

HOLLOW QUASI-FATOU COMPONENTS OF QUASIREGULAR MAPS

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ABSTRACT. We define a quasi-Fatou component of a quasiregular map as a connected component of the complement of the Julia set. A domain in \mathbb{R}^d is called *hollow* if it has a bounded complementary component. We show that for each $d \geq 2$ there exists a quasiregular map of transcendental type $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with a quasi-Fatou component which is hollow.

Suppose that U is a hollow quasi-Fatou component of a quasiregular map of transcendental type. We show that if U is bounded, then U has many properties in common with a multiply connected Fatou component of a transcendental entire function. On the other hand, we show that if U is not bounded, then it is completely invariant and has no unbounded boundary components. We show that this situation occurs if $J(f)$ has an isolated point, or if $J(f)$ is not equal to the boundary of the fast escaping set. Finally, we deduce that if $J(f)$ has a bounded component, then all components of $J(f)$ are bounded.

1. INTRODUCTION

In the study of complex dynamics, the first example of a transcendental entire function with a multiply connected Fatou component was given by Baker [1] over half a century ago. We refer to the survey [4] for definitions and further background on complex dynamics. Since Baker's paper many authors have studied multiply connected Fatou components; see, for example, the papers [2, 3, 11, 23, 26].

In this paper we extend this study to more than two (real) dimensions, for the first time. Suppose that $d \geq 2$, and that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a *quasiregular map of transcendental type*. We refer to Section 2 for a definition of a quasiregular map; intuitively, this is a map with a bounded amount of local distortion. A quasiregular map is said to be of transcendental type if it has an essential singularity at infinity. In this setting we need a different definition of the Julia set to that used in complex dynamics. Following [6] and [10], we define the *Julia set* $J(f)$ to be the set of all $x \in \mathbb{R}^d$ such that

$$(1.1) \quad \text{cap} \left(\mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} f^k(U) \right) = 0,$$

for every neighbourhood U of x . Here, if $S \subset \mathbb{R}^d$, then $\text{cap } S = 0$ means that S is, in a precise sense, a “small” set; we refer to Section 2 for a full definition. It is known [10, Theorem 1.1] that if f is a quasiregular map of transcendental type, then $\text{card } (J(f)) = \infty$.

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We are not aware that any author has studied the complement of the Julia set of a general quasiregular map. We define the *quasi-Fatou set* $QF(f)$ as the complement in \mathbb{R}^d of the Julia set. Thus the quasi-Fatou set is an open set which, if non-empty, has the Julia set as its boundary. Observe that we make no assumptions about the normality of the family of iterates $(f^k)_{k \geq 0}$ in a *quasi-Fatou component*, which is a connected component of the quasi-Fatou set. A set S is said to be *completely invariant* if $x \in S$ implies that $f(x) \in S$ and $f^{-1}(x) \subset S$. It follows from the definitions that both the Julia and quasi-Fatou sets are completely invariant. Note that it follows from [10, Theorem 1.2] that if f is a transcendental entire function, then $QF(f)$ is simply the usual Fatou set.

If $G \subset \mathbb{R}^d$ is a domain, then we denote by $T(G)$ the *topological hull* of the domain, in other words the union of G with its bounded complementary components. Note that unlike, for example, in [7], we allow G to be unbounded in this definition. It is well known that if $G \subset \mathbb{R}^d$ is a domain, then $T(G)$ is also a domain; see, for example [18, Exercise 2.13.6(c)]. If $G = T(G)$, then we say that G is *full*. Observe that a domain in the complex plane is full if and only if it is simply connected. If $G \neq T(G)$, then we say that G is *hollow*.

Our first result is a generalisation of Baker's result in [1].

Theorem 1.1. *For each $d \geq 2$, there exists a quasiregular map of transcendental type $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $QF(f)$ has a component which is hollow.*

Remark 1. If $\mathbb{R}^d = \mathbb{R}^2$, which we can identify with \mathbb{C} , we can take f to be a transcendental entire function with a multiply connected Fatou component. Our construction in dimensions greater than two is an analogue of Baker's original construction, applied using the functions in [12].

Remark 2. When Baker constructed a multiply connected Fatou component in [1] he did not know if it was bounded. In fact, it was over ten years before he showed [2] that this is indeed the case. He later showed [3] that, in fact, this is *always* the case for a transcendental entire function. We do not know, however, if the hollow quasi-Fatou component of the function in Theorem 1.1 is bounded or not. It is an open question whether there exists a quasi-Fatou component that is both hollow and unbounded.

We prove two results, which concern the cases that a hollow quasi-Fatou component is either bounded or unbounded. The first shows that quasi-Fatou components of quasiregular maps of transcendental type that are both bounded and hollow have properties very similar to those of multiply connected Fatou components of transcendental entire functions. In order to state this result we need to give a number of definitions, all of which are familiar from complex dynamics.

If U_0 is a quasi-Fatou component of f , then we let U_k be the quasi-Fatou component of f containing $f^k(U_0)$, for $k \in \mathbb{N}$. (Note that, even for entire f , we cannot assume that $U_k = f^k(U_0)$, although this equality can hold; see Lemma 5.1 below and [19, Lemma 4.1]). If $U_n \neq U_m$, for $n \neq m$, then we say that U_0 is *wandering*.

The *escaping set*

$$I(f) = \{x : f^k(x) \rightarrow \infty \text{ as } k \rightarrow \infty\}$$

was first studied for a general transcendental entire function in [13], and plays an important role in complex dynamics. This set was first studied for a quasiregular map in [8].

We also use two subsets of the escaping set which themselves now have an important role in complex dynamics. The *fast escaping set* $A(f)$ can be defined by

$$(1.2) \quad A(f) = \{x : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{k+\ell}(x)| \geq M^k(R, f), \text{ for } k \in \mathbb{N}\}.$$

Here $M(R, f)$ denotes the *maximum modulus function*

$$M(R, f) = \max_{|x|=R} |f(x)|, \quad \text{for } R > 0,$$

$M^k(R, f)$ denotes the k th iterate of $M(R, f)$ with respect to the first variable, and $R > 0$ can be taken to be any value such that $M^k(R, f) \rightarrow \infty$ as $k \rightarrow \infty$. For a transcendental entire function this form of the definition of $A(f)$ was first used in [21]. This definition was first used for a quasiregular map of transcendental type in [7], where it was shown to be independent of R , and equivalent to two other definitions.

We also define the set

$$A_R(f) = \{x : |f^k(x)| \geq M^k(R, f), \text{ for } k \in \mathbb{N}\}.$$

It is known that for a transcendental entire function f , the sets $A_R(f)$, $A(f)$ and $I(f)$ can have a structure known as a spider's web. We refer to [21, 22] for more background on this structure, which is defined as follows.

Definition 1.2. *A set $E \subset \mathbb{R}^d$ is a spider's web if E is connected, and there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of bounded full domains with $G_n \subset G_{n+1}$, for $n \in \mathbb{N}$, $\partial G_n \subset E$, for $n \in \mathbb{N}$, and $\cup_{n \in \mathbb{N}} G_n = \mathbb{R}^d$.*

We are now able to state our result. Here we denote the Euclidean distance from a point x to a set $U \subset \mathbb{R}^d$ by $\text{dist}(x, U) = \inf_{y \in U} |x - y|$. We say that a set $U \subset \mathbb{R}^d$ *surrounds* a set $V \subset \mathbb{R}^d$ if V is contained in a bounded component of the complement of U .

Theorem 1.3. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Suppose that U_0 is a quasi-Fatou component of f which is bounded and hollow. Let $R > 0$ be such that $M^k(R, f) \rightarrow \infty$ as $k \rightarrow \infty$. Then:*

- (a) *each U_k is bounded and hollow, U_{k+1} surrounds U_k for all large k , and also $\text{dist}(0, U_k) \rightarrow \infty$ as $k \rightarrow \infty$;*
- (b) *there exists $\ell \in \mathbb{N}$ such that $\overline{U}_\ell \subset A_R(f)$, and so $\overline{U}_0 \subset A(f)$;*
- (c) *the sets $A_R(f)$, $A(f)$ and $I(f)$ are spiders' webs.*

Remark 3. If f is a transcendental entire function and U_0 is a Fatou component of f , then the assumption that U_0 is bounded is not required. In this case part (a) is a result of Baker [3, Theorem 3.1], and part (b) and part (c) are due to Rippon and Stallard [21, Theorem 4.4 and Corollary 6.1]. We note that it follows from part (a) that U_0 is wandering.

Our second result, which concerns quasi-Fatou components of quasiregular maps of transcendental type which are unbounded and hollow, is as follows.

Theorem 1.4. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Suppose that U is a quasi-Fatou component of f which is unbounded and hollow. Then U has no unbounded boundary components and is completely invariant, and all other quasi-Fatou components of f are full.*

Remark 4. If f is a transcendental entire function and U is a Fatou component of f , then this result is due to Töpfer [24], although Baker [3, Theorem 3.1] later showed that all multiply connected Fatou components of a transcendental entire function are, in fact, bounded.

The following corollary of Theorem 1.3 and Theorem 1.4 is immediate, and is a generalisation of a well-known fact in complex dynamics.

Corollary 1.5. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. If $J(f)$ has a bounded component, then all components of $J(f)$ are bounded.*

Finally we consider the implications of two possible configurations of the Julia set of a quasiregular map, f , of transcendental type. Firstly, it is known [9, Theorem 1.1] that

$$(1.3) \quad J(f) \subset \partial A(f).$$

It is also known [9, Theorem 1.2] that if

$$\liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log \log r} = \infty,$$

then $J(f) = \partial A(f)$. However, it is not known if it is possible for $QF(f) \cap \partial A(f)$ to be non-empty. Secondly, it is known that for many quasiregular maps of transcendental type, $J(f)$ is perfect. The paper [10] shows that this holds in a variety of circumstances. It is not known, however, if it is ever possible for $J(f)$ to have an isolated point.

The following theorem considers the case that either $J(f) \neq \partial A(f)$, or that $J(f)$ has an isolated point. Note that if f is a transcendental entire function, then $J(f) = \partial A(f)$ [21, Theorem 5.1] and $J(f)$ is perfect [4, Theorem 3].

Theorem 1.6. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Suppose that $QF(f) \cap \partial A(f) \neq \emptyset$, or that $J(f)$ has an isolated point. Then f has a quasi-Fatou component which is unbounded and hollow.*

A key tool in the proof of these results is the following theorem, which may be of independent interest. Here we denote by $B(a, r)$ the open ball of Euclidean radius r , centred at a point $a \in \mathbb{R}^d$.

Theorem 1.7. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Let $r > 0$ be sufficiently large that $M^k(r, f) \rightarrow \infty$ as $k \rightarrow \infty$. Suppose that G is a domain, and that there exist $x, y \in G$ and $\ell_0, k_0 \in \mathbb{N}$ such that*

$$(1.4) \quad |f^{k+\ell_0}(y)| \leq M^{k-1}(r, f) < M^k(r, f) \leq |f^{k+\ell_0}(x)|, \quad \text{for } k \geq k_0.$$

Then there exists $\ell \in \mathbb{N}$ such that

$$(1.5) \quad B(0, M^k(r, f)) \subset T(f^{k+\ell}(G)), \quad \text{for } k \in \mathbb{N}.$$

Note that the hypothesis (1.4) is satisfied if $G \cap \partial A(f) \neq \emptyset$. We give this slightly more general result in view of potential applications.

The proof of Theorem 1.7 uses a certain conformally invariant metric. Very roughly, the theorem holds since f cannot increase this metric too much, whereas the points x and y must iterate far apart in the Euclidean metric. This can be used to show that the boundary of the topological hull of $f^k(G)$ must be far from these

iterates, for large values of k .

The subsequent article [19] contains further results about hollow quasi-Fatou components of quasiregular maps.

The structure of this paper is as follows. First, in Section 2 we recall the definitions of quasiregularity, capacity and the modulus of a curve family, and we give some known results required in the rest of the paper. Next, in Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.7. Finally in Section 5 we prove Theorem 1.3, Theorem 1.4 and Theorem 1.6.

2. QUASIREGULAR MAPS, CAPACITY AND THE MODULUS OF A CURVE FAMILY

We refer to the monographs [20] and [25] for a detailed treatment of quasiregular maps. Here we only recall the definition and the main properties used.

If $d \geq 2$, $G \subset \mathbb{R}^d$ is a domain and $1 \leq p < \infty$, then the *Sobolev space* $W_{p,loc}^1(G)$ consists of the functions $f : G \rightarrow \mathbb{R}^d$ for which all first order weak partial derivatives exist and are locally in L^p . If $f \in W_{d,loc}^1(G)$ is continuous, then f is *quasiregular* if there exists a constant $K_O \geq 1$ such that

$$(2.1) \quad |Df(x)|^d \leq K_O J_f(x) \quad a.e.,$$

where $Df(x)$ denotes the derivative,

$$|Df(x)| = \sup_{|h|=1} |Df(x)(h)|$$

denotes the norm of the derivative, and $J_f(x)$ denotes the Jacobian determinant.

We also let

$$\ell(Df(x)) = \inf_{|h|=1} |Df(x)(h)|.$$

The condition that (2.1) holds for some $K_O \geq 1$ is equivalent to the condition that

$$(2.2) \quad K_I \ell(Df(x))^d \geq J_f(x) \quad a.e.,$$

for some $K_I \geq 1$. The smallest constants K_O and K_I such that (2.1) and (2.2) hold are called the *outer and inner dilatation* of f and are denoted by $K_O(f)$ and $K_I(f)$. The *dilatation* of f is denoted by

$$K(f) = \max\{K_I(f), K_O(f)\}.$$

We say that f is K -quasiregular if $K(f) \leq K$, for some $K \geq 1$.

If f and g are quasiregular, with f defined in the range of g , then $f \circ g$ is quasiregular and [20, Theorem II.6.8]

$$(2.3) \quad K_I(f \circ g) \leq K_I(f)K_I(g).$$

Many properties of holomorphic functions extend to quasiregular maps. In particular, we frequently use the fact that non-constant quasiregular maps are open and discrete.

We need a result on the growth of the maximum modulus of a quasiregular map of transcendental type; see [5, Lemma 3.4], [15, Corollary 4.3].

Lemma 2.1. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Then*

$$\lim_{r \rightarrow \infty} \frac{\log M(r, f)}{\log r} = \infty.$$

Lemma 2.1 has the following corollary, which is immediate.

Corollary 2.2. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. If $r > 0$ is sufficiently large that $M^k(r, f) \rightarrow \infty$ as $k \rightarrow \infty$, then*

$$\lim_{k \rightarrow \infty} \frac{\log \log M^k(r, f)}{k} = \infty.$$

An important tool in the study of quasiregular mappings is the capacity of a condenser, and we recall this concept briefly. Suppose that $A \subset \mathbb{R}^d$ is open, and $C \subset A$ is non-empty and compact. The pair (A, C) is called a *condenser*, and its *capacity*, which we denote by $\text{cap}(A, C)$, is defined by

$$\text{cap}(A, C) = \inf_u \int_A |\nabla u|^d dm,$$

where the infimum is taken over all non-negative functions $u \in C_0^\infty(A)$ which satisfy $u(x) \geq 1$, for $x \in C$.

If C is compact and $\text{cap}(A, C) = 0$ for some bounded open set A containing C , then $\text{cap}(A', C) = 0$ for every bounded open set A' containing C ; see [20, Lemma III.2.2]. In this case we write $\text{cap } C = 0$. Otherwise we write $\text{cap } C > 0$. For an unbounded closed set $C \subset \mathbb{R}^d$, we write $\text{cap } C = 0$ if $\text{cap } C' = 0$ for every compact subset C' of C .

If C is closed and $\text{cap } C = 0$, then C is, in a sense, a small set. In particular we use the fact [20, VII.1.15] that the Hausdorff dimension of C is zero. In particular, if C has an interior point, then $\text{cap } C > 0$.

A second important tool in the study of quasiregular maps is the concept of the modulus of a curve family; we refer to [20, Chapter II] and [25, Chapter 2] for a detailed discussion. Suppose that Γ is a family of paths in \mathbb{R}^d . A non-negative Borel function $\rho : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ is called *admissible* if $\int_\gamma \rho ds \geq 1$, for all locally rectifiable paths $\gamma \in \Gamma$. We let $\mathcal{F}(\Gamma)$ be the family of all admissible Borel functions, and let the *modulus* of Γ be given by

$$M(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^d} \rho^d dm.$$

Suppose that $G \subset \mathbb{R}^d$ is a domain, and E, F are subsets of \overline{G} . We denote by $\Delta(E, F; G)$ the family of all paths which have one endpoint in E , one endpoint in F , and which otherwise are in G .

A connection between the capacity of a condenser and the modulus of a path family is the fact [20, Proposition II.10.2] that if E is a compact subset of G , then

$$\text{cap}(G, E) = M(\Delta(E, \partial G; G)).$$

Next we introduce a conformal invariant which is a useful alternative to the hyperbolic metric when working with quasiregular maps. Let $G \subset \mathbb{R}^d$ be a domain, and define a function μ_G by

$$\mu_G(x, y) = \inf_{C_{xy}} M(\Delta(C_{xy}, \partial G; G)), \quad \text{for } x, y \in G,$$

where the infimum is taken over curves C_{xy} which are contained in G and join x and y ; see [25, p.103] for this definition and more background. It is known that

$\mu_G(x, y)$ is a conformal invariant, and is a metric if $\text{cap } \partial G > 0$. It is noted in [25] that if $D \subset G$ is a domain, then

$$(2.4) \quad \mu_D(x, y) \geq \mu_G(x, y), \quad \text{for } x, y \in D.$$

The “transformation formula” for μ_G is as follows [25, Theorem 10.18].

Lemma 2.3. *Suppose that $f : G \rightarrow \mathbb{R}^d$ is a non-constant quasiregular mapping. Then*

$$\mu_{f(G)}(f(a), f(b)) \leq K_I(f) \mu_G(a, b), \quad \text{for } a, b \in G.$$

We require the following estimate for μ_G [25, Theorem 8.31]. (Here we write $\partial_\infty G$ for the boundary of G taken in $\overline{\mathbb{R}^d}$.)

Lemma 2.4. *Suppose that G is a proper subdomain of \mathbb{R}^d such that $\partial_\infty G$ is connected. Then there exists a constant c_d , depending only on d , such that*

$$\mu_G(a, b) \geq c_d \log \left(1 + \frac{|a - b|}{\min\{\text{dist}(a, \partial G), \text{dist}(b, \partial G)\}} \right), \quad \text{for } a, b \in G.$$

3. PROOF OF THEOREM 1.1

We construct our function using a theorem of Drasin and Sastry in [12]. We need to introduce some terminology before we can state the result we use from that paper.

Suppose that $\nu : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, positive and increasing, and that $\nu(r) \rightarrow \infty$ as $r \rightarrow \infty$. Let n_0 be an integer greater than $\nu(0)$, and define a sequence of integers $(r_n)_{n \geq n_0}$ by

$$(3.1) \quad r_n = \max\{r : \nu(r) = n\}.$$

In [12] it is assumed that

$$(3.2) \quad r\nu'(r) < \nu(r)/2 \quad \text{and} \quad r\nu'(r) = o(\nu(r)) \quad \text{as } r \rightarrow \infty.$$

In fact (3.2) is only used in [12] to deduce that

$$(3.3) \quad n \log \frac{r_{n+1}}{r_n} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

In our case it is somewhat easier to check (3.3) directly, and so we use this equation in the statement of Drasin and Sastry’s theorem.

It is helpful to write down a formula for an exceptional set. For $\epsilon \in (0, 1)$, we define a union of closed intervals

$$E_\epsilon = \bigcup_{n > n_0} [\epsilon r_n, r_n].$$

Define a function

$$(3.4) \quad L(r) = \exp \left(\int_1^r \frac{\nu(t)}{t} dt \right).$$

Finally, it is helpful to work with the *maximum norm*, which is defined by

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_d|\}, \quad \text{for } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

A statement of Drasin and Sastry’s result, which includes part of their construction, is as follows.

Theorem 3.1. *Suppose that $d \geq 2$, that $\nu(r)$ and $L(r)$ are as above, and that (3.3) is satisfied. Then there exist a quasiregular map of transcendental type $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and constants $c, C, R_0 > 0$ and $\epsilon \in (0, 1)$ such that*

$$(3.5) \quad cL(\|x\|_\infty) < \|f(x)\|_\infty < CL(\|x\|_\infty), \quad \text{for } \|x\|_\infty \in [R_0, \infty) \setminus E_\epsilon.$$

The form of the exceptional set E_ϵ in the statement of Theorem 3.1, which is not made explicit in [12], can be obtained as follows. First, we can deduce from [12, Equation (2.10)], together with the paragraph preceding it, that there exists $\epsilon \in (0, 1)$ such that if $\epsilon r_{n+1} > r_n$, then the interval $[r_n, \epsilon r_{n+1}]$ lies in the interval labelled J_n^0 by Drasin and Sastry, for all sufficiently large values of n . Second, the fact that (3.5) holds when $\|x\|_\infty \in J_n^0$ follows from [12, Equation (3.6)].

Proof of Theorem 1.1. Roughly speaking, we construct a quasiregular function f of transcendental type with the following properties:

- (a) the function f behaves like a power map in very large ‘‘square rings’’;
- (b) subrings of these can be defined in such a way that f maps these subrings into each other.

Note that this is, essentially, the same idea as Baker’s original construction of a multiply connected Fatou component of a transcendental entire function using an infinite product. Properties (a) and (b) are achieved iteratively: if r_n is defined, then ν and r_{n+1} are chosen so that r_{n+1} is much larger than r_n (to satisfy (a)), and so that if $\|x\|_\infty = r_n$, then $\|f(x)\|_\infty$ is approximately equal to r_{n+1} (which leads to (b)). Property (b) then ensures that these subrings lie in $QF(f)$, and the theorem follows.

We now give the full detail of the proof. Let $d \geq 2$ be an integer. We first construct a function ν with certain properties. We then invoke Theorem 3.1 to obtain a quasiregular map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that (3.5) is satisfied. Finally we prove that this function has a hollow quasi-Fatou component.

The construction of ν is as follows. First choose an integer $n_0 \geq 2$ and a real number $R' > 4$. Set $\nu(r) = n_0$, for $0 \leq r \leq R'$. Since we shall ensure that $\nu(r) > n_0$, for $r > R'$, this implies that $r_{n_0} = R'$.

We complete the definition of ν , and the sequence $(r_n)_{n > n_0}$ iteratively. Suppose that $n \geq n_0$, that r_n has been defined, and that $\nu(r)$ has been defined for $0 \leq r \leq r_n$. We set

$$(3.6) \quad r_{n+1} = L(r_n),$$

and let $\nu(r)$ be linear; in other words, we set

$$(3.7) \quad \nu(r) = \frac{r - r_n}{r_{n+1} - r_n} + n, \quad \text{for } r_n \leq r \leq r_{n+1}.$$

Note that if $n \geq n_0$, then

$$(3.8) \quad r_{n+1} = L(r_n) = \exp\left(\int_1^{r_n} \frac{\nu(t)}{t} dt\right) \geq \exp\left(\int_1^{r_n} \frac{n_0}{t} dt\right) \geq r_n^{n_0}.$$

It follows from the choices of n_0 and R' that ν is positive, continuous and increasing, tends to infinity, and that (3.3) holds. Hence we can let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and constants $c, C, R_0 > 0$ and $\epsilon \in (0, 1)$ be as in Theorem 3.1.

We complete the proof of the theorem by showing that f has a hollow quasi-Fatou component. For each sufficiently large integer $n \geq n_0$, define a “square ring”

$$A_n = \{x : 2r_n < \|x\|_\infty < \epsilon r_{n+1}\}.$$

We claim that for all sufficiently large values of n , we have that $f(A_n) \subset A_{n+1}$. It then follows from (1.1) that $A_n \subset QF(f)$. It is easy to see that the theorem follows from this fact, since the Julia set of f is not empty.

Choose $N_0 \geq n_0$ sufficiently large that, for $n \geq N_0$ we have that $2r_n < \epsilon r_{n+1}$, that $c2^n \geq 2$, and that $C\epsilon^n \leq \epsilon$.

Suppose that $n \geq N_0$, and that $x \in A_n$. Then, by (3.5), (3.6), (3.7) and since $L(r)$ is increasing

$$\begin{aligned} \|f(x)\|_\infty &\geq cL(2r_n) \\ &= c \exp\left(\int_1^{r_n} \frac{\nu(t)}{t} dt\right) \exp\left(\int_{r_n}^{2r_n} \frac{\nu(t)}{t} dt\right) \\ &\geq cL(r_n) \exp\left(\int_{r_n}^{2r_n} \frac{n}{t} dt\right) \\ &= c2^n L(r_n) \\ &\geq 2L(r_n) = 2r_{n+1}. \end{aligned}$$

For the same reasons,

$$\begin{aligned} \|f(x)\|_\infty &\leq CL(\epsilon r_{n+1}) \\ &= C \exp\left(\int_1^{r_{n+1}} \frac{\nu(t)}{t} dt\right) \exp\left(-\int_{\epsilon r_{n+1}}^{r_{n+1}} \frac{\nu(t)}{t} dt\right) \\ &\leq CL(r_{n+1}) \exp\left(-\int_{\epsilon r_{n+1}}^{r_{n+1}} \frac{n}{t} dt\right) \\ &= \epsilon^n CL(r_{n+1}) \leq \epsilon L(r_{n+1}) = \epsilon r_{n+2}. \end{aligned}$$

Hence $f(x) \in A_{n+1}$, and this completes the proof. \square

Remark 5. For the function f constructed above, we do not know if $J(f) = \partial A(f)$. The growth of f is too slow to use the result [9, Theorem 1.2] mentioned in the introduction. In fact, it follows from (3.8) that

$$\log r_{n+n_0} \geq n_0^n \log r_{n_0}, \quad \text{for } n \in \mathbb{N}.$$

We deduce, by (3.4) and (3.5), that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log \log r} &\leq \liminf_{n \rightarrow \infty} \frac{\log \log M(2r_n, f)}{\log \log 2r_n} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\log \log (C\sqrt{d}L(2r_n))}{\log \log 2r_n} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\log \log (C\sqrt{d}(2r_n)^{n+1})}{\log \log 2r_n} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\log 2(n+1) + \log \log 2r_n}{\log \log 2r_n} = 1. \end{aligned}$$

It does not seem possible to modify the construction to obtain sufficiently fast growth to apply the result of [9, Theorem 1.2], and we currently have no example of a

quasiregular map with a hollow quasi-Fatou component which does have sufficiently fast growth.

4. PROOF OF THEOREM 1.7

In general, if H is a domain, then we let the *outer boundary* of H be denoted by $\partial_{out}H = \partial T(H)$.

Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type, and that G is a domain. We can assume we have $T(f^k(G)) \neq \mathbb{R}^d$, for $k \in \mathbb{N}$, as otherwise there is nothing to prove.

Let $x, y \in G$, $r > 0$ and $\ell_0, k_0 \in \mathbb{N}$ be as in the statement of the theorem. By Lemma 2.1 and (1.4), we can assume that

$$(4.1) \quad 2|f^k(y)| \leq |f^k(x)|, \quad \text{for all sufficiently large } k.$$

We claim that there exists $k_2 \in \mathbb{N}$ such that

$$(4.2) \quad \text{dist}(0, \partial_{out}f^{k+\ell_0}(G)) \geq M^{k-1}(r, f), \quad \text{for } k \geq k_2.$$

Since $|f^{k+\ell_0}(y)| \leq M^{k-1}(r, f)$, for all $k \geq k_0$, it is easy to see that this claim implies that

$$B(0, M^k(r, f)) \subset B(0, M^{k+k_2-1}(r, f)) \subset T(f^{k+k_2+\ell_0}(G)), \quad \text{for } k \in \mathbb{N},$$

from which Theorem 1.7 follows with $\ell = k_2 + \ell_0$.

To prove (4.2), we first deduce from Lemma 2.3, Lemma 2.4, (2.3), (2.4) and (4.1) that, for all large k ,

$$\begin{aligned} K_I(f)^k \mu_G(x, y) &\geq K_I(f^k) \mu_G(x, y) \\ &\geq \mu_{f^k(G)}(f^k(x), f^k(y)) \\ &\geq \mu_{T(f^k(G))}(f^k(x), f^k(y)) \\ &\geq c_d \log \left(1 + \frac{|f^k(x) - f^k(y)|}{\text{dist}(f^k(y), \partial_{out}f^k(G))} \right) \\ &\geq c_d (\log(|f^k(x)|) - \log(\text{dist}(f^k(y), \partial_{out}f^k(G)))) - \log 2. \end{aligned}$$

We deduce by (1.4) that there exist $c > K_I(f)$ and $k_1 \geq k_0$ such that

$$(4.3) \quad \begin{aligned} \log M^k(r, f) &\leq \log |f^{k+\ell_0}(x)| \\ &\leq c^k + \log(\text{dist}(f^{k+\ell_0}(y), \partial_{out}f^{k+\ell_0}(G))) \\ &\leq c^k + \log(\text{dist}(0, \partial_{out}f^{k+\ell_0}(G)) + |f^{k+\ell_0}(y)|), \\ &\leq c^k + \log(\text{dist}(0, \partial_{out}f^{k+\ell_0}(G)) + M^{k-1}(r, f)), \quad \text{for } k \geq k_1. \end{aligned}$$

By Lemma 2.1 and Corollary 2.2, we can choose $k_2 \geq k_1$ sufficiently large that all the following hold for $k \geq k_2$:

- (i) $\log \log M^{k-1}(r, f) \geq 2k \log c$;
- (ii) $\log M^k(r, f) \geq 2 \log M^{k-1}(r, f)$;
- (iii) $c^{2k} \geq c^k + \log 2$.

We deduce by (4.3) that, for $k \geq k_2$,

$$\begin{aligned} c^k + \log(\text{dist}(0, \partial_{\text{out}} f^{k+\ell_0}(G)) + M^{k-1}(r, f)) &\geq \log M^k(r, f) \\ &\geq 2 \log M^{k-1}(r, f) \\ &\geq \log M^{k-1}(r, f) + c^{2k} \\ &\geq \log M^{k-1}(r, f) + c^k + \log 2. \end{aligned}$$

Equation (4.2) follows, and this completes the proof.

5. PROOFS OF THEOREM 1.3, THEOREM 1.4 AND THEOREM 1.6

We require the following topological lemma.

Lemma 5.1. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function, and that U is a quasi-Fatou component of f which is bounded and full. Then $f(U)$ is bounded and full, and the quasi-Fatou component of f containing $f(U)$ is equal to $f(U)$.*

Proof. Suppose that U is a quasi-Fatou component of f which is bounded and full. Clearly $f(U)$ is bounded. We claim that $\partial f(U)$ is connected. For, since f is an open map and U is bounded, $\partial f(U) \subset f(\partial U)$. Moreover, since f is continuous we have

$$(5.1) \quad f(\partial U) \subset f(\overline{U}) \subset \overline{f(U)}.$$

Since $f(\partial U) \subset J(f)$ and $f(U) \subset \overline{QF(f)}$, it follows from (5.1) that

$$f(\partial U) \subset \overline{f(U)} \setminus f(U) = \partial f(U).$$

We deduce that $f(\partial U) = \partial f(U)$. Since U is bounded and full, $\mathbb{R}^d \setminus U$ is connected and so ∂U is connected; see [14, Theorem 1.6(3)] or [16, §57, I–II]. Hence $\partial f(U)$ is connected, as claimed.

Since $f(U)$ is bounded, if it were also hollow, then $\partial f(U)$ would have at least two components. Hence $f(U)$ is full.

Since $\partial f(U) = f(\partial U) \subset J(f)$, we also have that the quasi-Fatou component of f containing $f(U)$ is equal to $f(U)$. \square

We now prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type, and that U_0 is a quasi-Fatou component of f which is bounded and hollow.

Since $T(U_0) \cap J(f) \neq \emptyset$, it follows from (1.3) that $T(U_0) \cap A(f) \neq \emptyset$. Since U_0 is bounded, it follows by the maximum principle that U_0 is wandering.

Our proof now splits into two parts. First we show that there exists a bounded complementary component of U_0 which contains a point of $J(f)$ in its interior. We then use this fact to prove the theorem.

Since $T(U_0) \cap J(f) \neq \emptyset$, it follows from (1.1) that there exist $x \in T(U_0)$ and $k \geq 1$ such that $f^k(x) \in U_0$. Let V be the quasi-Fatou component of f containing x . Then $V \subset T(U_0)$ and, since U_0 is wandering, we have $V \cap U_0 = \emptyset$.

Moreover, V is hollow by Lemma 5.1. Hence $T(V)$ contains a point of $J(f)$. Let E be the component of the complement of U_0 containing $T(V)$. It follows that $\text{int } E \cap J(f) \neq \emptyset$, as claimed.

Let $G_0 = \text{int } E$. It follows by the maximum principle that $f(G_0) \subset T(f(U_0))$. Moreover, $\partial G_0 \subset J(f)$, and so $f(\partial G_0)$ cannot meet $f(U_0)$. It follows that we

can let G_1 be the interior of the bounded complementary component of $f(U_0)$ which contains $f(G_0)$. In general, we can let G_k be the interior of the bounded complementary component of $f^k(U_0)$ which contains $f^k(G_0)$, for $k \in \mathbb{N}$. Clearly G_k is full, for $k \in \mathbb{N}$.

Suppose that $k \in \mathbb{N}$. Since $\partial G_0 \subset J(f)$, we have $f^k(U_0) \cap f^k(\partial G_0) = \emptyset$, and so $f^k(\partial G_0) \subset \overline{G_k}$. Clearly $f^k(\partial G_0) \cap G_k = \emptyset$, since otherwise G_k contains a point of $f^k(U_0)$. Hence $f^k(\partial G_0) \subset \partial G_k$. Thus, since f^k is an open map and G_0 is bounded,

$$\partial f^k(G_0) \subset f^k(\partial G_0) \subset \partial G_k.$$

Since G_k is full, we deduce that

$$(5.2) \quad f^k(G_0) = G_k, \quad \text{for } k \in \mathbb{N}.$$

Let $R > 0$ be sufficiently large that $M^k(R, f) \rightarrow \infty$ as $k \rightarrow \infty$. Since we have that $G_0 \cap J(f) \neq \emptyset$, it follows by (1.3) that $G_0 \cap \partial A(f) \neq \emptyset$, and so we can apply Theorem 1.7 with $G = G_0$. We deduce, by Theorem 1.7 and (5.2) that there exists $\ell \in \mathbb{N}$ such that

$$B(0, M^k(R, f)) \subset T(f^{k+\ell}(G_0)) = T(G_{k+\ell}) = G_{k+\ell}, \quad \text{for } k \in \mathbb{N}.$$

Theorem 1.3 part (a) and part (b) follow.

To prove part (c), we note first that it follows from Theorem 1.3 part (b) and (1.3) that the complement of $A_R(f)$ has a bounded component. The fact that the sets $A_R(f)$ and $A(f)$ are both spiders' webs follows immediately from [7, Proposition 6.2].

It remains to show that $I(f)$ is a spider's web. Since $A(f)$ is a spider's web, and so connected, we can let I_0 be the component of $I(f)$ which contains $A(f)$. By definition I_0 is a spider's web. We show that, in fact, $I(f) = I_0$.

First, suppose that $x \in I(f) \cap J(f)$. Then $x \in \partial A(f)$, by (1.3), and so $x \in \overline{I_0}$. Thus $I_0 \cup \{x\}$ is connected and, since $x \in I(f)$ and I_0 is a component of $I(f)$, it follows that $x \in I_0$. Thus

$$(5.3) \quad (I(f) \cap J(f)) \subset I_0.$$

Next, suppose that $x \in I(f) \cap QF(f)$. Let V_0 be the component of $QF(f)$ containing x . We claim that $\overline{V_0} \subset I(f)$. It is easy to see that the fact that $I(f) = I_0$ follows from this claim, by (5.3).

Recall that V_n (resp. U_n) denotes the Fatou component containing $f^n(V_0)$ (resp. $f^n(U_0)$). If $V_n = U_m$, for some $n, m \in \mathbb{N}$, then $\overline{V_n} \subset A(f)$, by Theorem 1.3 part (b). So we can assume that

$$(5.4) \quad V_n \neq U_m, \quad \text{for } n, m \in \mathbb{N}.$$

For all sufficiently large values of n , let B_n denote the intersection of the unbounded complementary component of U_n with $T(U_{n+1}) \setminus U_{n+1}$. Since $f^n(x) \rightarrow \infty$, it follows by (5.4) that there is a sequence $(k_n)_{n \in \mathbb{N}}$ such that $f^n(x) \in B_{k_n}$, for all sufficiently large $n \in \mathbb{N}$, and $k_n \rightarrow \infty$ as $n \rightarrow \infty$.

It follows by (5.4) that $\overline{V_n} \subset \overline{B_{k_n}}$, for all sufficiently large $n \in \mathbb{N}$, and the result follows. \square

Next we prove Theorem 1.4.

Proof of Theorem 1.4. Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Suppose that U is a quasi-Fatou component of f which is unbounded and hollow.

Let E be a bounded complementary component of U . We claim that there is a bounded full domain, G , such that $E \subset G$ and $\partial G \subset U$. We prove this claim as follows. For each $n \in \mathbb{N}$, let V_n be the component of $\{y : \text{dist}(y, U^c) \leq 1/n\}$ that contains E .

Suppose that all the sets V_n are unbounded. Let $\rho > 0$ be sufficiently large that $E \subset B(0, \rho/2)$. For each $n \in \mathbb{N}$, let V_n^* be the component of $V_n \cap \overline{B(0, \rho)}$ containing E . It is easy to see that each V_n^* is a continuum. Let $V^* = \bigcap_{n \in \mathbb{N}} V_n^*$. Then V^* is a nested intersection of continua, and so [17, Theorem 1.8] is a continuum. In particular V^* is connected.

Since U is open, $V^* \subset U^c$. Now $E = V^*$, since $E \subset V^*$ and E is a component of U^c . Also, V^* contains the non-empty set $\bigcap_{n \in \mathbb{N}} (V_n^* \cap \partial B(0, \rho))$, and so also contains a point of modulus ρ . This is a contradiction.

It follows that there exists $n \in \mathbb{N}$ such that V_n is bounded. We let G be the topological hull of the component of $\{y : \text{dist}(y, U^c) < 1/n\}$ that contains E . This completes the proof of our claim.

It follows from [7, Proposition 2.4] that

$$(5.5) \quad \partial T(f^k(G)) \subset f^k(\partial T(G)) = f^k(\partial G) \subset f^k(U).$$

Since $E \cap J(f) \neq \emptyset$, we have by (1.3) that $G \cap \partial A(f) \neq \emptyset$. We apply Theorem 1.7 with $r > 0$ sufficiently large that $M^k(r, f) \rightarrow \infty$ as $k \rightarrow \infty$.

Since U is unbounded, we deduce from (1.5) that $\partial T(f^k(G)) \cap U \neq \emptyset$ for all sufficiently large values of k . Hence, by (5.5), U meets $f^k(U)$ and $f^{k+1}(U)$, for some k , and so U is the quasi-Fatou component containing $f(U)$. We deduce similarly by (1.5) and (5.5) that U has no unbounded boundary components.

Next, suppose that $V \neq U$ is a quasi-Fatou component of f such that $f(V) \subset U$. Since U has no unbounded boundary components, all complementary components of U are bounded, and so V is bounded. It follows by Lemma 5.1 that V is hollow. We deduce a contradiction by Theorem 1.3 part (a). Hence U is completely invariant.

Finally, it is straightforward to see that Theorem 1.3 part (a) implies that f has no other hollow quasi-Fatou components. \square

We note the following corollary, which is a generalisation of a well-known result for transcendental entire functions.

Corollary 5.2. *Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Suppose that U is a quasi-Fatou component of f , and V is the quasi-Fatou component of f containing $f(U)$. Then U is full if and only if V is full.*

Proof. If U is hollow, then V is hollow by Theorem 1.3 and Theorem 1.4. If U is bounded and full, then V is full, by Lemma 5.1. Finally, it is easy to see that if U is unbounded and full, then f has no hollow quasi-Fatou components. \square

Finally, we prove Theorem 1.6.

Proof of Theorem 1.6. Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a quasiregular function of transcendental type. Suppose first that there exists $x \in QF(f) \cap \partial A(f)$. Let G be

a domain containing x and contained in $QF(f)$. We apply Theorem 1.7 with $r > 0$ sufficiently large that $M^k(r, f) \rightarrow \infty$ as $k \rightarrow \infty$.

It follows from (1.5), and from the fact that $J(f) \neq \emptyset$, that there exists $k \in \mathbb{N}$ and a hollow quasi-Fatou component U such that $f^k(G) \subset U$. Since $G \cap A(f)^c \neq \emptyset$, it follows from Theorem 1.3 part (b) that U is unbounded. This completes the proof of the first part of the theorem.

Suppose second that $J(f)$ has an isolated point, x . Let G be a domain such that $G \cap J(f) = \{x\}$. By (1.3) we have that $G \cap A(f)^c \neq \emptyset$. Clearly f has a hollow quasi-Fatou component U which contains $G \setminus \{x\}$. Then Theorem 1.3 part (b) implies that U is unbounded. \square

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