpha - 1)\, |D^\alpha u|^{p_\alpha - 2} D^{\alpha+\gamma} u + r |D^\alpha u|^{p_\alpha - 2} D^\alpha u \xi^{-1}(x)\, D^\gamma \xi(x) +$$

$$\left. + |D^\alpha u|^{p_\alpha - 2} D^\alpha u \, \frac{1}{v(x)} , D^\gamma v(x) \right\} D^\beta u(x,\tau)\, \xi^{r_1}(x)\, \eta^q(\tau)\, v(x)\, dx\, d\tau .$$

First we prove the inequality (3.6) for $j = 2$. For this it suffices to estimate the terms in brackets in the last integral by using the Young inequality with ε. In particular, we have

$$(3.8)\qquad |D^{\alpha}u|^{p_\alpha-2}|D^{\gamma+\alpha}u|\cdot|D^{\beta}u|\leqslant\varepsilon\{|D^{\alpha}u|^{p_\alpha}+|D^{\alpha+\gamma}u|^{p_{\alpha+\gamma}}+|D^{\beta}u|^{p_\beta})+\varepsilon^{-a_2}$$

for $|\alpha|=2$, where a_2 is determined by the condition

$$\frac{2}{q}+\frac{1}{a_2}=\frac{1}{m-1}\,\frac{1}{p}+\frac{m-2}{m-1}\,\frac{1}{q_1}\,.$$

Similarly we can evaluate all terms in the right-hand side of (3.7) and we obtain the inequality (3.6) for $j=2$.

For $j>2$ we prove (3.6) by induction assuming that (3.6) holds for $j<j_0$ and proving it for $j=j_0$. Let us estimate the right-hand side of (3.7) for $j=j_0>2$. In this case we obtain the inequality

$$(3.9)\qquad |D^{\alpha}u|^{p_\alpha-2}|D^{\alpha+\gamma}u|\cdot|D^{\beta}u|\leqslant\varepsilon(|D^{\alpha}u|^{p_\alpha}+|D^{\alpha+\gamma}u|^{p_{\alpha+\gamma}})+\varepsilon^{-q}|D^{\beta}u|^{p_\beta}$$

instead of (3.8). We evaluate by Young inequality another terms on right hand side of (3.7) and obtain the estimate

$$I_{j_0}\leqslant C_3\{\varepsilon I_{j_0+1}+\varepsilon I_1+\varepsilon^{-q}I_{j_0-1}+$$
$$+\varepsilon^{-a_3}\int_{t_0-\theta}^{t_0}\int_{B(x_0,R)}\Bigg\{\frac{1}{(\sigma R)^q}[u(x,\tau)-k]_{\pm}^{s-1+q}\,\xi^{r_1-q}(x)+[u(x,\tau)-k]^{s-a_3-1}\,\xi^{r_1}(x)+$$
$$+[u(x,\tau)-k]_{\pm}^{s-1}\left(\frac{1}{v(x)}\left|\frac{\partial v(x)}{\partial x}\right|\right)^{\varrho}\xi^{r_1}(x)\Bigg\}\,\eta^q(\tau)\,v(x)\,dx\,d\tau$$

with some a_3 determined by m, p, q, q_1. We evaluate I_{j_0-1} by using (3.6) on the basis of the induction assumption (with ε^{q+1} instead of ε) and we have from the last inequality the estimate (3.6) for $j=j_0$. This completes the proof of Lemma 3.1.

By summing the inequality (3.6) over j we obtain the estimate

$$(3.10)\qquad \sum_{j=2}^{m-1}I_j\leqslant\varepsilon(I_{m+1}+I_1)+C_4\varepsilon^{-b'}\int_{t_0-\theta}^{t_0}\int_{B(x_0,R)}\Bigg\{[u(x,\tau)-k]_{\pm}^{s-b'}\,\xi^{r_1}(x)+$$
$$+\frac{1}{(\sigma R)^q}[u(x,\tau)-k]_{\pm}^{s+q-1}\,\xi^{r_1-q}(x)+[u(x,\tau)-k]_{\pm}^{s-1}\cdot$$
$$\cdot\left[\frac{1}{v(x)}\left|\frac{\partial v(x)}{\partial x}\right|\right]^{\varrho}\xi^{r_1}(x)\Bigg\}\,v(x)\,\eta^q(\tau)\,dx\,d\tau$$

where $b'=\max\{b_2,\dots,b_{m-1}\}$.

Choosing a sufficiently small ε we have from inequalites (3.4), (3.10) the estimate

$$\sup_{t_0-\theta\leqslant t\leqslant t_0} \int_{B(x_0,R)} [u(x,t)-k]_\pm^{s+1}\,\eta^q(t)\,\xi^{r_1}(x)\,dx + \tag{3.11}$$

$$+\frac{C'(s+1)\,s}{4}\int_{t_0-\theta}^{t_0}\int_{B(x_0,R)} [u(x,\tau)-k]_\pm^{s-1}\left\{\sum_{|\alpha|=m}|D^\alpha u(x,\tau)|^p+\sum_{|\alpha|=1}|D^\alpha u(x,\tau)|^q\right\}\cdot$$

$$\cdot\,\xi^{r_1}(x)\,\eta^q(\tau)\,v(x)\,dx\,d\tau\leqslant\int_{B(x_0,R)}[u(x,t_0-\theta)-k]_\pm^{s+1}\,\eta^q(t_0-\theta)\,\xi^{r_1}(x)\,dx+$$

$$+C_5\int_{t_0-\theta}^{t_0}\int_{B(x_0,R)}\left\{[u(x,\tau)-k]_\pm^{s+1}\,\xi^{r_1}(x)\,\eta^{q-1}(\tau)\,\frac{d\eta(\tau)}{d\tau}+\right.$$

$$\left.+\frac{1}{(\sigma R)^q}[u(x,\tau)-k]_\pm^{s+q-1}\,\xi^{r_1-q}(x)\,\eta^q(\tau)\,v(x)\right\}dx\,d\tau+$$

$$+C_5\int_{t_0-\theta}^{t_0}\int_{B(x_0,R)}[u(x,\tau)-k]_\pm^{s-a_4}\{1+f(x,t)+\tilde v(x)\}\,\xi^{r_1-a_1m}(x)\,\eta^q(\tau)\,dx\,d\tau$$

where $a_4=\max\{b',\,a_1(m+q)\}$, $\tilde v(x)$ is defined by (1.8).

We will assume that the inequalities

$$s\geqslant m\,,\qquad s>a_5\,,\qquad C'\,s(s+1)\geqslant 4\,,\qquad r_1>a_1m \tag{3.12}$$

are fulfilled. Let us estimate the last integral in (3.11) by Hölder and Young inequalities. We have

$$\int_{t_0-\theta}^{t_0}\int_{B(x_0,R)}[u(x,\tau)-k]_\pm^{s-a_4}\{1+f(x,t)+\tilde v(x)\}\,\xi^{r_1-a_1m}\,\eta^q(\tau)\,dx\,d\tau\leqslant \tag{3.13}$$

$$\leqslant C_6\int_{t_0-\theta}^{t_0}\tilde v A_{k,R}^{\pm}(\tau)\,d\tau+C_6\left\{\int_{t_0-\theta}^{t_0}|A_{k,R}^{\pm}(\tau)|^{r_0'/p_0'}\,d\tau\right\}^{1/\varrho_0'}$$

where $r_0' = r_0/(r_0 - 1)$, $\varrho_0' = \varrho_0/(\varrho_0 - 1)$ and r_0, ϱ_0 are the same as in (2.3). Setting

$$r = r_0' q(1+\kappa), \qquad \varrho = \varrho_0' q(1+\kappa), \qquad \kappa = \frac{\kappa_0 - 1}{\kappa_0}\kappa'$$

we obtain from (3.13) that $u(x, t)$ satisfies the inequality $(2.9)_\pm$.
The proof of the inequality $(2.10)_\pm$ for the solution $u(x, t)$ of the equation (1.1) follows in analogous way after substitution in identity (3.1) a test function

$$\varphi_2(x, t) = \pm \frac{1}{\widetilde{H}^\pm \mp ([u(x, t)]_h - k) + \tilde{\nu}} \left[\ln \frac{\widetilde{H}^\pm}{\widetilde{H}^\pm \mp ([u(x, t)]_h - k) + \tilde{\nu}} \right]_+^s \xi^{r_2}(x)$$

where s, r_2 are sufficiently large numbers, $\xi(x)$ is the same function as in (3.2). $\widetilde{H}^\pm$, $\tilde{\nu}$ are positive numbers satisfying inequalities

$$\operatorname{ess\,sup}\{[[u(x, t)]_h - k]_\pm : (x, t) \in Q(R, \theta)\} \le \widetilde{H}^\pm \le \delta,$$

$$\tilde{\nu} \le \min\{\widetilde{H}^\pm, 1\}, \qquad \delta = \min\left\{\frac{C'}{4C''}, M\right\}.$$

This completes the proof of Theorem 2.3.

4. - Proof of the Theorem 2.2.

First we note some estimates for weighted functions. From the definition of class $A_{1/\kappa_0 + q/n}$ and Hölder inequality the estimate

$$v(B(x_0, R_2)) \le C^{(1)} \left(\frac{R_2}{R_1}\right)^{\frac{n}{\kappa_0} + q} v(B(x_0, R_1)) \tag{4.1}$$

follows for arbitrary positive numbers R_1, R_2 such that $R_1 \le R_2$ with some constant $C^{(1)}$ independent on R_1, R_2.

LEMMA 4.1. - *There exist positive numbers $C^{(2)}$, a_0 such that the inequality*

$$w(B(x_0, R)) \le C^{(2)} R^{a_0 - q} v(B(x_0, R)) \tag{4.2}$$

is valid for arbitrary $R \in (0.1)$, $x_0 \in \Omega$.

PROOF. - From definition of class A_∞ the inequality

$$w(B(x_0, R)) \le C^{(3)} w(B(x_0, 1)) \cdot \left(\frac{|B(x_0, R)|}{|B(x_0, 1)|}\right)^{a'}, \qquad 0 < R < 1 \tag{4.3}$$

follows with some positive numbers $C^{(3)}$, a'. We obtain the inequality (4.2) from (2.2) and (4.3):

$$\frac{w(B(x_0, R))}{w(B(x_0, 1))} = \left[\frac{w(B(x_0, R))}{w(B(x_0, 1))}\right]^{1/\kappa_1} \cdot \left[\frac{w((x_0, R))}{w(B(x_0, 1))}\right]^{1-1/\kappa_1} \leqslant$$

$$\leqslant \left(\frac{C}{R}\right)^q \frac{v(B(x_0, R))}{v(B(x_0, 1))} [C^{(3)} R^{a'}]^{1-1/\kappa_1}$$

and this completes the proof of Lemma 4.1.

Let $u(x, t)$ be an arbitrary function from the class $B_{q,s}(Q_T, v, w)$. In all further considerations we fixe point $(x_0, t_0) \in Q_T$ and define

$$f(R) \equiv f(x_0, R) = \frac{|B(x_0, R)|}{v(x_0, R))}. \tag{4.4}$$

For $0 < R' \leqslant 2R$, positive number λ, ξ we introduce cylinders

$$\begin{cases} Q_\lambda^{(1)}(R', R) = B(x_0, R') \times (t_0 - \lambda[R']^q f(R), t_0), & Q_\lambda^{(1)}(R) = Q_\lambda^{(1)}(R, R), \\ Q_\xi^{(2)}(R', R, \bar{t}) = B(x_0, R') \times (\bar{t} - \xi[R']^q f(R), \bar{t}), & Q_\xi^{(2)}(R, \bar{t}) = Q_\xi^{(2)}(R, R, \bar{t}). \end{cases} \tag{4.5}$$

Further we will assume that

$$\lambda = \left(\frac{K_1}{\omega}\right)^{q-2}, \quad \xi = \left(\frac{2M}{\delta\omega}\right)^{q-2}, \quad t_0 - \lambda R^q f(R) \leqslant \bar{t} - \xi R^q f(R) \leqslant \bar{t} \leqslant t_0, \tag{4.6}$$

K_1 is a sufficiently large number chosen below and such that

$$K_1 > \frac{2M}{\delta}, \quad K_1 > 2M + 1. \tag{4.7}$$

In (4.6) ω is an arbitrary positive number satisfying the inequality

$$\omega \leqslant 2M. \tag{4.8}$$

We assume that R is so small that $Q_\lambda^{(1)}(R) \subset Q_T$.
Denote

$$\begin{cases} \mu_+ = \text{ess sup}\{u(x, t): (x, t) \in Q_\lambda^{(1)}(R)\}, \\ \mu_- = \text{ess inf}\{u(x, t): (x, t) \in Q_\lambda^{(1)}(R)\}. \end{cases} \tag{4.9}$$

The proof of the Theorem 2.2 is based on the assertions formulated below in Propositions 4.1 and 4.2 which will be proved under the following assumptions

$$\text{ess osc}\{u(x, t): (x, t) \in Q_\lambda^{(1)}(R)\} \leqslant \omega, \tag{4.10}$$

$$\omega \geqslant KR^{\sigma_0} \tag{4.11}$$

where K is a sufficiently large positive number, σ_0 is a number defined by the equality

$$\sigma_0 = \tag{4.12}$$

$$= \min\left\{\frac{a_0}{s+q-1}, \frac{n(\kappa_0-1)}{\kappa_0(q-1)}, \frac{n\kappa}{s+q-1+(q-2)\max\{(q(1+\kappa))/r-1, 0\}}\right\}$$

where a_0 is a number introduced in Lemma 4.1, all another parameters were introduced by definition of class $B_{q,s}(Q_T, v, w)$.

PROPOSITION 4.1. – *There exist number $\alpha_0 \in (0, 1)$ depending only on known parameters and positive number K_2 depending on K_1 and known parameters such that from the inequalities* (4.10), (4.11), *with $K = K_2$ and from the inequality*

$$\text{meas}\left\{(x, t) \in Q_\xi^{(2)}(R, \bar{t}): u(x, t) < \mu_- + \frac{\delta\omega}{2M}\right\} \leqslant \alpha_0 |Q_\xi^{(2)}(R, \bar{t})| \tag{4.13}$$

for some $\bar{t} \in [t_0 - (\lambda - \xi) R^q f(R), t_0]$ the estimate

$$\text{ess osc}\left\{u(x, t): (x, t) \in Q_\xi^{(1)}\left(\frac{R}{8}, R\right)\right\} \leqslant \omega\left[1 - \frac{1}{K_2}\right] \tag{4.14}$$

follows.

PROPOSITION 4.2. – *There exists a positive number K_1 depending only on known parameters such that from the inequalities* (4.10), (4.11), *with $K = 2K_1$ and from the inequality*

$$\text{meas}\left\{(x, t) \in Q_\xi^{(2)}(R, \bar{t}): u(x, t) < \mu + \frac{\delta\omega}{2M}\right\} > \alpha_0 |Q_\xi^{(2)}(R, \bar{t})| \tag{4.15}$$

for every $\bar{t} \in [t_0 - (\lambda - \xi) R^q f(R), t_0]$ the estimate

$$\text{ess osc}\left\{u(x, t): (x, t) \in Q_{\lambda'}^{(1)}\left(\frac{R}{2}, R\right)\right\} \leqslant \omega\left(1 - \frac{1}{2K_1}\right) \tag{4.16}$$

follows with $\lambda' = (\alpha_0/2)\lambda$ and with α_0 defined in Proposition 4.1.

We will prove Propositions 4.1, 4.2 in two next section. Now we prove the Theorem 2.2 using these Propositions.

PROOF OF THE THEOREM 2.2. – First we define a number ω which satisfies all assumptions in Propositions 4.1, 4.2 for given small number R. Let

$$\bar{K} = \max\{2K_1, K_2\} \tag{4.17}$$

where K_1, K_2 are defined in Propositions 4.2, 4.1. Introduce a cylinder

$$Q^{(0)}(R) = B(x_0, R) \times \left(t_0 - \left[\frac{K_1}{\overline{K}} \right]^{q-2} \cdot R^{q - \sigma_0(q-2)} f(R),\ t_0 \right) \tag{4.18}$$

and we choose

$$\omega = \text{ess osc} \{ u(x, t) : (x, t) \in Q^{(0)}(R) \} . \tag{4.18}$$

By this we can assume that R is such that $Q^{(0)}(R) \subset Q_T$.

In order to prove the Theorem 2.2 it is sufficient to establish that for an arbitrary number R such that $Q_R^{(0)} \cup Q_\lambda^{(1)}(R) \subset Q_T$ there exists a number $R' \in (R^2, R)$ for which the inequality

$$\text{ess osc} \{ u(x, t) : (x, t) \in Q_{R'}(x_0, t_0) \} \leqslant B'(R')^\beta \tag{4.20}$$

holds where B', β are positive constants dependent only on known parameters.

Let us considered two possibilities:

a) $Q^{(0)}(R) \subseteq Q_\lambda^{(1)}(R)$;

b) $Q^{(0)}(R) \supset Q_\lambda^{(1)}(R)$.

In the case *a*), from the definitions of the cylinders $Q^{(0)}(R)$, $Q_\lambda^{(1)}(R)$ it is followed the inequality

$$\left[\frac{K_1}{\overline{K}} \right]^{q-2} \cdot R^{-\sigma_0(q-2)} \leqslant \left[\frac{K_1}{\omega} \right]^{q-2} .$$

We obtain $\omega \leqslant \overline{K} R^{\sigma_0}$ and from (4.19) and last inequality the estimate (4.20) follows.

In the case *b*) we have the inequalities

$$\omega > \overline{K} R^{\sigma_0} , \tag{4.21}$$

$$\text{ess osc} \{ u(x, t) : (x, t) \in Q_\lambda^{(1)}(R) \} \leqslant \omega \tag{4.22}$$

and consequently ω satisfies all assumptions in Propositions 4.1, 4.2.

We define a number N independent on R such that $N \geqslant 8$ and for $R \in (0, 1)$ the inequalities

$$N^{-\sigma_0} < \theta , \qquad K_1^{q-2} \left(\frac{R}{N} \right)^q f\left(\frac{R}{N} \right) \leqslant \tag{4.23}$$

$$\leqslant \theta^{q-2} \cdot \min \left\{ \left(\frac{1}{8} \right)^q \left(\frac{2M}{\delta} \right)^{q-2} , \left(\frac{1}{2} \right)^q \frac{\alpha_0}{2} K_1^{q-2} \right\} R^q f(R)$$

hold where

$$\theta = \max\left\{1 - \frac{1}{2K_1}, 1 - \frac{1}{K_2}\right\}. \tag{4.24}$$

The possibility of indicated choice of N follows from the extimate

$$\frac{(R/N)^q f(R/N)}{R^q f(R)} = (1/N)^{n+q} \frac{v(B(x_0, R))}{v(B(x_0, R/N))} \leqslant C^{(1)}\left(\frac{1}{N}\right)^{n(1-1/\kappa_0)} \tag{4.25}$$

which is obtained by use of the inequality (4.1).

Now we introduce the sequences

$$R_j = R\left(\frac{1}{N}\right)^{j-1}, \quad \omega_j = \omega\theta^{j-1}, \quad \xi_j = \left(\frac{2M}{\delta\omega}\right)^{q-2}, \quad \lambda_j = \left(\frac{K_1}{\omega_j}\right)^{q-2}, \quad j = 1, 2, \ldots. \tag{4.26}$$

Let us consider two possibilities:

i) for all values of $j = 1, 2, \ldots$ the inequality

$$\omega_j > \overline{K} R_j^{\sigma_0} \tag{4.27}$$

holds;

ii) there exists a number $J \geqslant 2$ such that the inequality (4.27) holds for $j < J$ and

$$\omega_J \leqslant \overline{K} R_J^{\sigma_0}. \tag{4.28}$$

For $j = 1$ the inequality (4.27) is coincided with the inequality (4.21). We will prove that is case *i*) the inequality

$$\operatorname{ess\,osc}\{u(x, t) : (x, t) \in Q_{\lambda_j}^{(1)}(R_j)\} \leqslant \omega_j \tag{4.29}$$

holds for all values of $j = 1, 2, \ldots$. Last inequality for $j = 1$ follows from (4.22).

We prove (4.29) by induction. Assume that (4.29) is valid for $j \leqslant j_0$ and check it for $j = j_0 + 1$. From the inequalities (4.27), (4.29) for $j = j_0$ and Propositions 4.1, 4.2 for the cylinder $Q_{\lambda_{j_0}}^{(1)}(R_{j_0})$ we obtain

$$\operatorname{ess\,osc}\left\{u(x, t) \in Q_{\xi_{j_0}}^{(1)}\left(\frac{R_{j_0}}{8}, R_{j_0}\right) \cap Q_{\lambda'_{j_0}}^{(1)}\left(\frac{R_{j_0}}{2}, R_{j_0}\right)\right\} \leqslant \omega_{j_0+1} \tag{4.30}$$

where $\lambda'_{j_0} = \alpha_0/2)\,\lambda_{j_0}$. Now it is sufficient to remark that from (4.23) the inclusion

$$Q_{\lambda_{j_0+1}}^{(1)}(R_{j_0+1}) \subset Q_{\xi_{j_0}}^{(1)}\left(\frac{R_{j_0}}{8}, R_{j_0}\right) \cap Q_{\lambda'_{j_0}}^{(1)}\left(\frac{R_{j_0}}{2}, R_{j_0}\right) \tag{4.31}$$

holds. So the inequality (4.29) for $j = j_0 + 1$ follows from (4.30), (4.31) and we prove (4.29) for all $j = 1, 2, \ldots$ in case *i*).

In considered case i) the inequality (4.20) follows from (4.29) and the choice of R_j, ω_j. We can assume that $R < 1/N$ and choose the number j_1 such that $R^2 \leqslant R_{j_1} < NR^2$. Then for $\beta_1 = -\ln\theta/\ln N$ we have the estimate

$$\omega_{j_1} = \frac{\omega}{R^{\beta_1}} R_{j_1}^{\beta_1} \leqslant 2MN^{\frac{\beta_1}{2}} R_{j_1}^{\beta_1/2}$$

from which and (4.29) the inequality (4.20) follows.

In case *ii*) from (4.28), (4.26) and the choice of N we have

$$\omega \leqslant \overline{K}R^{\sigma_0}\left[\frac{1}{\theta N^{\sigma_0}}\right]^{J-1} \leqslant \overline{K}R^{\sigma_0}$$

what it impossible by virtue of (4.21).

So we establish the inequality (4.20) and the proof of the Theorem 2.2 is completed.

5. - Proof of the Proposition 4.1.

We formulate at first auxiliarly statements which are connected with weighted functions.

LEMMA 5.1. - *Let weighted functions $v(x)$, $w(x)$ satisfy conditions of the Theorem 2.2 and denote $w_1(x) = 1$, $w_2(x) = v(x)$, $w_3(x) = w(x)$. Then there exists $\kappa(i) > 1$, $i = 1, 2, 3$ and a positive constant $C^{(4)}$ such that the inequality*

$$\frac{1}{w_i(B(x_0, R))}\int\limits_{B(x_0,R)} |g(x)|^{q\kappa(i)} w_i(x)\,dx \leqslant \tag{5.1}$$

$$\leqslant C^{(4)}\left\{\frac{1}{|B(x_0,R)|}\int\limits_{B(x_0,R)} |g(x)|^q\,dx\right\}^{\kappa(i)-1}\cdot$$

$$\cdot\left\{\frac{R^q}{v(B(x_0,R))}\int\limits_{B(x_0,R)}\left|\frac{\partial g(x)}{\partial x}\right|^q v(x)\,dx + \frac{1}{|B(x_0,R)|}\int\limits_{B(x_0,R)}|g(x)|^q\,dx\right\}$$

holds for $i = 1, 2, 3$ for every ball $B(x_0, R) \subset B(x_0, 2R) \subset \Omega$ and every function $g(x) \in L_q(\Omega) \cap W_q^1(\Omega, v)$. By this $\kappa(1) = (2\kappa_0 - 1)/\kappa_0$ where κ_0 is the number from the inequality (4.1).

For $i = 1$ the estimate (5.1) follows from the Poincaré inequality (Theorem 1.3 in [2]) and Hölder inequality. For $i = 2, 3$ the estimate (5.1) follows from Theorem 1.3 in [2], Corollary and remarks in end of Section 1 in [4].

For the same weigthed function $w_i(x)$ as in Lemma 5.1 the inequality

$$\frac{w_i(E)}{w_i(B(x_0, R))} \leq C^{(5)}\left[\frac{|E|}{|B(x_0, R)|}\right]^a, \qquad i=2, 3 \tag{5.2}$$

holds for an arbitrary set $E \subset B(x_0, R) \subset B(x_0, 2R) \subset \Omega$ with positive constants $C^{(5)}$, a independent on E, x_0, R. This inequality follows from Lemma 3 in [3].

LEMMA 5.2. – *There exists a positive number* $\alpha_0 \in (0, 1)$ *depending only on known parameters such that from the inequalities* (4.11), (4.13) *with* $K = K_1$ *the estimate*

$$u(x, t) \geq \mu_- + \frac{\delta\omega}{4M} \quad \textit{for} \quad (x, t) \in Q_\xi^{(2)}\left(\frac{R}{2}, R, \bar{t}\right) \tag{5.3}$$

holds.

PROOF. – We introduce for $j = 1, 2, \ldots$ the following sequences

$$\left\{\begin{array}{l} R(j) = \dfrac{R}{2} + \dfrac{R}{2^j}, \quad \sigma(j)\, R(j) = \dfrac{1}{4}\dfrac{R}{2^j}, \quad \overline{R}(j) = R(j) - \sigma(j)\, R(j), \\ \theta(j) = \xi[R(j)]^q f(R), \quad \overline{\theta}(j) = \xi[\overline{R}(j)]^q f(R), \\ B(j) = B(x_0, R(j)), \quad \overline{B}(j) = B(x_0, \overline{R}(j)), \quad Q(j) = B(j) \times (\bar{t} - \theta(j), \bar{t}), \\ Q(j) = \overline{B}(j) \times (\bar{t} - \overline{\theta}(j), \bar{t}). \end{array}\right. \tag{5.4}$$

For $R = R(j)$, $\sigma = \sigma(j)$, $\theta = \theta(j)$ we write the inequality $(2.9)_-$. The admissibility of this choice of values of parameters is guaranteed by next inequality

$$\text{ess sup}\,\{[u(x, t) - k(j)]_- : (x, t) \in Q(j)\} \leq \frac{\delta\omega}{2M}$$

which ensures that the condition $(2.14)_-$ is satisfied. The function $\eta(t)$ is assumed to be equal to one for $t \geq \bar{t} - \overline{\theta}(j)$ and to zero for $t \leq \bar{t} - \theta(j)$; also we assume that

$$0 \leq \frac{d\eta(t)}{dt} \leq \frac{2^{2+j+q}}{q\xi R^q f(R)}\,.$$

From $(2.9)_-$ and (5.5) we obtain the estimate

$$\xi \cdot \sup_{\bar{t} - \overline{\theta}(j) \leq t \leq \bar{t}} \int_{\overline{B}(j)} [u(x, t) - k(j)]_-^{s+q-1}\, dx + \tag{5.5}$$

$$+ \int\int_{\overline{Q}(j)} [u(x, t) - k(j)]_-^{s-1} \left| \frac{\partial u(x, t)}{\partial x} \right|^q v(x)\, dx\, dt \leq C_7 D_j^-$$

where

(5.6) $$D_j^- = \frac{2^{jq}}{R^q}\left(\frac{\delta\omega}{2M}\right)^{s+q-1}\left[V_j^- + \frac{1}{f(R)} M_j^-\right] + W_j^- + \left[\int_{\bar t - \theta(j)}^{\bar t} [m_j^-(t)]^{r/\varrho}\, dt\right]^{(q(1+\kappa))/r}$$

and

(5.7) $$\begin{cases} V_j^- = \int_{\bar t - \theta(j)}^{\bar t} v(A^-_{k(j),R(j)}(t))\, dt\,, & W_j^- = \int_{\bar t - \theta(j)}^{\bar t} w(A^-_{k(j),R(j)}(t))\, dt\,, \\ M_j^- = \int_{\bar t - \theta(j)}^{\bar t} |A^-_{k(j),R(j)}(t)|\, dt\,, & m_j^-(t) = |A^-_{k(j),R(j)}(t)|\,. \end{cases}$$

In (5.5) and further by C_i we denote constanst depending only on known parameters.

Define a function

(5.8) $$g(x, t) = [u(x, t) - k(j)]_-^{(s-1)/q+1}$$

and apply to this function the inequality (5.1) with $i = 1$. After the integration on t we obtain from (5.5):

(5.9) $$\iint_{\overline{Q}(j)} [u(x, t) - k(j)]_-^{(s+q-1)\kappa(1)}\, dx\, dt \leq C_8\, |\overline{B}(j)|^{1-\kappa(1)} \left[\frac{D_j^-}{\xi}\right]^{\kappa(1)-1} R^q f(R)\, D_j^-\,.$$

Using the Hölder inequality we get

(5.10) $$[k(j) - k(j+1)]^{s+q-1} \cdot M_{j+1}^- \leq \iint_{\overline{Q}(j+1)} [u(x, t) - k(j)]_-^{s+q-1}\, dx\, dt \leq$$

$$\leq \{M_j^-\}^{1-1/\kappa(1)} \cdot \left\{\iint_{\overline{Q}(j)} [u(x, t) - k(j)]_-^{(s+q-1)\kappa(1)}\, dx\, dt\right\}^{1/\kappa(1)}.$$

From the inequalities (5.9), (5.10), (4.11), (4.1) the choice of σ_0 we obtain the estimate

(5.11) $$Y_{j+1}^{(1)} \leq C_9 2^{j(s+2q)} \{Y_j^{(1)}\}^{1-1/\kappa(1)} \{Y_j^{(1)} + Y_j^{(2)} + Y_j^{(3)} + [Y_j^{(4)}]^{1+\kappa}\}$$

where

$$(5.12)\qquad \begin{cases} Y_j^{(1)} = \dfrac{M_j^-}{\xi R^{n+q} f(R)}, & Y_j^{(2)} = \dfrac{V_j^-}{\xi R^q f(R)\, v(B(x_0, R))}, \\[2ex] Y_j^{(3)} = \dfrac{W_j^-}{\xi R^q f(R)\, w(B(x_0, R))}, & Y_j^{(4)} = \dfrac{1}{R^n} \left[\dfrac{1}{\xi} \displaystyle\int_{\bar t - \theta(j)}^{\bar t} [m_j^-(t)]^{r/\varrho}\, dt \right]^{q/r}. \end{cases}$$

Now we apply to the function $g(x, t)$ defined by (5.8) the Lemma 5.1 for $i = 2, 3$. After the integration on t and the use of (5.5) we get the inequality

$$(5.13)\qquad \iint_{\overline{Q}(j)} [u(x, t) - k(j)]_-^{(s+q-1)\kappa(1)}\, w_i(x)\, dx\, dt \leq$$

$$\leq C_9 \frac{w_i(\overline{B}(j))}{|\overline{B}(j)|^{\kappa(i)}} \left[\frac{D_j^-}{\xi} \right]^{\kappa(i)-1} R^q f(R)\, D_j^- .$$

Evaluating analogously to (5.10) only with weighted function $w_i(x)$ and using (5.13), (4.1), (4.11) and the choice of σ_0 we obtain the estimate

$$(5.14)\qquad Y_j^{(i)} \leq C_{10} 2^{j(s+2q)} \{Y_j^{(i)}\}^{1-1/\kappa(i)} \{Y_j^{(1)} + Y_j^{(2)} + Y_j^{(3)} + [Y_j^{(4)}]^{1+\kappa}\}, \qquad i = 2, 3 .$$

In order to evaluate $Y_j^{(4)}$ we use the inequality

$$(5.15)\qquad \int_{B(x_0, R)} |g(x)|^{q\kappa_0}\, dx \leq C^{(6)} |B(x_0, R)|^{1-\kappa_0} \cdot$$

$$\cdot \left\{ \int_{B(x_0, R)} \left[R^q f(R) \left| \frac{\partial g(x)}{\partial x} \right|^q v(x) + |g(x)|^q \right] dx \right\}^{\kappa_0}$$

which follows from the Poincaré inequality (Theorem 1.3 in [2]). We apply the inequality (5.15) to the function $g(x, t)$ defined by the equality (5.8) and use the Hölder inequality. After the integration on t and using of (4.1) we obtain the estimate

$$\int_{\bar t - \bar\theta(j)}^{\bar t} \left\{ \int_{\overline{B}(j)} [u(x, t) - k(j)]_-^{(s+q-1)\varrho/q}\, dx \right\}^{r/\varrho} dt \leq C_{11} [\xi]^{1-r/q} [D_j^-]^{r/q}$$

from which the inequality

$$(5.16)\qquad Y_{j+1}^{(4)} \leq C_{12} 2^{j(s+2q)} \{Y_j^{(1)} + Y_j^{(2)} + Y_j^{(3)} + [Y_j^{(4)}]^{1+\kappa}\}$$

follows.

Inequalities (5.11), (5.14), (5.15) imply that $Y_j^{(i)} \to 0$ as $j \to \infty$, $i = 1, 2, 3, 4$ if the condition

$$Y_j^{(i)} \leqslant \alpha^{(1)}, \qquad i = 1, \ldots, 4 \tag{5.17}$$

is valid with some $\alpha^{(1)}$ which depends only on known parameters. This is possible to verify analogously to the proof of Lemma 5.7, Chapter 2 in [4]. Using the estimate (5.2) we obtain that the condition (5.17) is satisfied by the choice of sufficiently small number α_0 in the inequality (4.13).

From $Y_j^{(1)} \to 0$ the equality

$$\operatorname{meas}\left\{(x, t) \in Q_\xi^{(2)}\left(\frac{R}{2}, R, \bar{t}\right): \mu_- + \frac{\delta\omega}{4M} - u(x, t) > 0\right\} = 0$$

follows. This proves the estimate (5.3) and we conclude the proof of Lemma 5.2.

Define ζ, $\tilde{\theta}$ and H^- by the equalities

$$\begin{cases} t_0 - \zeta\left(\dfrac{R}{2}\right)^q f(R) = \bar{t} - \xi\left(\dfrac{R}{2}\right)^q f(R), \qquad \tilde{\theta} = \zeta\left(\dfrac{R}{2}\right)^q f(R) \\[2ex] H^- = \operatorname{ess\,sup}\left\{\left[u(x, t) - \mu_- - \dfrac{\delta\omega}{4M}\right]_- : (x, t) \in Q_\zeta^{(1)}\left(\dfrac{R}{2}, R\right)\right\} \end{cases} \tag{5.18}$$

Evidently that $H^- \leqslant \delta\omega/4M$.

LEMMA 5.3. – *Assume that the inequality*

$$H^- \geqslant \frac{\delta\omega}{4M} \tag{5.19}$$

is valid and let α_1 be an arbitrary number from the interval (0, 1). *Then there exists a positive number K' depending only on α_1, K_1 and known parameters such that from the inequalities* (4.11), (4.13) *with $K \geqslant K'$ the estimate*

$$\operatorname{meas}\left\{x \in B\left(x_0, \frac{R}{4}\right): u(x, t) < \mu_- + \frac{\omega}{K}\right\} < \alpha_1 \operatorname{mes} B\left(x_0, \frac{R}{4}\right) \tag{5.20}$$

follows for all $t \in [t_0 - \zeta(R/2)^q f(R), t_0]$.

PROOF. – Let β_0 be a number from the interval $(0, 1/4)$ which will be chosen later. We choose

$$K' = \max\left\{K_1, \frac{4M}{\beta_0 \delta}\right\} \tag{5.21}$$

and apply the inequality $(2.10)_-$ for $k = \mu_- + \delta\omega/4M$, $\sigma = 1/2$, $\nu = \beta_0(\delta\omega/4M)$ and the cylinder $Q_\zeta^{(1)}(R/2, R)$. By this choice the integral corresponding to the first summand on the right-hand side of $(2.10)_-$ is equal to zero by the inequality (5.3). We evaluate another integrals corresponding to summands on the right-hand side of $(2.10)_-$

and obtain the inequality

$$(5.22)\qquad \sup_{t_0-\tilde\theta\leq t\leq t_0}\int_{B(x_0,R/4)}\left[\ln\frac{H^-}{H^-+u(x,t)-\mu_-\delta\omega/4M+\beta_0(\delta\omega/4M)}\right]_+^{s+1}dx\leq$$

$$\leq C_{13}\left[\ln\frac{1}{\beta_0}\right]^s\left\{\frac{1}{R^q}\left(\frac{\delta\omega}{4M}\right)^{q-2}\cdot\tilde\theta v\left(B\left(x_0,\frac{R}{2}\right)\right)+\right.$$

$$\left.+\left(\frac{4M}{\beta_0\delta\omega}\right)^b\left[\tilde\theta w\left(B\left(x_0,\frac{R}{2}\right)\right)+\tilde\theta^{q/r(1+\kappa)}R^{nq/\varrho(1+\kappa)}\right]\right\}.$$

From the definition of ζ, the choice of $K(\alpha_1)$ in (5.21) and the inequality (4.11) with $K\geq K'$ the following estimate

$$(5.23)\qquad \zeta\leq\left(\frac{K_1}{\omega}\right)^{q-2}\leq\frac{1}{R^{\sigma_0(q-2)}}$$

holds. Evaluate the right hand side of (5.22) by the use of inequalities (5.23), (4.11) and the choice of σ_0 and we obtain that the right-hand side of (5.22) is not greater than

$$C_{14}(1+K_1^{q-2})R^n.$$

Let us find the lower bound for the integral on the left-hand side of (5.22) by replacing the integration over $B(x_0,R/4)$ by the integration over set $\{x\in\in B(x_0,R/4)\colon u(x,t)<\mu_-+\beta_0\delta\omega/4M\}$. We obtain that the left-hand side of (5.22) is not less than

$$C_{15}\left[\ln\frac{1}{4\beta_0}\right]^{s+1}\operatorname{meas}\left\{x\in B\left(x_0,\frac{R}{4}\right)\colon u(x,t)<\mu_-+\frac{\beta_0\delta\omega}{4M}\right\}.$$

By comparing of the estimates for the left-hand and right-hand sides of the inequality (5.22) we have the inequality

$$(5.24)\qquad \frac{1}{|B(x_0,R/4)|}\operatorname{meas}\left\{x\in B\left(x_0,\frac{R}{4}\right)\colon u(x,t)<\mu_-+\frac{\beta_0\delta\omega}{4M}\right\}\leq$$

$$\leq\frac{C_{16}(1+K_1^{q-2})(\ln 1/\beta_0)^s}{[\ln 1/\beta_0-\ln 4]^{s+1}}$$

whence it follows that the right-hand side of (5.24) can by made less than α_1 for sufficiently small β_0 dependent only on α_1, K_1 and known parameters. This and (5.21) completes the proof of Lemma 5.3.

LEMMA 5.4. – *Assume that the inequality* (5.19) *is valid. Then there exists a positive number K'' dependent only on K_1 and known parameters such that the inequali-*

ties (4.11), (4.13) *with* $K = K''$ *imply the estimate*

$$u(x, t) \geq \mu_- + \frac{\omega}{K''} \quad \textit{for} \quad (x, t) \in Q_\xi^{(1)}\left(\frac{R}{8}, R\right). \tag{5.25}$$

Proof of the Lemma 5.4 it is possible to carry out analogously to the proof of Lemma 5.2 with the use of Lemma 5.3.

PROOF OF THE PROPOSITION 1. – We determine the required number α_0 in the accordance with Lemma 5.2. For the number H^- determined by (5.18) we consider two possibilities:

$$\text{i)} \quad H^- \geq \frac{\delta\omega}{8M}; \qquad \text{ii)} \quad H^- < \frac{\delta\omega}{8M}.$$

In case i) from (4.10) and (5.25) we obtain the inequality

$$\operatorname{ess\,osc}\left\{u(x, t): (x, t) \in Q_\xi^{(1)}\left(\frac{R}{8}, R\right)\right\} \leq \omega\left(1 - \frac{1}{K''}\right). \tag{5.26}$$

In case ii) from (4.10) and (5.18) we have

$$\operatorname{ess\,osc}\left\{u(x, t): (x, t) \in Q_\xi^{(1)}\left(\frac{R}{8}, R\right)\right\} \leq \omega\left(1 - \frac{\delta}{8M}\right). \tag{5.27}$$

We can choose now $K_2 = \max\{K'', 8M/\delta\}$ and the inequality (4.14) follows from (5.26), (5.27). The proof of the Proposition 4.1 is completed.

6. – Proof of the Proposition 2.

We assume in this section that for every $\bar{t} \in [t_0 - (\lambda - \xi) R^q f(R), t_0]$ the inequality (4.15) is valid. Since we assumed that $\delta < M$ then from (4.15) the inequality

$$\operatorname{meas}\left\{(x, t) \in Q_\xi^{(2)}(R, \bar{t}): u(x, t) > \mu_+ - \frac{\delta\omega}{2M}\right\} < (1 - \alpha_0)\,|Q_\xi^{(2)}(R, \bar{t})| \tag{6.1}$$

follows for all cylinders $Q_\xi^{(2)}(R, \bar{t}) \subset Q_\lambda^{(1)}(R)$. From last inequality it is followed the existence $t^* \in [\bar{t} - \xi R^q f(R), \bar{t} - (\alpha_0/2)\, \xi R^q f(R)]$ such that the inequality

$$\operatorname{meas}\left\{x \in B(x_0, R): u(x, t^*) > \mu_+ - \frac{\delta\omega}{2M}\right\} < (1 - \alpha_0^2)\,|B(x_0, R)| \tag{6.2}$$

holds.

We choose further in this section

$$H^+ = \operatorname{ess\,sup}\left\{\left[u(x, t) - \mu_+ + \frac{\delta\omega}{2M}\right]_+ : (x, t) \in Q_\lambda^{(1)}(R)\right\}. \tag{6.3}$$

LEMMA 6.1. – *Assume that the inequality*

$$H^+ > \frac{\delta\omega}{4M} \tag{6.4}$$

is valid. Then there exists a possitive number K''' depending only on known parameters such that from the inequalities (4.11), (6.1) *with $K = K'''$ and every $\bar{t} \in [t_0 - (\lambda - \xi) R^q f(R), t_0]$ the estimate*

$$\operatorname{meas}\left\{x \in B(x_0, R): u(x, t) > \mu_+ - \frac{\omega}{K'''}\right\} < \left(1 - \frac{\alpha_0^2}{2}\right) |B(x_0, R)| \tag{6.5}$$

follows for $t \in (t_0 - \lambda R^q f(R), t_0)$, $\lambda' = \alpha_0 / 2\lambda$.

For the proof the of Lemma 6.1 we employ the inequality $(2.10)_+$ over cylinder $B(x_0, R) \times (t^*, \bar{t})$ and use discussions anologous to the proof of Lemma 5.3. In such way we prove the inequality (6.5) for $t \in (t^*, \bar{t})$. Since $\bar{t}$ is an arbitrary number from the interval $[t_0 - (\lambda - \xi) R^q f(R), t_0]$ we obtain the assertion of Lemma 6.1.

LEMMA 6.2. – *Assume that the inequality* (6.5) *holds. Then for every $\beta_1 \in (0, 1)$ there exists a possitive number $K(\beta_1)$ depending only on known parameters and β_1 such that the condition $K_1 \geq K(\beta_1)$ and the estimate* (4.11) *with $K = K(\beta_1)$ yield the inequality*

$$\operatorname{meas}\left\{(x, t) \in Q_{\lambda'}^{(1)}(R): u(x, t) > \mu_+ - \frac{\omega}{K(\beta_1)}\right\} \leq \beta_1 |Q_{\lambda'}^{(1)}(R)| . \tag{6.6}$$

PROOF. – We write the inequality $(2.9)_+$ for cylinder $Q_{\lambda'}^{(1)}(2R, R)$, $\sigma = 1/2$. We choose a cutoff function $\eta(t)$ such that $\eta(t) \equiv 1$ on $Q_{\lambda'}^{(1)}(R)$, $\eta(t_0 - \lambda'(2R)^q f(R)) = 0$, $0 \leq d\eta(t)/dt \leq 2/\alpha_0 (\lambda R^q f(R))^{-1}$. As for the levels k we take $k = \bar{k}(j) = \mu_+ - \omega/2^j K'''$, $j = 0, 1, 2, \ldots$ where K''' is the number claimed by Lemma 6.1.

Evaluating the terms corresponding to the summands on the right-hand side of $(2.9)_+$ we obtain the inequality

$$\begin{aligned} &\int\int_{Q_{\lambda'}^{(1)}(R)} [u(x, t) - \bar{k}(j)]_+^{s-1} \cdot \left| \frac{\partial u(x, t)}{\partial x} \right|^q v(x)\, dx\, dt \leq \\ &\leq C_{17} \lambda' f(R)\, v(B(x_0, R)) \left\{ \left(\frac{\omega}{2^j}\right)^{s+q-1} + \left(\frac{\omega}{2^j}\right)^{s+1} \left(\frac{\omega}{K_1}\right)^{q-2} + \right. \\ &\left. + R^q \frac{w(B(x_0, R))}{v(B(x_0, R))} + \left[\frac{K_1}{\omega}\right]^{(q-2)[q(1+\kappa)/r - 1]_+} [R^{q + nr/\varrho} f(R)]^{q/r(1+\kappa)} \frac{1}{R^n} \right\}. \end{aligned} \tag{6.7}$$

We will assume that $0 \leq j \leq \bar{J}$ where $\bar{J}$ is sufficient great and chosen further number. And óur choise of $K(\beta_1)$ will be by equality $K(\beta_1) = K''' \cdot 2^{\bar{J}}$. Using inequalities (4.1), (4.2), the condition (2.11) and the choice of σ_0 we evaluate the right hand side of

(6.7) and obtain the estimate

$$(6.8)\qquad \int\int_{Q_{\lambda'}^{(1)}(R)} [u(x,t)-\bar{k}(j)]_+^{s-1} \left| \frac{\partial u(x,t)}{\partial x} \right|^q v(x)\,dx\,dt \leqslant$$

$$\leqslant C_{18}\lambda' f(R)\, v(B(x_0,R)) \left(\frac{\omega}{2^j} \right)^{s+q-1}.$$

Now we introduce a function $\bar{g}(x,t)$ by the equality

$$\bar{g}(x,t) = \min\{[u(x,t)-\bar{k}(j)]_+,\ \bar{k}(j+1)-\bar{k}(j)\}^{s+q-1/q}$$

for $(x,t) \in Q_{\lambda'}^{(1)}(R)$ and apply to this function the inequality

$$(6.9)\qquad \int_{B(x_0,R)} |\bar{g}(x,t)|^{q/2} v(x)\,dx \leqslant C^{(6)} R^{q/2} \int_{B(x_0,R)} \left| \frac{\partial \bar{g}(x,t)}{\partial x} \right|^{q/2} v(x)\,dx\,.$$

Last inequality follows immediately from Theorem 1.6 in [2], the estimate (6.5) and Hölder inequality.

Denote

$$E_j = \{(x,t) \in Q_{\lambda'}^{(1)}(R):\ u(x,t) > \bar{k}(j)\}$$

$$v(E) = \int\int_E v(x)\,dx\,dt \qquad \text{for} \qquad E \subset Q_T\,.$$

Using the Hölder inequality and (6.8) we obtain from (6.9)

$$(6.10)\qquad \int\int_{Q_{\lambda'}^{(1)}(R)} |\bar{g}(x,t)|^{q/2} v(x)\,dx\,dt \leqslant$$

$$\leqslant C_{19}\left\{ v(Q_{\lambda'}^{(1)}(R)) \cdot \left(\frac{\omega}{2^j} \right)^{s+q-1} \right\}^{1/2} \cdot \{v(E_j) - v(E_{j+1})\}^{1/2}.$$

We evaluate the left hand side of (6.10) below by

$$[\bar{k}(j+1)-\bar{k}(j)]^{1/2(s+q-1)}\, v(E_{j+1}) = \left[\frac{\omega}{2^{j+1}K'''} \right]^{1/2(s+q-1)} v(E_{j+1})\,.$$

So we obtain from (6.10) the inequality

$$[v(E_{j+1})]^2 \leqslant C_{20}\, v(Q_{\lambda'}^{(1)}(R)) \cdot \{v(E_j) - v(E_{j+1})\}\,.$$

We add last inequalities for $j = 0, 1, \ldots, \bar{J}-1$ and we obtain

$$\bar{J}[v(E_{\bar{J}})]^2 \leqslant C_{21}[v(Q_{\lambda'}^{(1)}(R))]^2$$

from which it is followed the estimate

$$v\left\{(x,t)\in Q_{\lambda'}^{(1)}(R):\, u(x,t)>\mu_+-\frac{\omega}{K(\beta_1)}\right\}\leqslant\left\{\frac{C_{21}}{\bar{J}}\right\}^{1/2}v(Q_{\lambda'}^{(1)}(R))\,. \tag{6.11}$$

From the definition of class $A_{1+q/n}$ it is simple to obtain the inequality

$$|E|\leqslant C^{(7)}\cdot\left[\frac{v(E)}{v(B(x_0,R))}\right]^{n/(n+q)}\cdot|B(x_0,R)| \tag{6.12}$$

for an arbitrary set $E\subset B(x_0,R)$ with a constant $C^{(7)}$ independent on x_0, R, E. The assertion of the Lemma 6.2 follows from (6.11), (6.12) if we take $\bar{J}$ so large that it is valid the estimate

$$C^{(7)}\left\{\frac{C_{21}}{\bar{J}}\right\}^{n/(2(n+q))}<\beta_1\,.$$

LEMMA 6.3. – *Assume that the inequality* (6.5) *holds. Then there exists a constant* K_1 *depending only on known parameters such that the estimate* (4.11) *with* $K=2K_1$ *yields the inequality*

$$u(x,t)\leqslant\mu_+-\frac{\omega}{2K_1}\quad \textit{for}\quad (x,t)\in Q_{\lambda'}^{(1)}\left(\frac{R}{2},R\right). \tag{6.13}$$

The proof of Lemma 6.3 is analogous to the proof of Lemma 5.2.
Using Lemmas 6.1-6.3 we prove the Proposition 4.2 analogously to the proof of the Proposition 4.1.

7. – Example.

In this section we give the example of weighted function $v(x)$ of such that precending assumptions are satisfied. We take

$$v(x)=v_d(x_0)=|x-x_0|^d \tag{7.1}$$

where x_0 is some point of domain Ω, $q-n<d<\min\{q,n(p-1)\}$. In this case the function $v_d(x)$ belongs to the class $A_{1+q/n}$ and the function $\tilde{v}_d(x)$ defined by $v_d(x)$ in accordance with (1.8) is integrable.

Now we verify that the condition v_2 is satisfied. It is simple to verify that $\tilde{v}_d(x)\in A_\infty$. In order to satisfy the inequality (2.2) it is sufficient to choose κ_1 such that the estimate

$$\frac{n+d}{q}\left(1-\frac{1}{\kappa_1}\right)+\frac{\varrho}{q\kappa_1}\leqslant 1 \tag{7.2}$$

holds. This is possible because the inequality $\varrho < q$ follows from the condition $q_1 > mp$.

Consequently from Theorem 3.1 in [7] and Theorem 2.1 we obtain next result.

THEOREM 7.1. – *Assume that conditions* (1.2)-(1.5), (2.3), (2.4) *and satisfied with the function* $v(x) = v_d(x)$ *defined by* (7.1) *and with* $\kappa_0 = n\{\max(n-q, n+d-\varrho)\}$. *Assume that the inequality* $q - n < d < \min\{q, n(p-1)\}$ *is valid. Then an arbitrary generalized solution of the equation* (1.1) *is locally Hölder continuous.*

REFERENCES

[1] E. DI BENEDETTO, *On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients*, Ann. Sc. Norm. Sup., **13** (1986), no. 3, pp. 485-535.

[2] S. CHANILLO - R. L. WHEEDEN, *Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions*, Amer. Journ. Math., **107** (1985), pp. 1191-1226.

[3] R. R. COIFMAN - C. FEFFERMAN, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math., **51** (1974), pp. 241-250.

[4] C. GUTIERREZ - R. L. WHEEDEN, *Sobolev interpolation inequalities with weights*, Trans. Amer. Math. Soc., **323** (1990), pp. 263-281.

[5] O. A. LADYZHENSKAYA - V. A. SOLONNIKOV - N. N. URAL'ZEVA, *Linear and Quasilinear Parabolic Equations*, Nauka, Moscow (1967).

[6] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal functions*, Trans. Amer. Math. Soc., **165** (1972), pp. 207-226,

[7] F. NICOLOSI - I. V. SKRYPNIK, *On existence and boundedness of solution for degenerate quasilinear parabolic equations of higher order*, Commun. Part. Differ. Equat. (to appear).

[8] F. NICOLOSI - I. V. SKRYPNIK, *On the behaviour near the boundary of the solutions of nonlinear degenerate parabolic equations of higher order*, Nonlinear Analysis: Theory, Meth. Appl. (to appear).

[9] I. I. SKRYPNIK, *On Hölder continuity of solution for degenerate parabolic equations, weighted case*, Ukrainian Math. Journ. (to appear).

[10] I. V. SKRYPNIK, *On quasilinear papabolic higher-order equations with Hölder solutions*, Different. Equations, **29** (1993) no. 3, pp. 501-514.

[11] I. V. SKRYPNIK, *On the Hölder property for functions from the class* $B_{q,s}$, Ukrainian Math. Journ., **45** (1993), no. 7, pp. 1020-1029.