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HÖLDER DOMAINS AND POINCARÉ DOMAINS

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ABSTRACT. A domain $D \subset R^d$ of finite volume is said to be a p-Poincaré domain if there is a constant $M_p(D)$ so that

$$\int_D |u - u_D|^p \, dx \le M_p^p(D) \int_D |\nabla u|^p \, dx$$

for all functions $u \in C^1(D)$. Here u_D denotes the mean value of u over D. Techniques involving the quasi-hyperbolic metric on D are used to establish that various geometric conditions on D are sufficient for D to be a p-Poincaré domain. Domains considered include starshaped domains, generalizations of John domains and Hölder domains. D is a Hölder domain provided that the quasi-hyperbolic distance from a fixed point $x_0 \in D$ to x is bounded by a constant multiple of the logarithm of the euclidean distance of x to the boundary of D. The terminology is derived from the fact that in the plane, a simply connected Hölder domain has a Hölder continuous Riemann mapping function from the unit disk onto D. We prove that if D is a Hölder domain and $p \ge d$, then D is a p-Poincaré domain. This answers a question of Axler and Shields regarding the image of the unit disk under a Hölder continuous conformal mapping. We also consider geometric conditions which imply that the imbedding of the Sobolev space $W^{1,p}(D) \to L^p(D)$ is compact, and prove that this is the case for a Hölder domain D.

1. INTRODUCTION

We consider proper open connected subdomains D of euclidean d-space R^d, $d \ge 2$. Following [GO] we define the quasi-hyperbolic metric k_D in D by

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta_D(x)} \tag{1.1}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. Here we denote by $\delta_D(x)$ the euclidean distance between x and ∂D.

Fix a point $x_0 \in D$. We say that D is a Hölder domain if

$$k_D(x_0, x) \le c_1 \log \frac{\delta_D(x_0)}{\delta_D(x)} + c_2, \qquad x \in D, \tag{1.2}$$

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holds for some finite constants c_1, c_2. Our terminology is motivated by a result of Becker and Pommerenke.

Suppose that D is a bounded simply connected domain in R^2. A theorem in [BP] implies that D is a Hölder domain if and only if there is a Riemann mapping function g satisfying the Hölder condition

$$(1.3) \qquad |g(z_1) - g(z_2)| \le c|z_1 - z_2|^\alpha, \qquad |z_1| \le 1, \quad |z_2| \le 1,$$

for some $\alpha > 0$. In fact, if k_D is replaced by the comparable hyperbolic metric then the exponent $\alpha = c_1^{-1}$. See [SS1] for a localization of this. In [GM] Gehring and Martio established an R^d version of the Becker and Pommerenke result by showing that a domain D is the image of a ball under a Hölder continuous k-quasiconformal mapping if and only if condition (1.2) holds. Domains satisfying (1.2) are said to satisfy a quasi-hyperbolic boundary condition in [GM].

On the otherhand, Hölder domains are closely related to the BMO-Sobolev extension domains studied by Jones in [J1], [J2] and uniform domains. In [GO], Gehring and Osgood show that Jones' extension domains are equivalent to uniform domains. Another closely related type of domain is the John domain. Fix a point $x_0 \in D$, we say that D is a John domain provided that for each $x_1 \in D$ there is an arc γ joining x_0 to x_1 in D along which

$$(1.4) \qquad \delta_D(x) \ge \alpha|\gamma(x, x_1)|, \qquad x \in \gamma.$$

Here α is a positive constant, $\gamma(x, x_1)$ is the portion of γ joining x to x_1 and $|\gamma(x, x_1)|$ is its arc length.

The definition of uniform domains given in [GO] shows that uniform domains are John domains, but not conversely. An elementary exercise shows that John domains are Hölder domains. But the thickness condition (1.4), which can be visualized as a twisted cone condition, does not hold in general for Hölder domains. In [SS1], an example of a Hölder domain is constructed which contains a sequence of tubes of width $\varepsilon_n > 0$ and length $\varepsilon_n \log \varepsilon_n^{-1}$ where ε_n tends to zero. Thus, (1.4) is violated and hence Hölder domains are not necessarily John domains. See also the example in [BP].

Our interest in Hölder domains is motivated by a question of Axler and Shields [AS]. Suppose that g is a Riemann mapping function mapping the unit disk onto $D \subset R^2$ and satisfying (1.3). They asked whether D necessarily satisfied the analytic Poincaré inequality

$$(1.5) \qquad \iint_D |F|^2\, dxdy \le M \iint_D |F'|^2\, dxdy$$

whenever F is holomorphic in D and vanishes at $g(0) \in D$. Here M is a finite constant. Our main result provides an affirmative answer to this question. This result was known to be true provided the α in (1.3) was greater than $\frac{1}{2} - \frac{1}{320}$.

Let $D \subset R^d$ be a domain with finite volume, $m(D) < \infty$, and $1 \le p < \infty$. We denote by $W^{1,p}(D)$ the usual Sobolev space of functions on D that together with their first order weak partial derivatives are in $L^p(D)$. The norm for $W^{1,p}(D)$ is given by

$$\|u\|_{W^{1,p}(D)} = \left(\int_D |u|^p\,dx + \int_D |\nabla u|^p\,dx\right)^{1/p}.$$

We say that D is a p-Poincaré domain provided

$$\sup_u \frac{\int_D |u - u_D|^p\,dx}{\int_D |\nabla u|^p\,dx} = M_p^p(D) < \infty \tag{1.6}$$

holds, where the supremum is taken over all nonconstant functions $u \in W^{1,p}(D)$. Here u_D denotes the average of u over D, $u_D = \frac{1}{m(D)}\int u\,dx$. Meyers and Serrin [MS] have shown that $C^1(D)$ is dense in $W^{1,p}(D)$, so one only needs to consider functions in $C^1(D)$ to establish that a domain D is a p-Poincaré domain. Hamilton [H] has shown that for simply connected planar domains D of finite area, the analytic Poincaré inequality (1.5) is equivalent to (1.6) for $p = 2$.

Therefore, the Axler-Shields question is answered by our main results:

Theorem 1. *If $D \subset R^d$ is a Hölder domain, then D is a p-Poincaré domain for all $p \geq d$.*

Theorem 2. *If a domain D is a Hölder domain, then*

$$\int_D k_D^p(x_0, x)\,dx < \infty \tag{1.7}$$

for all $p < \infty$.

The restriction $p \geq d$ is necessary, as we show by an example at the end of §10. Nevertheless, it is surprising compared to a recent result of Martio [Mar] where he proves that John domains are p-Poincaré domains for all $p \geq 1$. On the otherhand, this restriction compares favorably with a result of Staples [S] that L^p-averaging domains are p-Poincaré domains for $p \geq d$. In fact, condition (1.7) implies that Hölder domain are L^p-averaging domains for all $p \geq 1$, see [S]. The proofs of Theorem 1 and Theorem 2 appear in §4, while preliminary work is contained in §§2 and 3.

§§5, 6 and 7 contain additional conditions which are shown to be sufficient for the Poincaré inequality to hold for a domain D. We show in §5 that a bounded starshaped domain is a p-Poincaré domain for $p \geq 1$, while in §7 generalizations of John domains that have cusps are considered. In §6 a Whitney decomposition of the domain D along with a family of curves in D is used to obtain an estimate of $M_p(D)$. This estimate involves integration over the "shadow" of an arbitrary Whitney cube with respect to the curve family. In §7 we introduce the k_p metric, which is a generalization of the quasi-hyperbolic metric, and we use this in our study of p-Poincaré domains.

The imbedding $W^{1,p}(D) \to L^p(D)$ is studied in §8. We show that some of our sufficient conditions for the p-Poincaré inequality to hold actually imply the stronger result that this imbedding is compact. In particular this is shown to be the case when D is a Hölder domain. An example showing that this is not true in general is given in §9.

The final section concerns a class of domains with simple geometry. This class contains the "rooms and corridors" type domains that have been used by several authors to study the Poincaré inequality. In §10 we use the k_p metric to provide a complete description of such domains for which the Poincaré inequality holds, and we partially characterize those for which the imbedding $W^{1,p}(D) \to L^p(D)$ is compact.

2. LENGTHS OF GEODESICS IN HÖLDER DOMAINS

Let D be a Hölder domain. Definition (1.2) does not make it clear that D is even a bounded domain. Let $x_1 \in D$, then by Lemma 1 in [GO] there exists a quasi-hyperbolic geodesic γ joining x_0 to x_1. Thus, γ is a rectifiable arc in D and

$$k_D(y_1, y_2) = \int_{\gamma(y_1, y_2)} \frac{ds}{\delta_D(x)}$$

for each pair of points $y_1, y_2 \in \gamma$. We will show that these geodesics are bounded in length by a multiple of $\delta_D(x_0)$.

Theorem 3. *Suppose that D satisfies (1.2). Then there is a finite constant c_3 so that whenever γ is a quasi-hyperbolic geodesic joining x_0 to x_1 in D, the inequality*

$$k_D(x_0, x) \le c_1 \log \frac{\delta_D(x_0)}{|\gamma(x, x_1)|} + c_3 \tag{2.1}$$

holds for all $x \in \gamma$.

Proof. Fix $x_1 \in D$ and let γ be a quasi-hyperbolic geodesic joining x_0 to x_1 in D. Assume that (2.1) is false, so that for every $c_3 < \infty$, there exists an $a_0 \in \gamma$ with $L = |\gamma(a_0, x_1)|$ and satisfying

$$c_1 \log \frac{\delta_D(x_0)}{L} + c_3 < k_D(x_0, a_0). \tag{2.2}$$

We will show that this is impossible if c_3 is sufficiently large.

Define $a_k \in \gamma(a_{k-1}, x_1)$ by $|\gamma(a_{k-1}, a_k)| = 2^{-k}L$, where $k = 1, 2, \dots$. Let $\lambda_k = \sup\{\delta_D(x)/L \mid x \in \gamma(a_k, x_1)\}$, where $k \ge 0$. Combining (1.2) and (2.2) we get that

$$c_1 \log \frac{\delta_D(x_0)}{L} + c_3 < c_1 \log \frac{\delta_D(x_0)}{\delta_D(x)} + c_2$$

for all $x \in \gamma(a_0, x_1)$ and hence that $\lambda_0 \le \exp(\frac{c_2 - c_3}{c_1})$. We choose c_3 large enough so that $c_3 > c_2$ and $\lambda_0 < 1/2$.

We prove by induction that $\lambda_{k-1} \le \lambda_0^k$. This is trivially true for $k = 1$ so assume that it is true for some $k \ge 1$. If $x \in \gamma(a_k, x_1)$, then by (2.2)

$$\begin{aligned} c_1 \log \frac{\delta_D(x_0)}{L} + c_3 + \frac{2^{-k}}{\lambda_{k-1}} &\le \int_{\gamma(x_0, a_0)} \frac{ds}{\delta_D(y)} + \int_{\gamma(a_{k-1}, a_k)} \frac{ds}{\delta_D(y)} \\ &\le \int_{\gamma(x_0, x)} \frac{ds}{\delta_D(y)} = k_D(x_0, x) \\ &\le c_1 \log \frac{\delta_D(x_0)}{\delta_D(x)} + c_2 \end{aligned}$$

and hence

$$\frac{c_1^{-1}}{(2\lambda_0)^k} \le \frac{c_1^{-1} 2^{-k}}{\lambda_{k-1}} \le \log \frac{1}{\lambda_k}. \tag{2.3}$$

Now, c_3 can be chosen (depending only on c_1) so that λ_0 is so small that

$$\log \frac{1}{\lambda_0^{k+1}} \le \frac{c_1^{-1}}{(2\lambda_0)^k}, \qquad k \ge 1. \tag{2.4}$$

Combining (2.3) and (2.4) we see that $\lambda_k \le \lambda_0^{k+1}$ and hence the induction hypothesis is satisfied. Since $0 < \delta_D(x_1) \le L\lambda_k \le L\lambda_0^{k+1}$ holds for all $k \ge 1$ and $\lambda_0 < 1$ we have a contradiction which proves the theorem.

Corollary 1. *Suppose that D is Hölder domain and that γ is a quasi-hyperbolic geodesic joining x_0 to x_1 in D. Then*

(a) *There is a constant $c_4 = c_4(c_1, c_2)$ so that $|\gamma| \le c_4 \delta_D(x_0)$. In particular, D is bounded.*

(b) *There is a constant $c_5 = c_5(c_1, c_2)$ so that whenever γ_1 is a subarc of γ and $L = |\gamma_1|/2\delta_D(x_0)$, then*

$$\frac{L}{\log \frac{1}{L}} \le c_5 \max\left\{ \frac{\delta_D(x)}{\delta_D(x_0)} \,\middle|\, x \in \gamma_1 \right\}. \tag{2.5}$$

Proof. The proof of (a) follows immediately from (2.1). To prove (b), assume that $\gamma_1 = \gamma(a, c)$ and that $b \in \gamma_1$ satisfies $|\gamma(a, b)| = |\gamma_1|/2$. Let $\lambda = \max\{\delta_D(x)/\delta_D(x_0) \mid x \in \gamma_1\}$. It follows from (2.1) that

$$\frac{L}{\lambda} = \frac{|\gamma_1|}{2\lambda\delta_D(x_0)} \le k(a, b) \le c_1 \log \frac{2\delta_D(x_0)}{|\gamma_1|} + c_3$$

and hence that

$$\frac{L}{\log \frac{1}{L}} \le \left[c_1 + \frac{c_3}{\log \frac{1}{L}} \right] \lambda \le \left[c_1 + \frac{c_3}{\log \frac{2}{c_4}} \right] \lambda = c_5 \lambda$$

which is (2.5). This completes the proof.

We observe that (2.5) places a restriction on the length of a tube in D. If a tube has a diameter of ε, then its length is $O(\varepsilon \log \varepsilon^{-1})$.

3. THE EXPONENTIAL INTEGRABILITY OF k_D IN JOHN DOMAINS

Let $W = \{Q\}$ be a Whitney decomposition of D into closed dyadic cubes with disjoint interiors. This means that the coordinates of the vertices of each cube are dyadic rational numbers and that the diameter of each cube $Q \in W$, which we denote by $d(Q)$, is comparable to its distance to ∂D. See Chapter 6 of Stein's book [St] for the existence of such a decomposition. Let x_0 be the center of some fixed cube $Q_0 \in W$. Suppose that $Q \in W$, $Q \neq Q_0$ and that x_Q is the center of Q. If $x \in Q$, then $k_D(x_0, x)/k_D(x_0, x_Q)$ is bounded from above and from below by positive constants (depending on W, which in turn only depends on d). We use the notation $a \approx b$ and $a \preceq b$ to denote that a, b are either comparable or satisfy an inequality with a constant depending only on the dimension d.

Lemma 1. *Suppose that* $a_n > 0$ *and that*

$$\sum_{k=n}^{\infty} a_k \leq c a_n, \qquad n = 1, 2, \ldots, \tag{3.1}$$

then $a_n \leq 2ca_1 \exp(-n/2c)$.

Proof. We compute for $k \geq 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^k a_n \leq k \sum_{n=1}^{\infty} a_n \sum_{m=1}^{n} m^{k-1} &= k \sum_{m=1}^{\infty} m^{k-1} \sum_{n=m}^{\infty} a_n \\ &\leq ck \sum_{m=1}^{\infty} m^{k-1} a_m\end{aligned}$$

and hence $\sum n^k a_n \leq c^{k+1} k! a_1$. Therefore,

$$\sum_{n=1}^{\infty} e^{n/2c} a_n \leq \sum_{k=0}^{\infty} \frac{ca_1}{2^k} \leq 2ca_1$$

and the result follows.

In order to provide a clearer picture of our proof of Theorem 2 we first consider the simpler case of John domains. The following theorem can also be derived using results in [MV].

Theorem 4. *If D is a John domain, then*

$$\int_D e^{\tau k_D(x_0, x)}\, dx < \infty$$

for some $\tau > 0$.

Proof. Assume that (1.4) holds and put $\beta = \delta_D(x_0)/\alpha$. Then $|\gamma| \leq \beta$ for all arcs γ satisfying (1.4). Since W is a Whitney decomposition we can choose

$a > 0$ so that

$$a \cdot \max(d(Q_1), d(Q_2)) < \min(d(Q_1), d(Q_2)) \tag{3.3}$$

whenever $Q_1, Q_2 \in W$ and $Q_1 \cap Q_2$ is nonempty. Let $D_n = \bigcup\{Q \in W \mid a^n < d(Q)/2\beta \leq a^{n-1}\}$ for $n = 1, 2, \ldots$.

Since $D \subset B(x_0, \beta)$ we see that D is the disjoint union of the D_n's and by making the a in (3.3) smaller we may assume that $x_0 \in D_1$. Suppose that $x_1 \in \bigcup_n^\infty D_k$ and that γ is an arc in D joining x_0 to x_1 along which (1.4) holds. Because of (3.3), there must be a point $x \in \gamma$ and a cube $Q \in W$ with $x \in Q \subset D_n$. Since $\delta_D(x) \approx \delta_D(x_Q)$, there is a dilation $\tilde{Q}$ of Q with $m(\tilde{Q}) \approx m(Q)/\alpha^d$ and so that $x_1 \in \tilde{Q}$.

Let $\tilde{D}_n = \bigcup\{\tilde{Q} \mid Q \in W, Q \subset D_n\}$. Then we have shown that

$$\sum_{k=n}^{\infty} m(D_k) = m\left(\bigcup_n^{\infty} D_k\right) \leq m(\tilde{D}_n) \preceq \frac{m(D_n)}{\alpha^d}.$$

Applying Lemma 1 we see that

$$m(D_n) \leq c\varepsilon^n, \qquad n = 1, 2, \ldots,$$

for some constant c and $0 < \varepsilon < 1$. Since D must also be a Hölder domain we have that for $x \in Q \subset D_n$,

$$k_D(x_0, x) \leq c_1 \log \frac{\delta_D(x_0)}{\delta_D(x)} + c_2 \leq c_1 \log \frac{2\beta}{d(Q)} + c_3 \leq c_4 n$$

where the c_i's are appropriate constants.

Finally, combining the above estimates we obtain that

$$\int_D e^{\tau k_D(x_0, x)}\, dx = \sum_{n=0}^{\infty} \int_{D_n} e^{\tau k_D(x_0, x)}\, dx \leq \sum_{n=0}^{\infty} c e^{\tau c_4 n} \varepsilon^n < \infty$$

provided τ is sufficiently small. This completes the proof.

In [J1] it is essentially shown that $m(\partial D) = 0$ whenever D is a John domain. Later Martio and Vourinen showed the stronger result that the Hausdorff dimension of ∂D is less than d. Carl Sundberg observed that this result follows from Theorem 4 and we thank him for allowing us to include this corollary.

Corollary 2 [MV, Corollary 6.4]. *If D is a John domain, then the Hausdorff dimension of ∂D is less than d.*

Proof. Assume that D is a John domain with fixed point $x_0 \in D$ and α satisfying (1.4). We cover ∂D with a collection of balls $B(y_1, r_1), \ldots, B(y_n, r_n)$, where each $y_i \in \partial D$. Assume that r_i is small enough so that $x_0 \notin B(y_i, 3r_i)$. Now it follows from (1.4) that there is a ball $B(x_i, \alpha r_i) \subset B(y_i, 3r_i) \cap D$. Hence $m(B(y_i, 3r_i) \cap D) \approx r_i^d$.

From Lemma 2.1 in [GP] we have

$$\left|\log \frac{\delta_D(x_0)}{\delta_D(x)}\right| \leq k_D(x_0, x), \qquad x \in D, \tag{3.4}$$

and it follows that

$$r_i^{-\tau} \leq \left(\frac{3}{\delta_D(x_0)}\right)^{\tau} e^{\tau k_D(x_0, x)}, \qquad x \in B(y_i, 3r_i) \cap D. \tag{3.5}$$

By a standard covering lemma, we can find disjoint balls $\{B(y_{n_i}, 3r_{n_i})\}$ whose triples cover ∂D. Integrating (3.5) over $B(y_{n_i}, 3r_{n_i}) \cap D$ yields that

$$\begin{aligned}\sum_i (9r_{n_i})^{d-\tau} &\preceq \sum_i \int_{B(y_{n_i}, 3r_{n_i}) \cap D} e^{\tau k_D(x_0, x)}\, dx \\ &\leq \int_D e^{\tau k_D(x_0, x)}\, dx.\end{aligned}$$

By the theorem, $\tau > 0$ can be chosen small enough so that this last integral is convergent. Hence the dimension of ∂D is no greater than $d - \tau$ and the proof is complete.

4. THE L^p-INTEGRABILITY OF K_D IN HÖLDER DOMAINS

We continue to assume that W is a Whitney decomposition of D with x_0 the center of some cube Q_0. In order to prove Theorem 2 and consequently Theorem 1, we modify the argument given in §3 for John domains. The idea is that while D is not a John domain, it still will have $\delta_D(x)$ comparable to $|\gamma(x, x_1)|$ at many points along a quasi-hyperbolic geodesic γ joining x_0 to x_1 in D.

We start with a modification of Lemma 1 where the right-hand side is replaced with an average.

Lemma 2. *Suppose that* $a_k \geq 0$ *and that*

$$\sum_{k=4n}^{\infty} a_k \leq \frac{c_1}{n} \sum_{k=n}^{4n-1} a_k, \qquad n = 1, 2, \ldots, \tag{4.1}$$

for a finite constant c_1. *Then there is a finite constant* $c_2 \preceq c_1$ *so that for all* $p \geq 1$:

$$\sum_{k=1}^{\infty} k^p a_k \leq c_2^p 2^{p^2} (a_1 + a_2 + a_3). \tag{4.2}$$

Proof. Let $b_n = \sum\{a_k \mid 4^n \leq k < 4^{n+1}\}$ for $n = 0, 1, \ldots$. By hypothesis,

$$b_n \leq \sum_{k=4^n}^{\infty} a_k \leq \frac{c_1}{4^{n-1}} b_{n-1}, \qquad n > 0.$$

For integers $p > 0$, we therefore have that

$$\sum_{k=0}^{\infty} 4^{pk} b_k \leq b_0 + c_1 \sum_{k=1}^{\infty} 4^{pk+1-k} b_{k-1} \leq 4^{p+1} c_1 \sum_{k=0}^{\infty} 4^{(p-1)k} b_k$$

assuming that $c_1 \geq 1$. Hence

$$\sum_{k=0}^{\infty} k^p a_k \leq \sum_{n=0}^{\infty} 4^{(n+1)p} b_n \leq 4^p c_1^{p+1} 2^{(p+1)(p+2)} b_0 \leq 2^{p^2} c_2^p b_0$$

and the proof is complete.

Lemma 3. *Suppose that D is a Hölder domain, that $x_1 \in D$ and that γ is a quasi-hyperbolic geodesic joining x_0 to x_1 in D. If $\delta_D(x_1) \leq \delta_D(x_0) 2^{-3n}$ for some integer $n \geq 1$, then there are n distinct integers $m_1, \dots, m_n$, with $n \leq m_i \leq 3n$, and points $y_1, \dots, y_n$ in D so that*

$$|\gamma(y_i, x_1)| \approx 2^{-m_i} \delta_D(x_0), \qquad |\delta_D(y_i)| \approx 2^{-m_i} \delta_D(x_0). \tag{4.3}$$

Proof. Assume that $\delta_D(x_0) = 1$. Choose consecutive points $a_0, \dots, a_{2n}$ satisfying

$$|\gamma(a_m, x_1)| = 2^{-(n+m)}, \qquad m = 0, \dots, 2n. \tag{4.4}$$

This is always possible because if $|\gamma| < 2^{-n}$, then $1 = \delta_D(x_0) \leq 2^{-n} + 2^{-3n} < 1$. Let I_m be the interval $\gamma(a_{m-1}, a_m)$ for $m = 1, \dots, 2n$.

Suppose that $\delta_D(x) \leq \beta |\gamma(x, x_1)|$ for all $x \in I_m$ where $m = m_1, \dots, m_n$ and the $\{m_i\}$ are distinct integers in the interval $[1, 2n]$. Using Theorem 3, it follows that

$$\begin{aligned} \frac{n}{\beta} \log 2 = \sum_{i=1}^{n} \int_{2^{-n-m_i}}^{2^{-n-m_i+1}} \frac{ds}{\beta s} &\leq \sum_{i=1}^{n} \int_{I_{m_i}} \frac{ds}{\delta_D(x)} \\ &< k_D(x_0, a_{2n}) \leq c_1 \log |\gamma(a_{2n}, x_1)|^{-1} + c_3 \leq c_4 n \end{aligned}$$

and hence that $\beta > c_4^{-1} \log 2$. In otherwords, if we choose $\beta = c_4^{-1} \log 2$, then we cannot find n of these intervals.

Thus, there must be n distinct intervals $I_{m_1}, \dots, I_{m_n}$ which contain a point y_i satisfying $\delta_D(y_i) \geq \beta |\gamma(y_i, x_1)|$. We therefore have a lower bound for $\delta_D(y_i)$ which is comparable to 2^{-n-m_i}. On the other hand, we have that $\delta_D(y_i)$ can not exceed $|\gamma(y_i, x_1)| + \delta_D(x_1) \preceq 2^{-n-m_i}$. This proves (4.3) and the proof is complete.

Proof of Theorem 2. We continue to assume that $\delta_D(x_0) = 1$. Define

$$D_m = \bigcup \left\{ Q \in W \mid \frac{b^{-1}}{2^m} \leq d(Q) \leq \frac{b}{2^m} \right\}, \qquad m = 1, 2, \dots, \tag{4.5}$$

where b is chosen large enough so that $D = \bigcup D_m$, which is possible since D is bounded, and so that the points y_i satisfying (4.3) in Lemma 3 must belong to D_{m_i}. This can be done since W is a Whitney decomposition. We also choose b large enough so that $m(D_m) > 0$ for all m.

Let $Q \in W$, then one easily sees that there is only a fixed number of sets D_m that Q can belong to and hence the function $\sum_m \chi_{D_m}$ is bounded on D. Here χ_{D_m} denotes the characteristic function of the set D_m.

Let $x_1 \in Q \in W$ and suppose that $Q \subset \bigcup_{3n+n_0}^{\infty} D_m$. Since $\delta_D(x_1) \approx d(Q) \le b2^{-3n-n_0}$ we can choose n_0 so that $\delta_D(x_1) \le 2^{-3n}$. Hence by Lemma 3 we can find n distinct integers $m_1, \dots, m_n$ with $n \le m_i \le 3n$ and points $y_i \in Q_i \in W$ satisfying (4.3), where $Q_i \subset D_{m_i}$. Using the constants in (4.3) we see that there is a dilation $\tilde{Q}_i$ of Q_i, so that $x_1 \in \tilde{Q}_i$ for all i, and $m(\tilde{Q}_i) \preceq m(Q_i)$. This means that

$$1 \le \frac{1}{n} \sum_{m=n}^{3n} \chi_{\tilde{D}_m(x_1)}, \qquad x_1 \in \bigcup_{3n+n_0} D_m, \tag{4.6}$$

where $\tilde{D}_m = \bigcup \{\tilde{Q} \mid Q \in W \text{ and } Q \subset D_m\}$.

Let $a_m = m(D_m)$ for $m \ge 0$. Using (4.6) and the fact that $\sum \chi_{D_m}$ is bounded we get

$$\begin{aligned} \sum_{3n+n_0}^{\infty} a_m &= \int_D \sum_{3n+n_0}^{\infty} \chi_{D_m}\, dx \preceq m\left(\bigcup_{3n+n_0}^{\infty} D_m \right) \\ &\le \int_D \frac{1}{n} \sum_{m=n}^{3n} \chi_{\tilde{D}_m}\, dx \le \frac{1}{n} \sum_{m=n}^{3n} m(\tilde{D}_m) \\ &\preceq \frac{1}{n} \sum_{m=n}^{3n} m(D_m) = \frac{1}{n} \sum_{m=n}^{3n} a_m. \end{aligned}$$

Therefore, if we take $N = n \ge n_0$, then $3n + n_0 \le 4N$ and so

$$\sum_{m=4N}^{\infty} a_m \le \sum_{m=3n+n_0}^{\infty} a_m \le \frac{c}{n} \sum_{m=n}^{3n} a_m \le \frac{c}{N} \sum_{m=N}^{4N-1} a_m.$$

Thus, the hypothesis to Lemma 2 is satisfied for large N. Since $a_m > 0$ for all m we can increase the constant in (4.1) so that it is true for all N. As in the proof of Theorem 4, $k_D(x_0, x) \preceq m$ on D_m and hence (1.7) follows from (4.2). This completes the proof of Theorem 2.

In a recent paper by S. Staples, L^p-averaging domains are characterized by the L^p-integrability of $k_D(x_0, x)$. Hence Hölder domains are L^p-averaging domains, see Theorem 2.6 in [S]. Furthermore, Theorem 3.4 in that paper proves that D is a p-Poincaré domain provided that k_D is in $L^p(D)$ and $p \ge d$. (See Theorem 9 below for a sharpening of this result.) Thus, the proof of Theorem 1 is completed and we also have the following corollary:

Corollary 3. *If D is a Hölder domain and $1 \le p < \infty$, then*

$$\int_D |u - u_D|^p\, dx \le c_D \sup_{B \subset D} \frac{1}{m(B)} \int_B |u - u_B|^p\, dx, \qquad u \in L^p(D), \tag{4.7}$$

where c_D is a constant and the supremum is taken over all balls B contained in D.

As a consequence of Lemma 3 we also have the following result.

Corollary 4. *If D is a Hölder domain, then $m(\partial D) = 0$.*

Proof. Suppose that $m(\partial D) > 0$. Let $y_0 \in \partial D$ be a point of density for ∂D. This means that for any $\varepsilon > 0$,

$$\frac{m(B(y_0, r) \cap \partial D)}{m(B(y_0, r))} \geq 1 - \varepsilon \tag{4.8}$$

provided r is sufficiently small.

Let $x_1 \in D \cap B(y_0, r)$. Assume that $x_0 \notin B(y_0, r)$ and that $|x_1 - y_0|$ is very small. By following a quasi-hyperbolic geodesic from x_1 to x_0, we can find a point $x \in D \cap B(y_0, r)$ with $\delta_D(x)$ comparable to $|x - y_0|$. Shrinking r to $|x - y_0|$ results in a contradiction to (4.8) provided ε is sufficiently small. The proof is complete.

Remark. Domains are constructed in [S] where k_D is in $L^{d-1}(D)$ and yet $m(\partial D) > 0$.

Question. Does there exist a Hölder domain in R^d whose boundary has Hausdorff dimension d?

5. STARSHAPED DOMAINS IN R^d

It is well known that balls are p-Poincaré domains and that $M_p(B(a, r)) \approx r$, for $1 \leq p < \infty$. Recall that $a \approx b$ means that a/b is bounded from above and from below by positive dimensional constants. See Chapter 7 in [GT] for generalization to convex domains and Chapter 2 of [M] for domains which are starshaped with respect to an open set. In this section, we give new and simpler proofs of these theorems in addition to extending the generality to the class of bounded starshaped domains.

Lemma 4. *If $Q = [0, a_1] \times \cdots \times [0, a_d]$, then $M_p(Q) \leq d \cdot \max(a_1, \dots, a_d)$.*

Proof. Assume that $\max(a_1, \dots, a_d) = 1$ and that $u \in C^1(Q)$. For $x, y \in Q$ and $1 \leq p < \infty$ we have that

$$|u(x) - u(y)|^p \leq d^{p-1} \sum_{i=1}^{d} \int_0^{a_i} |\nabla u(y_1, \dots, y_{i-1}, t, x_{i+1}, \dots, x_d)|^p \, dt$$

and hence that

$$\begin{aligned} \int_Q \left| u(x) - \frac{1}{m(Q)} \int_Q u(y)\, dy \right|^p dx &\leq \frac{1}{m(Q)} \int_Q dx \int_Q |u(x) - u(y)|^p \, dy \\ &\leq d^{p-1} \sum_{i=1}^{d} \frac{1}{m(Q)} \int_Q \int_Q \int_0^{a_i} |\nabla u(y_1, \dots, t, \dots, x_d)|^p \, dt dx dy \\ &= d^{p-1} \sum_{i=1}^{d} a_i \int_Q |\nabla u|^p \, dx \leq d^p \int_Q |\nabla u|^p \, dx. \end{aligned}$$

The proof is now completed with a change of variables argument.

Remark 1. In view of the Poincaré inequality given on page 164 (page 157 in the first edition) in [GT] it is surprising that the Poincaré constant, $M_p(Q)$, is independent of the small a_i's. Even for the unit cube Q, the constant in [GT] grows exponentially with d.

Remark 2. In general, one can modify the above argument to show that

$$M_p(D_1 \times D_2) \leq 3\max(M_p(D_1), M_p(D_2)).$$

Denote by $Q(x, r)$ the cube with center x and side length $2r$.

Lemma 5. *Suppose* $Q(0, 2a) \subset D$ *and* $1 \leq p < \infty$. *Let*

$$N_p(D) = \sup_u \left\{ \frac{\|u\|_{L^p(D)}}{\|\nabla u\|_{L^p(D)}} \,\middle|\, u \in C^1(D) \text{ is nonconstant and } u = 0 \text{ on } B(0, a) \right\}.$$

Then

$$2^{-d-6} M_p(D) \leq N_p(D) \leq 2\left(1 + \frac{m(D)}{m(B(0, a))}\right)^{1/p} M_p(D). \tag{5.1}$$

Proof. Assume that $N_p(D)$ is finite. Let $u = 0$ on $B(0, a)$, $u = a$ off $B(0, 2a)$ and $u(x) = |x| - a$ at all other points. Then

$$\frac{\|u\|_{L^p(D)}}{\|\nabla u\|_{L^p(D)}} \geq \left(\frac{\int_a^{2a} (r-a)^p r^{d-1}\, dr}{\int_a^{2a} r^{d-1}\, dr}\right)^{1/p} \geq \left(\frac{a^{d+p}/(p+1)}{2^d a^d / d}\right)^{1/p}$$

and hence a limit argument yields the lower bound

$$a(d/2^d(p+1))^{1/p} \leq N_p(D). \tag{5.2}$$

Put $\varphi = 1 - u/a$ and suppose that v is in $C^1(D)$ and satisfies $v_{Q(0,2a)} = 0$. Write $v = v_1 + v_2$ where $v_1 = \varphi v$. Since $v_{Q(0,2a)} = 0$ we have by Lemma 4, that

$$\|v\|_{L^p(Q(0,2a))} \leq 4ad\|\nabla v\|_{L^p(Q(0,2a))}. \tag{5.3}$$

Now $v_2 = 0$ on the set $B(0, a)$ and $|\nabla\varphi| = 0$ off the set $B(0, 2a)$, thus it follows from (5.3) that

$$\begin{aligned} \|v_2\|_{L^p(D)} &\leq N_p(D)\|\nabla v_2\|_{L^p(D)} \\ &\leq N_p(D)(\|(1-\varphi)\nabla v\|_{L^p(D)} + \|v\nabla\varphi\|_{L^p(D)}) \\ &\leq N_p(D)(\|\nabla v\|_{L^p(D)} + \tfrac{1}{a}\|v\|_{L^p(Q(0,2a))}) \\ &\leq 5dN_p(D)\|\nabla v\|_{L^p(D)}. \end{aligned}$$

Finally, since $v_1 = 0$ off the set $Q(0, 2a)$ we use (5.3) again and (5.2) to obtain that

$$\begin{aligned} \|v - v_D\|_{L^p(D)} &\leq 2\|v - (v_1)_D\|_{L^p(D)} \\ &\leq 2(\|v_1 - (v_1)_D\|_{L^p(D)} + \|v_2\|_{L^p(D)}) < 10d(a + N_p(D))\|\nabla v\|_{L^p(D)} \\ &\leq 10d((2^d(p+1)/d)^{1/p} + 1)N_p(D)\|\nabla v\|_{L^p(D)} \\ &\leq 10(2^d e + d)N_p(D)\|\nabla v\|_{L^p(D)} \leq 2^{d+6}N_p(D)\|\nabla v\|_{L^p(D)}. \end{aligned}$$

This completes the proof of the first inequality in (5.1).

To prove the other inequaility in (5.1) notice first that if $u \in C^1(D)$ and $u_{B(0,a)} = 0$, then

$$|u_D|^p \le \left(\frac{1}{m(B(0,a))}\int_{B(0,a)} |u-u_D|\,dx\right)^{1/p} \le \frac{1}{m(B(0,a))}\int_D |u-u_D|^p\,dx.$$

Thus

$$\begin{aligned} 2^{-p}\int_D |u|^p\,dx &\le \int_D |u-u_D|^p\,dx + |u_D|^p m(D) \\ &\le \left(1+\frac{m(D)}{m(B(0,a))}\right)\int_D |u-u_D|^p\,dx \\ &\le \left(1+\frac{m(D)}{m(B(0,a))}\right) M_p^p(D)\int_D |\nabla u|^p\,dx, \end{aligned}$$

and the result follows.

Theorem 5. *If* $Q(0,2a) \subset D \subset B(0,b)$, $1 \le p < \infty$ *and* D *is starshaped with respect to the origin, then*

$$M_p(D) \le 2^{d+6} b \left(\tfrac{b}{a}\right)^{\frac{d-1}{p}}. \tag{5.4}$$

Proof. Let $u \in C^1(D)$ and assume that $u = 0$ on $B(0,a)$. For $|x'| = 1$ we assume that $rx' \in D$ for $0 \le r < b(x')$ and that $b(x')x' \in \partial D$. By hypothesis, $2a \le b(x') \le b$. For such x' we have the inequality

$$\begin{aligned} \int_0^{b(x')} |u(rx')|^p r^{d-1}\,dr &\le b^p \int_0^{b(x')} \left(\frac{1}{r}\int_a^r |\nabla u(tx')|\,dt\right)^p r^{d-1}\,dr \\ &\le b^p \int_0^b r^{d-2}\,dr \int_a^{b(x')} |\nabla u(tx')|^p\,dt \\ &\le b^p \left(\frac{b}{a}\right)^{d-1} \int_0^{b(x')} |\nabla u(tx')|^p t^{d-1}\,dt \end{aligned}$$

and hence integration over the sphere yields that

$$N_p(D) \le b(\tfrac{b}{a})^{\frac{d-1}{p}}. \tag{5.5}$$

By Lemma 5 the result follows and the proof is complete.

For $1 < p < \infty$, this last theorem can be improved by using Muckenhoupt's weighted norm inequality for the Hardy-Littlewood maximal function as we did earlier for the analytic Poincaré inequality in the plane, see Theorem 2.1 in [SS2].

Theorem 6. *If* $Q(0,2a) \subset D \subset B(0,b)$ *and* $1 \le p < \infty$ *and* D *is starshaped with respect to the origin, then there are constants* $c_p = c_p(p,d)$ *such that*

$$M_p^p(D) \le c_p b^p \begin{cases} (\frac{b}{a})^{d-p}, & 1 \le p < d, \\ (\log(\frac{b}{a}+1))^{d-1}, & p = d, \\ 1, & p > d. \end{cases} \tag{5.6}$$

Proof. The case $p = 1$ is contained in Theorem 5. Fix $1 < p < \infty$ and define $w(t) = \min((a + |t|)^{d-1}, b^{d-1})$ for $-\infty < t < \infty$. Then a straightforward calculation shows that

$$A_p = \sup_I \frac{1}{|I|} \int_I w\, dt \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}}\, dt \right)^{p-1} \tag{5.7}$$

is comparable to the quantities on the right hand side of (5.6).

We modify the proof of Theorem 5 slightly to obtain that whenever $|x'| = 1$ and $u = 0$ on $B(0, a)$, we have that

$$\begin{aligned} \int_0^{b(x')} |u(rx')|^p r^{d-1}\, dr &\le b^p \int_a^{b(x')} \left(\frac{1}{r-a} \int_a^r |\nabla u(tx')|\, dt \right)^p r^{d-1}\, dr \\ &\le c_p b^p A_p \int_0^{b(x')} |\nabla u(rx')|^p r^{d-1}\, dr \end{aligned}$$

where this last inequality is due to Muckenhoupt, see [Mu]. The proof is then completed as previously.

The constants on the right-hand side of (5.6) can be improved if the domain D is such that integration with respect to polar coordinates in the proof can be replaced by integration with respect to rectangular coordinates. For $p > 1$, this is because the weight $w(t)$ in (5.7) reduces to $w(t) = 1$, so that $A_p = 1$. A similar modification of the proof of Theorem 5 works for $p = 1$. Using these ideas we obtain the following theorem; we omit further details. It generalizes Theorem 2.2 of [SS2], and will be required in section 10. We use Q^{d-1} to denote a cube in R^{d-1}.

Theorem 7. *Suppose that* $h(x)$ *is a lower semicontinuous function defined on* $Q^{d-1}(0, a)$ *and satisfies* $0 < a \le h(x) \le b$ *for all* x. *If*

$$D = B(0, 2a) \cup \{(x, t) : x \in Q^{d-1}(0, a),\ -a < t < h(x)\},$$

then $M_p^p(D) \le c_p b^p$, $1 \le p < \infty$.

6. A SUFFICIENT CONDITION FOR THE POINCARÉ INEQUALITY

We assume throughout that D is a domain in R^d with finite volume and Whitney decomposition W. Fix a cube $Q_0 \in W$ and let x_0 be its center. We assume that, for each $Q \in W$, there is a set $P(Q) \subset D$ containing a chain $Q_0, \dots, Q_n$ of cubes in W, starting with Q_0 and ending with $Q_n = Q$. This means that Q_i is adjacent to Q_{i+1} in W, so that a face of the smaller cube is contained in a face of the larger cube, and $Q_i \subset P(Q)$ for $0 \le i \le n$. Now define

$$S(Q) = \bigcup \{Q_1 \in W \mid Q \subset P(Q_1)\}. \tag{6.1}$$

As an example of how one might construct the the sets $P(Q)$, we could simply take the cubes in W which intersect a quasi-hyperbolic geodesic joining

x_0 to the center of Q. A geometric interpretation of $S(Q)$, in this case, is that it is the points in the *shadow* of Q, assuming that light travels from x_0 along the quasi-hyperbolic geodesics in D. For $x \in D$, let Q_x denote a cube in W containing x.

Theorem 8. *If D satisfies the above conditions and λ is a real number, then*

$$M_p(D) \preceq \sup_{Q\in W} \left\{ \int_{S(Q)} \left(\int_{P(Q_x)} \frac{dy}{\delta_D(y)^{p'(1-\lambda)(d-1)}} \right)^{p-1} \frac{dx}{d(Q)^{\lambda p(d-1)}} \right\}^{1/p} \tag{6.2}$$

for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and

$$M_1(D) \preceq \sup_{Q\in W} \frac{m(S(Q))}{d(Q)^{d-1}} \tag{6.3}$$

when $p = 1$.

Remark. Although we have stated Theorem 8 for an arbitrary real number λ, the result is only of interest for λ satisfying $\frac{d-1-p}{p(d-1)} < \lambda \le \frac{d}{p(d-1)}$. The right side of (6.2) will be infinite for any other choice of λ. We omit the details of this computation.

Lemma 6. *Let Q_1, Q_2 be adjacent cubes in W. Then $M_p(Q_1)$ and $M_p(Q_1 \cup Q_2)$ are both comparable to $d(Q_1)$, for $1 \le p < \infty$.*

Proof. The first assertion is Lemma 4 and the second follows from Theorem 5.

Remark. Lemma 6 is well known and follows from the uniform cone condition (see §1.1.11 of [M]). We give a new proof of this classical result in Theorem 10.

Lemma 7. *For $1 \le p < \infty$,*

$$M_p(D) \preceq \sup_{Q\in W} d(Q) + \sup_{u} \frac{\left(\sum_{Q\in W} |u_Q - u_{Q_0}|^p m(Q)\right)^{1/p}}{\|\nabla u\|_{L^p(D)}}. \tag{6.4}$$

Proof. For $u \in C^1(D)$, we have that

$$\begin{aligned} \int_D |u - u_D|^p\,dx &\le 2^p \int_D |u - u_{Q_0}|^p\,dx = 2^p \sum_{Q\in W} \int_Q |u - u_{Q_0}|^p\,dx \\ &\le 4^p \sum_{Q\in W} \left(\int_Q |u - u_Q|^p\,dx + |u_Q - u_{Q_0}|^p m(Q) \right) \\ &\le 4^p \left(\sup_{Q\in W} M_p^p(Q) \int_D |\nabla u|^p\,dx + \sum_{Q\in W} |u_Q - u_{Q_0}|^p m(Q) \right). \end{aligned}$$

The result now follows from Lemma 6.

Lemma 8. *If $u \in C^1(D)$ and $Q \in W$, then*

$$|u_Q - u_{Q_0}| \preceq \int_{P(Q)} \frac{|\nabla u|}{\delta_D(x)^{d-1}}\,dx. \tag{6.5}$$

Proof. Let $Q_0, \dots, Q_n$ be a minimal chain of cubes in $P(Q)$ which joins Q_0 to Q. The Q_i's must then be distinct, for if this were not the case, we could remove a subsequence of adjacent cubes and still have a chain from Q_0 to Q. Using Lemma 6 we then have that

$$\begin{aligned}
|u_Q - u_{Q_0}| &\le \sum_{i=1}^{n} |u_{Q_i} - u_{Q_{i-1}}| \\
&\preceq \sum_{i=1}^{n} \frac{1}{m(Q_i \cup Q_{i-1})} \int_{Q_i \cup Q_{i-1}} |u - u_{Q_i \cup Q_{i-1}}|\,dx \\
&\le \sum_{i=1}^{n} \frac{M_1(Q_i \cup Q_{i-1})}{m(Q_i \cup Q_{i-1})} \int_{Q_i \cup Q_{i-1}} |\nabla u|\,dx \\
&\preceq \sum_{i=1}^{n} \frac{1}{d(Q_i)^{d-1}} \int_{Q_i \cup Q_{i-1}} |\nabla u|\,dx \\
&\preceq \int_{\bigcup Q_i} \frac{|\nabla u|}{\delta_D(x)^{d-1}}\,dx \le \int_{P(D)} \frac{|\nabla u|}{\delta_D(x)^{d-1}}\,dx
\end{aligned}$$

and the proof is complete.

Proof of Theorem 8. Let M denote the right-hand side of (6.2) and assume that it is finite. If $Q \in W$, then $Q \subset S(Q) \cap P(Q)$ and hence for some λ we have that

$$\frac{m(Q)}{d(Q)^{\lambda p(d-1)}} \left(\int_Q \frac{dy}{\delta_D(y)^{p'(1-\lambda)(d-1)}} \right)^{p-1} \le M^p$$

and it follows that

$$d(Q) \preceq M, \qquad Q \in W. \tag{6.6}$$

Suppose that $u \in C_1(D)$ and put

$$F_p(x, \lambda) = \left(\int_{P(Q_x)} \frac{dy}{\delta_D(y)^{p'(1-\lambda)(d-1)}} \right)^{p-1} \tag{6.7}$$

for $x \in D$, $-\infty < \lambda < \infty$ and $1 < p < \infty$. For $p = 1$, let $F_1 = 1$ and set $\lambda = 1$. Using Lemma 8 and Hölder's inequality we see that

$$\begin{aligned}
\sum_{Q \in W} |u_Q - u_{Q_0}|^p m(Q) &\preceq \sum_{Q \in W} \left(\int_{P(Q)} \frac{|\nabla u| dx}{\delta_D(x)^{d-1}} \right)^p m(Q) \\
&\le \sum_{Q \in W} \int_{P(Q)} \frac{|\nabla u|^p dx}{\delta_D(x)^{\lambda p(d-1)}} F_p(x_Q, \lambda) m(Q) \\
&= \int_D |\nabla u|^p \left\{ \sum_{Q \in W} \frac{\chi_{P(Q)}(x) F_p(x_Q, \lambda)}{\delta_D(x)^{\lambda p(d-1)}} m(Q) \right\} dx.
\end{aligned}$$

Since $m(\bigcup\{\partial Q \mid Q \in W\}) = 0$, we need only consider $x \in D$ with x in the interior of some $Q_1 \in W$. For such x, $\chi_{P(Q)}(x) \ne 0$ if and only if $Q \subset S(Q_1)$.

In addition, $d(Q_1) \leq c\delta_D(x)$, for $x \in Q_1$, where $c \approx 1$. Thus, we simplify the above to get that

$$\begin{aligned}\sum_{Q\in W} |u_Q - u_{Q_0}|^p m(Q) &\leq c^{\lambda p(d-1)} \int_D |\nabla u|^p \left(\int_{S(Q_x)} F_p(y, \lambda) \frac{dy}{d(Q_x)^{\lambda p(d-1)}} \right) dx \\ &\leq c^{\lambda p(d-1)} M^p \int_D |\nabla u|^p \, dx\end{aligned}$$

and hence that

$$\frac{(\sum_{Q\in W} |u_Q - u_{Q_0}|^p m(Q))^{1/p}}{\|\nabla u\|_{L^p(D)}} \preceq M \tag{6.7}$$

Finally, Lemma 7 combined with (6.6) and (6.8) give the required bound on $M_p(D)$ and the proof is complete.

7. John domains with cusps and integrability conditions for K_D

We give some applications of Theorem 8 in this section by constructing sets $P(Q)$ from families of arcs in D generated by a new distance function on D. For $1 < p < \infty$ the metric k_p in D is defined by

$$k_p(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta_D(x)^{\frac{d-1}{p-1}}}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. Notice that $k_d = k_D$. Martin has shown that geodesics exist for k_p if $1 < p \leq d$ [Ma, 2.11], but we will not need this fact.

Theorem 9. *For* $1 \leq p < \infty$,

$$M_p(D) \preceq \begin{cases} \sup_{Q\in W} \left(\frac{1}{d(Q)^{d-p}} \int_{S(Q)} k_D^{p-1}(x_0, x)\, dx \right)^{1/p}, & 1 \leq p < d, \\ \left(\int_D k_D^{p-\frac{p}{d}}(x_0, x)\, dx \cdot m(D)^r \right)^{1/p}, & d \leq p, \end{cases} \tag{7.1}$$

where $r = \frac{p-d}{d(p-1)}$. *For* $d - 1 < p < \infty$,

$$M_p(D) \preceq \left(\int_D k_p^{p-1}(x_0, x)\, dx \right)^{1/p}. \tag{7.2}$$

Remarks. The statement of Theorem 9 has $p > d - 1$ in (7.2) since otherwise the right-hand side is infinite, as is easily checked.

For $d \leq p$

$$k_p^{p-1}(x_0, x) \preceq k_D^{p-\frac{p}{d}}(x_0, x) \cdot m(D)^r$$

where r is as in Theorem 9. This follows from the application of Hölder's inequality in (7.5) below. Thus (7.2) is stronger than (7.1) if $p \geq d$.

Proof. Let W be a Whitney decomposition of D and for each $Q \in W$, let γ_Q be a fixed arc joining x_0 to the center of Q, such that

$$\int_{\gamma_Q} \frac{ds}{\delta_D(x)^{\frac{d-1}{p-1}}} \leq 2k_p(x_0, x_Q). \tag{7.3}$$

Define $P(Q)$ to be the union over the cubes in W which intersect γ_Q.

Lemma 9. *For* $1 < p < \infty$,

$$\int_{P(Q)} \frac{dy}{\delta_D(y)^{p'(d-1)}} \preceq k_p(x_0, x) \tag{7.4}$$

whenever $x \in Q$, $Q \in W$ *and* $Q \neq Q_0$.

Proof. Fix $x \in Q$ with $Q \in W$ and $Q \neq Q_0$. By the triangle inequality, $k_p(x_0, x_Q) \leq k_p(x_0, x) + k_p(x, x_Q)$. Since W is a Whitney decomposition it is obvious from the definition that $k_p(x, x_Q) \preceq d(Q)^{1-(d-1)/(p-1)}$. Let γ be an arc in D joining x_0 to x. Since $Q \neq Q_0$]we can find a cube $Q_1 \in W$ with $d(Q_1) \approx d(Q)$ and $d(Q_1) \preceq |\gamma \cap Q_1|$. Hence

$$\int_\gamma \frac{ds}{\delta_D(y)^{\frac{d-1}{p-1}}} \succeq \frac{|\gamma \cap Q_1|}{d(Q_1)^{\frac{d-1}{p-1}}} \succeq d(Q)^{1-\frac{d-1}{p-1}} \succeq k_p(x, x_Q).$$

Thus, $k_p(x_0, x) \succeq k_p(x, x_Q)$ and so $k_p(x_0, x_Q) \preceq k_p(x_0, x)$ whenever $x \in Q$. Hence it suffices to prove (7.4) with $x = x_Q$.

Let γ_Q be the arc in (7.3). For $R \in W$, denote by $\tilde{R}$, the union of all cubes in W which have a nonempty intersection with R. It is easily seen that there are cubes $Q_1, \dots, Q_k$ in W for which (a) $\gamma_Q \cap Q_i$ is nonempty for $1 \leq i \leq k$, (b) $Q_i \not\subset \tilde{Q}_j$ for $i \neq j$ and (c) $P(Q) \subset \cup \tilde{Q}_i$. Clearly, we then have that

$$\begin{aligned}
\int_{P(Q)} \frac{dy}{\delta_D(y)^{p'(d-1)}} &\leq \sum_{i=1}^{k} \int_{\tilde{Q}_i} \frac{dy}{\delta_D(y)^{p'(d-1)}} \\
&\approx \sum_{i=1}^{k} d(Q_i)^{d-p'(d-1)} = \sum_{i=1}^{k} d(Q_i)^{1-\frac{d-1}{p-1}} \\
&\preceq \sum_{i=1}^{k} \int_{\gamma_Q \cap \tilde{Q}_i} \frac{ds}{\delta_D(y)^{\frac{d-1}{p-1}}} \\
&\preceq \int_{\gamma_Q} \frac{ds}{\delta_D(y)^{\frac{d-1}{p-1}}} \leq 2 \cdot k_p(x_0, x_Q).
\end{aligned}$$

This completes the proof of Lemma 9.

Returning to the proof of Theorem 9, we let $d-1<p<\infty$, apply Theorem 8 with $\lambda=0$ and use Lemma 9 to conclude (7.2):

$$\begin{aligned} M_p(D) &\preceq \left(\int_D \left(\int_{P(Q_x)} \frac{dy}{\delta_D(y)^{p'(d-1)}}\right)^{p-1} dx\right)^{1/p} \\ &\preceq \left(\int_D k_p^{p-1}(x_0, x)\,dx\right)^{1/p}. \end{aligned}$$

For $p \geq d$ we have that $p'(d-1) \leq d$. Arguing in a similar manner, but using the Hölder inequality we get that

$$\begin{aligned} M_p(D) &\preceq \left(\int_D \left(\int_{P(Q_x)} \frac{dy}{\delta_D(y)^{p'(d-1)}}\right)^{p-1} dx\right)^{1/p} \\ &\leq \left(\int_D \left(\int_{P(Q_x)} \frac{dy}{\delta_D(y)^{d}}\right)^{p-\frac{p}{d}} m(P(Q_x))^r\,dx\right)^{1/p} \qquad (7.5) \\ &\preceq \left(\int_D k_D^{p-\frac{p}{d}}(x_0, x)\,dx \cdot m(D)^r\right)^{1/p}, \end{aligned}$$

and the proof of (7.1) is complete in this case.

Observe that Lemma 9, with $p=d$, was used in this last argument. We choose to use quasi-hyperbolic geodesics γ_Q in this case, to generate the sets $P(Q)$.

Finally, for the case $1<p<d$ in (7.1) we apply Theorem 8 with $\lambda = (d-p)/p(d-1)$ to obtain the inequality

$$\begin{aligned} M_p(D) &\preceq \sup_{Q\in W} \left(\frac{1}{d(Q)^{d-p}} \int_{S(Q)} \left(\int_{P(Q_x)} \frac{dy}{\delta_D^d(y)}\right)^{p-1} dx\right)^{1/p} \\ &\preceq \sup_{Q\in W} \left(\frac{1}{d(Q)^{d-p}} \int_{S(Q)} k_D^{p-1}(x_0, x)\,dx\right)^{1/p}, \end{aligned}$$

which proves (7.1) for $1<p<d$. The case $p=1$ is part of Theorem 8 and thus the proof of Theorem 9 is complete.

Definition. Let D be a domain and $x_0 \in D$. We say that D is an η-John domain for $\eta \geq 1$ provided there is a constant $\alpha > 0$ such that, for each $x_1 \in D$, there is an arc γ joining x_0 to x_1 in D along which

$$\delta_D(x) \geq \alpha|\gamma(x, x_1)|^\eta, \qquad x \in \gamma. \tag{7.6}$$

Notice that when $\eta = 1$ this definition agrees with the definition of John domains given in the introduction. Power cusps are allowed in η-John domains for $\eta > 1$, and so these domains comprise a larger class of domains than John domains. Martio has shown that a John domain is a p-Poincaré domain for all $p \geq 1$ [Mar]. We now give an extension of this result to η-John domains.

Theorem 10. *Let* $1 \le p < \infty$. *If* D *is an* η*-John domain with*

$$\eta < \frac{d}{d-1} + \frac{p-1}{d-1} \tag{7.7}$$

then D *is a* p*-Poincaré domain. In particular, if* $\eta < \frac{d}{d-1}$, *then* D *is a* p*-Poincaré domain for all* $1 \le p < \infty$.

Proof. Let W be a Whitney decomposition for D. For each $Q \in W$, let γ be an arc joining x_0 to x_Q along which (7.6) holds. For each pair of adjacent cubes in W we fix an arc joining their centers which is contained in their union and whose length is comparable to their side lengths. We replace replace γ by an arc γ_Q constructed from these special arcs. This can be done so as to still satisfy (7.6) provided we account for an increase of the arc length by a constant factor. We therefore have that

$$\begin{aligned} &\delta_D(x) \ge \frac{\alpha}{c^\eta}|\gamma_Q(x, x_Q)|^\eta, \qquad x \in \gamma_Q, \\ &d(Q_1) \preceq |\gamma_Q \cap Q_1|, \qquad Q_1 \cap \gamma_Q \ne \varnothing, \quad Q_1 \in W, \end{aligned} \tag{7.8}$$

where c is a dimensional constant. Let $P(Q)$ be the union of the cubes in W which intersect γ_Q.

Let $0 < \lambda \le 1$ be determined by

$$\lambda p = \frac{d}{d+p-1} \le 1. \tag{7.9}$$

Using the property that $d(Q_1) \preceq |\gamma_Q \cap Q_1|$ for each Q_1 in $P(Q)$ we can replace volume measure dy on Q_1 by $\delta_D^{d-1}(y)ds$ on $\gamma_Q \cap Q_1$ and hence we get that

$$\left(\int_{P(Q)} \frac{dy}{\delta_D(y)^{p'(1-\lambda)(d-1)}}\right)^{p-1} \preceq \left(\int_{\gamma_Q} \frac{ds}{\delta_D(y)^{\frac{(1-p\lambda)(d-1)}{p-1}}}\right)^{p-1}. \tag{7.10}$$

Let $L = c(\delta_D(x_0)/\alpha)^{1/\eta}$, then from (7.8) it follows that $\gamma_Q \le L$. If we use euclidean distance as a lower bound for arc length and use that (7.9) implies that $\eta(1 - p\lambda)(d - 1) < p - 1$ we obtain that

$$\left(\int_{\gamma_Q} \frac{ds}{\delta_D(y)^{\frac{(1-p\lambda)(d-1)}{p-1}}}\right)^{p-1} \le \left(\frac{c^\eta}{\alpha}\right)^{(1-p\lambda)(d-1)} \left(\int_0^L \frac{ds}{s^{\frac{\eta(1-p\lambda)(d-1)}{p-1}}}\right)^{p-1} = A < \infty \tag{7.11}$$

and hence the right-hand side of (7.10) is bounded by a constant independent of Q.

In a similar manner we obtain an upper bound for $m(S(Q))$. If x is in the interior of $Q_1 \subset S(Q)$, then the arc γ_{Q_1} passes through the center of Q. By (7.8) we have that

$$d(Q) \ge \frac{\alpha}{c_1^\eta}|x - x_Q|^\eta$$

for a constant c_1 and therefore

$$\begin{aligned}\frac{m(S(Q))}{d(Q)^{\lambda p(d-1)}} &\leq \frac{c_1^{\eta\lambda p(d-1)}}{\alpha^{\lambda p(d-1)}} \int_{S(Q)} \frac{dx}{|x-x_Q|^{\eta\lambda p(d-1)}} \\ &\leq \frac{c_1^{\eta\lambda p(d-1)}}{\alpha^{\lambda p(d-1)}} \int_{B(0,L)} \frac{dx}{|x|^{\eta\lambda p(d-1)}} = B < \infty\end{aligned}$$

since (7.7) implies that $\eta\lambda p(d-1) < d$.

Finally, by combining (7.10), (7.11) and the above we have that

$$\sup_Q \int_{S(Q)} \left(\int_{P(Q)} \frac{dy}{\delta_D(y)^{p'(1-\lambda)(d-1)}} \right)^{p-1} \frac{dx}{d(Q)^{\lambda p(d-1)}} \leq A \cdot B < \infty$$

and the proof is completed by applying Theorem 8.

8. COMPACTNESS OF THE IMBEDDING $W^{1,p}(D) \to L^p(D)$

In this section we are concerned with the question of when the imbedding of a Sobolev space into L^p is compact and the implications this has for the Poincaré inequality. Questions of compactness are important for applications, in part because compact operators have discrete spectra. The study of compact imbeddings began with Rellich [R]. See also Chapter 6 of [Ad].

For a domain D in R^d and $1 \leq p < \infty$, recall that $W^{1,p}(D)$ is the usual Sobolev space of functions on D (see the introduction). Obviously, the natural imbedding of $W^{1,p}(D)$ into $L^p(D)$ is continuous. The next theorem gives an equivalent condition for the imbedding to be compact. Amick considered a closely related condition and the question of compactness for a bounded domain with $p = 2$ in [Am].

Theorem 11. *Let $1 \leq p < \infty$ and let D be an open set in R^d. The imbedding $W^{1,p}(D) \to L^p(D)$ is compact if and only if*

$$\lim_{n\to\infty} \sup \left\{ \int_{D\setminus D_n} |u|^p \, dx \mid \|u\|_{W^{1,p}(D)} \leq 1 \right\} = 0, \tag{8.1}$$

where $D_n = \{x \in D \mid \delta_D(x) > 1/n \text{ and } |x| < n\}$.

This theorem is an easy consequence of the following lemma. We follow the convention of extending $u \in L^p(D)$ to be defined in R^d by setting u equal to 0 on $R^d \setminus D$.

Lemma 10 [Ad, 2.21, p. 31]. *Let $1 \leq p < \infty$. A bounded subset X in $L^p(D)$ is precompact in $L^p(D)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ and a compact subset K of D such that*

$$\int_D |u(x+h) - u(x)|^p \, dx < \varepsilon^p, \qquad |h| < \delta, \; u \in X, \tag{8.2}$$

and

$$\int_{D\setminus K} |u|^p\,dx < \varepsilon^p, \qquad u \in X. \tag{8.3}$$

Proof of Theorem 11. First suppose that the imbedding is compact. We apply Lemma 10 with $X = \{u \in W^{1,p}(D) \mid \|u\|_{W^{1,p}(D)} \le 1\}$. Thus, given $\varepsilon > 0$ there is a compact set $K \subset D$ such that (8.3) holds. Since $K \subset D_n$ for all sufficiently large n, (8.1) follows.

For the converse, we adapt an argument from [Ad, p. 147]. Suppose that (8.1) holds. Let $X = C^1(D) \cap \{u \in W^{1,p}(D) \mid \|u\|_{W^{1,p}(D)} \le 1\}$. Then if $u \in X$ and $h < 1/n$, we have

$$\begin{aligned}\int_{D_n} |u(x+h) - u(x)|^p\,dx &\le \int_{D_n} \left(\int_0^1 \left|\frac{d}{dt}u(x+th)\right| dt\right)^p dx \\ &\le |h|^p \int_{D_n} dx \int_0^1 |\nabla u(x+th)|^p\,dt \\ &\le |h|^p \int_D |\nabla u|^p\,dx.\end{aligned}$$

This fact together with (8.1) clearly shows that (8.2) holds for $u \in X$. Furthermore, (8.3) follows immediately from (8.1) by letting K be the closure of D_n for n sufficiently large. The compactness of the imbedding now follows from the Lemma 10, and the proof is complete.

Corollary 5. *Let* $d - 1 < p < \infty$ *and* $x_0 \in D$. *If*

$$\int_D k_p^{p-1}(x_0, x)\,dx < \infty, \tag{8.4}$$

then the imbedding $W^{1,p}(D) \to L^p(D)$ *is compact.*

Proof. Recall that $d - 1 < p$ is necessary for (8.4) to hold. Let W be a Whitney decomposition of D and let $\{D_n\}$ be as in (8.1). Then there are positive constants c_1 and c_2 such that

$$D \setminus D_n \subset \bigcup\{Q \in W \mid d(Q) \le c_1/n\} \subset D \setminus D_{nc_2}, \qquad n \ge 1.$$

Suppose that $\|u\|_{W^{1,p}(D)} \le 1$. We then have

$$\begin{aligned}\int_{D\setminus D_n} |u|^p\,dx &\le \sum_{d(Q)\le c_1/n} \int_Q |u|^p\,dx \\ &\le 3^p \sum_{d(Q)\le c_1/n} \left\{\int_Q |u - u_Q|^p\,dx + |u_Q - u_{Q_0}|^p m(Q)\right\} \\ &\quad + 3^p |u_{Q_0}|^p m(D \setminus D_{nc_2}).\end{aligned} \tag{8.5}$$

Now,

$$\sum_{d(Q)\le c_1/n} \int_Q |u - u_Q|^p\,dx \le \sup_{d(Q)\le c_1/n} M_p^p(Q) \int_{D\setminus D_{nc_2}} |\nabla u|^p\,dx \le \left(\frac{c_3}{n}\right)^p,$$

where c_3 is a dimensional constant by Lemma 6. Also, as in the proof of Theorem 8 (with $\lambda = 0$) we get that

$$\sum_{d(Q)\le c_1/n} |u_Q - u_{Q_0}|^p m(Q) \preceq \int_D |\nabla u|^p\,dx \int_{D\setminus D_{nc_2}} k_p^{p-1}(x_0, x)\,dx = o(1),$$

since $k_p(x_0, x) \in L^{p-1}(D)$. The last term on the right side of (8.5) also goes to zero as n tends to infinity since D clearly has finite volume by condition (8.4). So we have that (8.1) holds and we are done by Theorem 11.

The following result is an immediate consequence of Theorem 2, Corollary 5 and the Remark in §7.

Corollary 6. *If $p \ge d$ and D is a Hölder domain, then the imbedding $W^{1,p}(D) \to L^p(D)$ is compact.*

We next consider the relationship between the Poincaré inequality and the compactness of the imbedding $W^{1,p}(D) \to L^p(D)$. The following result may well be known. We include a proof since we have been unable to find a reference.

Theorem 12. *Suppose that $1 \le p < \infty$ and $m(D) < \infty$. If the imbedding $W^{1,p}(D) \to L^p(D)$ is compact, then $M_p(D) < \infty$.*

Proof. Arguing by contradiction we assume that the imbedding is compact but that $M_p(D)$ is infinite. Then there is a sequence $\{u_n\} \subset C^1(D)$ such that

$$\int_D |u_n|^p\,dx = 1, \quad \int_D u_n\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_D |\nabla u_n|^p\,dx = 0. \tag{8.6}$$

By the compactness of the imbedding there is a subsequence which we continue to denote $\{u_n\}$ and $u \in L^p(D)$ such that $u_n \to u$ in $L^p(D)$. Now let α be a multi-index with $|\alpha| = 1$ and let $\phi \in C_0^\infty(D)$, where this denotes functions compactly supported on D and having continuous partial derivatives of all orders. Then

$$D^\alpha u(\varphi) = -\int_D u D^\alpha \varphi\,dx = -\lim_{n\to\infty} \int_D u_n D^\alpha \varphi\,dx = \lim_{n\to\infty} \int_D \varphi D^\alpha u_n\,dx = 0,$$

since $\|\nabla u_n\|_{L^p(D)} \to 0$ by (8.6). Thus u has a weak gradient in D and $\nabla u = 0$. Since $M_p(B) < \infty$ for each ball $B \subset D$, we see that this means that u is locally constant in D. Since D is a connected set, u is almost everywhere equal to a constant in D. This is a contradiction since it follows from (8.6) that $\|u\|_{L^p(D)} = 1$ and $\int_D u\,dx = 0$, and the proof is complete.

9. EXAMPLE OF A NONCOMPACT EMBEDDING

In §8 we showed that the sufficient condition for the Poincaré inequality from Theorem 10 actually implies that the imbedding $W^{1,p}(D) \to L^p(D)$ is compact. We now show that this does not extend to all of our other sufficient conditions for the Poincaré inequality. More precisely, we construct a domain

D for which the imbedding $W^{1,p}(D) \to L^p(D)$ is not compact, and we use (6.2) of Theorem 8 with $p = 2$, $d = 3$ and $\lambda = 1/4$ to show $M_2(D) < \infty$.

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in R^3 such that $|x_i| = 1$ and the balls $\{B(x_i, 2^{-i})\}$ are pairwise disjoint. Now set $y_i = (1 + 2^{-i+1})x_i$ and define

$$C_i = \bigcup\{B(x, 2^{-2i}) | x \in [x_i, (1 + 2^{-i})x_i]\},$$

$$R_i = B(y_i, 2^{-i}) \quad \text{and} \quad D = B(0, 1) \cup \bigcup_{i=1}^{\infty}(C_i \cup R_i).$$

Thus D consists of a central room $B(0, 1)$ and a sequence of rooms $\{R_i\}$ connected to $B(0, 1)$ by narrow corridors $\{C_i\}$.

General rooms and corridors type domains such as D are studied in detail in §10. We could apply Theorem 15 from that section to easily see that $M_2(D) < \infty$ and that the imbedding $W^{1,2}(D) \to L^2(D)$ is not compact. Our goal now, however, is to show that the hypotheses of Theorem 8 do not imply that such an imbedding is compact. We also wish to demonstrate how Theorem 8 can be used to show that a specific domain is a p-Poincaré domain. Accordingly, we proceed to show that D satisfies the hypotheses of Theorem 8.

To each point $x \in D$ we associate an arc γ_x in D with initial point 0 and terminal point x. These arcs will be used to determine the sets $P(Q)$ in this example. The arc associated with $x \in B(0, 1)$ traces the line segment $[0, x]$. For $x \in C_i \backslash \{B(0, 1) \cup R_i\}$ let $\tilde{x}$ be the point on $[0, (1+2^{-i})x_i]$ which is closest to x, and define γ_x to be the arc that traces the line segments $[0, \tilde{x}]$ followed by $[\tilde{x}, x]$. For $x \in R_i$ we define γ_x so that it traces the line segments $[0, y_i]$ followed by $[y_i, x]$. This completes the description of the family of arcs.

Let W be a Whitney decomposition for D. For $Q \in W$, denote by γ_Q the arc from our family of arcs associated with x_Q, the center of Q. Define $P(Q)$ to be the union of all cubes in W that intersect γ_Q. We shall show that D is a 2-Poincaré domain by applying Theorem 8 with $\lambda = 1/4$ and this definition of the sets $P(Q)$. Thus it suffices to show that

$$\sup_{Q \in W} \frac{1}{d(Q)} \int_{S(Q)} \int_{P(Q_x)} \frac{dy}{\delta_D(y)^3}\, dx < \infty. \tag{9.1}$$

We sketch a proof that (9.1) holds. It is easily checked that if $x \in D$, then

$$\int_{\gamma_x} \frac{ds}{\delta_D(y)} \preceq k_D(0, x).$$

Thus, by Lemma 9, it suffices to show that

$$\sup_{Q \in W} \frac{1}{d(Q)} \int_{S(Q)} k_D(0, x)\, dx < \infty. \tag{9.2}$$

First consider a cube $Q \subset C_i \cup R_i$ with the property that the diameter of $S(Q)$ satisfies $d(S(Q)) \leq 10\delta_D(x_Q)$. An easy estimate shows that for such a cube $k_D(0, x_Q) \preceq 2^i + \log(1/d(Q))$, so $m(Q) \cdot k_D(0, x_Q) \preceq d(Q)^2$. Also it is easily seen that

$$\int_{S(Q)} k_D(x_Q, x)\, dx \preceq d(Q)^2 \int_0^{d(Q)} \log \frac{1}{t}\, dt \preceq d^2(Q).$$

Thus

$$\frac{1}{d(Q)} \int_{S(Q)} k_D(0, x)\, dx \leq \frac{1}{d(Q)} \left(m(S(Q)) \cdot k_D(0, x_Q) + \int_{S(Q)} k_D(x_Q, x)\, dx \right)$$
$$\preceq 1.$$

A similar computation establishes that the same estimate holds for a cube $Q \subset C_i \cup R_i$ with the property that $d(S(Q)) \geq 10\delta_D(x_Q)$. The key point is showing that

$$\int_{C_i \cup R_i} k_D(x_i, x)\, dx \preceq \int_{R_i} k_D(x_i, x)\, dx \preceq 2^{-2i}. \tag{9.3}$$

Thus we have shown that if $Q \subset C_i \cup R_i$, then

$$\frac{1}{d(Q)} \int_{S(Q)} k_D(0, x)\, dx \preceq 1. \tag{9.4}$$

We now consider a cube $Q \subset B(0, 1)$. As above we have that

$$\frac{1}{d(Q)} \int_{S(Q) \cap B(0,1)} k_D(0, x)\, dx \preceq 1. \tag{9.5}$$

Also, since $\{B(x_i, 2^{-i})\}$ are pairwise disjoint,

$$\sum_{R_i \subset S(Q)} 2^{-2i} \preceq d(Q)^2.$$

This with (9.3) shows

$$\int_{S(Q) \setminus B(0,1)} k_D(0, x)\, dx \leq \sum_{R_i \subset S(Q)} \left(k_D(0, x_i) \cdot m(C_i \cup R_i) + \int_{C_i \cup R_i} k_D(x_i, x)\, dx \right)$$
$$\preceq \sum_{R_i \subset S(Q)} (i 2^{-3i} + 2^{-2i})$$
$$\preceq d(Q)^2.$$

This estimate together with (9.5) shows that if $Q \subset B(0, 1)$, then

$$\frac{1}{d(Q)} \int_{S(Q)} k_D(0, x)\, dx \preceq 1.$$

Now (9.2) follows from this and (9.4), so we have shown that $M_2(D) < \infty$.

To finish this example we now need to demonstrate that the imbedding $W^{1,2}(D) \to L^2(D)$ is not compact. For $1 \leq i < \infty$, define

$$u_i(x) = \begin{cases} 0, & x \in D \setminus (C_i \cup R_i), \\ 2^{5i/2}(|x| - 1), & x \in C_i \cap B(0, 1 + 2^{-i}), \\ 2^{3i/2}, & x \in (R_i \cup C_i) \setminus B(0, 1 + 2^{-i}). \end{cases}$$

A straightforward calculation shows that $\|u_i\|_{L^2(D)} \approx 1$ and $\|u_i\|_{W^{1,2}(D)} \approx 1$, for all i.

If the imbedding $W^{1,2}(D) \to L^2(D)$ were compact it would follow that there is a subsequence of $\{u_i\}$ converging to $u \in L^2(D)$ in norm. But then $\|u\|_{L^2(D)} \approx 1$, which is a contradiction since $u_i(x) \to 0$ almost everywhere. Hence the imbedding $W^{1,2}(D) \to L^2(D)$ is not compact, and we are done with the example.

10. ROOMS AND CORRIDORS TYPE EXAMPLES

It is a well-known elementary fact that a finite union of Poincaré domains is again a Poincaré domain. In this section we take up the study of infinite unions of Poincaré domains. A "rooms and corridors" type domain consists of a central cube shaped room along with an infinite disjoint collection of cube shaped rooms which are connected to the central room by narrow corridors (or tubes if $d > 2$), such as the domain constructed in §9. The resulting domain may or may not have M_p finite. The use of rooms and corridors type examples in the study of the Poincaré inequality can found in [CH], [M], [S] and [SS2], and variants of these domains are used in [Am] and [AS].

In this section, we characterize those rooms and corridors type examples which are p-Poincaré domains by using the k_p metric introduced in §7. We then construct a specific example to show that the condition $p \geq d$ in Theorem 1 is necessary. Results on compact imbeddings of rooms and corridors type domains are also obtained. The theorems in this section again demonstrate the important relationship between the k_p metric and the p-Poincaré inequality. Some of these results generalize earlier results of ours in [SS2].

Definition. Let T be a domain in R^d. For $x_1, x_2 \in T$ and $1 < p < \infty$, we define $h_{p,T}(x_1, x_2) = k_{p,T}^{p-1}(x_1, x_2)$. For $p = 1$, we put

$$h_{1,T}(x_1, x_2) = \inf\left\{\sup_{x\in\gamma} \delta_T^{1-d}(x) \mid \gamma \text{ is a path from } x_1 \text{ to } x_2 \text{ in } T\right\}. \tag{10.1}$$

This definition is motivated by the fact that for a fixed arc γ in T,

$$\lim_{p\to 1}\left(\int_\gamma \frac{ds}{\delta_T(y)^{\frac{d-1}{p-1}}}\right)^{p-1} = \sup_{y\in\gamma} \delta_T^{1-d}(y).$$

Lemma 11. *Let T be a domain in R^d. Suppose that W is a Whitney decomposition of T and that $1 \leq p < \infty$. Then*

$$|u_{Q_1} - u_{Q_2}|^p \preceq h_{p,T}(x_1, x_2) \cdot \int_T |\nabla u|^p\, dx \tag{10.2}$$

whenever Q_1, $Q_2 \in W$, with $x_i \in Q_i$ for $i = 1, 2$, and $u \in C^1(T)$.

Proof. For $p > 1$, let γ be an arc in T joining x_1 to x_2 and satisfying

$$\int_\gamma \frac{ds}{\delta_T(y)^{\frac{d-1}{p-1}}} \le 2k_{p,T}(x_1, x_2). \tag{10.3}$$

Let P be the collection of Whitney cubes in W which intersect γ. By Lemmas 8 and 9 and (10.3) we obtain that

$$\begin{aligned}|u_{Q_1} - u_{Q_2}|^p &\preceq \left(\int_P \frac{|\nabla u|}{\delta_T(y)^{d-1}}\right)^p \\ &\le \int_P |\nabla u|^p\, dx \cdot \left(\int_P \frac{dy}{\delta_T(y)^{p'(d-1)}}\right)^{p-1} \\ &\preceq k_{p,T}^{p-1}(x_1, x_2) \cdot \int_P |\nabla u|^p\, dx \\ &= h_{p,T}(x_1, x_2) \cdot \int_P |\nabla u|^p\, dx\end{aligned}$$

which verifies (10.2) in this case. An analogous argument for the case $p = 1$ establishes (10.2) for all $1 \le p < \infty$. This completes the proof.

See [SS2] where this lemma is proved for the special case of simply connected planar domains with $p = 2$ and u an analytic function on T.

We now consider a more general configuration of a connected domain $D \subset R^d$ with finite volume. We assume that $D = \bigcup_{n=0}^\infty G_n$ where each G_n is an open connected subdomain of D. Moreover, we assume that $\{G_n\}_{n=1}^\infty$ is a disjoint collection and that $G_n \cap G_0$ is a nonempty set for each n. Corresponding to each region G_n, with $n \ge 1$, there is a subregion $\tilde{G}_n$ of G_0 for which $G_n \cap \tilde{G}_n$ is a nonempty set. Put $T_n = G_n \cup \tilde{G}_n$ for $n \ge 1$, so that T_n is an open connected subdomain of D. Using the construction on page 167 of Stein's book [St], we construct a Whitney decompositions W_n of T_n with defining parameters independent of n. Finally, assume that $Q_n, \tilde{Q}_n \in W_n$ with $Q_n \subset G_n$, $\tilde{Q}_n \subset \tilde{G}_n$ and that x_n, $\tilde{x}_n$ are the centers of Q_n, $\tilde{Q}_n$.

Theorem 13. *Let* $1 \le p < \infty$ *and let* M, K *be finite constants. If*

(a) $m(G_n)$, $m(Q_n)$, $m(\tilde{G}_n)$ *and* $m(\tilde{Q}_n)$ *are all comparable for each* $n \ge 1$, *with constants independent of* n,

(b) $\{\tilde{Q}_n\}_{n\ge1}$ *is disjoint and*

$$\sup_{x\in D} \sum_{n=1}^\infty \chi_{T_n}(x) \le K,$$

(c) $M_p^p(G_n) \le M$ *for all* $n \ge 0$ *and*

(d) $h_{p,T_n}(\tilde{x}_n, x_n) \cdot m(G_n) \le M$ *for all* $n \ge 1$,

then $M_p(D) < \infty$.

Proof. Let u be a function defined on D satisfying $\nabla u \in L^p(D)$ and $u \equiv 0$ on Q_0. For $n \geq 1$, we use (a) to compute that

$$\begin{aligned}\int_{G_n} |u|^p\,dx &\preceq \int_{G_n} |u - u_{G_n}|^p\,dx + |u_{G_n} - u_{Q_n}|^p m(Q_n) \\ &\quad + |u_{Q_n} - u_{\tilde{Q}_n}|^p m(G_n) + |u_{\tilde{Q}_n}|^p m(\tilde{Q}_n) \\ &\leq 2\int_{G_n} |u - u_{G_n}|^p\,dx + \int_{\tilde{Q}_n} |u|^p\,dx + |u_{Q_n} - u_{\tilde{Q}_n}|^p m(G_n) \\ &= 2\cdot A_n + B_n + C_n.\end{aligned}$$

By (c), we have that

$$\sum_{n=1}^{\infty} A_n \leq M \sum_{n=1}^{\infty} \int_{G_n} |\nabla u|^p\,dx \leq M \int_D |\nabla u|^p\,dx.$$

Since $u \equiv 0$ on Q_0 it follows from Lemma 5 and conditions (b), (c) that

$$\sum_{n=1}^{\infty} B_n \leq \int_{G_0} |u|^p\,dx \preceq M \int_D |\nabla u|^p\,dx.$$

Combining these estimates we obtain that

$$\int_D |u|^p\,dx \leq \sum_{n=0}^{\infty} \int_{G_n} |u|^p\,dx \preceq M \int_D |\nabla u|^p\,dx + \sum_{n=1}^{\infty} C_n.$$

Using Lemma 11 we obtain that

$$|u_{Q_n} - u_{\tilde{Q}_n}|^p \preceq h_{p,T_n}(\tilde{x}_n, x_n) \cdot \int_{T_n} |\nabla u|^p\,dx, \qquad n \geq 1. \tag{10.4}$$

Finally, we have by conditions (b), (d) and (10.4) that

$$\begin{aligned}\sum_{n=1}^{\infty} C_n &\leq \sum_{n=1}^{\infty} h_{p,T_n}(\tilde{x}_n, x_n) \cdot m(G_n) \cdot \int_{T_n} |\nabla u|^p\,dx \\ &\leq M \int_D |\nabla u|^p \sum_{n=1}^{\infty} \chi_{T_n}\,dx \leq MK \int_D |\nabla u|^p\,dx,\end{aligned}$$

and hence that

$$\int_D |u|^p\,dx \preceq M(K+1) \int_D |\nabla u|^p\,dx$$

whenever u vanishes on Q_0. Thus, the proof is complete by Lemma 5.

We now consider an infinite union of domains $\{G_n\}$ with the property that each of the imbeddings $W^{1,p}(G_n) \to L^p(G_n)$ is compact. The assumptions and the notation introduced prior to Theorem 13 are still in force.

Theorem 14. *Let* $1 \leq p < \infty$ *and let* M, K *be finite constants. If*

(a) $m(G_n)$, $m(Q_n)$, $m(\tilde{G}_n)$ *and* $m(\tilde{Q}_n)$ *are all comparable for each* $n \geq 1$, *with constants independent of* n,

(b) $\{\tilde{Q}_n\}_{n\geq 1}$ *is disjoint and*

$$\sup_{x\in D}\sum_{n=1}^{\infty}\chi_{T_n}(x)\leq K,$$

(c) $\lim_{n\to\infty} M_p^p(G_n)=0$,

(d) $\lim_{n\to\infty} h_{p,T_n}(\tilde{x}_n, x_n)\cdot m(G_n)=0$,

(e) $\lim_{n\to\infty}\delta_D(\tilde{x}_n)=0$, *and*

(f) *the imbedding* $W^{1,p}(G_n)\to L^p(G_n)$ *is compact for all* $n\geq 0$,

then the imbedding $W^{1,p}(D)\to L^p(D)$ *is compact.*

Proof. Theorem 11 will be used to show the compactness of the imbedding. As a first step, we show that

$$\lim_{N\to\infty}\sup\left\{\sum_{n=N+1}^{\infty}\int_{G_n}|u|^p\,dx \mid \|u\|_{W^{1,p}(D)}=1\right\}=\lim_{N\to\infty}S_N=0. \tag{10.6}$$

Arguing as in the proof of Theorem 13, by using (a) we see that

$$S_N\preceq A'_N+B'_N+C'_N,$$

where

$$A'_N=\sup\left\{\sum_{n=N+1}^{\infty}\int_{G_n}|u-u_{G_n}|^p\,dx \mid \|u\|_{W^{1,p}(D)}=1\right\},$$

$$B'_N=\sup\left\{\sum_{n=N+1}^{\infty}\int_{\tilde{Q}_n}|u|^p\,dx \mid \|u\|_{W^{1,p}(D)}=1\right\},$$

$$C'_N=\sup\left\{\sum_{n=N+1}^{\infty}|u_{Q_n}-u_{\tilde{Q}_n}|^p m(G_n) \mid \|u\|_{W^{1,p}(D)}=1\right\}.$$

By (c) we have that $\lim_{N\to\infty}A'_N=0$, since

$$\int_{G_n}|u-u_{G_n}|^p\,dx\leq M_p^p(G_n)\int_{G_n}|u|^p\,dx.$$

For any $k\geq 1$, by (e) and the fact that $\tilde{Q}_n\in W_n$, we can pick N such that

$$\bigcup_{n=N+1}^{\infty}\tilde{Q}_n\subset D\setminus D_k,$$

where D_k is as in (8.1). Thus by applying (f) with $n=0$ and Theorem 11, we have that $\lim_{N\to\infty}B'_N=0$. Now, using (10.4) and (b) as is the proof of Theorem 13, we estimate

$$\begin{aligned}C'_N&\leq\sup\left\{\sum_{n=N+1}^{\infty}h_{p,T_n}(\tilde{x}_n, x_n)\cdot m(G_n)\cdot\int_{T_n}|\nabla u|^p\,dx \mid \|u\|_{W^{1,p}(D)}=1\right\}\\&\leq K\sup_{n\geq N+1}h_{p,T_n}(\tilde{x}_n, x_n)\cdot m(G_n).\end{aligned}$$

Thus $\lim_{N\to\infty} C'_N = 0$, by (d). The proof of (10.6) is now complete.

It is an easy consequence of the definition of a compact imbedding that, since each imbedding $W^{1,p}(G_n) \to L^p(G_n)$ is compact, the imbedding

$$W^{1,p}\left(\bigcup_{n=0}^{N} G_n\right) \to L^p\left(\bigcup_{n=0}^{N} G_n\right)$$

is compact for each $N \geq 0$. (Alternately, one can use Theorem 11 to see this.) Thus by Theorem 11 again, and with D_k as in (8.1),

$$\lim_{k\to\infty} \sup\left\{\int_{\bigcup_{n=0}^{N} G_n \setminus D_k} |u|^p \, dx \mid \|u\|_{W^{1,p}(G_N)} = 1\right\} = 0, \qquad N \geq 0. \tag{10.7}$$

But since

$$D \setminus D_k \subset \left(\bigcup_{n=0}^{N} G_n \setminus D_k\right) \cup \left(\bigcup_{n=N+1}^{\infty} G_n\right),$$

(10.6) and (10.7) imply that

$$\lim_{k\to\infty} \sup\left\{\int_{D\setminus D_k} |u|^p \, dx \mid \|u\|_{W^{1,p}(D)} = 1\right\} = 0,$$

so an application of Theorem 11 completes the proof.

We now proceed to simplify the above geometric configurations. Let R_n denote the ball $B(x_n, c_n)$ with center x_n and radius c_n, where $n = 0, 1, \ldots$. We assume that $x_0 = 0$, $c_0 = 1$, $1 < |x_n| < 2$, $a_n \leq c_n$ and that the collection of balls $\{R_n\}_{n=0}^{\infty}$ is disjoint. For $n \geq 1$, let $x'_n = x_n/|x_n|$, $b_n = |x_n - x'_n| - c_n$ and $C_n = \bigcup\{B(x, a_n) \mid 0 \leq |x - x'_n| \leq b_n\}$. We further assume, for $n \geq 1$, that the sets $\{C_n \cup R_n\}$ are disjoint and that D is constructed using the rooms R_n and the corridors C_n, i.e.

$$D = R_0 \cup \left(\bigcup_{n=1}^{\infty} (R_n \cup C_n)\right).$$

Theorem 15. *Let $1 \leq p < \infty$ and let D be the domain constructed above.*

(i) *$M_p(D)$ is finite if and only if*

$$\sup_n h_p(x_0, x_n) \cdot m(R_n) < \infty. \tag{10.8}$$

(ii) *Further suppose that*

$$\sup_n \frac{a_n^{d-1} b_n}{c_n^d} \preceq 1. \tag{10.9}$$

Then the imbedding $W^{1,p}(D) \to L^p(D)$ is compact if and only if

$$\lim_{n\to\infty} h_p(x_0, x_n) \cdot m(R_n) = 0. \tag{10.10}$$

Remark. The geometric interpretation of (10.9) is that $m(C_n) \preceq m(R_n)$. Without a restriction of this type it is easy to construct a counterexample with a starshaped domain.

Proof. (i) Assume that (10.8) holds. We establish that $M_p(D)$ is finite by showing that conditions (a)–(d) in Theorem 13 hold. Let $G_0 = R_0 \cup (\bigcup_{n=1}^{\infty} C_n)$. Then G_0 is starshaped with respect to the origin and hence is a p-Poincaré domain by Theorem 6. For $n \geq 1$ put $G_n = R_n$. Then by Theorem 6 again we see that condition (c) holds for some finite constant M.

For $n \geq 1$, let $\tilde{R}_n$, $\tilde{C}_n$ denote the reflections of the sets R_n, C_n about the sphere $|x| = 1$ in R^d. We put $\tilde{G}_n = \tilde{R}_n \cup \tilde{C}_n \cup C_n$ and construct a Whitney decomposition W_n of $T_n = G_n \cup \tilde{G}_n$ with constant defining parameters. Finally, we put Q_n to be a cube in W_n containing x_n and let $\tilde{Q}_n$ be a cube in W_n containing the reflection $\tilde{x}_n$ of x_n. Clearly, conditions (a) and (b) hold, with $K = 1$.

To prove that condition (d) holds, we will use the symmetry of T_n. For $p > 1$, let Γ_n be an arc in D joining x_0 to x_n so that

$$\int_{\Gamma_n} \frac{ds}{\delta_D(y)^{\frac{d-1}{p-1}}} \leq 2k_p(x_0, x_n). \tag{10.11}$$

Let γ_n be the subarc of Γ_n starting from the last exit from R_0 to the endpoint x_n. Let $\tilde{\gamma}_n$ be the reflection of γ_n about the sphere $|x| = 1$. Clearly, $\tilde{\gamma}_n \cup \gamma_n$ is an arc in T_n joining $\tilde{x}_n$ to x_n. Hence by symmetry,

$$\begin{aligned} k_{p,T_n}(x_n, \tilde{x}_n) &\leq \int_{\tilde{\gamma}_n \cup \gamma_n} \frac{ds}{\delta_{T_n}(y)^{\frac{d-1}{p-1}}} \preceq \int_{\gamma_n} \frac{ds}{\delta_{T_n}(y)^{\frac{d-1}{p-1}}} \\ &= \int_{\gamma_n} \frac{ds}{\delta_D(y)^{\frac{d-1}{p-1}}} \leq \int_{\Gamma_n} \frac{ds}{\delta_D(y)^{\frac{d-1}{p-1}}} \leq 2k_p(x_0, x_n). \end{aligned} \tag{10.12}$$

Thus, condition (d) holds for the case $1 < p < \infty$. A similar argument yields (d) in case $p = 1$. Hence $M_p(D)$ is finite by Theorem 13.

To prove the converse suppose that $M_p(D)$ is finite. We must find a bound $M < \infty$ so that

$$h_p(x_0, x_n) \cdot m(R_n) \leq M \tag{10.13}$$

holds for all n. Fix n and let γ_n be the line segment $[x_0, x_n]$. Integration over γ_n gives an upper bound for $k_p(x_0, x_n)$.

We first assume that $1 < p < d$. If $b_n + c_n < 2a_n$, then

$$k_p^{p-1}(x_0, x_n) \cdot m(R_n) \preceq \left(a_n^{1-\frac{d-1}{p-1}}\right)^{p-1} \cdot a_n^d = a_n^p \leq 1$$

and so (10.13) holds with some M which is independent of n. If $b_n + c_n \geq 2a_n$, we construct a function u_n which vanishes on D except for points in $C_n \cup R_n$.

For $x \in C_n \cup R_n$, we define

$$u_n(x) = \min\left(1, \frac{|x - x_n'| - a_n}{2a_n + b_n}\right), \qquad |x - x_n'| \geq a_n.$$

Clearly, $u_n = 1$ on at least half of R_n and hence

$$m(R_n) \preceq \int_{R_n} |u_n|^p \, dx \leq M_p^p(D) \int_D |\nabla u_n|^p \, dx.$$

If $b_n \leq a_n$, then it is easily seen that $k_p^{p-1}(x_0, x_n) \preceq a_n^{p-d}$ and $|\nabla u_n| \preceq a_n^{-1}$. Thus,

$$\int_D |\nabla u_n|^p \, dx \preceq \frac{m(\{|\nabla u_n| \neq 0\})}{a_n^p} \preceq a_n^{d-p} \preceq k_p^{1-p}(x_0, x_n).$$

Thus, (10.13) holds in this case.

If $b_n \geq a_n$, then $k_p^{p-1}(x_0, x_n) \preceq b_n^{p-1}/a_n^{d-1}$ and $|\nabla u_n| \preceq b_n^{-1}$. Again,

$$\int_D |\nabla u_n|^p \, dx \preceq \frac{m(\{|\nabla u_n| \neq 0\})}{b_n^p} \preceq \frac{b_n a_n^{d-1}}{b_n^p} \preceq k_p^{1-p}(x_0, x_n)$$

and so (10.13) holds. We have therefore completed the proof for the case $1 < p < d$.

For $d < p$, the same proof works, only the estimates are more elementary since the integral

$$\int_0^1 \frac{dt}{t^{\frac{d-1}{p-1}}} < \infty$$

is convergent. The case $p = 1$ can also be done with a similar argument. Finally, the case $p = d$ requires a slightly different construction for u_n. If $b_n \geq a_n$, then the proof is as before. But if $b_n \leq a_n$, then we define

$$u_n(x) = \min\left(1, \frac{\log(|x - x_n'|/a_n)}{\log((2a_n + b_n)/a_n)}\right), \qquad |x - x_n'| \geq a_n,$$

whenever $x \in C_n \cup R_n$. Analogous estimates show that (10.13) holds in this case also and the proof of part (i) is complete.

(ii) Assume that (10.10) holds. We will use Theorem 14 to show that the imbedding $W^{1,p}(D) \to L^p(D)$ is compact. Define $G_0 = R_0$, and for $n \geq 1$, $G_n = R_n \cup C_n$ and $\tilde{G}_n = \tilde{R}_n \cup \tilde{C}_n$, where $\tilde{R}_n$ and $\tilde{C}_n$ are as in the proof of (i). Let Q_n, $\tilde{Q}_n$, x_n and $\tilde{x}_n$ also be defined as in the proof of (i). Using (10.9) and estimating $k_p(x_0, x_n)$ as in (i), we get that

$$b_n^p \preceq b_n^{p-1} c_n^d / a_n^{d-1} \preceq h_p(x_n, \tilde{x}_n) \cdot m(R_n).$$

Hence $b_n \to 0$, by (10.10) and (10.12). We also have that $c_n \to 0$ since $m(D) < \infty$.

As in the proof of (i), conditions (a) and (b) of Theorem 14 are immediate by the definition of Q_n, $\tilde{Q}_n$ and x_n. Condition (c) holds as a consequence of Theorem 7, since $b_n + c_n \to 0$. Condition (d) is immediate from (10.10) and (10.12), while condition (e) follows since $b_n + c_n \to 0$. Finally, it is well known that the imbeddings in condition (f) are compact. See Chapter 6 of [Ad], for example, or alternately Corollary 6 could be used. Thus the imbedding $W^{1,p}(D) \to L^p(D)$ is compact by Theorem 14.

To prove the converse, suppose that (10.10) fails. By passing to a subsequence, assume without loss of generality that $\inf_n h_p(x_0, x_n) \cdot m(R_n) \succeq 1$. Thus

$$\left(\left(\log\frac{1}{a_n}\right)^{p-1} + \frac{b_n^{p-1}}{a_n^{d-1}}\right) \cdot c_n^d \succeq 1, \quad n \geq 1.$$

First consider the case that $\limsup b_n^{p-1} c_n^d / a_n^{d-1} \succeq 1$. By passing to a subsequence again, without lossed generality assume that

$$b_n^{p-1} c_n^d / a_n^{d-1} \succeq 1, \quad n \geq 1. \tag{10.14}$$

To finish the proof of this case, we demonstrate that the imbedding $W^{1,p}(D) \to L^p(D)$ is not compact in a manner similiar to that used in §9. For $1 \leq n \leq \infty$, define

$$u_n(x) = \begin{cases} 0, & x \in D \setminus G_n, \\ b_n^{-1} c_n^{-d/p}(|x| - 1), & x \in C_n \setminus \{1 < |x| < 1 + b_n\}, \\ c_n^{-d/p}, & x \in R_n \setminus B(0, 1 + b_n). \end{cases}$$

A straightforward calculation shows that $\|u_n\|_{L^2(D)} \approx 1$ and $\|\nabla u_n\|_{L^p(D)} \preceq 1$, for all n. (10.14) was used in the second estimate.

If the imbedding $W^{1,p}(D) \to L^p(D)$ were compact, it would follow that there is a subsequence of $\{u_n\}$ converging to $u \in L^p(D)$ in norm. This is a contradiction, since it would follow that $\|u\|_{L^p(D)} \approx 1$, but $u_n(x) \to 0$ for all $x \in D$.

The final case, with $\limsup(\log \frac{1}{a_n})^{p-1} \cdot c_n^d \succeq 1$, is handled similarly, with the functions defined as in the proof of (ii). This completes the proof of Theorem 15.

Example. Fix $m \geq 1$ and let D_m be constructed by attaching to the unit ball in R^d a disjoint sequence of balls, of radius $c_n = 2^{-n}$, and connecting tubes of length $b_n = (c_n)^{-m}$ and radius $a_n = b_n/n$. It is easily seen that condition (1.2) holds since

$$\sup_n \left(\frac{b_n}{a_n} - \log b_n\right) \cdot \frac{1}{\log c_n^{-1}} = \frac{1 + m\log 2}{\log 2} < \infty$$

and hence D_m is a Hölder domain. Thus, Theorem 1 shows that $M_p(D_m)$ is finite for all $d \leq p < \infty$.

On the other hand, Theorem 15(i) implies that for $1 \leq p < d$, $M_p(D_m)$ is finite if and only if

$$\sup_n \frac{b_n^{p-1}}{a_n^{d-1}} \cdot c_n^d < \infty. \tag{10.15}$$

Since the supremum in (10.15) is clearly infinity for $p \leq d(1 - 1/m)$, we see that $M_p(D_m) = \infty$ for this range of p's. Thus, the range of p in Theorem 1 is best possible.

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