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Hölder Regularity Theorem for a Class of Linear Nonuniformly Elliptic Operators with Measurable Coefficients.

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1. – The purpose of this note is to extend the classical De Giorgi's theorem ([5], see also [17] and [15]) by proving the Hölder regularity of the weak solutions of $Lu = 0$, where $L = \sum_{i,j=1}^{n} \partial_i(a_{i,j}\partial_j)$ is a linear degenerate elliptic operator in divergence form.

Many authors ([14], [16], [18], [11], [6]) proved the same result for different classes of operators which are degenerate but uniformly elliptic (i.e. the ratio Λ/λ is bounded; here Λ and λ are the greatest and the lowest eigenvalue of the quadratic form associated to the operator). In this paper, even if in a particular situation, we drop such a hypothesis, if the integral curves of the vector fields $\pm\lambda_1\partial_1, \ldots, \pm\lambda_n\partial_n$ satisfy a suitable condition (here λ_j, $j, \ldots, n$, is a real continuous nonnegative function such that the quadratic form $\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2$ is equivalent to $\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j$). Roughly speaking, we suppose that R^n is $(\lambda_1, \ldots, \lambda_n)$-connected, i.e., for every $x, y \in R^n$, it is possible to join x and y by a continuous curve which is « a piecewise integral curve » of $\pm\lambda_1\partial_1, \ldots, \pm\lambda_n\partial_n$. This condition enables us to construct a metric d in R^n which is « natural » for L as the euclidean metric is « natural » for the Laplace operator. By a similar geometrical approach, we proved in [10] the Harnack inequality for a wide class of degenerate non uniformly elliptic operators. If some additional hypotheses on the λ_j's are satisfied, we get more precise information on the structure of the d-balls (see [9]) and on the constants appearing in Harnack inequality. Thus, we obtain the Hölder regularity of the weak solutions of $Lu = 0$, arguing as in the nondegenerate case. The main result of this paper has been announced in [8]. Moreover, in [8] (see also [10]) we showed that $(\lambda_1, \ldots, \lambda_n)$-con-

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nectedness can be viewed as a « weak extention » to the non-smooth case of the usual Hörmander condition ([12]) on the rank of the Lie algebra generated by $\lambda_1 \partial_1, \dots, \lambda_n \partial_n$.

The scheme of the proof follows Moser's [15] technique. In Section 2 we formulate our hypotheses and state some properties of the d-balls which are essential for Moser's machinery. In particular, we get a « doubling condition » implying that (R^n, d) is a metric space of homogeneous type with respect to Lebesgue measure in the sense of [3]. Moreover, we construct a class of homotethical transformations which are « natural » for the operator L.

In Section 3, we prove a Sobolev embedding theorem and a Poincaré inequality.

Finally, in Section 4, we prove our Hölder regularity theorem.

2. – In what follows, L will be the differential operator $\sum_{i,j=1}^{n} \partial_i(a_{i,j}\partial_j)$, where $a_{ij} = a_{ji}$ are real functions belonging to $L^\infty(R^n)$ and $\partial_j = \partial/\partial x_j$. We shall suppose that

(2.a) *there exists* $m \in R_+$ *such that*

$$m^{-1}\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2 \leqslant \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leqslant m\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2$$

$\forall x \in R^n$, $\forall \xi \in R^n$, *where* $\lambda_j(x) = \lambda_j^{(1)}(x_1)\dots\lambda_j^{(n)}(x_n)$ *and the* $\lambda_j^{(k)}$*'s are nonnegative continuous real functions with continuous first derivatives outside the origine such that*

(2.b) $\lambda_j^{(j)}$ *is Lipschitz-continuous*;

(2.c) $0 \leqslant t(\lambda_j^{(k)})'(t) \leqslant \varrho_{j,k}\lambda_j^{(k)}(t)$, $\forall t \neq 0$, *for suitable positive constants* $\varrho_{j,k}$, $j, k = 1, \dots, n, j \neq k$;

(2.d) $\lambda_j^{(k)}(t) = \lambda_j^{(k)}(|t|)$, $\forall t \in R$, $j, k = 1, \dots, n$, $j \neq k$.

The meaning of hypotheses (2.b) and (2.c) is illustrated in [10] and [9].

If Ω is an open subset of R^n, we shall denote by $W_\lambda^2(\Omega)$ $(\mathring{W}_\lambda^2(\Omega))$ the completion of $\{u \in C^\infty(\Omega);\ \|u;\ W_\lambda^2(\Omega)\| < +\infty\}$ $(C_0^\infty(\Omega))$ with respect to the norm

$$\|u;\ W_\lambda^2(\Omega)\| = \left(\|u;\ L^2(\Omega)\|^2 + \sum_{j=1}^{n}\|\lambda_j\partial_j u;\ L^2(\Omega)\|^2\right)^{\frac{1}{2}}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$. For the sake of brevity, we shall omit the index 2 and we shall write $W_\lambda(\Omega)$ $(\mathring{W}_\lambda(\Omega))$. Furthermore, we shall say that u belongs to $W_\lambda^{\mathrm{loc}}(\Omega)$ if $\varphi u \in \mathring{W}_\lambda(\Omega)$ for every test function φ supported in Ω.

The following assertion is straightforward.

PROPOSITION 2.1. *The bilinear form* $\mathfrak{L}$ *on* $C^\infty(\Omega) \cap W_\lambda(\Omega)$ *defined as follows*

$$\mathfrak{L}(u, v) = \int_\Omega \sum_{i,j=1}^{n} a_{i,j}\, \partial_i u \partial_j v \, dx ,$$

can be continued on all of $W_\lambda(\Omega)$.

DEFINITION 2.2. *Let u be a function belonging to $W_\lambda^{\mathrm{loc}}(\Omega)$. We shall say that $Lu \geqslant 0$ $(Lu \leqslant 0)$ if $\mathfrak{L}(u, \varphi) \leqslant 0$ $\big(\mathfrak{L}(u, \varphi) \geqslant 0\big)$ for every nonnegative test function φ supported in Ω. Moreover we shall say that $Lu = 0$ if $\mathfrak{L}(u, \varphi) = 0$ for every test function supported in Ω.*

In order to formulate our regularity theorem, the following definition is a basic step.

DEFINITION 2.3. *An open subset Ω of R^n will be said λ-connected if for every $x, y \in \Omega$, there exists a continuous curve lying in Ω which is piecewise an integral curve of the vector fields $\pm \lambda_1 \partial_1, \dots, \pm \lambda_n \partial_n$ connecting x to y.*

We note that, by our hypotheses, a λ-connected open subset of R^n is connected and locally λ-connected in the sense of Definition 2.2 in [10]. This is a straightforward consequence of the following result.

THEOREM 2.4. *Let Ω be a λ-connected open subset of R^n. Then, for every $\bar{x} \in \Omega$ there exists a neighbourhood V of $\bar{x}$ such that, up to a reordering of the variables, the inequalities* (2.*a*) *hold in V (for a new choice of the constant m) with $\lambda_1(x) = 1$, $\lambda_j(x) = \lambda_j^{(1)}(x_1) \dots \lambda_j^{(j-1)}(x_{i-1})$, $j = 2, \dots, n$.*

PROOF. Let $\bar{x}$ be fixed; by the λ-connectedness and by (2.*b*), there exists at least one of the λ_j's which is different from zero in $\bar{x}$, and hence in a neighbourhood V of $\bar{x}$. Without loss of generality, we may suppose that $c_1^{-1} \geqslant \lambda_1(x) \geqslant c_1 > 0$, $\forall x \in V$. Analogously, there is at least one of the λ_j's $(j = 2, \dots, n)$ not identically vanishing on

$$\{\bar{x} + te_1, t \in R\}, \qquad \text{where } e_1 = (1, 0, \dots, 0).$$

Without loss of generality, we may suppose $\lambda_2(\bar{x} + t^* e_1) \neq 0$, for a suitable

$t^* \in R$. But, since $\lambda_2(\bar{x} + t^* e_1) = \lambda_2^{(1)}(\bar{x} + t^*)\, \lambda_2^{(2)}(\bar{x}_2) \dots \lambda_2^{(n)}(\bar{x}_n)$, shrinking, if necessary, V, we may suppose $c_2^{-1} \geqslant \lambda_2^{(2)}(x_2) \dots \lambda_2^{(n)}(x_n) \geqslant c_2 > 0$, $\forall x \in V$; so $c_2^{-1} \geqslant \lambda_2(x)/\lambda_2^{(1)}(x_1) \geqslant c_2$, $\forall x \in V$.

Repeating this argument, we can prove our assertion.

Since we are dealing with local properties, in what follows, we shall suppose that the λ_j's have everywhere the particular structure which is locally obtained in Theorem 2.4. So, we may suppose that R^n is λ-connected.

Using the technique we introduced in [9], we shall denote by $P(\lambda_1, \dots, \lambda_n)$ the set of all continuous curves which are piecewise integral curves of the vector fields $\pm \lambda_1 \partial_1, \dots, \pm \lambda_n \partial_n$. If $\gamma\colon [0, T] \to R^n$, $\gamma \in P$, we shall put $l(\gamma) = T$; by the λ-connectedness, we can give the following definition.

DEFINITION 2.5. *If $x, y \in R^n$, put*

$$d(x, y) = \inf \left\{ l(\gamma),\ \gamma \in P,\ \gamma \textit{ connecting } x \textit{ and } y \right\}.$$

Obviously, d is a metric in R^n.

DEFINITION 2.6. *If $x \in R^n$, $t \in R$, put $H_0(x, t) = x$, $H_{k+1}(x, t) = H_k(x, t) + t\lambda_{k+1}\big(H_k(x, t)\big)\, e_{k+1}$, $k = 0, \dots, n-1$. Here $e_k = (\underset{1}{0}, \dots, \underset{k}{1}, \dots, \underset{n}{0})$. Denoting by R_j^n the set of the points $x = (x_1, \dots, x_n) \in R^n$ such that $x_k \geqslant 0$, $k = 1, \dots, j-1$, if $x \in R_j^n$, the function $s \to F_j(x, s) = s\lambda_j\big(H_{j-1}(x, s)\big)$ is strictly increasing on $]0, +\infty[$; thus, we can put $\varphi_j(x, \cdot) = \big(F_j(x, \cdot)\big)^{-1}$, $j = 1, \dots, n$.*

If $x \in R^n$, we shall denote by x^* the point $\big(|x_1|, \dots, |x_n|\big)$ and, if $y \in R^n$, we shall put

$$\varrho(x, y) = \sum_{j=1}^{n} \varphi_j\big(x^*, |x_j - y_j|\big).$$

In [9] we proved the following estimates.

THEOREM 2.7 ([9], Theorems 2.6 and 2.7). *There exists $a \in R_+$ (depending only on the $\varrho_{j,k}$'s) such that*

$$a^{-1} \leqslant d(x, y)/\varrho(x, y) \leqslant a\,, \qquad \forall x, y \in R^n\,;$$

$$a^{-1} \leqslant \mu\big(S_d(x, r)\big)\Big/\prod_{j=1}^{n} F_j(x^*, r) \leqslant a\,, \qquad \forall x \in R^n\,,\ \forall r > 0\,,$$

where $S_d(x, r)$ is the d-ball $\left\{ y \in R^n;\ d(x, y) < r \right\}$.

THEOREM 2.8 ([10], Proposition 4.3). *Put* $G_1 = 1$, $G_k = 1 + \sum_{l=1}^{K-1} G_l \varrho_{k,l}$, $k = 2, \dots, n$ *and* $\varepsilon_k = (G_k)^{-1}$, $k = 1, \dots, n$. *Then*, $\forall x \in R^n$, $\forall s > 0$, $\forall \theta \in]0, 1[$

$$\theta^{G_j} \leqslant F_j(x^*, \theta s)/F_j(x^*, s) \leqslant \theta \; ; \tag{2.8.a}$$

$$\theta \leqslant \varphi_j(x^*, \theta s)/\varphi_j(x^*, s) \leqslant \theta^{\varepsilon_j} . \tag{2.8.b}$$

A first consequence of Theorems 2.7 and 2.8 is the following estimate for the metric d.

PROPOSITION 2.9. *For every compact subset K of R^n, there exists $C_k > 0$ such that*

$$C_K^{-1} |x - y| \leqslant d(x, y) \leqslant C_K |x - y|^{\varepsilon_0}, \tag{2.9.a}$$

where $\varepsilon_0 = \min\{\varepsilon_1, \dots, \varepsilon_n\}$ (see also [7]).

Moreover, the metric space $(R^n; d)$ is a space of homogeneous type in the sense of [3], since the following « doubling condition » holds:

$$\mu(S_d(x, 2r)) \leqslant A \mu(S_d(x, r)) \tag{2.9.b}$$

$\forall x \in R^n$, $\forall r > 0$, where μ is Lebesgue measure in R^n and $A = a^2 2^{\sum_j G_j}$.

The following technical estimate will be used in the sequel.

PROPOSITION 2.10. *There exists $b \in R_+$ depending only on the constants $\varrho_{j,k}$ such that $\forall x \in R^n$, $\forall r$, $R > 0$, $r \leqslant 2R$, $\forall y \in S_d(x, R)$, we have*

$$b^{-1} \leqslant \mu(S_d(x, R) \cap S_d(y, r))/\mu(S_d(y, r)) \leqslant b . \tag{2.10.a}$$

PROOF. The first step is to prove that there exists $z \in R^n$ such that

$$d(x, z) + d(y, z) = d(x, y) \quad \text{and} \quad d(y, z) = \min\left\{d(x, y), \frac{r}{2}\right\}. \tag{2.10.b}$$

In fact, by (2.9.a), (R^n, d) is locally compact; so that, by the λ-connectedness of R^n, $\forall x, y \in R^n$ there exists a continuous curve γ such that, $\forall \xi \in \gamma$, $d(x, \xi) + d(\xi, y) = d(x, y)$ (see, e.g., [2] 5.18). Then (2.10.b) follows straightforwardly. Now, from (2.10.b) we get

$$S_d(z, r/2) \subseteq S_d(x, R) \cap S_d(y, r) . \tag{2.10.c}$$

To prove (2.10.a), by (2.9.b) we need only to prove that $\mu(S_d(z, r))$ is equivalent to $\mu(S_d(y, r))$, with equivalence constants depending only on the $\varrho_{j,k}$'s. But, since $d(y, z) < r$, by (2.9.b), we have:

$$\mu(S_d(z, r)) \leqslant \mu(S_d(y, 2r)) \leqslant A\mu(S_d(y, r)) \leqslant (A\mu(S_d(z, 2r)) \leqslant A^2\mu(S_d(z, r)) .$$

So, the assertion is proved.

In particular, from Proposition 2.10, it follows that every fixed d-ball is a space of homogeneous type.

The particular structure of the metric d appearing in Theorem 2.7 suggests the construction of a suitable set of homotethical transformations T_α which are « good transformations » for our operators, i.e. the class of the differential operators satisfying (2.a)-(2.b) is, in a suitable sense, invariant under T_α.

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in R^n$ be fixed; for $\alpha > 0$, put

$$T_\alpha(x) = \bar{x} + \sum_{j=1}^{n} (x_j - \bar{x}_j) F_j(\bar{x}^*, \alpha)\, e_j = (T^1_\alpha, \ldots, T^n_\alpha) \tag{2.e}$$

and

$$\lambda^{(k)}_{(\alpha)j} = \big(\alpha / F_j(\bar{x}^*, \alpha)\big) \lambda^{(k)}_j \circ T^k_\alpha . \tag{2.f}$$

Moreover if $\omega = T^{-1}_\alpha(0)$, put

$$\pi_\omega = \Big\{ x \in \mathbb{R}^n;\ \prod_{j=1}^{n} (x_j - \omega_j) = 0 \Big\} ; \tag{2.g}$$

$$x^*_\omega = \omega + (x - \omega)^* , \qquad \forall x \in \mathbb{R}^n . \tag{2.h}$$

Denote by L_α the differential operator $\sum_{i,j=1}^{n} \partial_i(a^{(\alpha)}_{i,j} \partial_j)$, where

$$a^{(\alpha)}_{i,j} = \big(\alpha^2 / F_i(\bar{x}^*, \alpha) F_j(\bar{x}^*, \alpha)\big)\, a_{i,j} \circ T_\alpha , \qquad i, j = 1, \ldots, n .$$

It is straightforward matter to prove (with an obvious meaning of the notations) that

$$m^{-1} \sum_{j=1}^{n} \lambda^2_{(\alpha)j} \xi_j^2 \leqslant \sum_{i,j=1}^{n} a^{(\alpha)}_{i,j}(x)\, \xi_i \xi_j \leqslant m \sum_{j=1}^{n} \lambda^2_{(\alpha)j}(x)\, \xi_j^2 ; \tag{2.a'}$$

$$0 \leqslant (t - \omega_j)(\lambda^{(k)}_{(\alpha)j})'(t) \leqslant \varrho_{i,k} \lambda^{(k)}_{(\alpha)j}(t) , \qquad \forall t \in \mathbb{R} \setminus \{\omega_j\},\ j, k = 1, \ldots, n ,\ k < j ; \tag{2.c'}$$

$$\lambda^{(k)}_{(\alpha)j}(t) = \lambda^{(k)}_{(\alpha)j}(\omega_k + |t - \omega_k|) , \qquad \forall t \in \mathbb{R} ,\ i, k = 1, \ldots, n, k < j , \tag{2.d'}$$

so that $\lambda_{(\alpha)j}(x) = \lambda_{(\alpha)j}(x^*_\omega)$.

If we denote by $F_j^{(\alpha)}$ the function we obtain from the $\lambda_{(\alpha)j}$'s as we obtained the F_j's from the λ_j's, we get the following identity.

$$F_j^{(\alpha)}(\bar{x}^*_\omega, \sigma) = F_j(\bar{x}^*, \alpha\sigma)/F_j(x^*, \alpha)\,, \qquad \forall\sigma > 0\,,\ j = 1, \ldots, n\,. \tag{2.i}$$

The assertion is obvious if $j = 1$. By induction, let us suppose that (2.i) holds for $k \leqslant j$ and let us prove it for $j + 1$. We note that, if $k \leqslant n$,

$$\bar{x}_k + (\bar{x}^*_\omega)_k F_k(\bar{x}^*, \alpha) - \bar{x}_k F_k(\bar{x}^*, \alpha) = (\bar{x}^*)_k;$$

then, by the inductive hypothesis, we have:

$$\begin{aligned}
F_{j+1}^{(\alpha)}(\bar{x}^*_\omega, \sigma) &= \sigma\lambda_{(\alpha)j+1}\big((\bar{x}^*_\omega)_1 + F_1^{(\alpha)}(\bar{x}^*_\omega, \sigma), \ldots, (\bar{x}^*_\omega)_j + F_j^{(\alpha)}(\bar{x}^*_\omega, \sigma)\big)\\
&= \big(\alpha\sigma/F_{j+1}(\bar{x}^*, \alpha)\big)\,\lambda_{j+1}\Big(\bar{x}_1 + \big((\bar{x}^*_\omega)_1 + F_1^{(\alpha)}(\bar{x}^*_\omega, \sigma) - \bar{x}_1\big)\,F_1(\bar{x}^*, \alpha), \ldots\Big)\\
&= \big(\alpha\sigma/F_{j+1}(\bar{x}^*, \alpha)\big)\,\lambda_{j+1}\Big(\bar{x}_1 + \big((\bar{x}^*_\omega)_1 + F_1(\bar{x}^*, \alpha\sigma)/F_1(\bar{x}^*, \alpha) - \bar{x}_1\big)\,F_1(\bar{x}^*, \alpha), \ldots\Big)\\
&= \big(\alpha\sigma/F_{j+1}(\bar{x}^*, \alpha)\big)\,\lambda_{j+1}\big((\bar{x}^*)_1 + F_1(\bar{x}^*, \alpha\sigma), \ldots\big) = F_{j+1}(\bar{x}^*, \alpha\sigma)/F_{j+1}(\bar{x}^*, \alpha)\,.
\end{aligned}$$

So, (2.i) is proved.

We note that, by (2.i), we have

$$\varphi_j^{(\alpha)}(\bar{x}^*_\omega, s) = \big(F_j^{(\alpha)}(\bar{x}^*_\omega, \cdot)\big)^{-1}(s) = \alpha^{-1}\varphi_j\big(\bar{x}^*, sF(\bar{x}^*, \alpha)\big) \tag{2.j}$$

so that $\varphi_j^{(\alpha)}(\bar{x}^\omega_*, 1) = 1$, $\forall\alpha > 0$, $j = 1, \ldots, n$.

Moreover, if we put

$$S_\varrho(\bar{x}, r) = \big\{x \in R^n;\ |x_j - \bar{x}_j| < F_j(\bar{x}^*, r),\ j = 1, \ldots, n\big\}$$

and, analogously,

$$S_\varrho^{(\alpha)}(\bar{x}, r) = \big\{x \in R^n;\ |x_j - \bar{x}_j| < F_j^{(\alpha)}(\bar{x}^*_\omega, r),\ j = 1, \ldots, n\big\}\,,$$

by (2.i), we have

$$T_\alpha\big(S_\varrho^{(\alpha)}(\bar{x}, r)\big) = S_\varrho(\bar{x}, \alpha r) \quad \forall\alpha, r > 0. \tag{2.k}$$

Finally we note that, if $u \in W_\lambda^{\mathrm{loc}}(\Omega)$ and $Lu \geqslant 0$ ($Lu \leqslant 0$) in the open set Ω, then $u_\alpha \in W^{\mathrm{loc}}_{\lambda(\alpha)}\big(T_\alpha^{-1}(\Omega)\big)$ and $L_\alpha u \geqslant 0$ ($L_\alpha u \leqslant 0$) in $T^{-1}(\Omega)$, where $u_\alpha = u \circ T_\alpha$.

3. – In this Section, we shall prove some fundamental results allowing us to adapt Moser's machinery to prove the Hölder regularity of our solutions.

Analogously to Remark 2.7 in [10], we can prove the following embedding theorem.

THEOREM 3.1. *There exist* $q \in]2, +\infty[$ *and* $C \in R_+$ *such that,* $\forall \bar{x} \in R^n$, $\forall u \in C_0^\infty(S_d(\bar{x}, 1))$,

$$\|u;\, L^q(\mathbf{R}^n)\| \leqslant C\Big(1 + \sum_{j=1}^n \varphi_j(\bar{x}^*, 1)\Big)\|u;\, W_\lambda(\mathbf{R}^n)\|$$

where q *and* C *depend only on the* $\varrho_{j,k}$'s.

PROOF. By classical Sobolev theorem, without loss of generality, we need only to prove that, if $0 < \varepsilon < \min\{\varepsilon_1, \dots, \varepsilon_n\}$, then

$$I = \int_0^1 h^{-1-2\varepsilon} \int_{\mathbf{R}_j^n} |u(x + he_j) - u(x)|^2\, dx\, dh \leqslant C_\varepsilon\Big(1 + \sum_{j=1}^n \varphi_j(\bar{x}^*, 1)\Big)\, \|u;\, W_\lambda(\mathbf{R}^n)\|^2\,,$$

where C_ε depends only on ε and the $\varrho_{j,k}$'s. Obviously, the integral with respect to the x-variable in I is computed in $R_j^n \cap K$, where

$$K = \bigcup_{0 \leqslant h \leqslant 1} \big(S_d(\bar{x}, 1) - he_j\big)\,.$$

Now, since $\forall x \in K$

$$\begin{aligned}
&|x_k - \bar{x}_k| \leqslant |x_k + h\delta_{j,k} - \bar{x}_k| + 1 < F_k(\bar{x}^*, a) + 1 \\
&= F_k(\bar{x}^*, a) + F_k\big(\bar{x}^*, \varphi_k(\bar{x}^*, 1)\big) \leqslant 2F_k\big(\bar{x}^*, \max\{a, \varphi_k(\bar{x}^*, 1)\}\big) \leqslant \qquad \text{(cfr. (2.8.}a\text{))} \\
&\leqslant F_k\big(\bar{x}^*, 2\max\{a, \varphi_k(\bar{x}^*, 1)\}\big)\,,
\end{aligned}$$

then $K \subseteq S_d(\bar{x}, ar(\bar{x}))$, where

$$r(\bar{x}) = 2\max\{a, \varphi_1(\bar{x}^*, 1), \dots, \varphi_n(\bar{x}^*, 1)\}\,.$$

Now, if $x \in S_d\big(\bar{x}, r(\bar{x})\big) \cap R_j^n$,

$$\begin{aligned}
&\varphi(x, 1) \leqslant \qquad \text{(by Theorem 2.7)} \\
&\leqslant ad(x, x + e_j) \leqslant a\big(d(x, \bar{x}) + d(\bar{x}, x + e_j)\big) \leqslant a\Big(r(\bar{x}) + a\sum_{l=1}^n \varphi_l\big(\bar{x}^*, |\bar{x}_l - x_l| + 1\big)\Big)\,;
\end{aligned}$$

but since

$$1 = F_l\big(x^*, \varphi_l(x, 1)\big) \leqslant F_l\big(\bar{x}^*, r(\bar{x})\big)\,,$$

$$|\bar{x}_l - x_l| + 1 < 2F_l\big(\bar{x}^*, r(\bar{x})\big) \leqslant F_l\big(\bar{x}^*, 2r(\bar{x})\big)\,,$$

so that $\varphi_j(x, 1) \leqslant a(1 + 2na)r(\bar{x}) = C(\bar{x})$, and then, by (2.8.b), $\forall x \in R_j^n \cap K$, $\forall h \in]0, 1[$, $\varphi_j(x, h) \leqslant C(\bar{x})\, h^{\varepsilon_j}$.

Arguing as in Section 3 of [10] I can be estimated by a sum of $2j-1$ integrals such as

$$\int_0^1 dh\, h^{-1-2\varepsilon} \int_{\mathbf{R}_j^n \cap K} dx \Bigg(\int_0^{\varphi_j(x,h)\lambda_k(H_{k-1}(x,\varphi_j(x,h)))} \Big|\partial_k u\big(H_{k-1}(x, \varphi_j(x, h)) + s e_k\big)\Big| ds\Bigg)^2$$

$$\leqslant \int_0^1 dh\, h^{-1-2\varepsilon} \int_{\mathbf{R}_j^n} dx \Bigg(\int_0^{C(\bar{x})h^{\varepsilon_j}\lambda_k(H_{k-1}(x,\varphi_j(x,h)))} \Big|\partial_k u\big(H_{k-1}(x, \varphi_j(x, h)) + s e_k\big)\Big| ds\Bigg)^2$$

$$\leqslant C(\bar{x}) \int_0^1 dh\, h^{-1-2\varepsilon} \int_{\mathbf{R}_j^n} dx \int_0^{C(\bar{x})h^{\varepsilon_j}\lambda_k(H_{k-1}(x,\varphi_j(x,h)))} \Big|(X_k u)\big(H_{k-1}(x, \varphi_j(x, h)) + s e_k\big)\Big|^2 h^{\varepsilon_j}\big(\lambda_k(\ldots)\big)^{-1} ds$$

$\leqslant\big($putting $y = H_{k-1}(x, \varphi_j(x, h)) + s e_k$ and keeping in mind that $|dx/dy| \leqslant G_j$, by [10], (4.3.g)$\big)$

$$\leqslant G_j C^2(\bar{x}) \int_0^1 dh\, h^{-1-2(\varepsilon-\varepsilon_j)} \int_{\mathbf{R}_j^n} |X_k u(y)|^2\, dy\,.$$

So, the assertion is proved.

An analogous technique can be used to prove the following Poincaré inequality.

THEOREM 3.2. *There exist $c, C \in R_+$ such that, $\forall u \in C^\infty(R^n)$,*

$$\Bigg(\int_{S_d(\bar{x},r)} |u - u_r|\, dx\Bigg)^2 \leqslant C r^2 \mu\big(S_d(\bar{x}, r)\big) \int_{S_d(\bar{x},cr)} |\nabla_\lambda u|^2\, dx\,, \tag{3.2.a}$$

$\forall \bar{x} \in R^n$, $\forall r > 0$, where μ is Lebesgue measure in R^n, $|\nabla_\lambda u|^2 = \sum_{j=1}^n \lambda_j^2\, |\partial_j u|^2$ and

$$u_r = \mu\big(S_d(\bar{x}, r)\big)^{-1} \int_{S_d(\bar{x},r)} u(y)\, dy\,.$$

We note explicitly that c and C depend only on the constants $\varrho_{j,k}$'s.

PROOF. In the sequel all constants appearing in the estimates will depend only on $\varrho_{j,k}$. By Theorem 2.7, $S_d(\bar{x}, r) \subseteq S_\varrho(\bar{x}, ar)$, so that

$$\Big(\int\limits_{S_d(\bar{x},r)} |u - u_r|\, dx\Big)^2 \leqslant \int\limits_{(S_d(\bar{x},r))^2} |u(y) - u(z)|^2\, dy\, dz \leqslant \int\limits_{(S_\varrho(\bar{x},ar))^2} |u(y) - u(z)|^2\, dy\, dz$$

$$\leqslant C_1 \sum_{j=1}^{n} \int\limits_{(S_\varrho(\bar{x},ar))^2} |u(z_1, \dots, z_{j-1}, y_j, \dots, y_n) - u(z_1, \dots, z_j, y_{j+1}, \dots, y_n)|^2\, dy\, dz = C_1 \sum_{j=1}^{n} I_j .$$

Now,

$$I_j = \int\limits_{S_\varrho(\bar{x},ar)} \Big(\int\limits_{S_\varrho(\bar{x},ar)} |u(x) - u(x + (z_j - x_j) e_j)|^2\, dx \Big) dy_1 \dots dy_{j-1}\, dz_j \dots dz_n$$

$$\leqslant C_2 \prod_{k \neq j} F_k(\bar{x}^*, ar) \int\limits_{-2F_j(\bar{x}^*, ar)}^{2F_j(\bar{x}^*, ar)} dh \int\limits_{S_\varrho(x,ar)} |u(x + he_j) - u(x)|^2\, dx$$

$$= C_2 \prod_{k \neq j} F_k(\bar{x}^*, ar) \int\limits_{-2F_j(\bar{x}^*, ar)}^{2F_j(\bar{x}^*, ar)} dh \Big(\sum_{\alpha \in \mathcal{A}_j} \int\limits_{S_\alpha(ar)} |u(x + he_j) - u(x)|^2\, dx \Big),$$

where

$$\mathcal{A}_j = \{\alpha = (\alpha_1, \dots, \alpha_n);\ \alpha_k = \pm 1,\ k < j,\ \alpha_j = \dots = \alpha_n = 0\}$$

and

$$S_\alpha(ar) = \{x = (x_1, \dots, x_n) \in S_\varrho(\bar{x}, ar);\ \alpha_k x_k \geqslant 0,\ k = 1, \dots, n\} .$$

Let us now estimate

$$I_\alpha = \int\limits_{S_\alpha(ar)} |u(x + he_j) - u(x)|^2\, dx .$$

Without loss of generality, we may suppose that $\alpha = (1, \dots, 1, 0, \dots, 0)$ and $h > 0$; thus

$$I_\alpha \leqslant C_3 \Big(\sum_{k=1}^{j-1} \int\limits_{S_\alpha(ar)} \big|u\big(H_{k-1}(x, \varphi) + he_j\big) - u\big(H_k(x, \varphi) + he_j\big)\big|^2\, dx$$

$$+ \int\limits_{S_\alpha(ar)} \big|u\big(H_j(x, \varphi)\big) - u\big(H_{j-1}(x, \varphi)\big)\big|^2\, dx$$

$$+ \sum_{k=1}^{j-1} \int\limits_{S_\alpha(ar)} \big|u\big(H_{k-1}(x, \varphi)\big) - u\big(H_k(x, \varphi)\big)\big|^2\, dx \Big) = C_3 \Big(\sum_{k=1}^{j-1} J'_k + J_0 + \sum_{k=1}^{j-1} J_k \Big),$$

where $\varphi = \varphi_j(x, h)$. We have (by the very definition of φ)

$$J_0 = \int_{S_\alpha(ar)} dx \left| \int_0^h (\partial_j u)(H_{j-1}(x, \varphi) + se_j)\, ds \right|^2$$
$$\leqslant \int_{S_\alpha(ar)} h^{-1}\big(h/\lambda_j(H_{j-1}(x, \varphi))\big)^2 \left(\int_0^h |X_j u(H_{j-1}(x, \varphi) + se_j)|^2\, ds \right) dx$$
$$= \int_{S_\alpha(ar)} h^{-1} \varphi^2 \left(\int_0^h |X_j u(H_{j-1}(x, \varphi) + se_j|)^2\, ds \right) dx .$$

Now, by Theorem 2.7, for every $x \in S_\alpha(ra)$, we get

$$\text{(3.2.}b) \qquad \varphi_j(x, h) \leqslant a d(x, x + he_j) \leqslant a\big(d(x, \bar{x}) + d(\bar{x}, x + he_j)\big)$$
$$\leqslant a^2\big(\varrho(\bar{x}, x) + \varrho(\bar{x}, x + he_j)\big) \leqslant (n + 3) a^3 r = C_3 r ,$$

since $|\bar{x}_k - (x + he_j)_k| = |\bar{x}_k - x_k| < F_k(\bar{x}^*, ar)$, for every $k \neq j$ and

$$|\bar{x}_j - (x + he_j)_j| \leqslant |\bar{x}_j - x_j| + h \leqslant F_j(\bar{x}^*, ar) + 2F_j(\bar{x}^*, ar) \leqslant F_j(\bar{x}^*, 3ar) ,$$

so that $\varrho(\bar{x}, x + he_j) \leqslant (n + 2) ar$.

Then

$$J_0 \leqslant C_3^2 r^2 \int_{S_\alpha(ar)} h^{-1} \left(\int_0^h |X_j u(H_{j-1}(x, \varphi) + se_j)|^2\, ds \right) dx$$

$\leqslant$ (putting $y = H_{j-1}(x, \varphi) + se_j$ and keeping in mind that, by [10] (4.3.g). $|dx/dy| \leqslant G_j) \leqslant C_4 r^2 \int_{S_\alpha(c_5 r)} |X_j u(y)|^2\, dy$.

In fact, for every fixed $x \in S_\alpha(ar)$, if we denote by γ the polygonal

$$[x, x + F_1(x, \varphi) e_1] \cup [x + F_1(x, \varphi) e_1, x + F_1(x, \varphi) e_1 + F_2(x, \varphi) e_2]$$
$$\ldots \cup [x + F_1(x, \varphi) e_1 + \ldots + F_{j-1}(x, \varphi) e_{j-1}, y] ,$$

we have $d(x, y) \leqslant l(\gamma) = j\varphi_j(x, h) \leqslant C_3 jr$, so that

$$d(y, \bar{x}) \leqslant d(x, \bar{x}) + d(x, y) \leqslant a^2 r + C_3 nr = C_5 a^{-1} r ,$$

and hence $\varrho(y, x) \leqslant C_5 r$.

So, J_0 is estimated.

Let us now estimate J_k, $1 \leqslant k \leqslant j-1$. Analogously as above, we have:

$$J_k = \int_{S_\alpha(ar)} dx \left| \int_0^{\varphi\lambda_k(H_{k-1}(x,\varphi))} (\partial_k u)(H_{k-1}(x,\varphi) + se_k)\, ds \right|^2 \leqslant (\text{by } (3.2.b))$$

$$\leqslant \int_{S_\alpha(ar)} dx \left(\int_0^{C_3 r \lambda_k(H_{k-1}(x,\varphi))} \left|(\partial_k u)(H_{k-1}(x,\varphi) + se_k)\right| ds \right)^2$$

$$\leqslant C_3 r \int_{S_\alpha(ar)} dx\, \lambda_k(H_{k-1}(x,\varphi)) \int_0^{C_3 r \lambda_k(H_{k-1}(x,\varphi))} \left|(\partial_k u)(H_{k-1}(x,\varphi) + se_k)\right|^2 ds$$

$$(\text{putting } y = H_{k-1}(x,\varphi) + se_k) \leqslant C_6 r^2 \int_{S_\alpha(c_5 r)} |X_k u(y)|^2\, dy\,.$$

The terms J_k', $1 \leqslant k \leqslant j-1$ can be handled analogously. Then, if we put $c = aC_5$, we get

$$I_\alpha \leqslant C_7 r^2 \int_{S_d(\bar{x}, cr)} |\nabla_\lambda u|^2\, dx\,, \quad \text{so that } I_j \leqslant C_8 r^2 \prod_{k=1}^{n} F_k(\bar{x}^*, ar) \int_{S_d(\bar{x}, cr)} |\nabla_\lambda u|^2\, dx$$

$$\leqslant C_9 r^2 \prod_{k=1}^{n} F_k(\bar{x}^*, r) \int_{S_d(\bar{x}, cr)} |\nabla_\lambda u|^2\, dx \leqslant \quad (\text{by Theorem 2.7})$$

$$\leqslant C_{10} r^2 \mu(S_d(\bar{x}, r)) \int_{S_d(\bar{x}, cr)} |\nabla_\lambda u|^2\, dx\,.$$

So, the assertion is proved.

REMARK 3.3. Let $x_0 \in R^n$ and $r, R \in R_+$ be fixed, $r \leqslant 2R$; if $\bar{x} \in S_d(x_0, R)$, we shall denote by u_r^* the mean value of u on the relative ball $S_d^*(\bar{x}, r) = S_d(x_0, R) \cap S_d(\bar{x}, r)$. Then, we have

$$\left(\int_{S_d^*(\bar{x}, r)} |u - u_r^*|\, dx \right)^2 \leqslant \int_{(S_d^*(\bar{x}, r))^2} |u(y) - u(z)|^2\, dy\, dz \leqslant (\text{by Theorem 3.2})$$

$$\leqslant C r^2 \mu(S_d(\bar{x}, r)) \int_{S_d(\bar{x}, cr)} |\nabla_\lambda u|^2\, dx \leqslant \quad (\text{by Proposition 2.10})$$

$$\leqslant Cb\, r^2 \mu(S_d^*(\bar{x}, r)) \int_{S_d(\bar{x}, cr)} |\nabla_\lambda u|^2\, dx\,.$$

4. – In this Section, we shall prove the Hölder regularity of the weak solutions of $Lu = 0$ via Moser's technique ([15]; see also [11], Section 8.6).

To this end, preliminarily, we note that if $f\colon R \to R$ is a continuous function with piecewise continuous first derivative $f' \in L^\infty(R)$, then $f\circ u$ belongs to $W_\lambda(\Omega)$ for every $u \in W_\lambda(\Omega)$. Moreover, if Ω is λ-connected and if $u \in W_\lambda(\Omega)$, then $\partial_j u \in L^2_{\text{loc}}(\Omega \setminus \Pi)$, where

$$\Pi = \left\{x = (x_1, \dots, x_n) \in R^n,\ \prod_{j=1}^n x_j = 0\right\},$$

so that

$$x \to q(u, v) = \sum_{i,j=1}^n a_{i,j}(x)\, \partial_i u(x)\, \partial_j u(x)$$

belongs to $L^1(\Omega)$, $\forall u, v \in W_\lambda(\Omega)$. In the sequel, we shall put $|\nabla_A u|^2 = q(u, u)$.

The first step is to prove the local boundedness of the solutions.

THEOREM 4.1. *Let Ω be a λ-connected open subset of R^n and let $u \in W^\lambda_{\text{loc}}(\Omega)$ be such that $Lu \geq 0$. Then, $\forall \bar{x} \in \Omega$ $\exists R_0 > 0$ such that, $\forall R > 0$, $R \leq R_0$, we have:*

$$\text{(4.1.}a\text{)} \qquad \sup_{B(\bar{x}, R)} u \leq C_R \|u^+;\ L^2(B(\bar{x}, 2R))\|,$$

where $B(\bar{x}, R) = \{x \in R^n;\ |x - \bar{x}| < R\}$ is the usual euclidean ball,

$$u_+ = \max\{0, u\}$$

and R_0, C_R are independent of u.

PROOF. First, let us suppose $u \geq 0$. Analogously to the elliptic case (see, e.g., [11], Section 8.5), with a suitable choice of the test function in the inequality $\mathfrak{L}(u, v) \leq 0$, we get:

$$\text{(4.1.}b\text{)} \qquad \int_\Omega |\nabla_A(\psi H(u))|^2\, dx \leq C_1^2 \int_\Omega |H'(u)u|^2\, |\nabla_A \psi|^2\, dx,$$

where $\psi \in C_0^\infty(B(\bar{x}, R))$ and, for fixed $\beta \geq 1$ and $N > 0$, $H(t) = t^\beta$ for $t \in [0, N]$ and $H(t) = N^\beta + (t - N)\beta N^{\beta - 1}$ for $t \geq N$. The constant C_1 is independent of u, β, N. Let $R_0 \in R_+$ be fixed in such a way that $B(\bar{x}, 3R_0) \subseteq \Omega$. Then, by Theorem 3.1 and (2.a), there exist $q > 2$, $C_2 = C_2(R_0)$ independent of β and N such that, if $R \leq R_0$, $r < R$ and $\psi / B(\bar{x}, r) \equiv 1$,

$$\left(\int_{\mathbf{R}^n} |\psi H(u)|^q\, dx\right)^{1/q} \leq C_2\big(\|\psi H(u);\ L^2(\mathbf{R}^n)\| + \|\,|\nabla_A(\psi H(u))|;\ L^2(\mathbf{R}^n)\|\big);$$

hence

$$\begin{aligned}\|H(u);\, L^q(B(\bar{x}, r))\| &\leqslant \|\psi H(u);\, L^q(B(\bar{x}, R))\|\\ &\leqslant C_2\big(\|\psi H(u);\, L^2(\mathbb{R}^n)\| + \|\,|\nabla_A(\psi H(u))|;\, L^2(\mathbb{R}^n)\|\big) \leqslant \quad \text{(by (4.1.}b\text{) and (2.}a\text{))}\\ &\leqslant C_2\big(\|\psi H(u);\, L^2(\mathbb{R}^n)\| + C_1 m\, \|H'(u)u|\nabla_\lambda \psi|;\, L^2(\mathbb{R}^n)\|\big).\end{aligned}$$

Now, since it is possible to choice ψ such that $|\nabla_\lambda \psi| \leqslant 2(R-r)^{-1}$, for $N \to +\infty$, we get:

$$\|u;\, L^{\beta q}(B(\bar{x}, r))\| \leqslant \big(C_4\beta/(R-r)\big)^{1/\beta}\, \|u;\, L^{2\beta}(B(\bar{x}, R))\|\,,$$

where C_4 is independent of u and β.

Now, (4.1.a) follows via Moser's iteration technique (see [15] and [11], Section 8.5) if $u \geqslant 0$.

Finally, we can handle the general case in the following way. Let $(f_k)_{k\in N}$ be a sequence of C^2-functions such that: i) $f_k\colon R \to R$; ii) f_k is an increasing, nonnegative convex function which is linear outside of a compact set; iii) $f_k(t) \leqslant 2(1 + |t|)$, $\forall t \in R$; iv) $f_k(t) \to \max\{0, t\}$ as $k \to +\infty$. Then $f_k(u) \in W^{\mathrm{loc}}_\lambda(\Omega)$ and $L(f_k(u)) \geqslant 0$ (see [15]). Thus, since $f_k(u) \geqslant 0$, we get

$$\sup_{B(\bar{x}, R)} f_k(u) \leqslant C_{\mathbf{R}} \|f_k(u);\, L^2(B(\bar{x}, 2R))\|\,, \qquad \forall k \in N\,.$$

So, if $k \to +\infty$, (4.1.a) follows.

LEMMA 4.2. *Let Ω be an open λ-connected subset of R^n and let u be a nonnegative solution of $Lu = 0$ belonging to $W^{\mathrm{loc}}_\lambda(\Omega)$. Moreover, let $\bar{x}$ be a fixed point of Ω such that $\overline{S_\varrho(\bar{x}, 3a^2c)} \subseteq \Omega$, where c is the constant appearing in Theorem 3.2. Then*

i) $\forall p > 1,\ \sup\limits_{S(\bar{x}, \frac{1}{2})} u \leqslant M'_p \|u;\, L^p(S_\varrho(\bar{x}, 1))\|$;

ii) $\exists \sigma > 1$ *such that,* $\forall p \in [1, \sigma[,\ \inf\limits_{S_\varrho(\bar{x}, \frac{1}{2})} u \geqslant M''_p \|u;\, L^p(S^p{}_\varrho(\bar{x}, 1))\|$,

where σ, M'_p, M''_p depend only on the constant m of (2.a), on $\varrho_{j,k}$ and on $\varphi_j(\bar{x}^*, 1)$, $F_j(\bar{x}^*, 1)$, $j = 1, \ldots, n$.

PROOF. Obviously, we need only to prove the assertion if $u \geqslant k > 0$. In this case, by the local boundedness of u (Theorem 4.1), $\forall \beta \in R$ and $\forall \eta \in C^\infty_0(\Omega)$, the function $v = \eta u^\beta$ belongs to $\overset{\circ}{W}_\lambda(\Omega)$; so that $\mathfrak{L}(u, v) = 0$.

Then, arguing as in [11], Section 8.6, if $\beta \neq 0$, we get

$$(4.2.a)\qquad \int_{\mathbf{R}^n} |\eta \nabla_\lambda w|^2\, dx \leqslant \begin{cases} C_1((\beta+1)/\beta)^2 \displaystyle\int_{\mathbf{R}^n} |\nabla_\lambda \eta|^2 w^2\, dx\,, & \text{if } \beta \neq -1\,,\\ C_1 \displaystyle\int_{\mathbf{R}^n} |\nabla_\lambda \eta|^2\, dx\,, & \text{if } \beta = -1\,,\end{cases}$$

where C_1 depends only on the constant m and

$$(4.2.b)\qquad w = \begin{cases} u^{(\beta+1)/2}\,, & \text{if } \beta \neq -1\,,\\ \log u\,, & \text{if } \beta = -1\,.\end{cases}$$

Let now r_1 and r_2 be fixed real positive numbers such that $r_1 < r_2 < 3a^2 c$. Preliminarily, let us prove that it is possible to choice $\eta = \eta(\bar{x}, r_1, r_2, \cdot) \in C_0^\infty(S_\varrho(\bar{x}, r_2))$ in such a way that $\eta = 1$ on $S_\varrho(\bar{x}, r_1)$ and $|\nabla_\lambda \eta| \leqslant 2(r_2 - r_1)^{-1}$. Let $\psi \in C_0^\infty(R, R)$ be such that: i) $0 \leqslant \psi \leqslant 1$; ii) $\psi(t) = \psi(-t)$, $\forall t \in R$; iii) $\psi \equiv 1$ on $[-r_1/r_2, r_1/r_2]$; iv) $\psi = 0$ outside of $]-1, 1[$; v) $|\psi'(t)| \leqslant 2(1 - r_1/r_2)^{-1}$, $\forall t \in R$.

We put $\eta(x) = \prod_{j=1}^n \psi(|x_j - \bar{x}_j|/F_j(\bar{x}^*, r_2))$; obviously, η is a smooth function supported in $S_\varrho(\bar{x}, r_2)$. Moreover, since

$$F_j(\bar{x}^*, r_1) \leqslant (r_1/r_2) F_j(\bar{x}^*, r_2)\,, \qquad j = 1, \ldots, n \ \big(\text{see } (2.8.a)\big)\,,$$

if $x \in S_\varrho(\bar{x}, r_1)$, then $\eta(x) = 1$. Finally, if $1 \leqslant j \leqslant n$ and $x \in S_\varrho(\bar{x}, r_2)$,

$$\begin{aligned} |\lambda_j(x)\, \partial_j \eta(x)| = \prod_{r \neq j} \psi\big(|x_k - \bar{x}_k|/F_k(\bar{x}^*, r_2)\big)\, \lambda_j(x) \big|\psi'\big(|x_j - \bar{x}_j|/F_j(\bar{x}^*, r_2)\big)\big(F_j(\bar{x}^*, r_2)\big)^{-1} \\ \leqslant 2 r_2 (r_2 - r_1)^{-1} \lambda_j(x) \big(F_j(\bar{x}^*, r_2)\big)^{-1}\,. \end{aligned}$$

Then, the assertion follows if we note that

$$\begin{aligned} r_2 \lambda_j(x) &= r_2 \lambda_j(|x_1|, \ldots, |x_{j-1}|) \\ &\leqslant r_2 \lambda_j\big(|\bar{x}_1| + F_1(\bar{x}^*, r_2), \ldots, |\bar{x}_{j-1}| + F_{j-1}(x^*, r_2)\big) = F_j(\bar{x}^*, r_2)\,. \end{aligned}$$

Now, by Theorem 3.1 (with the constants q and C_q appearing therein), we get:

$$\|\eta w; L^q(\mathbf{R}^n)\| \leqslant C_q\Big(1 + \sum_{j=1}^n \varphi_j(\bar{x}^*, 1)\Big) \cdot \big(\|\eta w; L^2(\mathbf{R}^n)\| + \|\, |\nabla_\lambda(\eta w)|; L^2(\mathbf{R}^n)\|\big)\,.$$

So, by (4.2.a) and (4.2.b), if $\beta > 0$, we have

$$\|u; L^{\sigma p}(S_\varrho(\bar{x}, r_1))\| \leqslant \left[C'_\varrho\left(1 + \sum_{j=1}^{n} \varphi_j(\bar{x}^*, 1)\right)(1 + p/(p-1)(r_2 - r_1))\right]^{2/p} \|u; L^p(S_\varrho(\bar{x}, r_2))\|, \tag{4.2.c}$$

where $p = \beta + 1$ and $\sigma = q/2$.

From (4.2.c), by Moser's iteration technique, we get i). Moreover, by (4.2.a) and (4.2.b) with $\beta \in]-1, 0[$ and $\beta \in]-\infty, -1[$, we obtain, respectively $\forall p, p_0$, $0 < p_0 < p < \sigma$,

$$\left(\int_{S_\varrho(\bar{x}, 1)} u^p \, dx\right)^{1/p} \leqslant C_2 \left(\int_{S_\varrho(\bar{x}, \frac{3}{2})} u^{p_0}\right)^{1/p_0}; \tag{4.2.d}$$

$$\inf_{S_\varrho(\bar{x}, \frac{3}{2})} u \geqslant C_3 \left(\int_{S_\varrho(\bar{x}, \frac{1}{2})} u^{-p_0} \, dx\right)^{-1/p_0}, \tag{4.2.e}$$

where C_2, C_3 depend only on p, p_0, m, $\varrho_{j,k}$, $\varphi_j(\bar{x}^*, 1)$, $j, k = 1, \ldots, n$.

Now, the proof of ii) will be accomplished if we show that there exists $p_0 \in]0, 1[$ such that

$$\left(\int_{S_\varrho(\bar{x}, \frac{3}{2})} u^{p_0} \, dx\right)\left(\int_{S_\varrho(\bar{x}, \frac{3}{2})} u^{-p_0} \, dx\right) \leqslant C_4, \tag{4.2f}$$

where p_0, C_4 depend only on m, $\varrho_{j,k}$ and $F_j(\bar{x}^*, 1)$, $j = 1, \ldots, n$. Indeed, if we put $w = \log u$, we have:

$$\left(\int_{S_\varrho(\bar{x}, \frac{3}{2})} u^{p_0} \, dx\right)^{\frac{1}{2}}\left(\int_{S_\varrho(\bar{x}, \frac{3}{2})} u^{-p_0} \, dx\right)^{\frac{1}{2}} \leqslant \int_{S_d(\bar{x}, 3a/2)} \exp\left(p_0 |w - w_{3a/2}|\right) dx = p_0 \int_0^{+\infty} \nu(s) \exp(p_0 s) \, ds + \mu\left(S_d(\bar{x}, 3a/2)\right),$$

where $w_{3a/2}$ is the mean value of w in $S_d(\bar{x}, 3a/2)$ (see Theorem 3.2) and $\nu(s) = \mu\left(\{x \in S_d(\bar{x}, 3a/2);\ |w(x) - w_{3a/2}| > s\}\right)$.

Now, the function ν can be estimated as follows:

$$\nu(s) \leqslant C_5 \exp(-C_6 s) \mu\left(S_d(\bar{x}, 3a/2)\right), \tag{4.2.g}$$

where C_5 and C_6 depend only on $\varrho_{j,k}$ and m. In order to prove (4.2.g), we note preliminarily that w is a bounded mean oscillation (BMO) function with respect to the d-balls in the space of homogeneous type $S_d(\bar{x}, 3a/2)$. Let y belong to $S_d^*(\bar{x}, 3a/2)$; first, let us suppose $r \geq 3a$; then, obviously, $S_d^*(y, r) = S_d(y, r) \cap S_d(\bar{x}, 3a/2) = S_d(\bar{x}, 3a/2)$. Then, by Theorem 3.1, (4.2.a) and (4.2.b) with $\eta = \eta(\bar{x}, 3a^2 c/2, 3a^2 c, \cdot)$, we have ($w_r^*$ is the mean value of u on $S_d^*(y, r)$):

$$\Bigl(\int_{S_d^*(y,r)} |w - w_r^*|\, dx\Bigr)^2 = \Bigl(\int_{S_d(\bar{x},3a/2)} |w - w_{3a/2}|\, dx\Bigr)^2 \leq (9Ca^2/4)\,\mu\bigl(S_d(\bar{x}, 3a/2)\bigr)\int_{S_d(\bar{x},3ac/2)} |\nabla_\lambda w|^2\, dx$$

$$\leq C_7\, \mu\bigl(S_d^*(y, r)\bigr)\,\mu\bigl(S_d(\bar{x}, 3a^3 c)\bigr) \qquad \leq \text{(by the doubling condition)}$$

$$\leq C_8\, \mu^2\bigl(S_d^*(y, r)\bigr)\,,$$

here C_8 depends only on m and $\varrho_{j,k}$.

On the other hand, if $r < 3a$, by Remark 3.3, (4.2.a) and (4.2.b) with $\eta = \eta(y, acr, 2acr, \cdot)$,

$$\Bigl(\int_{S_d^*(y,r)} |w - w_r^*|\, dx\Bigr)^2 \leq C_9\, \mu\bigl(S_d^*(y, r)\bigr)\,\mu\bigl(S_d(y, 2a^2 cr)\bigr) \leq \quad \text{(by Proposition 2.10)}$$

$$\leq C_{10}\, \mu^2\bigl(S_d^*(y, 2a^2 cr)\bigr) \leq C_{11}\, \mu^2\bigl(S_d^*(y, r)\bigr)\,,$$

where C_{11} depends only on m and $\varrho_{j,k}$.

So, we proved that w is a BMO-function. Then, (4.2.g) follows by John-Nirenberg's theorem which holds in a metric space of homogeneous type, too ([4], p.594; see also [1]). Now, (4.4.f) follows by (4.2.g) and Theorem 2.7. Thus ii) is proved.

The careful estimate of the constants in Lemma 4.2 enables us to prove the following crucial result.

THEOREM 4.3. *Let Ω be a λ-connected open subset of R^n and let u be a nonnegative solution of $Lu = 0$ belonging to $W_\lambda^{\mathrm{loc}}(\Omega)$. Then, there exist c_1, M_p', $M_p'' \in R_+$ such that, $\forall \bar{x} \in \Omega$, $\forall R > 0$ such that $S_\varrho(\bar{x}, c_1 R) \subseteq \Omega$, we have*

i) $\forall p > 1$, $\sup\limits_{S_\varrho(\bar{x},R/2)} u \leq M_p'\bigl(\mu(S_\varrho(\bar{x}, R))\bigr)^{-1/p} \|u;\, L^p(S_\varrho(\bar{x}, R))\|$;

ii) $\forall p \in [1, \sigma[$, $\inf\limits_{S_\varrho(\bar{x},R/2)} u \geq M_p''\bigl(\mu(S_\varrho(\bar{x}, R))\bigr)^{-1/p} \|u;\, L^p(S_\varrho(\bar{x}, R)\|$.

PROOF. The proof will be carried out by using the homotethical transformations centred in $\bar{x}$ defined in Section 2; in the sequel we shall use the notations introduced therein. We have: $u_R \in W_{\lambda(R)}^{\mathrm{loc}}(T^{-1}(\Omega))$, $L_R u_R = 0$ in $T_R^{-1}(\Omega)$, and, obviously, $u_R \geq 0$. Moreover, if we put $c_1 = 3a^2 c$, $T_R^{-1}(S_\varrho(\bar{x}, R)) = S_\varrho^{(R)}(\bar{x}, 1)$, $T_R^{-1}(S_\varrho(\bar{x}, c_1 R)) = S_\varrho^{(R)}(\bar{x}, 3a^2 c) \subseteq T^{-1}(\Omega)$; so, we can apply the results of Lemma 4.2.

The essential point is that the constants M_p', M_p'' depend only on the constant m, on $\varrho_{j,k}$ (see (2.a') and (2.c')) and on $\varphi_j^{(R)}(\bar{x}_\omega^*, 1)$, $F_j^{(R)}(\bar{x}_\omega^*, 1)$, $j = 1, \ldots, n$; but the last constants are identically equal to 1, by (2.i) and (2.j); thus σ, M_p', M_p'' are independent of R. The proof of the Theorem can be accomplished by the change of variables $y = T_R(x)$.

Now, we can prove the following extention of De Giorgi Theorem.

THEOREM 4.4. *Let Ω be a λ-connected open subset of R^n. If $u \in W_\lambda^{\mathrm{loc}}(\Omega)$ and $Lu = 0$ in Ω, then u is locally Hölder-continuous in Ω.*

PROOF. Exactly as in the elliptic case (see, e.g., [11], Section 8.9), by Theorem 4.3 we have:

$$\underset{S_d(y,R)}{\operatorname{osc}} u \leqslant CR^\alpha, \qquad \forall R \leqslant R_0 \tag{4.4.a}$$

for a suitable R_0, C, $\alpha > 0$, that can be chosen independent on y if y belongs to a fixed compact subset K of Ω. Then, the assertion follows by (2.9.a).

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