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Hölder exponent of planar Julia sets associated with polynomials

Marta Kosek

Department of Mathematics
Uppsala University

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Marta Kosek
Institute of Mathematics, Jagiellonian University
ul. Łojasiewicza 6, 30-348 Kraków, Poland
E-mail: Marta.Kosek@im.uj.edu.pl

Abstract

If the Green function g_E of a compact set $E \subset \mathbb{C}$ is Hölder continuous, then the Hölder exponent of the set E is the supremum over all such α, that $|g_E(z) - g_E(w)| \leq M|z - w|^\alpha, \quad z, w \in \mathbb{C}$. We give a lower bound for the Hölder exponent of the Julia sets of polynomials. In particular we show that there are totally disconnected planar sets with the Hölder exponent greater than 3/5.

1 Introduction

The continuity of the Green function of a compact set was always the object of an intensive research. In what follows we investigate only compact planar sets with Hölder continuous Green function, i.e. we take such a compact set $E \subset \mathbb{C}$ that there exist positive constants M and α with

$$(1.1) \qquad |g_E(z) - g_E(w)| \leq M|z - w|^\alpha, \quad z, w \in \mathbb{C}.$$

In this situation there arises another problem: to find estimates for the exponent α. To this end we define the *Hölder exponent* of the set E to be

$$\lambda(E) := \sup\{\alpha : \text{ (1.1) holds with the exponent } \alpha\}$$

and look for its estimates.

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In this note we deal with polynomial Julia sets, i.e. the Julia sets for polynomials of degree $d \geq 2$. They are compact and regular. The proof of the Hölder continuity of their Green functions was provided by Sibony (see [CG, Chapter VIII, Theorem 3.2] and the comments near it).

The main result of this paper runs as follows

Main Theorem 1.1. *Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d \geq 2$. Denote by $J[p]$ the Julia set of p and by $K[p]$ the filled-in Julia set of p. Let F be a convex compact set containing the set $K[p]$. Then*

$$\lambda(J[p]) = \lambda(K[p]) \geq \frac{\log d}{\log(\max\{|p'(z)| : z \in F\})}. \tag{1.2}$$

The most known and investigated are Julia sets $J_c \subset \mathbb{C}$ and their filled-in counterparts K_c, defined by the quadratic polynomials of the form $z \mapsto z^2 + c$, $c \in \mathbb{C}$. We will obtain

Corollary 1.2. $\forall c \in \mathbb{C}: \qquad \lambda(J_c) = \lambda(K_c) \geq \dfrac{\log 2}{\log(1 + \sqrt{1 + 4|c|})}.$

Note that this estimate is sharp in the following sense. Namely $\lambda(K_0) \geq 1$ and $\lambda(K_{-2}) \geq 1/2$ in view of Corollary 1.2. Since $K_0 = \{z \in \mathbb{C} : \ |z| \leq 1\}$, it is known that $\lambda(K_0) = 1$. On the other hand $K_{-2} = [-2, 2]$ and hence it is known that $\lambda(K_{-2}) = 1/2$. The estimate given in Theorem 1.2 is especially interesting when K_c is totally disconnected or if $|c|$ is small (small enough to give the estimate greater than 1/2).

What is already known about the Hölder exponent $\lambda(E)$ of a compact set $E \in \mathbb{C}$ (with Hölder continuous Green function)? We have always (see [S, Remark 3.7])

$$\lambda(E) \leq 1. \tag{1.3}$$

Furthermore, we have also (see [RR, Theorem 2])

$$\lambda(E) \leq \dim_H(E), \tag{1.4}$$

where $\dim_H(E)$ denotes the Hausdorff dimension of the set E.

On the other hand if E is a non-singular continuum then (for the proof see e.g. [BCK, Corollary 2.2])

$$\lambda(E) \geq 1/2. \tag{1.5}$$

This inequality remains true if $E \subset \mathbb{C}$ is a compact set satisfying the following condition

$$\exists \delta > 0 \ \forall a \in E \ \exists F - \text{a continuum}: \ a \in F \subset E, \operatorname{diam}(F) > \delta$$

(diam(F) denotes here the diameter of the set F). This result follows from Leja Polynomial Lemma (cf. [L]). All compact sets with finitely many non-singular connected components satisfy this condition.

Finally let us mention one of the most important applications of the Hölder exponent in the theory of polynomial inequalities. Namely if the Green function g_E is Hölder continuous, then E is a Markov set (for the definition and the background see e.g. [P]) and

$$\lambda(E) \leq \frac{1}{m(E)}, \tag{1.6}$$

where $m(E)$ is so called *Markov exponent* of the set E defined in [BP] (for its importance, good references and some new results see [BBC]).

Note that while obtaining bounds for the Hölder exponent we gain also estimates for the Markov exponent by (1.6) and for the Hausdorff dimension of the set by (1.4).

This paper was stimulated by Baran, who asked whether it is possible to find the estimates for the filled-in Julia sets, and also by the paper [RR], where some estimates for the Hölder exponent of the Cantor ternary sets were given. The Cantor ternary sets can be viewed (and actually the authors of [RR] use this fact) as attractors of iterated function systems. Julia sets, the ones we talk here about, can actually be obtained in a similar way even if the methods for getting the estimates are different.

2 Preliminaries

Put $D(a,r) := \{z \in \mathbb{C} : |z-a| \leq r\}$ for $a \in \mathbb{C}$ and $r > 0$. For a compact set $F \subset \mathbb{C}$ and a positive number $r > 0$ define

$$F_r := \{z \in \mathbb{C} :\ \mathrm{dist}(z,F) \leq r\} = \bigcup_{a \in F} D(a,r).$$

The Green function g_E of the compact set $E \subset \mathbb{C}$ of positive logarythmic capacity can be defined in the same way as the function V_E in [K1, Chapter 5] or as the Green's function of the unbounded component of $\mathbb{C}_\infty \setminus E$ with pole at ∞ extended to be zero elsewhere on $\mathbb{C}$ (see e.g. [R, Chapter 4.4]). If $h : \mathbb{C} \ni z \mapsto az + b \in \mathbb{C}$ for some complex number b and non-zero complex number a then $g_E \circ h = g_{h^{-1}(E)}$ (see e.g. [K1, Theorem 5.3.1]).

If a set $E \in \mathbb{C}$ is compact, its polynomially convex hull is denoted by $\widehat{E}$. We have $g_E \equiv g_{\widehat{E}}$. By an argument due to Błocki, if the Green function g_E is Hölder continuous, then $g_{\widehat{E}}$ is Hölder continuous with the same exponent (see e.g. [Ko, Proposition 2.2]). Hence $\lambda(E) = \lambda(\widehat{E})$. Moreover, by the same Błocki argument, in order to prove the Hölder continuity (with the exponent $\alpha > 0$) of the Green function of a compact set E, it suffices to show that there exist constants $\varrho, M > 0$ such that the following inequality holds

$$g_E(z) \leq M(\mathrm{dist}(z,E))^\alpha, \qquad \text{if } \mathrm{dist}(z,E) \leq \varrho \tag{2.1}$$

(see [S, Proposition 3.5]). Inequality (2.1) is called the *Hölder Continuity Property.*

Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree $d \geq 2$. The Julia set $J[p]$ is usually defined in the terms of (non-)normality of the family $\{p^n\}$ (see e.g. [CG] or [B]). We will use another way. We define first the filled-in Julia set associated with p to be

$$K[p] := \{z \in \mathbb{C} : (p^n(z))_{n=1}^\infty \text{ is bounded}\}$$

and then $J[p]$ to be its boundary. First of all we have $\widehat{J[p]} = K[p]$. The Julia set $J[p]$ is non-empty, moreover, it is perfect and uncountable (see [B, Theorem 4.2.4]). Furthermore, if $J[p]$ is disconnected then it has uncountably many components and each point of $J[p]$ is an accumulation point of infinitely many distinct components of $J[p]$ ([B, Theorem 5.7.1]). Both sets are totally invariant under p, i.e. $p(J[p]) = J[p] = p^{-1}(J[p])$ and $p(K[p]) = K[p] = p^{-1}(K[p])$. The following transformation formula

$$g_{K[p]} = g_{p^{-1}(K[p])} = \frac{1}{d}\, g_{K[p]} \circ p. \tag{2.2}$$

can be obtained e.g. from [K1, Theorem 5.3.1].

3 Proof of the main result

Proof. (of the **Main Theorem 1.1**) Fix $\varepsilon > 0$ and define

$$A(\varepsilon) := \max\{|p'(z)| : \ z \in F_\varepsilon\};$$
$$\alpha(\varepsilon) := \frac{\log d}{\log A(\varepsilon)};$$
$$M(\varepsilon) := \varepsilon^{-\alpha(\varepsilon)} \max\{(g_{K[p]} \circ p)(z) : z \in F_\varepsilon\}.$$

Note that

$$(\ \forall j \in \{0,1,...,n-1\} : \ p^j(w) \in F_\varepsilon) \implies |(p^n)'(w)| \leq A(\varepsilon)^n \tag{3.1}$$

for any positive integer n.

Since $A(\varepsilon)^{\alpha(\varepsilon)} = d$, we derive from (2.2) by induction that

$$g_{K[p]} = (A(\varepsilon)^{-n})^{\alpha(\varepsilon)}\ g_{K[p]} \circ p^n, \qquad n \in \mathbb{N}. \tag{3.2}$$

Take $z \in F_\varepsilon \setminus K[p]$ and fix $z_0 \in K[p]$ with $|z - z_0| = \text{dist}(z, K[p])$. Then $p^n(z_0) \in K[p] \subset F$, $n \in \mathbb{N}$. Since $z \notin K[p]$, there exists $n_0 \in \mathbb{N}$ with $p^{n_0}(z) \notin F_\varepsilon$. Note that $[z_0, z] \subset F_\varepsilon$ since F is convex and define $E_n := p^n([z_0, z])$, $n \in \mathbb{N}$. Take the smallest integer $m = m(z)$ such that $E_m \not\subseteq F_\varepsilon$. Then we have $p^{m-1}(z) \in F_\varepsilon$, there exists however a point $z_1 \in [z_0, z]$ with $p^m(z_1) \notin F_\varepsilon$. In view of (3.1) by the mean value property

$$\varepsilon \le |p^m(z_1) - p^m(z_0)| \le A(\varepsilon)^m |z_1 - z_0| \le A(\varepsilon)^m \text{dist}(z, K[p]),$$

and consequently $\varepsilon^{-\alpha(\varepsilon)}(\text{dist}(z, K[p]))^{\alpha(\varepsilon)} \ge (A(\varepsilon)^{-m})^{\alpha(\varepsilon)}$. Hence in view of (3.2)

$$g_{K[p]}(z) = \varepsilon^{-\alpha(\varepsilon)}\ (g_{K[p]} \circ p)(p^{m-1}(z))\ (\text{dist}(z, K[p]))^{\alpha(\varepsilon)}$$

and the definition of $M(\varepsilon)$ yields

$$g_{K[p]}(z) \le M(\epsilon)(\text{dist}(z, K[p]))^{\alpha(\varepsilon)}.$$

Note that in the last inequality there is no $m = m(z)$.

We have proved thus that $\lambda(K[p]) \ge \alpha(\varepsilon)$. It suffices to note now that

$$\alpha(\varepsilon) \nearrow \alpha := \frac{\log d}{\log(\max\{|p'(z)| : z \in F\})}, \quad \text{as} \quad \varepsilon \searrow 0.$$

□

Remark 3.1. Note that it does not follow from the proof of the Main Theorem 1.1 that $g_{K[p]}$ is Hölder continuous with the exponent α, since if $\varepsilon \longrightarrow 0$, then $M(\varepsilon)$ tends to infinity. □

4 Quadratic polynomials

We start with the announced filled-in Julia sets for quadratic polynomials. For the background see e.g. [CG]. It is well known that it is enough to consider the polynomials of the form $Q_c : z \mapsto z^2 + c$, $c \in \mathbb{C}$, since any other quadratic polynomial is conjugated to one of this type. Let J_c denote the Julia set of Q_c and K_c – the filled-in one. We will now prove Corollary 1.2.

Proof. (of **Corollary 1.2**). It is well known that $J_c \subset D(0, r_c)$ with

$$r_c := \frac{1}{2}(1 + \sqrt{1 + 4|c|}).$$

Hence $K_c = \widehat{J_c} \subset D(0, r_c)$ too and the Main Theorem 1.1 applied to $F = D(0, r_c)$ yields

$$\lambda(K_c) \geq \frac{\log 2}{\log(2r_c)}.$$

□

Remark 4.1. Note that for $c \leq 0$ we take here the smallest possible value of r with $K_c \subset D(0, r)$ since $-r_c, r_c \in K_c$. We could take F from the Main Theorem 1.1 to be the ellipse

$$E_c = \left\{ x + iy : \frac{x^2}{r_c^2} + \frac{y^2}{r_c + 1} \leq 1 \right\}$$

for $c \in [-2, 0]$ and the interval $[-r_c, r_c]$ for $c < -2$ (instead of $D(0, r_c)$) (see [BR]) but the result would not change, since

$$\max\{|Q_c'(z)| : z \in D(0, r_c)\} = \max\{|Q_c'(z)| : z \in E_c\} =$$
$$= \max\{|Q_c'(z)| : z \in [-r_c, r_c]\} = 2r_c.$$

□

We shall start with the mentioned special cases

Example 4.2. $K_0 = D(0, 1)$ and $\lambda(K_0) = 1$.

Proof. Since $c = 0$, by Corollary 1.2

$$\lambda(K_0) \geq \frac{\log 2}{\log 2} = 1.$$

On the other hand we have (1.3). □

Now we turn to the other special case: the interval.

Example 4.3. If $|c| = 2$, then $\lambda(K_c) \geq 1/2$. In particular

$$\lambda([-2, 2]) = \lambda(K_{-2}) = \frac{1}{2}.$$

Proof. By Corollary 1.2

$$\lambda(K_c) \geq \frac{\log 2}{\log 4} = \frac{1}{2}.$$

By the famous Markov inequality it is known that $m([-2, 2]) = 2$, hence $\lambda(K_{-2}) = 1/2$ by (1.6). □

Remark 4.4. Note that K_{-2} is connected but if $|c| = 2$ and $c \neq -2$, then K_c is totally disconnected. However, its Hölder exponent satisfies the same inequality as all those of connected sets. □

Before we list the estimates for some other values of c let us recall some facts. The Mandelbrot set can be defined in two ways

$$\mathcal{M} := \{c \in \mathbb{C} : \ Q_c^n(0) \nrightarrow \infty\} = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$

We have $D(0, \frac{1}{4}) \subset \mathcal{M}$, thus in particular if $|c| \leq 1/4$ then the filled-in Julia set K_c is connected. What is more if c is in the interior of the main cardioid C_1 (see [D2, Fig.17.4]), the Julia set J_c is a simple closed curve which contains no smooth arcs (see [D1, Proposition 3.6.3 and the Remark after it]). Therefore if we get the bound greater than 1/2 of the Hölder exponent is also interesting for such c.

Corollary 4.5. *If $|c| < 2$, then $\lambda(K_c) > 1/2$. If $|c| < 2 - \sqrt{2}$, then $\lambda(K_c) > 2/3$. In particular if $|c| < 1/4$ (hence $c \in C_1$) then J_c is a simple closed curve which contains no smooth arcs and $\lambda(J_c) > 2/3$.*

Proof. This is the straightforward consequence of Corollary 1.2 and the facts above. □

The estimate given here is also interesting for all c outside the Mandelbrot set, since then $J_c = K_c$ is totally disconnected and we cannot use the bound (1.5).

Corollary 4.6. *There exist totally disconnected sets with the Hölder exponent greater than 3/5 (in particular greater than 1/2).*

Proof. It suffices to take $K_c = J_c$ for $c \in (1/4, 1]$. □

See [CG, Chapter VIII.1, Fig.5] for the picture of $K_{0,251}$, which is totally disconnected and has the Hölder exponent greater than 3/5 by Corollary 4.6. $K_{0,255}$ is given by [D2, Fig. 16.10] and $K_{0,5}$ by [D2, Fig. 16.13]. One can also visit e.g. [J] for pictures of Julia sets for polynomials Q_c, $c \in \mathbb{C}$.

We list one more example, one can count many others in a similar way.

Example 4.7. If $|c| = \frac{15}{2}$ then $\lambda(K_c) \geq \frac{\log 2}{\log 5} > 0,43$ and K_c is totally disconnected. □

To conclude this section let us recall that if $c > 1/4$, then $J_c = K_c$ is totally disconnected and has Lebesgue measure 0 and if $c < -2$, then the linear measure of $J_c = K_c$ (which is a subset of the interval $[-r_c, r_c]$) is equal to 0 ([Br, Theorem 12.1]).

5 Cubic polynomials

Every cubic polynomial can be conjugated to one of the form

$$C_{a,b} : \mathbb{C} \ni z \mapsto z^3 + az + b \in \mathbb{C}, \quad a, b \in \mathbb{C},$$

thus it suffices to consider cubic polynomials of this type. Denote by $J_{a,b}$ and $K_{a,b}$ the Julia set and the filled-in Julia set of $C_{a,b}$.

Corollary 5.1. *Let a, b be complex numbers. Then*

$$\lambda(J_{a,b}) = \lambda(K_{a,b}) \geq \frac{\log 3}{\log |3 \max(|b|^2, |a| + 2)) + a|}.$$

Proof. From the Escape Criterion for Cubics [D2, Chapter 18.2] we conclude that $K_{a,b} \subset D(0, \max(|b|, \sqrt{|a| + 2}))$ and we apply Main Theorem 1.1. □

We will improve this result in some special cases.

Proposition 5.2. *Let b be a complex number. Then*

$$\lambda(K_{0,b}) \geq \frac{\log 3}{\log\left(3\left(1 + \sqrt[3]{|b|}\right)^2\right)}.$$

Proof. Define $r := 1 + \sqrt[3]{|b|}$. For any $\varepsilon > 0$ put

$$r_\varepsilon := \sqrt{1 + \varepsilon} + \sqrt[3]{|b|}.$$

Note that $r_\varepsilon > 1$ and $r_\varepsilon \searrow r$ if $\varepsilon \searrow 0$.

Fix an $\varepsilon > 0$ and consider the function $h(t) = t^3 - (1 + \varepsilon)t - \alpha$, which is increasing in $[\sqrt{(1 + \varepsilon)/3}, \infty)$. Since $r_\varepsilon > \sqrt{(1 + \varepsilon)/3}$ and $h(r_\varepsilon) > 0$, we have

$$|C_{0,b}(z)| \geq |z|^3 - |b| > (1 + \varepsilon)|z|,$$

whenever $|z| > r_\varepsilon$. We conclude that $K_{0,b} \subset D(0, r)$.

Main Theorem 1.1 yields

$$\lambda(K_{0,b}) \geq \frac{\log 3}{\log(3r^2)}.$$

□

The estimate from Proposition 5.2 is better than the one from Corollary 5.1 if $|b| < 5\sqrt{2} - 7$ or $|b| \geq \sqrt{2}$. Let us count the estimates for two chosen values of b satisfying the first inequality.

Example 5.3.

$$|b| = \frac{1}{27} \quad \Longrightarrow \quad \lambda(K_{0,b}) \geq \frac{\ln 3}{\ln \frac{16}{3}} > 0,656,$$
$$|b| = \frac{1}{3\sqrt{3}} \quad \Longrightarrow \quad \lambda(K_{0,b}) \geq \frac{\ln 3}{\ln(4+2\sqrt{3})} > 0,546.$$

□

Recall that if $b \in [-2\sqrt{3}/9, 2\sqrt{3}/9]$ then $J_{0,b}$ is connected and for other real values of b the Julia set $J_{0,b}$ is totally disconnected and has Lebesgue measure 0 ([Br, Theorem 13.3]).

The following proposition improves the result from Corollary 5.1 for real a and $b = 0$.

Proposition 5.4. *Let $a \in \mathbb{R}$. Then*

$$\lambda(K_{a,0}) \geq \frac{\log 3}{\log\left(3 + 2|a|\right)}.$$

Proof. In view of [LB, Theorem 3.1 and Corollary 3.2]

$$a \leq -3 \Longrightarrow J_{a,0} \subset E(a) := \left[-\sqrt{1-a}, \sqrt{1-a}\right]$$
$$-3 < a \leq 0 \Longrightarrow J_{a,0} \subset E(a) := \left\{x + iy : \frac{x^2}{1-a} + \frac{y^2}{1+\frac{a}{3}} \leq 1\right\}$$
$$0 \leq a < 3 \Longrightarrow J_{a,0} \subset E(a) := \left\{x + iy : \frac{x^2}{1-\frac{a}{3}} + \frac{y^2}{1+a} \leq 1\right\}$$
$$a \geq 3 \Longrightarrow J_{a,0} \subset E(a) := \left[-i\sqrt{1+a}, i\sqrt{1+a}\right].$$

Apply Main Theorem 1.1. We have $\max\{|3z^2+a| : z \in E(a)\} = 3+2|a|$. □

Some pictures of cubic Julia sets can be find in [LB].

Recall here that if $a \in [-3, 3]$ then $J_{a,0}$ is connected, and for other real values of a the Julia set $J_{a,0}$ is totally disconnected and has linear measure 0 ([Br, Theorem 13.1]).

Let us finally count some other examples

Example 5.5. $J[z \mapsto z^2 - z^3/9]$ has infinitely many non-degenerate components and $\lambda(J[z \mapsto z^2 - z^3/9]) \geq 1/3$.

Proof. The first statement is from [B, Section 11.4]. The polynomial $z \mapsto z^2 - z^3/9$ is conjugated to $C_{3i,0}$, so it suffices to apply Corollary 5.1. □

Example 5.6. $J[z \mapsto (3\sqrt{3}/2)z(z+1)(z+2)]$ has infinitely many non-degenerate components and

$$\lambda(J[z \mapsto (3\sqrt{3}/2)z(z+1)(z+2)]) \geq \frac{\log 3}{\log(3\sqrt{3}+6)} \geq 0,4.$$

Proof. The first statement is from [B, Section 11.5]. The polynomial $z \mapsto (3\sqrt{3}/2)z(z+1)(z+2)$ is conjugated to $C_{-3\sqrt{3}/2,-1}$, so it suffices to apply Corollary 5.1. □

Example 5.7. $J[z \mapsto z^3 - 12z^2 + 36z]$ is totally disconnected and

$$\lambda(J[z \mapsto z^3 - 12z^2 + 36z]) \geq \frac{\log 3}{\log 444} \geq 0,18.$$

Proof. The first statement is from [B, Section 11.5]. The polynomial $z \mapsto z^3 - 12z^2 + 36z$ is conjugated to $C_{-12,12}$, so it suffices to apply Corollary 5.1. □

6 Other polynomials

Let us note first

Remark 6.1. $K[z \mapsto z^k] = D(0,1)$ for any $k \in \{2,3,4,...\}$. By the Main Theorem 1.1 for $r = 1$ and $K = K[z \mapsto z^k]$ we obtain once again $\lambda(D(0,1)) \geq \frac{\log k}{\log k} = 1$. □

Let us also recall that $\forall n \in \mathbb{N} : J[p] = J[p^n]$ ([D1, Proposition 3.5.4]), thus in the previous sections were already considered the Julia sets of some polynomials of higher degrees too.

We propose some more examples whose proofs we omit. They are more or less based on the same argument for a quadratic function.

Proposition 6.2.

$$\forall k \in \mathbb{N}\, \forall c \in \mathbb{C} : \quad \lambda(K[z \mapsto z^{2k} + c]) \geq \frac{\log(2k)}{\log\left(2k\left(\frac{1+\sqrt{1+4|c|}}{2}\right)^{\frac{2k-1}{k}}\right)}.$$

□

Proposition 6.3. *If $k \in \mathbb{N}$ and $c \in \mathbb{C}$, then* $\lambda(K[z \mapsto z^k(z^2+c)]) \geq$

$$\geq \frac{\log(2+k)}{\log\left(\left(\frac{1+\sqrt{1+4|c|}}{2}\right)^{k-1}\left((k+2)\frac{1+\sqrt{1+4|c|}}{2} + 2(k+1)|c|\right)\right)}.$$

□

Proposition 6.4. *If $k \in \mathbb{N}$ and $c_1, ..., c_k \in \mathbb{C}$ with $a := \max(|c_1|, ..., |c_k|)$, then*

$$\lambda\left(K\left[z \mapsto \prod_{j=1}^{k}(z^2 + c_j)\right]\right) \geq \frac{\log(2k)}{\log\left(k\left(1 + 4a + (1 + 2a)\sqrt{1 + 4a}\right)\right)}.$$

□

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