Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type

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1 Introduction and motivation

1.1 Introduction

The classical Hölder inequality for the Lebesgue spaces on $\mathbf{R}^n$ is given by

$$L_{r_1}\, L_{r_2} \subset L_r, \tag{1.1.1}$$

where

$$1 \le r_1 \le \infty, \quad 1 \le r_2 \le \infty \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \le 1. \tag{1.1.2}$$

Of course (1.1.1) is a short version of the pointwise multiplication inequality

$$\|fg|L_r(\mathbf{R}^n)\| \le c\,\|f|L_{r_1}(\mathbf{R}^n)\| \cdot \|g|L_{r_2}(\mathbf{R}^n)\|, \tag{1.1.3}$$

where in that special case $c = 1$ may be chosen. With exception of Subsection 1.2, all spaces in this paper are defined on $\mathbf{R}^n$. This justifies to omit $\mathbf{R}^n$ in the sequel. One of the main aims of the paper is to study the appropriate counterparts of (1.1.1) and (1.1.2) for the spaces B^s_{pq} and F^s_{pq}. That means for a given smoothness s we are looking for

$$B^s_{p_1q_1}\, B^s_{p_2q_2} \subset B^s_{pq} \tag{1.1.4}$$

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and

$$(1.1.5) \qquad F^s_{p_1q_1}\, F^s_{p_2q_2} \subset F^s_{pq},$$

interpreted similarly as in (1.1.3). Recall that the spaces B^s_{pq} and F^s_{pq} cover some well-known spaces, such as the (fractional) Sobolev spaces, the classical Besov spaces, the Hölder-Zygmund spaces and the (inhomogeneous) Hardy spaces. These spaces have been studied systematically in [24, 26]. Our interest in inequalities of type (1.1.4) and (1.1.5) comes from some recent work on eigenvalue distributions of degenerate elliptic differential operators, where (1.1.1), (1.1.4) and (1.1.5) play a decisive role. We refer to [5]. In Subsection 1.2 we outline roughly this motivation for (1.1.4) and (1.1.5). If we plot s against $1/p$, see Fig.1, then the distinguished strip

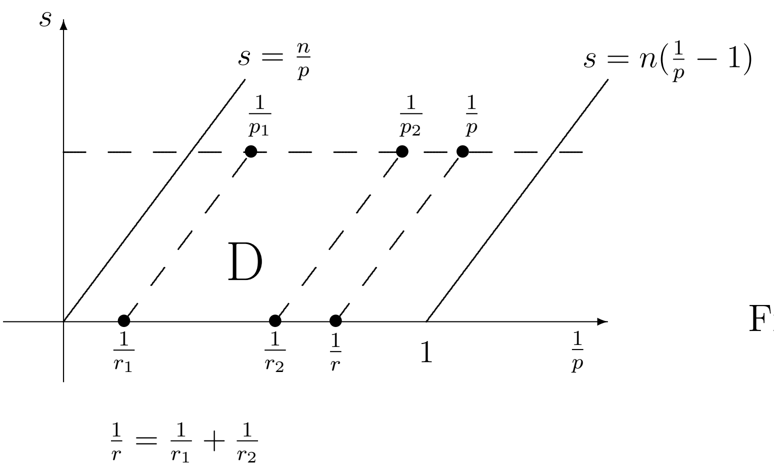

Fig.1

$$(1.1.6) \qquad D: \quad 0 < p < \infty, \quad n\,(\frac{1}{p} - 1)_+ < s < \frac{n}{p}$$

plays a crucial role. The lines of slope n indicate embeddings with constant differential dimension $s - n/p$. One of the main results of this paper reads as follows :
Let s, p_1, p_2 and p be given as indicated in Fig.1 and let $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Then

(i) (1.1.4) holds if and only if $0 < q_1 \leq r_1$, $0 < q_2 \leq r_2$ and $q \geq \max(q_1, q_2)$ and

(ii) (1.1.5) holds if and only if $q \geq \max(q_1, q_2)$.

In other words, the classical Hölder inequality (1.1.1) which corresponds to the bottom line $s = 0$ of the strip D in the way indicated in Fig.1 is shifted along the lines of slope n

to the level of smoothness s. This situation justifies to denote (1.1.4) and (1.1.5) under the just sketched values of the involved parameters as Hölder inequalities. As a special case of (1.1.5) we have

$$H^s_{p_1}\, H^s_{p_2} \,\subset\, H^s_p$$

under the conditions indicated in Fig.1, where $H^s_p = F^s_{p,2}$ are the (fractional) Sobolev-Hardy spaces. In accordance with the limiting cases in (1.1.1), (1.1.2) we pay some attention to related limiting cases with respect to the strip D, that means

$$s = \frac{n}{p_1} \quad \text{and} \quad s = n(\frac{1}{p} - 1)_+ \tag{1.1.7}$$

including especially the bottom line $s = 0$. These two limiting cases are connected with L_∞ and L_1, respectively. Inevitable linked with Hölder inequalities of type (1.1.4) and (1.1.5) are sharp embeddings with constant smoothness s of type

$$B^s_{pu} \,\subset\, F^s_{pq} \,\subset\, B^s_{pv},$$

and with constant differential dimension $s - n/p$. These embeddings correspond to the horizontal lines and the lines of slope n in Fig.1, respectively. We give final answers in these cases. In connection with the above sketched limiting cases (1.1.7) where L_∞ and L_1 make their natural appearance we complement the just mentioned two sharp embeddings by sharp assertions under which the spaces B^s_{pq} and F^s_{pq} are embedded in L_∞ (which is known) and in L_1^{loc}, respectively. Of course, the latter can be rephrased as the search for sharp conditions such that B^s_{pq} and F^s_{pq} consist solely of regular distributions. Some of the incorporated sharp embeddings are known, especially the "if"-parts. But we seal several gaps, mostly related to the "only if"-parts.

The plan of the paper is the following. As mentioned above we provide in Subsection 1.2 motivations for inequalities of type (1.1.4) and (1.1.5). Section 2 contains the necessary definitions and some preparations about paramultiplication. In Section 3 we present the results about sharp embeddings : with constant smoothness, with constant differential dimension, in L_∞, and in L_1^{loc}. Section 4 deals with Hölder inequalities. In Subsection 4.1 we describe the necessary conditions for s and $1/p$ such that we have (1.1.4) and (1.1.5), see Fig.3a and Fig.3b. The cases of our interest correspond to the heavy lines. In Subsection 4.2 we formulate the Hölder inequalities, whereas Subsection 4.3 deals with the

indicated limiting cases. Finally Subsection 4.4 contains further results connected with the shaded areas in Fig.3 covering the region of necessity for the inequalities (1.1.4) and (1.1.5) treated in Subsection 4.1 . It comes out that this is also the region of sufficiency with some peculiarities on the border lines. Proofs are presented in Section 5.
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1.2 Motivation

Let Ω be a bounded smooth domain in $\mathbf{R}^n$. Let Δ be the Laplacian and let

$$Au(x) \;=\; a(x)\,(id - \Delta)\,a(x)\,u(x) \qquad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0$$

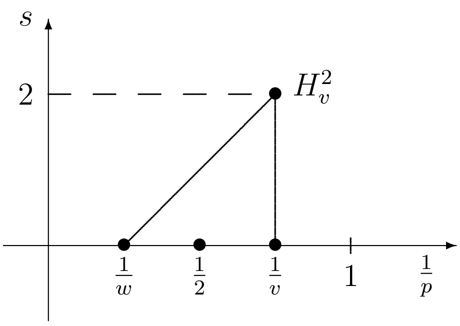
be a degenerate elliptic differential operator with non-smooth coefficients related to the Dirichlet problem. Assume that

$$Bu(x) \;=\; b(x)\,(id - \Delta)^{-1}\,b(x)\,u(x) \qquad \text{with } b(x) = a^{-1}(x) \in L_r(\Omega),$$

where $2 \le n < r \le \infty$, makes sense as the inverse of A. In accordance with well-known classical assertions we obtained in [5] sharp assertions for the distribution of the eigenvalues λ_k of A of type $\lambda_k \approx k^{2/n}$ based on two ingredients :
(i) Sharp assertions for the entropy numbers of the compact embeddings

$$id: \quad B^{s_1}_{p_1 q_1}(\Omega) \longrightarrow B^{s_0}_{p_0 q_0}(\Omega), \qquad s_1 - \frac{n}{p_1} > s_0 - \frac{n}{p_0}, \quad s_1 > s_0,$$

(and similarly with $F^s_{pq}(\Omega)$),
(ii) Sharp embeddings of type (1.1.1), (1.1.4) and (1.1.5).
To describe the flavour of this approach we start with $L_2(\Omega)$, multiplication with



$$\frac{1}{v} = \frac{1}{2} + \frac{1}{r}$$

$$\frac{1}{w} = \frac{1}{2} - \frac{1}{r}$$

Fig.2

$b(x)$ brings us to $L_v(\Omega)$, where we used (1.1.1). Then we apply $(id - \Delta)^{-1}$. We arrive at $H^2_v(\Omega)$. The embedding of $H^2_v(\Omega)$ in $L_w(\Omega)$ is compact since the slope of the corresponding line is steeper than n. Finally a second multiplication with $b(x)$ brings us back to $L_2(\Omega)$. The compact embedding is the point where the entropy numbers come in, whereas for the multiplications with $b(x)$ one needs inequalities of type (1.1.1) in the outlined case as sharp as possible. The interplay between the two ingredients is clear. Necessary explanations and details, especially about the role played by the entropy numbers, may be found in [5]. It is not necessary to begin with $L_2(\Omega)$ as the basic space. One can start with other suitable spaces in Fig.2. Then the triangle in Fig.2 is shifted, say in the distinguished strip D in Fig.1. Instead of the classical inequality (1.1.1) one has to work with the Hölder inequalities (1.1.4) and (1.1.5).

2 Definitions and preparations

2.1 Definitions

In general all functions, spaces, etc. are defined on the Euclidean n-space $\mathbf{R}^n$. So we omit $\mathbf{R}^n$ in notations. Further we shall use $\mathbf{N}$ to denote the set of natural numbers, $\mathbf{N}_0$ to denote $\mathbf{N} \cup \{0\}$, and a_+ instead of $\max(a, 0)$.

Let S be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions. By S' we denote its topological dual, the space of tempered distributions. If $\varphi \in S$ then

$$\mathcal{F}\varphi(x) \quad = \quad (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-i\,x\xi}\, \varphi(\xi)\, d\xi, \quad x \in \mathbf{R}^n,$$

denotes the Fourier transform $\mathcal{F}\varphi$ of φ. As usual, $\mathcal{F}^{-1}\varphi$ means the inverse Fourier transform of φ. Both, $\mathcal{F}, \mathcal{F}^{-1}$ are extended to S' in the standard way.

Let $\psi \in S$ be a non-negative function with

$$\text{(2.1.1)} \qquad \begin{cases} \psi(x) = 1 & \text{if} \quad |x| \le 1, \\ \psi(x) = 0 & \text{if} \quad |x| \ge \frac{3}{2}. \end{cases}$$

We define

$$\text{(2.1.2)} \qquad \begin{cases} \varphi_o(x) = \psi(x), \\ \varphi_1(x) = \psi(\frac{x}{2}) - \psi(x), \\ \varphi_k(x) = \varphi_1(2^{-k+1}x), \quad x \in \mathbf{R}^n, \quad k = 2, 3, \dots . \end{cases}$$

It follows

$$\sum_{j=0}^{M} \varphi_j(x) = \psi(2^{-M}x), \quad M = 0, 1, \ldots,$$

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \quad x \in \mathbf{R}^n,$$

(2.1.3) $$\operatorname{supp} \varphi_j \subset \{ x : \quad 2^{j-1} \le |x| \le 3 \cdot 2^{j-1} \}, \quad j = 1, \ldots, n,$$

and if

(2.1.4) $\frac{3}{2} \cdot 2^{j-1} \le |x| \le 2^j$ then it holds $\varphi_k(x) = \delta_{k,j}$, $j = 1, 2, \ldots$; $k = 0, 1, \ldots$.

L_p are the usual Lebesgue spaces on $\mathbf{R}^n$.

Definition 2.1.1. *Let $-\infty < s < \infty$, and $0 < q \le \infty$.*

(i) If $0 < p < \infty$ we put

$$F^s_{pq} = \left\{ f \in S' : \|f|F^s_{pq}\| = \| \left(\sum_{j=0}^{\infty} 2^{sjq} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot)|^q \right)^{1/q} |L_p\| < \infty \right\}$$

(usual modification if $q = \infty$).

(ii) If $0 < p \le \infty$ we put

$$B^s_{pq} = \left\{ f \in S' : \|f|B^s_{pq}\| = \left(\sum_{j=0}^{\infty} 2^{sjq} \|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot)|L_p\|^q \right)^{1/q} < \infty \right\}$$

(usual modification if $q = \infty$).

Remark 2.1.1. These types of spaces are studied systematically in [24, 26]. We always assume that the reader is familiar with it. Recall some special cases :

$F^0_{p,2} = L_p$, $1 < p < \infty$ (Lebesgue spaces),

$F^s_{p,2} = W^s_p$, s non-negative integer, $1 < p < \infty$ (Sobolev spaces),

$F^s_{p,2} = H^s_p$, $1 < p < \infty$, $s \in \mathbf{R}$ (fractional Sobolev spaces),

$F^0_{p,2} = h_p$, $0 < p < \infty$ (inhomogeneous Hardy spaces),

B^s_{pq}, $s > 0$, $1 < p < \infty$, $1 \le q \le \infty$ (classical Besov spaces),

$B^s_{\infty\infty} = \mathcal{C}^s$, $s > 0$ (Hölder-Zygmund spaces).

2.2 Pointwise multiplication

Let ψ be the function defined in (2.1.1) and let $\{\varphi_j\}_{j=0}^{\infty}$ be the corresponding system (cf. (2.1.2)). For brevity we put

$$f_j(x) \;=\; \mathcal{F}^{-1}[\varphi_j(\xi)\mathcal{F}f(\xi)]\,(x),$$

and

$$f^j(x) \;=\; \mathcal{F}^{-1}[\psi(2^{-j}\xi)\mathcal{F}f(\xi)]\,(x), \quad j=0,1,2,\ldots. \tag{2.2.1}$$

It is easily checked that $\lim_{j\to\infty} f^j \underset{S'}{=} f$ for any $f \in S'$. Moreover, f^j is an entire analytic function of exponential type. Hence, the product $f^j \cdot g^j$ makes sense for any j and any $f, g \in S'$. We define

$$f\cdot g \quad = \quad \lim_{j\to\infty} f^j\cdot g^j \tag{2.2.2}$$

whenever this limit exists. Note that

$$\begin{aligned} f\cdot g \;&=\; \lim_{\ell\to\infty}\left(\sum_{j=0}^{\ell} f_j\right)\,\left(\sum_{k=0}^{\ell} g_k\right) \\ &= \sum_{j=2}^{\infty} f^{j-2}\,g_j + \sum_{k=2}^{\infty} f_k\,g^{k-2} + \sum_{k=0}^{\infty}\sum_{j=k-1}^{k+1} f_k\,g_j \\ &= \sum{}'\,(f,g) + \sum{}''\,(f,g) + \sum{}'''(f,g) \qquad (\text{put } g_{-1}\equiv 0). \end{aligned}$$

The advantage of such a decomposition is based on

$$\operatorname{supp}\mathcal{F}\,(f^{k-2}\,g_k)\;\subset\;\{\xi\;:\;3\cdot 2^{k-3}\le|\xi|\le 11\cdot 2^{k-3}\,\} \tag{2.2.3}$$

and

$$\operatorname{supp}\mathcal{F}\,(\sum_{j=k-1}^{k+1} f_k\,g_j)\;\subset\;\{\xi\;:\;|\xi|\le 9\cdot 2^{k-1}\,\}. \tag{2.2.4}$$

Remark 2.2.1. Recall the Fatou property of the underlying spaces. Let A^s_{pq} denote either B^s_{pq} or F^s_{pq} . If $\{f^j\,g^j\}_j$ is a Cauchy sequence in S' with limit h and if

$$\sup_j \|f^j\,g^j|A^s_{p,q}\| = A < \infty$$

then it follows $h \in A^s_{pq}$ and $\|h|A^s_{pq}\| \le c\,A$, where c is independent of f and g, cf. [7].

Remark 2.2.2. The operator

$$\Pi_f\;:\quad g\;\longrightarrow\;\sum_{j=2}^{\infty} f^{j-2}\,g_j$$

is called paramultiplication operator. Estimates for this operator are the heart of several contributions to the problem of pointwise multiplication [24, 14, 30, 31, 32, 20, 21]. Further they are of importance in microlocal analysis and in the theory of Calderon-Zygmund singular integral operators [3, 16, 30, 31, 32].

The essence of the needed estimates are formulated in the following proposition, where we make use of the abbreviation $h_\infty = L_\infty$.

Proposition 2.2.1. *Let* $0 < p_1 \le \infty$, $0 < p_2 \le \infty$, $1/p = 1/p_1 + 1/p_2$ *and* $0 < q \le \infty$.
(i) *Let* $0 < p < \infty$. *Then*

$$(2.2.5)\quad \| \Big(\sum_{k=0}^{\infty} |\mathcal{F}^{-1}\Big[\varphi_k \mathcal{F}{\textstyle\sum}'(f,g)\Big](\cdot)|^q \Big)^{1/q} |L_p\| \le c\, \|f|h_{p_2}\|\, \| \Big(\sum_{k=0}^{\infty} |g_k(\cdot)|^q \Big)^{1/q} |L_{p_1}\|,$$

where c *is independent of* f *and* g *(usual modification if* $q = \infty$*).*
(ii) *Let* $0 < p \le \infty$. *Then*

$$(2.2.6)\quad \|\mathcal{F}^{-1}\Big[\varphi_k\, \mathcal{F}{\textstyle\sum}'''(f,g)\Big](\cdot)|L_p\| \le c \max_{-1\le j\le 1} \sum_{\ell=-2}^{\infty} \|f_{k+\ell}|L_{p_1}\|\, \|g_{k+\ell+j}|L_{p_2}\|$$

if $p \ge 1$ *and*

$$\|\mathcal{F}^{-1}\Big[\varphi_k\, \mathcal{F}{\textstyle\sum}'''(f,g)\Big](\cdot)|L_p\| \le c \max_{-1\le j\le 1} \Big(\sum_{\ell=-2}^{\infty} 2^{\ell n(1-p)} \|f_{k+\ell}|L_{p_1}\|^p\, \|g_{k+\ell+j}|L_{p_2}\|^p \Big)^{1/p}$$

if $p < 1$, *where* c *is independent of* f, g *and* $k \in \mathbf{N}_0$, *(put* $f_r = g_r = 0$ *if* $r < 0$*).*
(iii) *Let* $0 < p < \infty$ *and* $s > n\,(\frac{1}{p} - 1)_+$. *Then*

$$\|\sup_k 2^{ks}|\mathcal{F}^{-1}\Big[\varphi_k\, \mathcal{F}{\textstyle\sum}'''(f,g)\Big](\cdot)|\, |L_p\|$$
$$\le c \max_{-1\le j\le 1} \|\sup_k 2^{ks/2}\, |f_k|\, |L_{p_1}\|\, \|\sup_k 2^{ks/2}\, |g_{k+j}|\, |L_{p_2}\|,$$

where c *is independent of* f *and* g.

Remark 2.2.3. In the scalar case of (i), given by

$$\|\mathcal{F}^{-1}[\varphi_k\, \mathcal{F}{\textstyle\sum}'(f,g)](\cdot)\, |L_p\| \le c\, \|f|h_{p_2}\| \max_{-1\le j\le 1} \|g_{k+j}|L_{p_1}\|,$$

also $p = \infty$ is admissible. Part (iii) is taken from [30], Theorem 3.7, complemented by the use of the Hölder inequality with respect to $1/p = 1/p_1 + 1/p_2$. Proofs of (i) and (ii) will be given in Subsection 5.5.

3 Sharp embeddings

3.1 Embeddings with constant smoothness

Here "$\subset$" stands for continuous embedding. Recall that all spaces are defined on $\mathbf{R}^n$.

Theorem 3.1.1. *(i) Let $s \in \mathbf{R}$, $0 < p < \infty$, $0 < q \le \infty$, $0 < u \le \infty$ and $0 < v \le \infty$. Then*

(3.1.1) $$B^s_{pu} \subset F^s_{pq} \subset B^s_{pv}$$

if and only if

(3.1.2) $$0 < u \le \min(p,q) \quad and \quad \max(p,q) \le v \le \infty.$$

(ii) Let $0 < u \le \infty$ and $0 < v \le \infty$. Then

(3.1.3) $$B^0_{1,u} \subset L_1 \subset B^0_{1,v}$$

if and only if

$$0 < u \le 1 \quad and \quad v = \infty.$$

(iii) Let $0 < u \le \infty$. Then

(3.1.4) $$F^0_{1,u} \subset L_1$$

if and only if

$$0 < u \le 2.$$

Furthermore

(3.1.5) $$L_1 \not\subset F^0_{1,\infty}.$$

(iv) Let $0 < u \le \infty$ and $0 < v \le \infty$. Then

(3.1.6) $$B^0_{\infty,u} \subset L_\infty \subset B^0_{\infty,v}$$

if and only if

$$0 < u \le 1 \quad and \quad v = \infty.$$

Remark 3.1.1. Let C be the space of all complex-valued bounded and uniformly continuous functions on $\mathbf{R}^n$ normed in the usual way. In (3.1.6) one can replace L_∞ by C.

Remark 3.1.2. By (3.1.1) we know $F^0_{1,\infty} \subset B^0_{1,\infty}$. The assertion (3.1.5) shows that the second inequality in (3.1.3) can not improved by replacing $B^0_{1,\infty}$ by $F^0_{1,\infty}$.

Remark 3.1.3. The "if"-parts of the theorem are known, see [24], Proposition 2.3.2/2, p.47, Proposition 2.5.7, p.89 and Theorem 2.5.8/1, p.92. In other words, we have to complement these known assertions by the "only if"-parts and the proof of (3.1.5) .

3.2 Embeddings with constant differential dimension

Recall that $s - n/p$ is called the differential dimension both of B^s_{pq} and F^s_{pq} . It is a characteristic number which plays a crucial role in the theory of these spaces, see, for instance, Fig.1 and the accompanying remarks.

Theorem 3.2.1. *(i) Let $0 < p_0 < p < p_1 \le \infty$, $s \in \mathbf{R}$,*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1},$$

$0 < q \le \infty$, $0 < u \le \infty$ and $0 < v \le \infty$. Then

(3.2.1) $$B^{s_0}_{p_o u} \subset F^s_{pq} \subset B^{s_1}_{p_1 v}$$

if and only if

(3.2.2) $$0 < u \le p \le v \le \infty.$$

(ii) Let $0 < p < p_1 < \infty$, $s \in \mathbf{R}$,

(3.2.3) $$s - \frac{n}{p} = s_1 - \frac{n}{p_1}$$

and $0 < q \le \infty$. Then

(3.2.4) $$F^s_{p\infty} \subset F^{s_1}_{p_1 q}.$$

(iii) Let $0 < p < 1$ and $0 < q \le \infty$. Then

(3.2.5) $$B^{n(\frac{1}{p}-1)}_{pq} \subset L_1$$

if and only if

$$0 < q \le 1.$$

Remark 3.2.1. The "if"-part of (i) is due to Jawerth and Franke, see [10, 7], [24], p.131 and [25], p.191. Furthermore, (ii) is mentioned here for the sake of completeness, see [24], Theorem 2.7.1, p.129. Of course, by the monotonicity of the F^s_{pq} -spaces, ∞ in (3.2.4) can be replaced by any positive number. As mentioned in the introduction, embeddings in L_1 and L_∞ deserve special attention. The L_∞-counterpart of (3.2.5) will be described in the next subsection.

3.3 Embeddings in L_∞ and in L_1^{loc}

The space C has been defined in Remark 3.1.1.

Theorem 3.3.1. *(i) Let $s \in \mathbf{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Then the following three assertions are equivalent :*

$$(a) \qquad F^s_{pq} \subset L_\infty,$$

$$(b) \qquad F^s_{pq} \subset C,$$

$$(c) \qquad \textit{either } s > \frac{n}{p} \textit{ or } s = \frac{n}{p} \textit{ and } 0 < p \le 1.$$

(ii) Let $s \in \mathbf{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. Then the following three assertions are equivalent :

$$(a) \qquad B^s_{pq} \subset L_\infty,$$

$$(b) \qquad B^s_{pq} \subset C,$$

$$(c) \qquad \textit{either } s > \frac{n}{p} \textit{ or } s = \frac{n}{p} \textit{ and } 0 < q \le 1. \tag{3.3.1}$$

Remark 3.3.1. This theorem is known. We incorporate it both for sake of completeness and because it will be of great service later on in this paper. A proof of (i) may be found [7]. As far as (ii) is concerned we refer to [23], Theorem 1, p.133, see also [24], 2.8.3, p.146 and [25], 2.8.3, p.211 .

Remark 3.3.2. Let A^s_{pq} either B^s_{pq} or F^s_{pq} . Then A^s_{pq} is called a multiplication algebra if

$$A^s_{pq} \quad A^s_{pq} \ \subset \ A^s_{pq}$$

where the multiplication of two distributions is given by (2.2.2). Part (i) of the theorem can be complemented by

$$(d) \qquad F^s_{pq} \quad \text{is a multiplication algebra.}$$

A corresponding assertion for B^s_{pq} is only "almost" true. More precisely : B^s_{pq} is a multiplication algebra if and only if

$$(d) \qquad \text{either } s > \frac{n}{p} \quad \text{with} \quad 0 < p \leq \infty$$

$$\text{or} \quad s = \frac{n}{p} \quad \text{with} \quad 0 < p < \infty \quad \text{and } 0 < q \leq 1.$$

This assertion differs from (3.3.1) by the case $s = 0$, $p = \infty$. We refer to [7] and [23], see also [24], 2.8.3 (with the indicated correction as far as the case $s = 0$, $p = \infty$ is concerned) and [20], p.56 . The case $B^0_{\infty q}$ will be established in Remark 4.3.5 in the indicated way : It is not a multiplication algebra. Although the study of multiplication algebras fits quite well in the framework of our paper we shall not stress this point in the sequel. We are mostly interested in multiplication with essentially unbounded function.
Of course, L^{loc}_1 stands for the collection of all complex-valued functions which are locally integrable in $\mathbf{R}^n$. It is interpreted here as the set of all regular distributions on $\mathbf{R}^n$.

Theorem 3.3.2. *(i) Let $s \in \mathbf{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then the following two assertions (i_1) and (i_2) are equivalent :*

$$(i_1) \qquad F^s_{pq} \subset L^{loc}_1$$

$$(i_2) \qquad \textit{either} \quad 0 < p < 1, \quad s \geq n\,(\frac{1}{p} - 1), \quad 0 < q \leq \infty, \tag{3.3.2}$$

$$\textit{or} \quad 1 \leq p < \infty, \quad s > 0, \quad 0 < q \leq \infty,$$

$$\textit{or} \quad 1 \leq p < \infty, \quad s = 0, \quad 0 < q \leq 2. \tag{3.3.3}$$

(ii) Let $s \in \mathbf{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then the following two assertions (ii_1) and (ii_2) are equivalent :

$$(ii_1) \qquad B^s_{pq} \subset L^{loc}_1$$

$$(ii_2) \qquad \textit{either} \quad 0 < p \leq \infty, \quad s > n\,(\frac{1}{p} - 1)_+, \quad 0 < q \leq \infty,$$

$$\textit{or} \quad 0 < p \leq 1, \quad s = n(\frac{1}{p} - 1), \quad 0 < q \leq 1,$$

$$\textit{or} \quad 1 < p \leq \infty, \quad s = 0, \quad 0 < q \leq \min(p, 2). \tag{3.3.4}$$

Remark 3.3.3. If $s > n(\frac{1}{p}-1)_+$ then it is well-known that B^s_{pq} and F^s_{pq} consist of regular distributions. In other words, the interesting part of the theorem is the final classification what happens in the limiting case $s = n(\frac{1}{p}-1)_+$.

We compare the above theorem with the sharp embeddings described in Subsections 3.1 and 3.2. The case $p = \infty$ plays a special role. Without going in details we mention

$$B^0_{\infty,2} \subset F^0_{\infty,2} = bmo, \tag{3.3.5}$$

see [24], pp.37, 50, 93 for definitions and explanations. As far as the spaces $F^0_{\infty,q}$ are concerned we refer also to [14, 8].

Corollary 3.3.1. *(i) Let* $s \in \mathbf{R}$, $0 < p < \infty$ *and* $0 < q \le \infty$. *Let* A^s_{pq} *be either* B^s_{pq} *or* F^s_{pq} . *Then the following two assertions* (i_1) *and* (i_2) *are equivalent :*

$$(i_1) \qquad A^s_{pq} \subset L^{loc}_1$$

$$(i_2) \qquad A^s_{pq} \subset L_{\bar{p}} \quad \textit{with } \bar{p} = \max(1,p). \tag{3.3.6}$$

(ii) Let $s \in \mathbf{R}$ *and* $0 < q \le \infty$. *Then the following two assertions* (ii_1) *and* (ii_2) *are equivalent :*

$$(ii_1) \qquad B^s_{\infty q} \subset L^{loc}_1$$

$$(ii_2) \qquad B^s_{\infty q} \subset \ bmo. \tag{3.3.7}$$

Proof. Part (ii) is covered by (3.3.5) and the above theorem, especially (3.3.4). The F-case of part (i) related to (3.3.2) follows from (3.2.4), (3.2.3) and (3.1.3) :

$$F^{n\,(\frac{1}{p}-1)}_{pq} \subset B^0_{1,1} \subset L_1 \qquad (p < 1).$$

The F-case related to (3.3.3) is clear since $F^0_{p,2} = L_p$, $1 < p < \infty$ and

$$F^0_{1,2} = h_1 \subset L_1. \tag{3.3.8}$$

The B-case of part (i) follows immediately from the above theorem, (3.2.5), (3.1.1) and (3.3.8). ■

Remark 3.3.4. Let again A^s_{pq} be either B^s_{pq} or F^s_{pq} and let

$$A^s_{pq} \subset L_1. \tag{3.3.9}$$

Let ψ be a C^∞-function with $\psi(x) = 0$ near the origin and $\psi(y) = 1$ if, say, $|y| \geq 1$. Let

$$r_j \; : \quad f \longrightarrow \mathcal{F}^{-1}\left[\psi(\xi)\,\frac{\xi_j}{|\xi|}\,\mathcal{F}f(\xi)\right](\cdot)$$

be the inhomogeneous Riesz transforms, $j = 1, \ldots, n$. Recall that the (inhomogeneous) Hardy spaces $h_1 = F^0_{1,2}$ can also be characterized as the collection of all $f \in L_1$ with $r_j\, f \in L_1$ if $j = 1, \ldots, n$, see [24], pp.93/94. Since $r_j A^s_{pq} \subset A^s_{pq}$ we can improve (3.3.9) by

$$A^s_{pq} \subset h_1.$$

In other words, if $p = 1$ then (3.3.6) can be strengthened by

$$A^s_{1,q} \subset h_1.$$

Now (3.3.7) looks a little bit more natural since $bmo = h_1'$.

Remark 3.3.5. For better reference we formulate one consequence of Theorem 3.1.1, Theorem 3.2.1, Theorem 3.3.2, Remark 2.1.1 (first item) and (3.3.8) once again. Let $s > 0$, $0 < p < \infty$ and $0 < q \leq \infty$. If

$$1 \geq \frac{1}{r} = \frac{1}{p} - \frac{s}{n} > 0.$$

then holds

$$B^s_{pq} \subset L_r$$

if and only if

$$0 < q \leq r.$$

A corresponding assertion for F^s_{pq} holds without restrictions on q.

4 Hölder inequalities

4.1 Necessary conditions for s and p

Let $s \in \mathbf{R}$, $0 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 < p \leq \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. We ask under which conditions

$$B^s_{p_1 q_1} \quad B^s_{p_2 q_2} \subset B^s_{pq} \tag{4.1.1}$$

and

$$(4.1.2) \qquad F^s_{p_1q_1} \quad F^s_{p_2q_2} \subset F^s_{pq},$$

hold, where in case of (4.1.2) we assume, in addition, $p_1 \neq \infty$, $p_2 \neq \infty$ and $p \neq \infty$.

Theorem 4.1.1. *If either* (4.1.1) *or* (4.1.2) *hold under the indicated general conditions for the parameters then*

$$(4.1.3) \qquad \max\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \leq \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2},$$

$$(4.1.4) \qquad 2s \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right)_+,$$

and

$$(4.1.5) \qquad s \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right).$$

Remark 4.1.1. Of course, (4.1.5) can be rewritten as

$$(4.1.6) \qquad s - \frac{n}{p} \leq s - \frac{n}{p_1} + s - \frac{n}{p_2}.$$

Recall that $s - \frac{n}{p}$ is the differential dimension both of B^s_{pq} and F^s_{pq} . In other words, the differential dimension of the target spaces in (4.1.1) or (4.1.2) has to be less than or equal to the sum of the differential dimension of the spaces on the left-hand sides of (4.1.1) and (4.1.2). In the Figures 3a and 3b we summarized the above restrictions in dependence on whether $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ or $\frac{1}{p_1} + \frac{1}{p_2} > 1$. If one compares Figures 1,

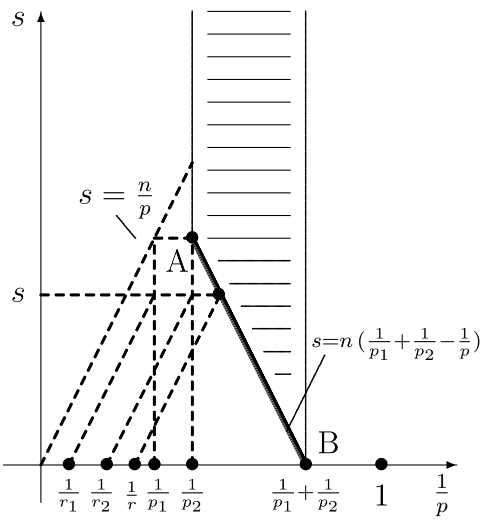


Fig. 3a $\qquad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$

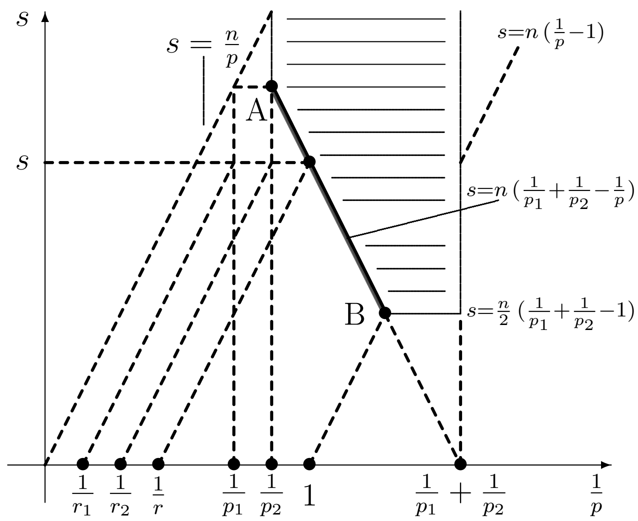


Fig. 3b

3a and 3b then we are mainly interested in those cases, where we have equality in (4.1.5) and (4.1.6). This corresponds to the heavy lines in Figures 3a and 3b and in that case

(4.1.6) is identical with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ in Fig.1 . The limiting cases (1.1.7) correspond to the points A and B in Figures 3a and 3b, respectively. However in the final Subsection 4.4 we sketch briefly what happens inside of the shaded regions.

4.2 The main results

Recall that all the spaces are defined on $\mathbf{R}^n$. Furthermore, the Hölder inequalities we are looking for are characterized by the situation sketched in Fig.1 and indicated by the heavy lines in Figures 3a and 3b. There is a significant difference between B-spaces and F-spaces acting as pointwise multiplier spaces which can be clearly seen by the theorem below and which will be prepared by the following proposition.

Proposition 4.2.1. *Let* $s \in \mathbf{R}$, $0 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < p < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ *and* $0 < q \leq \infty$. *Let*

$$\frac{1}{r_1} = \frac{1}{p_1} - \frac{s}{n} > 0, \qquad \text{and} \quad \frac{1}{p_2} + \frac{1}{r_1} = \frac{1}{p}. \tag{4.2.1}$$

Let independently

$$A^s_{p_2q_2} \quad \text{be either} \quad B^s_{p_2q_2} \quad \text{or} \quad F^s_{p_2q_2}$$

and

$$A^s_{pq} \quad \text{be either} \quad B^s_{pq} \quad \text{or} \quad F^s_{pq}.$$

If

$$B^s_{p_1q_1} \quad A^s_{p_2q_2} \subset A^s_{pq} \tag{4.2.2}$$

then

$$q_1 \leq r_1. \tag{4.2.3}$$

Remark 4.2.1. If one replaces the pointwise multiplier space $B^s_{p_1q_1}$ by $F^s_{p_1q_1}$ then the restriction of type (4.2.3) simply does not occur, see the theorem below. As far as r_1 is concerned we refer to Fig.1. If r_2 and r are defined in a similar way, then the second part of (4.2.1) can be reformulated as

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2},$$

which coincides with Figures 1, 3a and 3b. However it is not assumed that the involved spaces are characterized by the points within $\overline{D}$, where D is given by (1.1.6).

Theorem 4.2.1. *Let* $s > 0$, $0 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < p < \infty$, $0 < q_1 \le \infty$, $0 < q_2 \le \infty$ *and* $0 < q \le \infty$. *Let*

(4.2.4) $$\frac{1}{r_1} = \frac{1}{p_1} - \frac{s}{n} > 0, \quad \frac{1}{r_2} = \frac{1}{p_2} - \frac{s}{n} > 0 \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} = \frac{1}{p} - \frac{s}{n} < 1.$$

(i) *Then holds*

(4.2.5) $$B^s_{p_1 q_1} \quad B^s_{p_2 q_2} \subset B^s_{pq}$$

if and only if

(4.2.6) $$0 < q_1 \le r_1, \quad 0 < q_2 \le r_2 \quad \text{and} \quad \infty \ge q \ge \max(q_1, q_2).$$

(ii) *Then holds*

(4.2.7) $$F^s_{p_1 q_1} \quad F^s_{p_2 q_2} \subset F^s_{pq}$$

if and only if

(4.2.8) $$\infty \ge q \ge \max(q_1, q_2).$$

Remark 4.2.2. Both (4.2.5) and (4.2.7) are the Hölder inequalities in the distinguished strip D in (1.1.6) we are looking for. The situation described in Fig.1 is the same as in (4.2.4). Compared with Figures 3a and 3b condition (4.2.4) corresponds to the heavy lines where the endpoints A and B are excluded. Furthermore, (4.2.4) is connected with embeddings with constant differential dimensions, see Theorem 3.2.1 and the broken lines in the Figures 3a and 3b ending at $1/r_1$, $1/r_2$, and $1/r$.

Remark 4.2.3. One can try to mix B-spaces and F-spaces in (4.2.5) and (4.2.7). We do not go into detail. By the above proposition it is quite clear what can be expected.

4.3 Two limiting cases

We discuss two limiting cases connected with the point A, Figures 3a and 3b and point B in Fig.3a.

First we assume $s = n/p_1$. In agreement with (4.2.1) we have

$$r_1 = \infty \quad \text{and} \quad p_2 = p.$$

However it comes out that (4.2.3) is no longer the natural condition. In contrast to Proposition 4.2.1 we have now to handle the B-spaces and the F-spaces separately.

Proposition 4.3.1. *(i) Let* $0 < p_1 \leq \infty$, $0 < p \leq \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ *and* $0 < q \leq \infty$. *Let*

(4.3.1) $$s = \frac{n}{p_1}.$$

If

(4.3.2) $$B^s_{p_1 q_1} \quad B^s_{pq_2} \subset B^s_{pq}$$

then

(4.3.3) $$q_1 \leq 1.$$

(ii) Let $0 < p_1 < \infty$, $0 < p < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ *and* $0 < q \leq \infty$. *Let* s *be given by* (4.3.1). *If*

(4.3.4) $$F^s_{p_1 q_1} \quad F^s_{pq_2} \subset F^s_{pq}$$

then

(4.3.5) $$p_1 \leq 1.$$

Proof. If (4.3.2) holds then it follows by the same arguments as in [7], pp.38/39, $B^s_{p_1 q_1} \subset L_\infty$. Similarly if (4.3.4) holds then we have necessarily $F^s_{p_1 q_1} \subset L_\infty$. Now (4.3.3), respectively (4.3.5) follow immediately from Theorem 3.3.1. ■

Theorem 4.3.1. *Let* $0 < p_1 < \infty$, $0 < p < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ *and* $0 < q \leq \infty$. *Let*

$$s = \frac{n}{p_1} \quad \text{and} \quad 0 < \frac{1}{p} - \frac{s}{n} = \frac{1}{r} < 1.$$

(i) Then holds

$$B^s_{p_1 q_1} \, B^s_{pq_2} \subset B^s_{pq}$$

if and only if

$$0 < q_1 \leq 1, \, 0 < q_2 \leq r \quad \text{and} \quad \infty \geq q \geq \max(q_1, q_2).$$

(ii) Then holds

$$F^s_{p_1 q_1} \, F^s_{pq_2} \subset F^s_{pq}$$

if and only if

$$0 < p_1 \leq 1 \quad \text{and} \quad \infty \geq q \geq \max(q_1, q_2).$$

Remark 4.3.1. This theorem is connected with the points A in the Figures 3a and 3b. The formulation is chosen in such a way that it can be compared immediately with Theorem 4.2.1. Instead of (4.2.6) with the expected $0 < q_1 \leq \infty$ we have now $0 < q_1 \leq 1$, and (4.2.8) must now be complemented by $0 < p_1 \leq 1$.

The second limiting case is connected with the point B in Fig.3a, that means with

$$s = 0 \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1.$$

Theorem 4.3.2. *Let* $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ *and* $0 < q \leq \infty$. *Let*

$$1 \leq p_1 \leq \infty, \quad 1 \leq p_2 < \infty, \quad \textit{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1.$$

Let independently

$$A^0_{p_1q_1} \quad \textit{be either} \quad B^0_{p_1q_1} \quad \textit{or} \quad F^0_{p_1q_1},$$

$$A^0_{p_2q_2} \quad \textit{be either} \quad B^0_{p_2q_2} \quad \textit{or} \quad F^0_{p_2q_2}$$

and

$$A^0_{pq} \quad \textit{be either} \quad B^0_{pq} \quad \textit{or} \quad F^0_{pq}$$

(we assume $A^0_{p_1q_1} = B^0_{p_1q_1}$ *if* $p_1 = \infty$*). Then holds*

$$A^0_{p_1q_1}\, A^0_{p_2q_2} \subset A^0_{pq} \tag{4.3.6}$$

if and only if

$$A^0_{p_1q_1} \subset L_{p_1}, \quad A^0_{p_2q_2} \subset L_{p_2} \quad \textit{and} \quad L_p \subset A^0_{pq}. \tag{4.3.7}$$

Remark 4.3.2. We compare the above assertion with the classical Hölder inequality

$$L_{p_1}\, L_{p_2} \subset L_p.$$

Then the Theorem 4.3.1 states that, within the scales B^s_{pq} and F^s_{pq} with $s = 0$, the classical Hölder inequality is not improvable.

As a consequence of Theorem 4.3.2, Theorem 3.1.1 and $F^0_{p,2} = L_p$, $1 < p < \infty$, we obtain the following corollary.

Corollary 4.3.1. *Let* $0 < q_1 \le \infty$, $0 < q_2 \le \infty$ *and* $0 < q \le \infty$.
Let

$$1 < p_1 < \infty, \quad 1 < p_2 < \infty, \quad \textit{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1.$$

(i) Then holds

$$B^0_{p_1 q_1}\, B^0_{p_2 q_2} \subset B^0_{pq}$$

if and only if

$$q_1 \le \min(p_1, 2), \quad q_2 \le \min(p_2, 2) \quad \textit{and} \quad q \ge \max(p, 2).$$

(ii) Then holds

(4.3.8)
$$F^0_{p_1 q_1}\, F^0_{p_2 q_2} \subset F^0_{pq}$$

if and only if

$$q_1 \le 2, \quad q_2 \le 2 \quad \textit{and} \quad q \ge 2.$$

Remark 4.3.3. In case $1 \le p_1 < \infty$, $1 \le p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2 = 1$ Theorem 4.3.2 yields : there do not exist q_1, q_2 and q such that (4.3.8) holds. This follows from (3.1.5).

Remark 4.3.4. Of special interest is also the situation in case $p_1 = \infty$. Let $1 \le p < \infty$, $0 < q_1 \le \infty$ and $0 < q \le \infty$. By (4.3.7) and Theorem 3.1.1 we have

(4.3.9)
$$B^0_{\infty q_1}\, B^0_{pq} \subset B^0_{pq}$$

if and only if

$$0 < q_1 \le 1 \quad \text{and} \quad p = q = 2 \quad (\text{that means} \quad B^0_{2,2} = L_2.)$$

and

$$B^0_{\infty q_1}\, F^0_{pq} \subset F^0_{pq}$$

if and only if

$$0 < q_1 \le 1, \quad 1 < p < \infty \quad \text{and} \quad q = 2 \quad (\text{that means} \quad F^0_{p,2} = L_p).$$

Hence, with the obvious exception of L_p the space L_∞ is not contained in the set of pointwise multipliers of these spaces, which was proved earlier by [8]. (4.3.9) improves also some results of Bourdaud [4].

Not as a consequence of Theorem 4.3.2 but as a consequence of the proof of this theorem one obtains the following corollary. Here b^s_{pq} denotes the closure of S in B^s_{pq} , equipped with the same quasi-norm as B^s_{pq} .

Corollary 4.3.2. *Let* $0 < q_1 \le \infty$, $0 < q_2 \le \infty$ *and* $0 < q \le \infty$. *Then holds*

$$b^0_{\infty q_1}\, b^0_{\infty q_2} \subset b^0_{\infty q}$$

if and only if

$$b^0_{\infty q_1} \subset L_\infty, \quad b^0_{\infty q_2} \subset L_\infty \quad \textit{and} \quad L_\infty \subset b^0_{\infty q}.$$

Remark 4.3.5. Because of $b^0_{\infty q} \ne L_\infty$ one consequence of the corollary is the fact that $b^0_{\infty q}$ is not a multiplication algebra, cf. Remark 3.3.2. But this implies that also $B^0_{\infty q}$ can not be an algebra with respect to pointwise multiplication.

4.4 Further results

We complement our previous considerations by collecting some further results, mostly connected with the shaded areas in the Figures 3a and 3b. We refer also to Subsections 3.3 and 4.3, where we characterized the conditions under which B^s_{pq} or F^s_{pq} are multiplication algebras.

Theorem 4.4.1. *Let* $0 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < q_1 \le \infty$, $0 < q_2 \le \infty$ *and* $0 < q \le \infty$. *Let* $(s, \frac{1}{p})$ *be a point in the interior of the shaded areas in Figures* 3a *and* 3b, *that means*

$$\max(\frac{1}{p_1}, \frac{1}{p_2}) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2}, \tag{4.4.1}$$

$$2s > n\,(\frac{1}{p_1} + \frac{1}{p_2} - 1)_+, \tag{4.4.2}$$

and

$$s > n\,(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}). \tag{4.4.3}$$

Then holds

$$B^s_{p_1 q_1}\, B^s_{p_2 q_2} \,\subset\, B^s_{pq} \tag{4.4.4}$$

if and only if

(4.4.5) $$q \geq \max(q_1, q_2)$$

and

(4.4.6) $$F^s_{p_1q_1}\, F^s_{p_2q_2} \subset F^s_{pq}$$

if and only if

(4.4.7) $$q \geq \max(q_1, q_2).$$

In case of the B-spaces $p_1 = \infty$ *and* $p_2 = \infty$ *are admitted in* (4.4.4).

Remark 4.4.1. We used the same notations as in Theorem 4.2.1 and it is immediately clear that the above assertion complements that theorem, where we are now not restricted to the strip D. The necessity of (4.4.5), resp. (4.4.7), is proved in [7].

Remark 4.4.2. The above theorem has a lot of forerunners. Without going into detail we refer to [1], [2], [7], [9], [12], [13], [14], [15], [17], [20], [21], [22], [23], [24], [29], [30], [31], [32], [33], [34].

Remark 4.4.3. Again let us cast a look on the Figures 3a and 3b. The results given so far answer the question of the existence of inequalities of type (4.4.4) and (4.4.6) in the interior of the shaded area (yes) and partly also on the boundary. Whereas on the vertical lines the answer is again yes in any case, except maybe the points A and B itself (these claims can be proved by suitable modifications of the proof of the above theorem given in 5.7, cf. Remark 5.7.1) the answer on the horizontal line $2s = n(\frac{1}{p_1} + \frac{1}{p_2} - 1)$ may be yes or no. A partly positive answer for the existence of (4.4.6) on this line is given by [7]. A negative answer for the existence of (4.4.4) one obtains in case $1/q_1 + 1/q_2 < 1$ by replacing the simple counterexample used in proof of (4.1.4) by a more sophisticated one, cf. [11] or [18, Lemma 4.3.1/3].

5 Proofs

5.1 Proofs of the assertions in Subsection 3.1

Proof of Theorem 3.1.1. Step 1. Proofs of the "if"-parts of (i), (ii) and (iv) may be found in [24], Proposition 2.3.2/2, p.46 and Proposition 2.5.7, p.89. The "if"-part of (iii) is an immediate consequence of the identity $F^0_{1,2} = h_1$, cf. [24], Theorem 1, p.92. So in what follows we restrict ourselves to the "only if"-parts and to the proof of (3.1.5). ■

Step 2. (Proof of (3.1.1)). First note that it will be sufficient to prove (3.1.1) in case of a fixed s. By well-known lifting properties (cf. [24], Theorem 2.3.8, p.58) it can be extended to arbitrary s afterwards.

Substep 2.1 . Let $\psi \in S$ such that

$$\text{supp}\ \mathcal{F}\psi \subset \{ \xi \, : \, \xi_1 \le 0, \ \frac{7}{4} \le |\xi| \le 2 \}. \tag{5.1.1}$$

For given complex numbers a_j and $e = (1, 0, \ldots, 0)$ we put

$$f(x) \ = \ \sum_{j=3}^{\infty} a_j e^{i\,\lambda_j <e,x>} \, \psi(x).$$

Then

$$\mathcal{F}f(\xi) \ = \ \sum_{j=3}^{\infty} a_j \, (\mathcal{F}\psi)(\xi - \lambda_j e).$$

Choose $\lambda_j = 2^j - 2, \ j = 3, 4, \ldots$, we arrive at

$$\mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](x) = a_j \, e^{i\,\lambda_j <e,x>} \, \psi(x),$$

where we have used (2.1.3), (2.1.4) and (5.1.1). Consequently we have

$$\|f|F^0_{pq}\| \ = \ c \, \left(\sum_{j=3}^{\infty} |a_j|^q \right)^{1/q}$$

and

$$\|f|B^0_{pu}\| \ = \ c \, \left(\sum_{j=3}^{\infty} |a_j|^u \right)^{1/u} .$$

The monotonicity of the l_q-norms gives

$$u \le q \le v. \tag{5.1.2}$$

Substep 2.2 . We wish to prove the counterpart of (5.1.2) with p instead of q. For this purpose we use local means, cf. [26], 1.8.4 and 2.5.3. Let $k_0, \ k^0 \in S$ such that

$$\text{supp}\ k_0 \subset \{y \, : \, |y| \le 1\}, \quad \text{supp}\ k^0 \subset \{y \, : \, |y| \le 1\}, \quad \mathcal{F}k_0(0) \ne 0, \quad \mathcal{F}k^0(0) \ne 0.$$

Define

$$k(y) = \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right)^N \, k^0(y)$$

with $N \in \mathbf{N}$. We introduce the local means by

$$k(t,f)(x) = \int k(y)\, f(x+ty)\, dy, \quad t>0, \tag{5.1.3}$$

and similarly $k_0(t,f)$. Recall, for N large enough we have

$$\|f|B^s_{pu}\| \quad \approx \quad \|k_0(1,f)(\cdot)|L_p\| + \left(\sum_{m=1}^{\infty} 2^{msu}\|k(2^{-m},f)(\cdot)|L_p\|^u\right)^{1/u}. \tag{5.1.4}$$

Let $\varphi \in S$ be non-trivial with a compact support near the origin. Let ℓ be an integer, then we have

$$k(2^\ell, \Delta^N\varphi)(x) = \int \Delta^N k^0(y)\, (\Delta^N\varphi)(x+2^\ell y)\, dy$$

$$= 2^{2\ell N} \int k^0(y)\, (\Delta^{2N}\varphi)(x+2^\ell y)\, dy \;=\; 2^{-2\ell N} \int \Delta^{2N} k^0(y)\, \varphi(x+2^\ell y)\, dy.$$

In other words, we have

$$|k(2^\ell, \Delta^N\varphi)(x)| \le c_N\, 2^{-2|\ell|N}, \quad \ell \in \mathbf{Z}, \quad N \in \mathbf{N} \tag{5.1.5}$$

and similarly

$$|k_0(2^\ell, \Delta^N\varphi)(x)| \le c_N\, 2^{-2\ell N}, \quad \ell \in \mathbf{N}_0, \quad N \in \mathbf{N}. \tag{5.1.6}$$

Now we put

$$f \;=\; \sum_{j=0}^{\infty} a_j\, (\Delta^N\varphi)(2^j(x-x^j)), \tag{5.1.7}$$

with $a_j \in \mathbf{C}$ and, say, $x^j = c\,(j,0,\ldots,0)$ where $c>0$ is a suitable positive number. We insert f in (5.1.3) and calculate

$$\|k(2^{-m},f)(\cdot)|L_p\|^p = \sum_{j=0}^{\infty} |a_j|^p \|k(2^{-m},(\Delta^N\varphi)(2^j(\cdot - x^j)))|L_p\|^p, \tag{5.1.8}$$

where we used the construction of the local means and the fact, that the supports of the terms of f have a sufficiently large distance from each other. For the term with $j=m\in\mathbf{N}$ we have

$$\|k(2^{-m},(\Delta^N\varphi)(2^m\cdot - 2^m x^m))|L_p\|^p = 2^{-mn}\, \|k(1,\Delta\varphi)|L_p\|^p, \tag{5.1.9}$$

where we may assume that the last factor on the right-hand side is positive. Of course we have an obvious counterpart of (5.1.8) with $k_0(1,f)$ instead of $k(2^{-m},f)$ and of (5.1.9) for the term $j=m=0$. Hence, by (5.1.4) we have

$$\|f|B^s_{pu}\|^u \ge c \sum_{m=0}^{\infty} 2^{msu}\, |2^{-\frac{mn}{p}}\, a_m|^u \tag{5.1.10}$$

for some $c > 0$. To prove the converse estimate we apply (5.1.5) with $\ell = j - m$ to the corresponding term in (5.1.8), and (5.1.6) with $\ell = j$ to the k_0-counterpart of (5.1.8). For sake of convinience we put $a_j = 0$, for $-j \in N$. Then (5.1.5) and (5.1.8) yield

$$\|k(2^{-m}, f)(\cdot)|L_p\|^p \le c \sum_j |2^{-\frac{jn}{p}} a_j|^p \, 2^{-2N\,|j-m|\,p}$$

$$\le c' \sup_j |2^{-\frac{jn}{p}} a_j|^p \, 2^{-\alpha\,|j-m|\,p} \le c' \sum_j \left(|2^{-\frac{jn}{p}} a_j|^u \, 2^{-\alpha\,|j-m|\,u}\right)^{p/u}$$

for $0 < \alpha < 2N$ and appropriate $c > 0$ and $c' > 0$. We choose N and afterwards α sufficiently large such that

$$2^{msu} \|k(2^{-m}, f)(\cdot)|L_p\|^u \le c \sum_j 2^{jsu} \, |2^{-\frac{jn}{p}} \, a_j|^u \, 2^{-\beta\,|j-m|\,u} \tag{5.1.11}$$

with $\beta = \alpha - |s| > 0$. Now we insert (5.1.11) and its k_0-counterpart in (5.1.4), change the order of summation and arrive at the converse of (5.1.10). Hence we have

$$\|f|B^s_{pu}\|^u \approx \sum_{m=0}^{\infty} 2^{msu} \, |2^{-\frac{mn}{p}} \, a_m|^u. \tag{5.1.12}$$

On the other hand, by the localization property of F^s_{pq} (cf. [26], 2.4.7) we have

$$\|f|F^s_{pq}\|^p \approx \sum_{j=0}^{\infty} |a_j|^p \, \|(\Delta^N \varphi)(2^j \cdot - 2^j x^j)|F^s_{pq}\|^p. \tag{5.1.13}$$

To estimate the last factors we use the counterpart of (5.1.4) for the F^s_{pq} -space, see again [26], 2.4.6 . By (5.1.5), (5.1.6) and the same technique as above we have

$$\|(\Delta^N \varphi)(2^j \cdot - 2^j x^j)|F^s_{pq}\| \approx 2^{j(s-\frac{n}{p})}.$$

We insert this result in (5.1.13) and arrive at

$$\|f|F^s_{pq}\|^p \approx \sum_{m=0}^{\infty} 2^{msp} \, |2^{-\frac{mn}{p}} \, a_m|^p. \tag{5.1.14}$$

Now (3.1.1), (5.1.12) and (5.1.14) yield

$$u \le p \le v.$$

Together with (5.1.2) this proves (3.1.2). ■

Step 3. (Proof of (3.1.6)). Taking the characteristic function χ_Q of the cube $Q = \{x : |x_\ell| \le 1,\ \ell = 1, \ldots, n\}$ it is well-known that

$$\chi_Q \in B^0_{\infty q} \quad \Longleftrightarrow \quad q = \infty, \quad \text{cf. [23], pp. 142-145.} \tag{5.1.15}$$

There the one-dimensional case is treated only but the general result can be deduced by using some tensorproduct arguments. From the equivalence (5.1.15) it follows $v = \infty$. The remaining implication

$$B^0_{\infty u} \subset L_\infty \quad \Longrightarrow \quad u \le 1$$

can be derived from the existence of essentially unbounded functions in $B^0_{\infty u}$, $u > 1$ (cf. [23], pp. 134/135). This proves (3.1.6). ■

Step 4. (Proof of (3.1.3)). The proof of (3.1.3) ("only if"-part) can be reduced to (3.1.6) by using duality arguments. Suppose

$$B^0_{1,u} \subset L_1 \subset B^0_{1,v} \quad \text{for some} \quad 1 < u < \infty \quad \text{and/or} \quad v < \infty$$

then this would imply (cf. [24], 2.11.2, p.178)

$$B^0_{\infty v'} \subset L_\infty \subset B^0_{\infty u'}, \quad \frac{1}{u} + \frac{1}{u'} = \frac{1}{v} + \frac{1}{v'} = 1.$$

This contradicts (3.1.6). Hence, (3.1.3) is proved. ■

Step 5. (Proof of (3.1.4)). The proof of $F^0_{1,q} \not\subset L_1$, $q > 1$ we postpone to the proof of the stronger implication

$$F^0_{1,q} \subset L_1^{loc} \quad \Longrightarrow \quad q \le 2,$$

cf. Subsection 5.3. ■

Step 6. (Proof of (3.1.5)). To prove (3.1.5) we apply again duality arguments. Let $f^0_{1,\infty}$ be the closure of S in $F^0_{1,\infty}$. Assume $L_1 \subset F^0_{1,\infty}$ we conclude $L_1 \subset f^0_{1,\infty}$ using the density of S in L_1. This yields

$$(f^0_{1,\infty})' = F^0_{\infty,1} \subset L_\infty, \tag{5.1.16}$$

cf. [14]. But this is false since $F^0_{\infty,1}$ contains essentially unbounded functions. This can be derived from the embedding

$$B^{n/p}_{p\infty} \subset F^0_{\infty,1}, \quad 0 < p < \infty, \tag{5.1.17}$$

cf. again [14] and Theorem 3.3.1. ■

Remark 5.1.1. To avoid the technical difficulties occurring in Substep 2.2 one can use the following elegant argumentation. Let $n \ge 2$. If s is large enough then the trace on

$\mathbf{R}^{n-1}$ ($\approx$ $x_n = 0$) makes sense. Suppose (3.1.1) then it follows

$$B_{pu}^{s-\frac{1}{p}}(\mathbf{R}^{n-1}) \subset B_{pp}^{s-\frac{1}{p}}(\mathbf{R}^{n-1}) \subset B_{pv}^{s-\frac{1}{p}}(\mathbf{R}^{n-1}),$$

cf. [24], Theorem 2.7.2, p.132 . This proves $u \leq p \leq v$, unfortunately in case $n \geq 2$ only.

5.2 Proofs of the assertions in Subsection 3.2

Proof of Theorem 3.2.1. Step 1. The proof of (3.2.5) ("only if"-part) will be postponed to the proof of the stronger implication

$$B_{pq}^{n\,(\frac{1}{p}-1)} \subset L_1^{loc} \quad \Longrightarrow \quad q \leq 1,$$

given in Subsection 5.3. The "if"-part of (3.2.5) follows from

$$B_{pq}^{n(\frac{1}{p}-1)} \subset B_{1,q}^0,$$

see [24], Theorem 2.7.1, and (3.1.3). Furthermore as pointed out in Remark 3.2.1 both (ii) and the "if"-part of (i) are known. ■

Step 2. (Proof of the "only if"-part of (3.2.1)). Let f be given by (5.1.7). We put $b_j = 2^{j(s-\frac{n}{p})} a_j$. Since $s - n/p = s_0 - n/p_0 = s_1 - n/p_1$ we find by (5.1.12) and (5.1.14)

$$\|f|B_{p_0u}^{s_0}\|^u \quad \approx \quad \sum_{j=0}^{\infty} |b_j|^u, \quad \|f|B_{p_1v}^{s_1}\|^v \quad \approx \quad \sum_{j=0}^{\infty} |b_j|^v$$

and

$$\|f|F_{pq}^s\|^p \quad \approx \quad \sum_{j=0}^{\infty} |b_j|^p.$$

Then (3.2.1) implies $u \leq p$ and $p \leq v$. ■

5.3 Proofs of the assertions in Subsection 3.3

Proof of Theorem 3.3.2. Step 1. The implications $(i_2) \rightarrow (i_1)$ and $(ii_2) \rightarrow (ii_1)$ are known and follow directly from the sharper embedding

$$B_{pq}^s \subset L_{\bar{p}}, \quad F_{pq}^s \subset L_{\bar{p}}, \quad \bar{p} = \max(1, p),$$

(Theorem 3.1.1, Theorem 3.2.1, Remark 2.1.1 and the proof of Corollary 3.3.1). ■

Step 2. Let $p \le 1$ and $s = n(1/p - 1)$. We shall prove

$$B_{pq}^{n(\frac{1}{p}-1)} \subset L_1^{loc} \quad \Longrightarrow \quad q \le 1. \tag{5.3.1}$$

Let φ be a non-vanishing C^∞ function with support near to origin and $\mathcal{F}\varphi(0) = 0$. Let

$$f = \sum_{j=0}^{\infty} \lambda_j \, 2^{jn} \, \varphi(2^j x - x^j), \quad x^j \in \mathbf{R}^n, \ |x^j| \le 1, \tag{5.3.2}$$

where we assumed that the functions $\varphi(2^j x - x^j)$ have disjoints supports. Then (5.3.2) is an atomic representation of f with

$$\| \, f \, | B_{pq}^{n(\frac{1}{p}-1)} \| \le c \left(\sum_{j=0}^{\infty} |\lambda_j|^q \right)^{1/q} < \infty, \tag{5.3.3}$$

see [8] or [26], 1.9.2 . On the other hand we have

$$\| \sum_{j=0}^{M} \lambda_j \, 2^{jn} \, \varphi(2^j \cdot - x^j) \, | L_1 \| = c \sum_{j=0}^{M} |\lambda_j|, \quad c \ne 0. \tag{5.3.4}$$

If $1 < q < \infty$, then we find numbers λ_j with (5.3.3) such that (5.3.4) diverges. Hence f does not belong to L_1. This proves (5.3.1). ■

Step 3. Let $1 \le p < \infty$. We shall prove

$$F_{pq}^0 \subset L_1^{loc} \quad \Longrightarrow \quad q \le 2.$$

Assume $q > 2$. For technical reasons we switch temporarily to the one-dimensional periodic case. Let $\mathbf{T}^1$ be the 1-torus. Let $\{a_k\}_k \notin l_2$. Immediately it follows:

(i) The lacunary series $\sum_k a_k \, e^{i2^k t}$ belongs to $F_{pq}^0(\mathbf{T}^1)$ if and only if $\{a_k\}_k \in l_q$ (cf. [19], 3.5.1, 6.4.2).

(ii) $f \notin L_1(\mathbf{T}^1)$, hence is not a regular distribution on the 1-torus (cf. [6], 15.3.1 and 15.3.2).

This yields the result in the one-dimensional periodic case. The same argumentation works in the general non-periodic case if we start with

$$g(x) = f(x_1) \cdot \chi(x), \quad x = (x_1, \ldots, x_n).$$

Here χ denotes a compactly supported C^∞ function in $\mathbf{R}^n$ which is identically 1 in the cube $[-\pi, \pi]^n$. One can prove this claim by using the characterization of F_{pq}^s spaces via local means with kernels having a product structure, cf. [26], 1.8.4. ■

Step 4. Let $1 \le p \le \infty$. The same machinery as in the preceding step can be applied to prove

$$B^0_{pq} \subset L_1^{loc} \quad \Longrightarrow \quad q \le 2. \quad ■$$

Step 5. Let $1 \le p \le \infty$. It remains to check

$$B^0_{pq} \subset L_1^{loc} \quad \Longrightarrow \quad 0 < q \le p. \tag{5.3.5}$$

We shall prove the existence of a singular distribution in B^0_{pq}, $1 \le p < q \le \infty$.

Substep 5.1. Preparations. Let f be a smooth function, non-trivial, supported around the origin and $|f(x)| \le 1$, $\int f(x)\, dx = 0$. Let $\sigma > 1$. Define

$$\kappa_0 \; = \; 0, \quad \kappa_j \; = \; \sum_{\ell=1}^{j} \ell^{-1} (\log(\ell+1))^{-\sigma}.$$

Since $\sigma > 1$ there exists a real number κ with $\kappa_j \longrightarrow \kappa$ if $j \longrightarrow \infty$. Further we put

$$R_j = \{x = (x_1, \ldots, x_n) : \quad \kappa_{j-1} < x_1 \le \kappa_j, \; 0 < x_\ell < 1, \; \ell = 2, \ldots, n\},$$

$j = 1, 2, \ldots$.

Next we subdivide R_j in $N_j = 2^{j(n-1)} \, [2^j j^{-1} \, (\log(j+1))^{-\sigma}]$ ([ ] denotes the integer part) cubes of side-length 2^{-j}, centered at $x^{j,r}$.

Substep 5.2. The announced singular distribution is given by

$$g \; = \; \sum_{j=1}^{\infty} \sum_{r=1}^{N_j} (\, \log(j+1))^{\sigma} \, f(2^{j+1} \, (x - x^{j,r})).$$

To see this, first note, that $2^{jn/p} \, f(2^{j+1}(x - x^{j,r}))$ is an atom. More exactly, it is an $(Q_{j,r}, 0, p)$-atom (cf. [26], p.62), where $Q_{j,r}$ is an appropriate cube with volume $\approx 2^{-jn}$ and located around $x^{j,r}$. Using the characterization of Besov spaces via atoms, due to Frazier and Jawerth (cf. [26], Theorem 1.9.2, p.63) we obtain

$$\|g|B^0_{pq}\|^q \le c \sum_{j=1}^{\infty} 2^{-j\frac{n}{p}q} \, (\, \log(j+1))^{\sigma q} \left(\sum_{r=1}^{N_j} 1 \right)^{q/p}$$

$$\le c \sum_{j=1}^{\infty} j^{-q/p} \, (\, \log(j+1))^{\sigma q (1 - \frac{1}{p})} < \infty$$

since $q > p$. Hence,

$$g \in B^0_{pq} \quad \text{if} \quad 1 \le p < q < \infty. \tag{5.3.6}$$

By construction g has compact support. Furthermore

$$\int |g(x)|\,dx = \sum_{j=1}^{\infty} \int_{R_j} |g(x)|\,dx \approx \sum_{j=1}^{\infty} (\,\log(j+1))^{\sigma}\,|R_j| = \sum_{j=1}^{\infty} j^{-1} = \infty.$$

Hence,

$$g \notin L_1^{loc}. \tag{5.3.7}$$

The formulas (5.3.6) and (5.3.7) prove that g has the required properties, which gives (5.3.5). ■

5.4 Proofs of the assertions in Subsection 4.1

Proof of Theorem 4.1.1. Step 1. (Proof of (4.1.3). The right-hand side of this inequality was proved in [20], p.51. We prove the left-hand side and assume that (4.1.1) holds. Then we have

$$\|f|B^s_{pq}\| \leq\ c\,\|f|B^s_{p_1q_1}\| \tag{5.4.1}$$

for all $f \in B^s_{p_1q_1}$ with a compact support, say, in the unit cube. Assume that f is non-trivial and smooth and that

$$\int x^{\beta}\, f(x)\,dx = 0 \quad \text{ for } \quad |\beta| \leq L.$$

If L is sufficiently large then $2^{j(\frac{n}{p}-s)}\, f(2^j x)$ are atoms in B^s_{pq} , see [26], 1.9.2. Then we have

$$\|f(2^j\cdot)|B^s_{pq}\| \approx 2^{j(s-\frac{n}{p})}.$$

Similarly for $B^s_{p_1q_1}$. Then (5.4.1) yields $1/p_1 \leq 1/p$. By (3.1.1) the assumption that (4.1.2) holds yields a corresponding assertion of type (4.1.1). Hence we have again $1/p_1 \leq 1/p$. ■

Step 2. (Proof of 4.1.4). The necessity of $s \geq 0$ is proved in [7]. So it remains to check $2s \geq n/p_1 + n/p_2 - n$. Let φ_j be the functions defined in (2.1.2). Consider the sequences

$$H_j(x) = 2^{-j\alpha_1}\,\mathcal{F}^{-1}\varphi_j(x), \quad G_j(x) = 2^{-j\alpha_2}\,\mathcal{F}^{-1}\varphi_j(x), \quad j = 1, \dots . \tag{5.4.2}$$

Obviously,

$$\|H_j|B^s_{p_1q_1}\|\,\|G_j|B^s_{p_2q_2}\| \ \approx\ 2^{j(2s-(\alpha_1+\alpha_2)+2n-\frac{n}{p_1}-\frac{n}{p_2})}. \tag{5.4.3}$$

Let us assume $2s < n/p_1 + n/p_2 - n$. Then we may choose $\alpha_1 + \alpha_2 < n$ such that

$$2s - (\alpha_1 + \alpha_2) + 2n - \frac{n}{p_1} - \frac{n}{p_2} = 0. \tag{5.4.4}$$

Next we consider the sequence $(G_j \cdot H_j)(\varphi)$, where φ is taken from S. We find

$$\begin{aligned}(G_j \cdot H_j)(\varphi) &= \int G_j(x)\, H_j(x)\, \varphi(x)\, dx \\ &= 2^{-j(\alpha_1+\alpha_2)}\, 2^{2jn} \int\int \varphi_1(\tau)\, \varphi_1(\xi - \tau)\, d\tau\, \mathcal{F}\varphi(2^j \xi)\, d\xi.\end{aligned}$$

We choose φ such that $\mathcal{F}\varphi \geq 0$, $\xi \in \mathbf{R}^n$ and $\mathcal{F}\varphi(\xi) = 1$, $|\xi| \leq 1$. Then it follows

$$|(G_j \cdot H_j)(\varphi)| \geq c\, 2^{j(n-(\alpha_1+\alpha_2))}$$

for some appropriate positive constant c. From the continuous embedding $B^s_{pq} \subset S'$ (cf. [24], Theorem 2.3.2, p.48) and (5.4.3), (5.4.4) we obtain a contradiction to (4.1.1). Since (5.4.3) remains true if we replace the B-spaces by the F-spaces the same argumentation works also in this case. ■

Step 3. (Proof of (4.1.5)). Again we can make use of the sequences defined in (5.4.2). Now it will be sufficient to take $\alpha_1 = \alpha_2 = 0$. Observe that

$$\begin{aligned}\|G_j \cdot H_j | B^s_{p\infty}\| &\geq 2^{js}\, \|\mathcal{F}^{-1}[\varphi_j \mathcal{F}(\mathcal{F}^{-1}\varphi_j \cdot \mathcal{F}^{-1}\varphi_j)](x) | L_p\| \\ &\geq 2^{js}\, 2^{2jn}\, 2^{-jn/p}\, \|\mathcal{F}^{-1}[\varphi_1(\varphi_1 * \varphi_1)](x) | L_p\|.\end{aligned} \tag{5.4.5}$$

Comparing (5.4.5) with (5.4.3) the necessity of (4.1.5) in case (4.1.1) follows. As in the preceding step the same proof can be taken over to the case of the F-spaces. ■

Remark 5.4.1. Let $A^s_{p_1 q_1}$ denote either $B^s_{p_1 q_1}$ or $F^s_{p_1 q_1}$ and similar $A^s_{p_2 q_2}$ and A^s_{pq}. Without any changes the above proof can be taken over to the more general problem whether

$$A^s_{p_1 q_1}\, A^s_{p_2 q_2} \subset A^s_{pq} \tag{5.4.6}$$

holds or not. Then as above from (5.4.6) the necessity of (4.1.3)-(4.1.5) will follow.

Remark 5.4.2. There is a difference between (4.1.4) and (4.1.5). Whereas (4.1.5) is necessary to keep the product in $B^s_{p\infty}$, (4.1.4) saves the membership of the product to S'.

5.5 Proofs of the assertions in Subsection 4.2

Proof of Proposition 4.2.1. For sake of simplicity we always assume $n = 1$. Otherwise one has to modify the following in an obvious way.

Step 1. Preparations. We construct a smooth counterpart of Rademacher functions. To this end, let ϱ_0 be a C^∞ function supported near 1 and identical 1 in a certain neighbourhood of 1. Then we put $\varrho_1(x) = \varrho_0(x) - \varrho_0(x-1)$. Consequently $\int \varrho_1(x)\, dx = 0$. Next we define $\varrho_2(x) = \varrho_1(x) - \varrho_1(x-2)$. This function ϱ_2 has now two vanishing moments

$$\int \varrho_2(x)\, dx = 0, \qquad \int x\varrho_2(x)\, dx = \int x\varrho_1(x)\, dx - \int (x-2)\, \varrho_1(x-2)\, dx = 0.$$

Iteration of this construction yields a family of functions ϱ_k having the following properties:

$$\text{supp } \varrho_k \subset [0, 2^k+1],$$

$$|\mathcal{F}\varrho_k(\xi)| \le c_k\, |\xi|^k, \quad \text{if } |\xi| \le 1 \tag{5.5.1}$$

and

$$|\mathcal{F}\varrho_k(\xi)| \le c_{k,K}\, |\xi|^{-K}, \quad \text{if } 1 \le |\xi| \tag{5.5.2}$$

for arbitrary K and suitable constants c_k, $c_{k,K}$. Both k and K are at our disposal.

Step 2. We fix some k and denote the corresponding function ϱ_k simply by ϱ. In what follows we investigate linear combinations of some scaled versions of this function.

Let $\varrho^\ell(x) = \varrho(2^\ell x)$, $\ell \in \mathbf{N}$. Recall $\{\varphi_k\}_k$ denotes the decomposition of unity defined in (2.1.2) and ψ the function from (2.1.1). Let $\Phi \in S$ be a function with

$$\text{supp } \Phi \subset [-4, -1/2] \cup [1/2, 4] \quad \text{and} \quad \Phi(x) = 1 \text{ on supp } \varphi_1.$$

From (5.5.1) and (5.5.2) one derives

$$\|\Phi(\cdot)\, (\mathcal{F}\varrho)(2^{j-\ell}\cdot)|W_2^m\| \le c_m\, 2^{-a\,|j-\ell|} \tag{5.5.3}$$

and

$$\|\psi(\cdot)\, (\mathcal{F}\varrho)(2^{-\ell}\cdot)|W_2^m\| \le c_m\, 2^{-a\,\ell} \tag{5.5.4}$$

for any $m \ge 0$ and c_m does not depend on $j \in \mathbf{N}_0$ and $\ell \in \mathbf{N}_0$. Here $a > 0$ is at our disposal. The Fourier multiplier theorem [24], 1.5.2 and (5.5.3) yield in case $j > 0$

$$\|\mathcal{F}^{-1}[\varphi_j\, \mathcal{F}\varrho^\ell](\cdot)|L_p\| = 2^{-\ell+j\,(1-\frac{1}{p})}\|\mathcal{F}^{-1}[\varphi_1(\xi)\, \Phi(\xi)\, \mathcal{F}\varrho(2^{j-\ell}\xi)](\cdot)|L_p\| \tag{5.5.5}$$

$$\leq\ c\, 2^{-\ell+j\,(1-\frac{1}{p})} \|\Phi(\cdot)\,(\mathcal{F}\varrho)(2^{j-\ell}\cdot)|W_2^m\|\ \|\mathcal{F}^{-1}\varphi_1|L_p\| \leq\ c'\, 2^{-\frac{j}{p}}\, 2^{-a\,|j-\ell|},$$

where m has to be suffiently large. In case $j=0$ one has to apply (5.5.4) instead of (5.5.3) and obtains

$$\|\mathcal{F}^{-1}[\varphi_0\,\mathcal{F}\varrho^\ell](\cdot)|L_p\| \leq\ c\, 2^{-a\,\ell}. \tag{5.5.6}$$

Both (5.5.5) and (5.5.6) lead to

$$2^{js}\, \|\mathcal{F}^{-1}[\varphi_j\,\mathcal{F}\varrho^\ell](\cdot)|L_p\| \leq\ c\, 2^{j(s-\frac{1}{p})}\, 2^{-a\,|j-\ell|} \tag{5.5.7}$$

which may be assumed to be an equivalence if $j=\ell$. We introduce

$$\lambda^L(x) = \sum_{\ell=0}^{L} a_\ell\, \varrho^\ell(x-x_\ell), \tag{5.5.8}$$

where the points x_ℓ are chosen such that $\varrho^\ell(\cdot - x_\ell)$ have disjoint supports (that is not important in this step but it will be used later on). Next we wish to calculate $\|\lambda^L|B^s_{pq}\|$. We have

$$\|\mathcal{F}^{-1}[\varphi_j\,\mathcal{F}\lambda^L](\cdot)|L_p\|^{\min(1,p)}$$

$$\geq |a_j|^{\min(1,p)}\, \|\mathcal{F}^{-1}[\varphi_j\,\mathcal{F}\varrho^j](\cdot)|L_p\|^{\min(1,p)} - \sum_{l\neq j} |a_\ell|^{\min(1,p)} \|\mathcal{F}^{-1}[\varphi_j\,\mathcal{F}\varrho^l](\cdot)|L_p\|^{\min(1,p)}$$

if $L\geq j$. Using (5.5.7) this leads to

$$2^{js\,\min(1,p)}\, \|\mathcal{F}^{-1}[\varphi_j\,\mathcal{F}\lambda^L](\cdot)|L_p\|^{\min(1,p)}$$

$$\geq\ c_1|a_j|^{\min(1,p)}\, 2^{j(s-\frac{1}{p})\,\min(1,p)} - c_2 \sum_{l\neq j} |a_\ell|^{\min(1,p)} 2^{\ell(s-\frac{1}{p})\,\min(1,p)}\, 2^{-a\,|j-\ell|\,\min(1,p)},$$

where c_1 and c_2 are positive constants independent of j and ℓ. This implies

$$\|\lambda^L|B^s_{pq}\|^{\min(1,p,q)} \geq\ c_1' \left(\sum_{j=0}^{L} 2^{j(s-\frac{1}{p})\,q}\, |a_j|^q\right)^{\frac{1}{q}\,\min(1,p,q)}$$

$$-c_2' \left(\sum_{j=0}^{\infty} \left|\sum_{\ell\neq j} |a_\ell|^{\min(1,p)}\, 2^{\ell(s-\frac{1}{p})\,\min(1,p)}\, 2^{-a\,|\ell-j|\,\min(1,p)}\right|^{q/\min(1,p)}\right)^{\frac{1}{q}\,\min(1,p,q)}$$

Choosing a large we may assume that the second term on the right-hand side can be estimated from above by, say,

$$\frac{c_1'}{2} \left(\sum_j 2^{j(s-\frac{1}{p})\,q} |a_j|^q\right)^{\frac{1}{q}\,\min(1,p,q)}.$$

Then we obtain

$$\|\lambda^L|B^s_{pq}\| \geq c \left(\sum_{j=0}^{L} 2^{j(s-\frac{1}{p})\,q}\,|a_j|^q\right)^{1/q} \tag{5.5.9}$$

for an appropriate positive constant c. The reverse inequality to (5.5.9) can be derived in a similar way, again based on (5.5.7). Hence we have

$$\|\lambda^L|B^s_{pq}\| \approx \left(\sum_{j=0}^{L} 2^{j(s-\frac{1}{p})\,q}\,|a_j|^q\right)^{1/q} \tag{5.5.10}$$

and the corresponding constants do not depend on L and the sequences $\{a_j\}_j$ and $\{x_\ell\}_\ell$.
Step 3. Let $\ell = 0, \ldots, L$ and let $j > L$. Then we can choose points t^r, $r = 1, \ldots c\,2^{j-\ell}$ such that supp $\varrho^j(\cdot - t^r) \subset \{t\,:\,\varrho^\ell(t) = 1\,\}$ and the supports of $\varrho^j(\cdot - t^r)$ and $\varrho^j(\cdot - t^m)$, $r \neq m$ have a mutual distance of at least $c'2^{-j}$ for some positive numbers c and c'. We put

$$\mu^j_\ell(x) = \sum_{r=1}^{c2^{j-l}} \varrho^j(x - t^r).$$

Such functions are studied in [28], for partial results see also [27]. By the Theorem on p.183 in [28] it follows

$$\|\mu^j_\ell|F^s_{p_2q_2}\| \approx \|\mu^j_\ell|B^s_{p_2q_2}\| \approx 2^{js-\frac{\ell}{p_2}}.$$

Step 4. Of course, the results of Step 3 remains unchanged replacing ϱ^ℓ by $\varrho^\ell(\cdot - x^\ell)$. We put

$$\mu^j(x) = \sum_{\ell=0}^{L} b_\ell\,\mu^j_\ell(x - x^\ell). \tag{5.5.11}$$

Then all the ingredients have mutually disjoint supports. Moreover, from [28] we know

$$\|\mu^j|F^s_{p_2q_2}\| \approx \|\mu^j|B^s_{p_2q_2}\| \approx 2^{js}\left(\sum_{\ell=0}^{L} |2^{-\frac{\ell}{p_2}}\,b_\ell|^{p_2}\right)^{1/p_2}. \tag{5.5.12}$$

Step 5. We multiply λ^L from (5.5.8) with μ^j from (5.5.11). By construction

$$\mu^j(x)\,\lambda^L(x) = \sum_{\ell=0}^{L} b_\ell\,a_\ell\,\mu^j_\ell(x - x^\ell).$$

By (5.5.12) it follows

$$\|\mu^j\,\lambda^L|F^s_{pq}\| \approx \|\mu^j\,\lambda^L|B^s_{pq}\| \approx 2^{js}\left(\sum_{\ell=0}^{L} |2^{-\frac{\ell}{p}}\,b_\ell\,a_\ell|^p\right)^{1/p}. \tag{5.5.13}$$

Step 6. Assume (4.2.2) holds, then by (5.5.13), (5.5.12) and (5.5.10) (with p_1 instead of p and q_1 instead of q)

$$\left(\sum_{\ell=0}^{L} |2^{-\frac{\ell}{p}}\,b_\ell\,a_\ell|^p\right)^{1/p} \leq c\left(\sum_{\ell=0}^{L} |2^{-\frac{\ell}{p_2}}\,b_\ell|^{p_2}\right)^{1/p_2}\left(\sum_{\ell=0}^{L} 2^{\ell(s-\frac{1}{p_1})\,q_1}\,|a_\ell|^{q_1}\right)^{1/q_1} \tag{5.5.14}$$

with c independent of L and a_ℓ, b_ℓ. Let

$$\beta_\ell = 2^{-\frac{\ell}{p_2}}\, b_\ell \quad \text{and} \quad \alpha_\ell = 2^{-\frac{\ell}{r_1}}\, a_\ell, \quad \text{see (4.2.1).}$$

Applying (4.2.1) then (5.5.14) yields

$$\left(\sum_{\ell=0}^{L} |\beta_\ell\, \alpha_\ell|^p\right)^{1/p} \leq\, c \left(\sum_{\ell=0}^{L} |\beta_\ell|^{p_2}\right)^{1/p_2} \left(\sum_{\ell=0}^{L} |\alpha_\ell|^{q_1}\right)^{1/q_1}.$$

Let $\beta_\ell = \alpha_\ell = 1$, then

$$L^{\frac{1}{p}} \leq\, c\, L^{\frac{1}{p_2}+\frac{1}{q_1}}$$

and hence

$$\frac{1}{p_2} + \frac{1}{r_1} = \frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{q_1},$$

from which (4.2.3) follows. ■

To prove Theorem 4.2.1. we need Proposition 2.2.1. Therefore we start to prove Proposition 2.2.1 first.

Proof of Proposition 2.2.1. Step 1. (Proof of (i)). We shall use (2.2.3), (2.1.1), (2.1.3) together with a Fourier multiplier assertion (cf. [24], Theorem 1.6.3, p.31). This gives

$$\| \left(\sum_{k=0}^{\infty} |\mathcal{F}^{-1}\left[\varphi_k\, \mathcal{F}{\textstyle\sum}'(f,g)\right](\cdot)|^q\right)^{1/q} |L_p\| \leq\, c\, \| \left(\sum_{k=2}^{\infty} |f^{k-2}\, g_k|^q\right)^{1/q} |L_p\|.$$

Applying Hölder's inequality and

$$\| \sup_k |f^k(x)|\, |L_{p_2}\| \leq\, c\, \|f|h_{p_2}\|,$$

(cf. [24], p.37) we arrive at (2.2.5). ■

Step 2. (Proof of (ii)). We shall apply the following identity

$$\mathcal{F}^{-1}[\varphi_k \mathcal{F}({\textstyle\sum}'''(f,g))](x) = \sum_{\ell=-2}^{\infty} \sum_{j=-1}^{1} \mathcal{F}^{-1}[\varphi_k \mathcal{F}(f_{k+\ell} \cdot g_{k+\ell+j})](x). \tag{5.5.15}$$

Here we have used (2.2.4), (2.1.1) and (2.1.3). First, let $p \geq 1$. By the Michlin-Hörmander Fourier multiplier theorem, the triangle inequality and Hölder's inequality we get

$$\|\mathcal{F}^{-1}\left[\varphi_k\, \mathcal{F}{\textstyle\sum}'''(f,g)\right](\cdot)|L_p\| \leq\, c\, \| \sum_{\ell=-2}^{\infty} \sum_{j=-1}^{1} \mathcal{F}^{-1}[\varphi_k \mathcal{F}(f_{k+\ell} \cdot g_{k+\ell+j})](\cdot)|L_p\|$$

$$\leq\ c \max_{-1\leq j\leq 1}\sum_{\ell=-2}^{\infty}\ \|f_{k+\ell}|L_{p_1}\|\ \|g_{k+\ell+j}|L_{p_2}\|$$

This proves (2.2.6).

Let $p<1$. Let $\ell\geq -2$. Proposition 1.5.1 and Remark 1.5.2/3 in [24], p.25/28 give

$$\|\mathcal{F}^{-1}[\varphi_k\mathcal{F}(f_{k+\ell}\cdot g_{k+\ell})](\cdot)\,|L_p\|\leq\ c\,2^{\ell n(\frac{1}{p}-1)}\,\|f_{k+\ell}\cdot g_{k+\ell}|L_p\|.$$

Again we use (5.5.15). Hence

$$\|\mathcal{F}^{-1}\left[\varphi_k\,\mathcal{F}{\sum}'''(f,g)\right](\cdot)|L_p\|^p\leq\ c\max_{-1\leq j\leq 1}\sum_{\ell=-2}^{\infty}\ \|\mathcal{F}^{-1}[\varphi_k\mathcal{F}(f_{k+\ell}\cdot g_{k+\ell+j})](\cdot)|L_p\|^p$$

$$\leq\ c\max_{-1\leq j\leq 1}\sum_{\ell=-2}^{\infty}\ 2^{\ell n(\frac{1}{p}-1)p}\,\|f_{k+\ell}|L_{p_1}\|^p\ \|g_{k+\ell+j}|L_{p_2}\|^p.$$

This completes the proof of Proposition 2.2.1. ■

Proof of Theorem 4.2.1. Step 1 (necessity part). After application of Proposition 4.2.1 it remains to prove $q\geq\max(q_1,q_2)$. But this is stated in a more general context in [7]. ■

Sufficiency part. Step 2 (proof of 4.2.5). Therefore we use the preparations made in Subsection 2.2.

Substep 2.1 . Estimate of $\sum'$, $\sum''$. Our assumptions in (4.2.4) and (4.2.6) imply the embeddings $B^s_{p_1q_1}\subset L_{r_1}$ and $B^s_{p_2q_2}\subset L_{r_2}$, cf. Remark 3.3.5. Hence, we may use Proposition 2.2.1 (i) to obtain

(5.5.16) $$\|2^{js}\,\mathcal{F}^{-1}[\varphi_j\mathcal{F}({\sum}'(f,g)](\cdot)|L_p\|\leq\ c\,\left(2^{js}\,\|g_j|L_{p_2}\|\right)\,\|f|h_{r_1}\|$$

and

(5.5.17) $$\|2^{js}\,\mathcal{F}^{-1}[\varphi_j\mathcal{F}({\sum}''(f,g)](\cdot)|L_p\|\leq\ c\,\left(2^{js}\,\|f_j|L_{p_1}\|\right)\,\|g|h_{r_2}\|.$$

Taking the q-th power and summing up the desired estimates of $\sum'$ and $\sum''$ follow, where we used $h_{r_1}=L_{r_1}\supset B^s_{p_1q_1}$ and $h_{r_2}=L_{r_2}\supset B^s_{p_2q_2}$.

Substep 2.2 . Estimate of $\sum'''$. Let $1/t=1/p_1+1/p_2$. We assume $0<q\leq\min(1,t)=u$. Using Proposition 2.2.1 (ii) we derive

(5.5.18) $$\|{\sum}'''(f,g)|B^{2s}_{tq}\|^q$$

$$\leq\ c\max_{-1\leq j\leq 1}\sum_{k=0}^{\infty}2^{2ksq}\left(\sum_{\ell=-2}^{\infty}\ 2^{\ell n(\frac{1}{u}-1)u}\,\|f_{k+\ell}\cdot g_{k+\ell+j}|L_t\|^u\right)^{q/u}$$

$$\leq c \max_{-1\leq j\leq 1} \sum_{l=-2}^{\infty} 2^{\ell(n(\frac{1}{u}-1)-2s)q} \sum_{k=0}^{\infty} 2^{2(k+\ell)sq} \, \|f_{k+\ell}|L_{p_1}\|^q \, \|g_{k+\ell+j}|L_{p_2}\|^q .$$

Because of

$$n\,(\frac{1}{t}-1) = n\,(\frac{1}{p_1}+\frac{1}{p_2}-1) = n\,(\frac{1}{p}+\frac{s}{n}-1) \quad \text{(cf. (4.2.4))} \tag{5.5.19}$$

and $s > 0$ we have $2s > n\,(\frac{1}{u}-1)$ and hence the right-hand side can be estimated from above by $c\,\|f|B^s_{p_1q_1}\|^q\,\|g|B^s_{p_2q_2}\|^q$. In addition we have the embedding $B^{2s}_{tq} \subset B^s_{pq}$. This follows from Theorem 3.2.1 and (4.2.4), see also (5.5.19). This proves (4.2.5) in case $0 < q \leq \min(1,t)$. If $q > \min(1,t)$ one has to modify the above estimate by using the triangle inequality in $l_{q/u}$. (5.5.16)-(5.5.18) together with (2.2.2) prove (4.2.5). ■

Step 3 (proof of (4.2.7)). Substep 3.1. Estimate of $\sum'$, $\sum''$. Because of $F^s_{p_1q_1} \subset L_{r_1} = h_{r_1}$ and $F^s_{p_2q_2} \subset L_{r_2} = h_{r_2}$ without restrictions on q_1, q_2, cf. Theorem 3.2.1 (ii) and Remark 2.1.1 (first item) we may apply Proposition 2.2.1 (i) to obtain

$$\|{\textstyle\sum}'(f,g)|F^s_{pq}\| \leq c\,\|g|F^s_{p_2q}\|\,\|f|h_{r_1}\| \leq c'\,\|g|F^s_{p_2q_2}\|\,\|f|F^s_{p_1q_1}\|. \tag{5.5.20}$$

Here we have used in addition $q \geq q_2$. Similar one derives

$$\|{\textstyle\sum}''(f,g)|F^s_{pq}\| \leq c\,\|g|F^s_{p_2q_2}\|\,\|f|F^s_{p_1q_1}\|. \tag{5.5.21}$$

Substep 3.2. Estimate of $\sum'''$. Again we put $1/t = 1/p_1 + 1/p_2$. As in Substep 2.2 we know $2s > n\,(\frac{1}{t}-1)$, cf. (4.2.4) and (5.5.19). This ensures the applicability of Proposition 2.2.1 (iii), we arrive at

$$\|{\textstyle\sum}'''(f,g)|F^{2s}_{t\infty}\| \leq c\,\|f|F^s_{p_1q_1}\|\,\|g|F^s_{p_2q_2}\|. \tag{5.5.22}$$

The embedding $F^{2s}_{t\infty} \subset F^s_{pq}$ (cf. Theorem 3.2.1 (ii) and (4.2.4)) complements the estimate of $\sum'''$. Now (4.2.7) follows from (5.5.20)-(5.5.22). ■

5.6 Proofs of the assertions in Subsection 4.3

Proof of Theorem 4.3.1. Concerning the sufficiency part one can follow the same arguments as in proof of Theorem 4.2.1, where one has to use now Theorem 3.3.1. The necessity part is covered by Proposition 4.3.1 and [7], where the last reference is used to prove $q \geq \max(q_1, q_2)$. ■

Proof of Theorem 4.3.2. We have to prove the "only if"-part only, cf. Remark 4.3.2.
Step 1. Let $f \in A^0_{p_1q_1}$ and assume we have (4.3.6). Then the operator

$$T_f \,:\, g \longrightarrow f \cdot g$$

is bounded from $A^0_{p_2q_2}$ into A^0_{pq}. Using Fourier multiplier assertions, cf. [24], Theorem 2.3.7, p.57 one derives that T_{f^j} yields a uniformly bounded family of those operators, where f^j is given by (2.2.1).

Let $g \in L_{p_2}$, then $g^k \in L_{p_2}$ and $\|g^k|L_{p_2}\| \le c\|g|L_{p_2}\|$ with c independent of g and k. Note, that the Fourier image of $g^k\, e^{ic2^kx_1}$ is concentrated near $|\xi| \approx 2^k$ if c is large enough, cf. (2.2.1) and (2.1.3). Let $k > j$, then the same is true in case of $f^j\, g^k\, e^{ic2^kx_1}$. Hence

$$\text{(5.6.1)} \qquad \|f^j\, g^k|L_p\| = \|f^j\, g^k\, e^{ic2^kx_1}|L_p\| \approx \|f^j\, g^k\, e^{ic2^kx_1}|A^0_{pq}\|$$

$$\le\; c\,\|f^j|A^0_{p_1q_1}\|\;\|g^k\, e^{ic2^kx_1}|A^0_{p_2q_2}\| \quad \le\; c\,\|f|A^0_{p_1q_1}\|\;\|g^k\, e^{ic2^kx_1}|L_{p_2}\|.$$

Consequently we have

$$\|f^j\, g^k|L_p\| \le\; c(f)\,\|g|L_{p_2}\|$$

for all $g \in L_{p_2}$. Let j be fixed, then for k tends to ∞ we get

$$\|f^j\, g|L_p\| \le\; c(f)\,\|g|L_{p_2}\|$$

using the Fatou lemma. By standard arguments we conclude $f^j \in L_{p_1}$ and moreover, by (5.6.1)

$$\|f^j|L_{p_1}\| \le c(f) \le c\,\|f|A^0_{p_1q_1}\|.$$

Let $1 < p_1 < \infty$, then using again Fatou's lemma we obtain $f \in L_{p_1}$. The case $p_1 = \infty$ can be covered by a Lebesgue point argument. If $p_1 = 1$, then $\|\, f^j\, |L_1\|$ in the above inequality can be replaced by $\|\, f^j\, |h_1\|$, cf. Remark 3.3.4. Then again by Fatou's lemma we obtain $f \in h_1 \subset L_1$. Hence

$$\text{(5.6.2)} \qquad A^0_{p_1q_1} \subset L_{p_1}.$$

Step 2. It remains to check

$$\text{(5.6.3)} \qquad L_p \subset A^0_{pq}.$$

We may assume $q < \infty$. In case $q = \infty$ nothing more is to prove with the exception of $A^0_{pq} = F^0_{1,\infty}$. The latter case will be considered in Substep 2.5. Further, observe that Step 1 implies $q_2 < \infty$.

Substep 2.1. Let $A^0_{p_2q_2} = B^0_{p_2q_2}$ and $A^0_{pq} = B^0_{pq}$. By duality (cf. [24], Theorem 2.11.2, p.178) (4.3.6) leads to

$$A^0_{p_1q_1}\, B^0_{p'q'} \subset B^0_{p'_2q'_2}$$

(we put $q' = \infty$ if $q \leq 1$). As in Step 1 this yields

$$B^0_{p'q'} \subset L_{p'}. \tag{5.6.4}$$

Using either $B^0_{p'q'} \subset L_{p'}$ if and only if $q' \leq \min(p', 2)$, $p \neq 1$ or $B^0_{\infty q'} \subset L_\infty$ if and only if $q' \leq 1$, (5.6.4) gives

$$L_p \subset B^0_{pq} \quad \text{(cf. Theorem 3.1.1 and Theorem 3.3.2)}$$

which is (5.6.3) in our case here.

Substep 2.2. Let $A^0_{p_2q_2} = F^0_{p_2q_2}$ and $A^0_{pq} = F^0_{pq}$ and assume $p > 1$ and $p_2 > 1$. To use the duality argument is a little bit more complicated than in Substep 2.1. Temporarily we restrict us to $1 < q_2 < \infty$ and $1 < q < \infty$. Under these conditions the duality argument works, cf. again [24], Theorem 2.11.2, p.178 and we arrive at

$$A^0_{p_1q_1}\, F^0_{p'q'} \subset F^0_{p'_2q'_2}.$$

By Step 1 this implies

$$F^0_{p'q'} \subset L_{p'}$$

which gives (5.6.3) by using again duality . If $q_2 \leq 1$ and/or $q \leq 1$ then (4.3.6) yields

$$A^0_{p_1q_1}\, B^0_{p_2q_2} \subset F^0_{pq},$$

cf. Theorem 3.1.1. By duality we find

$$F^0_{p_1q_1}\, (F^0_{pq})' \subset B^0_{p'_2q'_2}.$$

Step 1 gives

$$(F^0_{pq})' \subset L_{p'}.$$

Using the monotonicity of the F-spaces with respect to q (5.6.3) follows.

Substep 2.3. Let $A^0_{p_2q_2} = F^0_{p_2q_2}$ and $A^0_{pq} = F^0_{1,q}$. The proof runs the same way as in Step 1 and Substep 2.2 if one takes into account

$$(F^0_{1,q})' = F^0_{\infty,q'}\,, \qquad (1 \le q < \infty),$$

and

$$(F^0_{1,q})' = B^0_{\infty\infty}\,, \qquad (0 < q \le 1),$$

cf. [14, 8] and [24], p.180. Furthermore

$$\|g^k\, e^{i\,c2^k x_1} | F^0_{\infty,q'}\| \approx \|g^k | B^0_{\infty\infty}\| \approx \|g^k | L_\infty\|,$$

where the latter one follows from $B^0_{\infty q'} \subset F^0_{\infty q'} \subset B^0_{\infty\infty}$.

Substep 2.4. Let $A^0_{p_2q_2} = F^0_{1,q_2}$ and $A^0_{pq} = F^0_{1,q}$ with $q < \infty$. Then necessarily we have $A^0_{p_1q_1} = B^0_{\infty,q_1}$. We use that (4.3.6) implies

$$B^0_{\infty q_1}\, B^0_{1,\min(1,q_2)} \subset F^0_{1,q}. \tag{5.6.5}$$

Now we can argue as in Substep 2.3.

Substep 2.5. It remains (5.6.5) with $q = \infty$. Restricting to completions of S in the involved spaces we may replace $F^0_{1,\infty}$ by $f^0_{1,\infty}$, see the end of Section 5.1. Then we can use the arguments in (5.1.16) and (5.1.17) which disprove this possibility. ■

Proof of Corollary 4.3.1. Theorem 3.1.1 and Corollary 3.3.1 imply

$$B^0_{pq} \subset L_p \quad \Longleftrightarrow \quad q \le \min(p,2) \quad (1 \le p < \infty)$$

and

$$F^0_{pq} \subset L_p \quad \Longleftrightarrow \quad q \le 2 \quad (1 \le p < \infty).$$

In view of these equivalences the corollary is a reformulation of Theorem 4.3.2. ■

Proof of Corollary 4.3.2. Our method in proving Theorem 4.3.2 depends on the duality procedure. If we switch from B^0_{pq} to b^0_{pq}, the closure of S in B^0_{pq} then the restriction $p_2 < \infty$ becomes superflous, cf. [24], Remark 2.11.2/2, p.180. The result is formulated as a consequence of (5.6.2) and (5.6.3). ■

5.7 Proofs of the assertions in Subsection 4.4

Proof of Theorem 4.4.1. Step 1 (Proof of (4.4.4), sufficiency part). Substep 2.1 . Estimate of $\sum'(f,g)$ and $\sum''(f,g)$. Thanks to Proposition 2.2.1 we know

$$\|2^{js}\,\mathcal{F}^{-1}[\varphi_j\mathcal{F}(\sum{}'(f,g)](\cdot)|L_p\| \le c\,\left(2^{js}\,\|g_j|L_{p_2}\|\right)\,\|f|h_{r_1}\|, \tag{5.7.1}$$

where $1/p = 1/p_2 + 1/r_1$. Theorem 3.1.1, Theorem 3.2.1 tell us that we may apply (5.7.1) with

$$\frac{1}{n}\left(\frac{n}{p_1} - s\right)_+ < \frac{1}{r_1} < \frac{1}{p_1} \tag{5.7.2}$$

in the same way as in proof of Theorem 4.2.1 given in Subsection 5.5. Similar we obtain

$$\|2^{js}\,\mathcal{F}^{-1}[\varphi_j\mathcal{F}(\sum{}''(f,g)](\cdot)|L_p\| \le c\,\left(2^{js}\,\|f_j|L_{p_1}\|\right)\,\|g|h_{r_2}\|, \tag{5.7.3}$$

and now $1/p = 1/p_1 + 1/r_2$ holds. As above we derive the restrictions for $1/r_2$

$$\frac{1}{n}\left(\frac{n}{p_2} - s\right)_+ < \frac{1}{r_2} < \frac{1}{p_2}. \tag{5.7.4}$$

(5.7.2) and (5.7.4) yield that (5.7.1) and (5.7.3) are applicable simultaneously if

$$\max_{i\neq j}\left(\frac{1}{n}\left(\frac{n}{p_i} - s\right)_+ + \frac{1}{p_j}\right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2}. \tag{5.7.5}$$

But this is ensured by (4.4.1) and (4.4.3).

Substep 2.2. Estimate of $\sum'''(f,g)$. Let $1/t = 1/p_1 + 1/p_2$. We put $\min(1,t) = u$. Since $2s > n(\frac{1}{u} - 1)$ (cf. (4.4.2)) we derive as in (5.5.18)

$$\|\sum{}'''(f,g)|B^{2s}_{tq}\| \le c\,\|f|B^s_{p_1q_1}\|\,\|g|B^s_{p_2q_2}\|.$$

Because of $B^{2s}_{tq} \subset B^s_{pq}$ (cf. Theorem 3.2.1 and (4.4.3), see also (5.5.19)) this proves the desired estimate in case of $\sum'''(f,g)$. This completes the proof of (4.4.4). ■

Step 2 (Proof of (4.4.6), sufficiency part). The proof is similar to that given in Step 1, cf. also the proof of Theorem 4.2.1 (ii) in Subsection 5.5. Proposition 2.2.1 (i) and $q \ge \max(q_1,q_2)$ yield

$$\|\sum{}'(f,g)|F^s_{pq}\| \le c\,\|g|F^s_{p_2q_2}\|\,\|f|h_{r_1}\| \le c\,\|g|F^s_{p_2q_2}\|\,\|f|F^s_{p_1q_1}\|$$

and

$$\|\sum{}''(f,g)|F^s_{pq}\| \le c\,\|f|F^s_{p_1q_1}\|\,\|g|h_{r_2}\| \le c\,\|f|F^s_{p_1q_1}\|\,\|g|F^s_{p_2q_2}\|$$

if (5.7.5) holds, which is guaranteed by (4.4.1) and (4.4.3). To estimate $\sum'''$ we may apply (5.5.22) once again. ■

Remark 5.7.1. The proof shows that (4.4.4) and (4.4.6) remain valid not only in the interior of the shaded area in Figures 3a and 3b, they are true also on the vertical lines of the boundary, may be with exception of the endpoints (note that (4.4.1) is used only to establish (5.7.5)). With the help of Theorem 3.1.1 and Theorem 3.2.1 one has to check under which conditions $<$ in (5.7.2), (5.7.4) and (5.7.5) can be replaced by $\leq$, cf. [11] and [18] for a more detailed explanation.

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