

## Holographic description of $\text{AdS}_2$ black holes

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**ABSTRACT:** We develop the holographic renormalization of  $\text{AdS}_2$  gravity systematically. We find that a bulk Maxwell term necessitates a boundary mass term for the gauge field and verify that this unusual term is invariant under gauge transformations that preserve the boundary conditions. We determine the energy-momentum tensor and the central charge, recovering recent results by Hartman and Strominger. We show that our expressions are consistent with dimensional reduction of the  $\text{AdS}_3$  energy-momentum tensor and the Brown-Henneaux central charge. As an application of our results we interpret the entropy of  $\text{AdS}_2$  black holes as the ground state entropy of a dual CFT.

**KEYWORDS:** Black Holes in String Theory, AdS-CFT Correspondence, 2D Gravity.

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## 1. Introduction

Extremal black hole spacetimes universally include an AdS<sub>2</sub> factor [1]. It is therefore natural to study quantum black holes by applying the AdS/CFT correspondence to the AdS<sub>2</sub> factor. There have been several interesting attempts at implementing this strategy [2–10] but AdS<sub>2</sub> holography remains enigmatic, at least compared with the much more straightforward case of AdS<sub>3</sub> holography.

Recently a new approach was proposed by Hartman and Strominger [11], in the context of Maxwell-dilaton gravity with bulk action

$$I_{\text{bulk}} = \frac{\alpha}{2\pi} \int_{\mathcal{M}} d^2x \sqrt{-g} \left[ e^{-2\phi} \left( R + \frac{8}{L^2} \right) - \frac{L^2}{4} F^2 \right]. \quad (1.1)$$

These authors pointed out that, for this theory, the usual conformal diffeomorphisms must be accompanied by gauge transformations, in order to maintain boundary conditions. They found that the combined transformations satisfy a Virasoro algebra with a specific central charge. These results suggest a close relation to the AdS<sub>3</sub> theory.

In this paper we develop the holographic description of AdS<sub>2</sub> for the theory (1.1) systematically, following the procedures that are well-known from the AdS/CFT correspondence in higher dimensions. Specifically, we consider:

1. **Holographic renormalization.** We apply the standard holographic renormalization procedure [12–15] to asymptotically AdS<sub>2</sub> spacetimes. In particular, we impose precise boundary conditions and determine the boundary counterterms needed for a consistent variational principle. These counterterms encode the infrared divergences of the bulk theory.
2. **Stress tensor and central charge.** The asymptotic  $SL(2, \mathbb{R})$  symmetry of the theory is enhanced to a Virasoro algebra, when the accompanying gauge transformation is taken into account. We determine the associated boundary stress tensor and its central charge. Our result for the central charge

$$c = \frac{3}{2} k E^2 L^4 \quad (1.2)$$

is consistent with that of Hartman and Strominger [11].

3. **Dimensional reduction from 3D to 2D.** We show that our results in two dimensions (2D) are consistent with dimensional reduction of standard results in three dimensions (3D). In particular, we verify that our result (1.2) agrees with the Brown-Henneaux central charge for AdS<sub>3</sub> spacetimes [16].
4. **Entropy of AdS<sub>2</sub> black holes.** We use our results to discuss the entropy of black holes in AdS<sub>2</sub>. To be more precise, we use general principles to determine enough features of the microscopic theory that we can determine its entropy, but we do not discuss detailed implementations in string theory. This is in the spirit of the well-known microscopic derivation of the entropy of the BTZ black hole in 3D [17], and also previous related results in AdS<sub>2</sub> [3, 18–20].

The main lesson we draw from our results is that, even for AdS<sub>2</sub>, the AdS/CFT correspondence can be implemented in a rather conventional manner.

In the course of our study we encounter several subtleties. First of all, we find that the coupling constant  $\alpha$  in (1.1) must be negative. We reach this result by imposing physical conditions, such as positive central charge, positive energy, sensible thermodynamics, and

a consistent 3D/2D reduction. The redundancy gives us confidence that we employ the physically correct sign.

Another surprise is that consistency of the theory requires the boundary term

$$I_{\text{new}} \sim \int_{\partial\mathcal{M}} dx \sqrt{-\gamma} m A^a A_a, \quad (1.3)$$

where  $m$  is a constant that we compute. The boundary term (1.3) takes the form of a mass term for the gauge field. This is remarkable, because it appears to violate gauge invariance. However, we demonstrate that the new counterterm (1.3) is invariant with respect to all gauge variations that preserve the boundary conditions.

Some other important issues relate to the details of the KK-reduction. In our embedding of asymptotically AdS<sub>2</sub> into AdS<sub>3</sub> we maintain Lorentzian signature and reduce along a direction that is light-like in the boundary theory, but space-like in the bulk. A satisfying feature of the set-up is that the null reduction on the boundary manifestly freezes the holomorphic sector of the boundary theory in its ground state, as it must since the global symmetry is reduced from  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  to  $\text{SL}(2, \mathbb{R})$ . The corollary is that the boundary theory dual to asymptotically AdS<sub>2</sub> necessarily becomes the chiral part of a CFT and such a theory is not generally consistent by itself [21, 22]. The study of the ensuing microscopic questions is beyond the scope of this paper.

This paper is organized as follows. In section §2 we set up our model, the boundary conditions, and the variational principle. We use this to determine the boundary counterterms and verify gauge invariance of the mass term (1.3). In section §3, we use the full action, including counterterms, to derive the renormalized energy-momentum tensor, and the central charge. We compare to the result of Hartman and Strominger, being careful to spell out conventions. In section §4 we present the reduction from 3D to 2D, give the identification between fields, and verify consistency with standard results in AdS<sub>3</sub> gravity. In section §5 we apply our results to black hole thermodynamics. This provides the setting for our discussion of black hole entropy. Section §6 generalizes our results to linear dilaton backgrounds and shows consistency with the constant dilaton sector. In section §7 we discuss a few directions for future research. Our conventions and notations are summarized in appendix 8, and some calculations concerning the dictionary between 3D and 2D are contained in appendix 9.

## 2. Boundary counterterms in Maxwell-Dilaton AdS gravity

In this section we study a charged version of a specific 2D dilaton gravity. We construct a well-defined variational principle for this model by adding boundary terms to the standard action, including a novel boundary mass term for the U(1) gauge field.

### 2.1 Bulk action and equations of motion

There exist many 2D dilaton gravity models that admit an AdS ground state (see [23, 24] and references therein). For the sake of specificity we pick a simple example — the Jackiw-

Teitelboim model [25] — and add a minimally coupled U(1) gauge field. The bulk action

$$I_{\text{bulk}} = \frac{\alpha}{2\pi} \int_{\mathcal{M}} d^2x \sqrt{-g} \left[ e^{-2\phi} \left( R + \frac{8}{L^2} \right) - \frac{L^2}{4} F^2 \right], \quad (2.1)$$

is normalized by the dimensionless constant  $\alpha$  which is left unspecified for the time being. For constant dilaton backgrounds we eventually employ the relation

$$\alpha = -\frac{1}{8G_2} e^{2\phi} \quad (2.2)$$

between the 2D Newton constant  $G_2$  and  $\alpha$ . While the factors in (2.2) are the usual ones (see appendix 8), the sign will be justified in later sections by computing various physical quantities.

The variation of the action with respect to the fields takes the form

$$\delta I_{\text{bulk}} = \frac{\alpha}{2\pi} \int_{\mathcal{M}} d^2x \sqrt{-g} \left[ \mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + \mathcal{E}_\phi \delta\phi + \mathcal{E}^\mu \delta A_\mu \right] + \text{boundary terms}, \quad (2.3)$$

with

$$\mathcal{E}_{\mu\nu} = \nabla_\mu \nabla_\nu e^{-2\phi} - g_{\mu\nu} \nabla^2 e^{-2\phi} + \frac{4}{L^2} e^{-2\phi} g_{\mu\nu} + \frac{L^2}{2} F_\mu^\lambda F_{\nu\lambda} - \frac{L^2}{8} g_{\mu\nu} F^2, \quad (2.4a)$$

$$\mathcal{E}_\phi = -2 e^{-2\phi} \left( R + \frac{8}{L^2} \right), \quad (2.4b)$$

$$\mathcal{E}_\mu = L^2 \nabla^\nu F_{\nu\mu}. \quad (2.4c)$$

Setting each of these equal to zero yields the equations of motion for the theory. The boundary terms will be discussed in section 2.2 below.

All classical solutions to (2.4) can be found in closed form [23, 24]. Some aspects of generic solutions with non-constant dilaton will be discussed in section 6, below. Until then we focus on solutions with constant dilaton, since those exhibit an interesting enhanced symmetry. This can be seen by noting that the dilaton equation  $\mathcal{E}_\phi = 0$  implies that all classical solutions must be spacetimes of constant (negative) curvature. Such a space is maximally symmetric and exhibits three Killing vectors, i.e. it is locally (and asymptotically) AdS<sub>2</sub>. A non-constant dilaton breaks the SL(2, ℝ) algebra generated by these Killing vectors to U(1), but a constant dilaton respects the full AdS<sub>2</sub> algebra.

With constant dilaton the equations of motion reduce to

$$R + \frac{8}{L^2} = 0, \quad \nabla^\nu F_{\nu\mu} = 0, \quad e^{-2\phi} = -\frac{L^4}{32} F^2. \quad (2.5)$$

The middle equation in (2.5) is satisfied by a covariantly constant field strength

$$F_{\mu\nu} = 2E \epsilon_{\mu\nu}, \quad (2.6)$$

where  $E$  is a constant of motion determining the strength of the electric field. The last equation in (2.5) determines the dilaton in terms of the electric field,

$$e^{-2\phi} = \frac{L^4}{4} E^2. \quad (2.7)$$

Expressing the electric field in terms of the dilaton, we can rewrite (2.6) as  $F_{\mu\nu} = \frac{4}{L^2} e^{-\phi} \epsilon_{\mu\nu}$ . Without loss of generality, we have chosen the sign of  $E$  to be positive. The first equation in (2.5) requires the scalar curvature to be constant and negative. Working in a coordinate and U(1) gauge where the metric and gauge field take the form

$$ds^2 = d\eta^2 + g_{tt} dt^2 = d\eta^2 + h_{tt} dt^2, \quad A_\mu dx^\mu = A_t(\eta, t) dt, \quad (2.8)$$

the curvature condition simplifies to the linear differential equation

$$\frac{\partial^2}{\partial \eta^2} \sqrt{-g} = \frac{4}{L^2} \sqrt{-g}, \quad (2.9)$$

which is solved by  $\sqrt{-g} = (h_0(t) e^{2\eta/L} + h_1(t) e^{-2\eta/L})/2$ . Therefore, a general solution to (2.5) is given by

$$g_{\mu\nu} dx^\mu dx^\nu = d\eta^2 - \frac{1}{4} \left( h_0(t) e^{2\eta/L} + h_1(t) e^{-2\eta/L} \right)^2 dt^2, \quad (2.10a)$$

$$A_\mu dx^\mu = \frac{1}{L} e^{-\phi} \left( h_0(t) e^{2\eta/L} - h_1(t) e^{-2\eta/L} + a(t) \right) dt, \quad (2.10b)$$

$$\phi = \text{constant}, \quad (2.10c)$$

where  $h_0$ ,  $h_1$ , and  $a$  are arbitrary functions of  $t$ . This solution can be further simplified by fixing the residual gauge freedom in (2.8). In particular, the U(1) transformation  $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(t)$  preserves the condition  $A_\eta = 0$ , and a redefinition  $h_0(t) dt \rightarrow dt$  of the time coordinate preserves the conditions  $g_{\eta\eta} = 1$  and  $g_{\eta t} = 0$ . This remaining freedom is fixed by requiring  $a(t) = 0$  and  $h_0(t) = 1$ . Thus, the general gauge-fixed solution of the equations of motion depends on the constant  $\phi$ , specified by the boundary conditions, and an arbitrary function  $h_1(t)$ .

Following the standard implementation of the AdS/CFT correspondence in higher dimensions, we describe asymptotically AdS<sub>2</sub> field configurations by (2.8) with the Fefferman-Graham expansions:

$$h_{tt} = e^{4\eta/L} g_{tt}^{(0)} + g_{tt}^{(1)} + e^{-4\eta/L} g_{tt}^{(2)} + \dots, \quad (2.11a)$$

$$A_t = e^{2\eta/L} A_t^{(0)} + A_t^{(1)} + e^{-2\eta/L} A_t^{(2)} + \dots, \quad (2.11b)$$

$$\phi = \phi^{(0)} + e^{-2\eta/L} \phi^{(1)} + \dots. \quad (2.11c)$$

Our explicit solutions (2.10) take this form with asymptotic values

$$g_{tt}^{(0)} = -\frac{1}{4}, \quad A_t^{(0)} = \frac{1}{L} e^{-\phi^{(0)}}, \quad \phi^{(0)} = \text{constant}, \quad (2.12)$$

and specific values for the remaining expansion coefficients in (2.11). The variational principle considers general off-shell field configurations with (2.12) imposed as boundary conditions, but the remaining expansion coefficients are free to vary from their on-shell values.

## 2.2 Boundary terms

An action principle based on (2.1) requires a number of boundary terms:

$$I = I_{\text{bulk}} + I_{\text{GHY}} + I_{\text{counter}} = I_{\text{bulk}} + I_{\text{boundary}} . \quad (2.13)$$

The boundary action  $I_{\text{GHY}}$  is the dilaton gravity analog of the Gibbons-Hawking-York (GHY) term [26, 27], and it is given by

$$I_{\text{GHY}} = \frac{\alpha}{\pi} \int_{\partial\mathcal{M}} dx \sqrt{-h} e^{-2\phi} K , \quad (2.14)$$

where  $h$  is the determinant of the induced metric on  $\partial\mathcal{M}$ , and  $K$  the trace of the extrinsic curvature (our conventions are summarized in appendix 8). This term is necessary for the action to have a well-defined boundary value problem for fields satisfying Dirichlet conditions at  $\partial\mathcal{M}$ . However, on spacetimes with non-compact spatial sections this is not sufficient for a consistent variational principle. We must include in (2.13) a set of ‘boundary counterterms’ so that the action is extremized by asymptotically AdS<sub>2</sub> solutions of the equations of motion. In order to preserve the boundary value problem these counterterms can only depend on quantities intrinsic to the boundary. Requiring diffeomorphism invariance along the boundary leads to the generic *ansatz*

$$I_{\text{counter}} = \int_{\partial\mathcal{M}} dx \sqrt{-h} \mathcal{L}_{\text{counter}}(A^a A_a, \phi) . \quad (2.15)$$

In the special case of vanishing gauge field the counterterm must reduce to  $\mathcal{L}_{\text{counter}} \propto e^{-2\phi}$ , cf. e.g. [28]. In the presence of a gauge field the bulk action contains a term that scales quadratically with the field strength. Therefore, the counterterm may contain an additional contribution that scales quadratically with the gauge field. This lets us refine the *ansatz* (2.15) to

$$I_{\text{counter}} = \frac{\alpha}{\pi} \int_{\partial\mathcal{M}} dx \sqrt{-h} \left[ \lambda e^{-2\phi} + m A^a A_a \right] . \quad (2.16)$$

The coefficients  $\lambda, m$  of the boundary counterterms will be determined in the following.

With these preliminaries the variation of the action (2.13) takes the form

$$\delta I = \int_{\partial\mathcal{M}} dx \sqrt{-h} \left[ (\pi^{ab} + p^{ab}) \delta h_{ab} + (\pi_\phi + p_\phi) \delta\phi + (\pi^a + p^a) \delta A_a \right] + \text{bulk terms} , \quad (2.17)$$

where the bulk terms were considered already in the variation of the bulk action (2.3). The boundary contributions are given by

$$\pi^{tt} + p^{tt} = \frac{\alpha}{2\pi} (h^{tt} n^\mu \partial_\mu e^{-2\phi} + \lambda h^{tt} e^{-2\phi} + m h^{tt} A^t A_t - 2 m A^t A^t) , \quad (2.18a)$$

$$\pi^t + p^t = \frac{\alpha}{2\pi} (-L^2 n_\mu F^{\mu t} + 4 m A^t) , \quad (2.18b)$$

$$\pi_\phi + p_\phi = -2 \frac{\alpha}{\pi} e^{-2\phi} (K + \lambda) . \quad (2.18c)$$

In our notation ‘ $\pi$ ’ is the part of the momentum that comes from the variation of the bulk action and the GHY term, and ‘ $p$ ’ represents the contribution from the boundary counterterms.

For the action to be extremized the terms in (2.17) must vanish for generic variations of the fields that preserve the boundary conditions (2.12). If we consider field configurations admitting an asymptotic expansion of the form (2.11), then the boundary terms should vanish for arbitrary variations of the fields whose leading asymptotic behavior is:

$$\delta h_{tt} = \delta g_{tt}^{(1)} = \text{finite} \tag{2.19a}$$

$$\delta A_t = \delta A_t^{(1)} = \text{finite} \tag{2.19b}$$

$$\delta\phi = e^{-2\eta/L} \delta\phi^{(1)} \rightarrow 0 \tag{2.19c}$$

We refer to variations of the form (2.19) as “variations that preserve the boundary conditions”.

Inserting the asymptotic behavior (2.11) in (2.18a)-(2.18c), the boundary terms in (2.17) become

$$\begin{aligned} \delta I \Big|_{\text{EOM}} = \frac{\alpha}{\pi} \int_{\partial\mathcal{M}} dt \left[ - e^{-2\phi} \left( \lambda + \frac{4}{L^2} m \right) e^{-2\eta/L} \delta h_{tt} - e^{-2\phi} \left( \frac{2}{L} + \lambda \right) e^{2\eta/L} \delta\phi \right. \\ \left. + 2e^{-\phi} \left( 1 - \frac{2}{L} m \right) \delta A_t + \dots \right], \end{aligned} \tag{2.20}$$

where ‘ $\dots$ ’ indicates terms that vanish at spatial infinity for any field variations that preserve the boundary conditions. The leading terms in (2.20) vanish for  $\lambda$  and  $m$  given by

$$\lambda = -\frac{2}{L}, \quad m = \frac{L}{2}. \tag{2.21}$$

As a consistency check we note that these *two* values cancel *three* terms in (2.20). Also, the value of  $\lambda$ , which is present for dilaton gravity with no Maxwell term, agrees with previous computations [28].<sup>1</sup> With the values (2.21) the variational principle is well-defined because the variation of the on-shell action vanishes for all variations that preserve the boundary conditions.

In summary, the full action

$$\begin{aligned} I = \frac{\alpha}{2\pi} \int_{\mathcal{M}} d^2x \sqrt{-g} \left[ e^{-2\phi} \left( R + \frac{8}{L^2} \right) - \frac{L^2}{4} F^{\mu\nu} F_{\mu\nu} \right] \\ + \frac{\alpha}{\pi} \int_{\partial\mathcal{M}} dx \sqrt{-h} \left[ e^{-2\phi} \left( K - \frac{2}{L} \right) + \frac{L}{2} A^a A_a \right], \end{aligned} \tag{2.22}$$

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<sup>1</sup>We also comment on the only previous example of  $A^2$  boundary terms that we are aware of [29]. That work employs the Einstein frame, which is not accessible in 2D, and many of the expressions appearing in that paper indeed diverge when applied to 2D. An exception is their equation (92), which determines the numerical factor  $N_0$  in the boundary mass term (90) for the gauge field  $B_i$ . Equation (92) has two solutions, and the authors of [29] exclusively consider the trivial one  $N_0 = 0$ , i.e. there is no boundary mass term. However, the other solution leads to a non-vanishing boundary mass term for the gauge field. Translating their notations to ours ( $d = 1$ ,  $N_0 = 2m\alpha/\pi$ ,  $K_0 = \alpha L^2/(2\pi)$ ,  $\ell = L/2$ ) we find perfect agreement between the non-trivial solution  $N_0 = K_0/\ell$  of their equation (90) and our result (2.21).



has a well-defined boundary value problem, a well-defined variational principle, and is extremized by asymptotically AdS<sub>2</sub> solutions of the form (2.11).

### 2.3 Boundary mass term and gauge invariance

The boundary term

$$I_{\text{new}} = \frac{\alpha L}{2\pi} \int_{\partial\mathcal{M}} dx \sqrt{-h} A^a A_a \quad (2.23)$$

is novel and requires some attention, because it would seem to spoil invariance under the gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda . \quad (2.24)$$

The purpose of this section is to show that the mass term (2.23) is in fact invariant under gauge transformations that preserve the gauge condition  $A_\eta = 0$  and the boundary condition specified in (2.19b).

The gauge parameter  $\Lambda$  must have the asymptotic form

$$\Lambda = \Lambda^{(0)}(t) + \Lambda^{(1)}(t) e^{-2\eta/L} + \mathcal{O}(e^{-4\eta/L}) \quad (2.25)$$

in order that the asymptotic behavior

$$A_t = A_t^{(0)} e^{2\eta/L} + \mathcal{O}(1), \quad (2.26a)$$

$$A_\eta = \mathcal{O}(e^{-2\eta/L}), \quad (2.26b)$$

of the gauge field is preserved. Indeed, allowing some positive power of  $e^{2\eta/L}$  in the expansion (2.25) of  $\Lambda$  would spoil this property.

Having established the most general gauge transformation consistent with our boundary conditions we can investigate whether the counterterm (2.23) is gauge invariant. Acting with the gauge transformation (2.24) and taking the asymptotic expansions (2.25) and (2.26) into account yields

$$\delta_\Lambda I_{\text{new}} = \frac{\alpha L}{\pi} \lim_{\eta \rightarrow \infty} \int_{\partial\mathcal{M}} dt \sqrt{-h} h^{tt} A_t \delta_\Lambda A_t = -\frac{2\alpha L}{\pi} A_t^{(0)} \int_{\partial\mathcal{M}} dt \partial_t \Lambda^{(0)} . \quad (2.27)$$

The same result holds for the full action (2.22), because all other terms in  $I$  are manifestly gauge invariant. The integral in (2.27) vanishes for continuous gauge transformations if  $\Lambda^{(0)}$  takes the same value at the initial and final times. In those cases the counterterm (2.23) and the full action (2.22) are both gauge invariant with respect to gauge transformations that asymptote to (2.25).

The “large” gauge transformations that do not automatically leave the action invariant are also interesting. As an example, we consider the discontinuous gauge transformation

$$\Lambda^{(0)}(t) = 2\pi q_m \theta(t - t_0), \quad (2.28)$$

where  $q_m$  is the dimensionless magnetic monopole charge with a convenient normalization. We assume that  $t_0$  is contained in  $\partial\mathcal{M}$ , so that the delta function obtained from  $\partial_t \Lambda^{(0)}$  is

supported. Inserting the discontinuous gauge transformation (2.28) into the gauge variation of the action (2.27) gives

$$\delta_\Lambda I = \delta_\Lambda I_{\text{new}} = -2\alpha L^2 E q_m, \quad (2.29)$$

which tells us that the full action is shifted by a constant. We investigate now under which conditions this constant is an integer multiple of  $2\pi$ .

The 2D Gauss law relates the electric field  $E$  to the dimensionless electric charge  $q_e$ :

$$E = -\frac{\pi q_e}{\alpha L^2}. \quad (2.30)$$

Again we have chosen a convenient normalization.<sup>2</sup> The Gauss law (2.30) allows to rewrite the gauge shift of the action (2.29) in a suggestive way:

$$\delta_\Lambda I = \delta_\Lambda I_{\text{new}} = 2\pi q_e q_m. \quad (2.31)$$

Thus, as long as magnetic and electric charge obey the Dirac quantization condition

$$q_e q_m \in \mathbb{Z}, \quad (2.32)$$

the action just shifts by multiples of  $2\pi$ . We shall assume that this is the case. Then  $I_{\text{new}}$  and  $I$  are gauge invariant modulo  $2\pi$  despite of the apparent gauge non-invariance of the boundary mass term  $m A^a A_a$ .

In conclusion, the full action (2.22) is gauge invariant with respect to all gauge variations (2.25) that preserve the boundary conditions (2.11) provided the integral in (2.27) vanishes (modulo  $2\pi$ ). This is the case if the Dirac quantization condition (2.32) holds.

### 3. Boundary stress tensor and central charge

The behavior of the on-shell action is characterized by the linear response functions of the boundary theory<sup>3</sup>

$$T^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h_{ab}}, \quad J^a = \frac{1}{\sqrt{-h}} \frac{\delta I}{\delta A_a}. \quad (3.1)$$

The response function for the dilaton, which is not relevant for the present considerations, is discussed in [28]. The general expressions (2.18a) and (2.18b) give

$$T_{tt} = \frac{\alpha}{\pi} \left( -\frac{2}{L} h_{tt} e^{-2\phi} - \frac{L}{2} A_t A_t \right), \quad (3.2a)$$

$$J^t = \frac{\alpha}{2\pi} \left( -L^2 n_\mu F^{\mu t} + 2L A^t \right). \quad (3.2b)$$

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<sup>2</sup>If we set  $\alpha L^2/(2\pi) = 1$  then the action (1.1) has a Maxwell-term with standard normalization. In that case our Gauss law (2.30) simplifies to  $2E = -q_e$ . The factor 2 appears here because in our conventions the relation between field strength and electric field contains such a factor,  $F_{\mu\nu} = 2E \epsilon_{\mu\nu}$ . Thus, apart from the sign, the normalization in (2.30) leads to the standard normalization of electric charge in 2D. The sign is a consequence of our desire to have positive  $E$  for positive  $q_e$  in the case of negative  $\alpha$ .

<sup>3</sup>These are the same conventions as in [12]. The boundary current and stress tensor used here is related to the definitions in [11] by  $J^a = \frac{1}{2\pi} J_{\text{HS}}^a$  and  $T^{ab} = \frac{1}{2\pi} T_{\text{HS}}^{ab}$ .

We want to find the transformation properties of these functions under the asymptotic symmetries of the theory; i.e. under the combination of bulk diffeomorphisms and U(1) gauge transformations that act non-trivially at  $\partial\mathcal{M}$ , while preserving the boundary conditions and the choice of gauge.

A diffeomorphism  $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$  transforms the fields as

$$\delta_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu, \quad \delta_\epsilon A_\mu = \epsilon^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu \epsilon^\nu. \quad (3.3)$$

The background geometry is specified by the gauge conditions  $g_{\eta\eta} = 1$ ,  $g_{\eta t} = 0$ , and the boundary condition that fixes the leading term  $g_{tt}^{(0)}$  in the asymptotic expansion (2.11a) of  $h_{tt}$ . These conditions are preserved by the diffeomorphisms

$$\epsilon^\eta = -\frac{L}{2} \partial_t \xi(t), \quad \epsilon^t = \xi(t) + \frac{L^2}{2} \left( e^{4\eta/L} + h_1(t) \right)^{-1} \partial_t^2 \xi(t), \quad (3.4)$$

where  $\xi$  is an arbitrary function of the coordinate  $t$ . Under (3.4), the boundary metric transforms according to

$$\delta_\epsilon h_{tt} = - \left( 1 + e^{-4\eta/L} h_1(t) \right) \left( h_1(t) \partial_t \xi(t) + \frac{1}{2} \xi(t) \partial_t h_1(t) + \frac{L^2}{4} \partial_t^3 \xi(t) \right). \quad (3.5)$$

Turning to the gauge field, the change in  $A_\eta$  due to the diffeomorphism (3.4) is

$$\delta_\epsilon A_\eta = -2 e^{-\phi} \frac{(e^{2\eta/L} - h_1(t) e^{-2\eta/L})}{(e^{2\eta/L} + h_1(t) e^{-2\eta/L})^2} \partial_t^2 \xi(t). \quad (3.6)$$

Thus, diffeomorphisms with  $\partial_t^2 \xi \neq 0$  do not preserve the U(1) gauge condition  $A_\eta = 0$ . The gauge is restored by the compensating gauge transformation  $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ , with  $\Lambda$  given by

$$\Lambda = -L e^{-\phi} \left( e^{2\eta/L} + h_1(t) e^{-2\eta/L} \right)^{-1} \partial_t^2 \xi(t). \quad (3.7)$$

The effect of the combined diffeomorphism and U(1) gauge transformation on  $A_t$  is

$$(\delta_\epsilon + \delta_\Lambda) A_t = -e^{-2\eta/L} e^{-\phi} \left( \frac{1}{L} \xi(t) \partial_t h_1(t) + \frac{2}{L} h_1(t) \partial_t \xi(t) + \frac{L}{2} \partial_t^3 \xi(t) \right). \quad (3.8)$$

This transformation preserves the boundary condition (2.11b) for  $A_t$ , as well as the condition  $A_t^{(1)} = 0$  that was used to fix the residual U(1) gauge freedom. Thus, the asymptotic symmetries of the theory are generated by a diffeomorphism (3.4) accompanied by the U(1) gauge transformation (3.7). Under such transformations the metric and gauge field behave as (3.5) and (3.8), respectively.

We can now return to our goal of computing the transformation of the linear response functions (3.2) under the asymptotic symmetries of the theory. The change in the stress tensor (3.2a) due to the combined diffeomorphism (3.4) and U(1) gauge transformation (3.7) takes the form

$$(\delta_\epsilon + \delta_\Lambda) T_{tt} = 2 T_{tt} \partial_t \xi + \xi \partial_t T_{tt} - \frac{c}{24\pi} L \partial_t^3 \xi(t). \quad (3.9)$$

The first two terms are the usual tensor transformation due to the diffeomorphism. In addition, there is an anomalous term generated by the U(1) component of the asymptotic symmetry. We included a factor  $L$  in the anomalous term in (3.9) in order to make the central charge  $c$  dimensionless. Using the expressions (3.5) and (3.8) for the transformation of the fields we verify the general form (3.9) and determine the central charge

$$c = -24 \alpha e^{-2\phi} . \tag{3.10}$$

The relation (2.2) allows us to rewrite (3.10) in the more aesthetically pleasing form

$$c = \frac{3}{G_2} . \tag{3.11}$$

The requirement that the central charge should be positive determines  $\alpha < 0$  as the physically correct sign. We shall see the same (unusual) sign appearing as the physically correct one in later sections.

Another suggestive expression for the central charge is

$$c = 3 \text{Vol}_L \mathcal{L}_{2D} , \tag{3.12}$$

where the volume element  $\text{Vol}_L = 2\pi L^2$  and Lagrangian density  $\mathcal{L}_{2D} = \frac{4\alpha}{\pi L^2} e^{-2\phi}$  is related to the on-shell bulk action (1.1) by

$$I_{\text{bulk}}|_{\text{EOM}} = - \int_{\mathcal{M}} d^2x \sqrt{-g} \mathcal{L}_{2D} . \tag{3.13}$$

The central charge (3.12) is the natural starting point for computation of higher derivative corrections to the central charge, in the spirit of [30, 31, 8, 10].

So far we considered just the transformation property of the energy momentum tensor (3.2a). We should also consider the response of the boundary current (3.2b) to a gauge transformation. Generally, we write the transformation of a current as

$$\delta_\Lambda J_t = -\frac{k}{4\pi} L \partial_t \Lambda , \tag{3.14}$$

where the level  $k$  parametrizes the gauge anomaly. The only term in (3.2b) that changes under a gauge transformation is the term proportional to  $A_t$ . The resulting variation of the boundary current takes the form (3.14) with the level

$$k = -4\alpha = \frac{1}{2G_2} e^{2\phi} . \tag{3.15}$$

Our definitions of central charge (3.9) and level in (3.14) are similar to the corresponding definitions in 2D CFT. However, they differ by the introduction of the AdS scale  $L$ , needed to keep these quantities dimensionless. We could have introduced another length scale instead, and the anomalies would then be rescaled correspondingly as a result. Since  $c$  and  $k$  would change the same way under such a rescaling we may want to express the central charge (3.11) in terms of the level (3.15) as

$$c = 6 k e^{-2\phi} . \tag{3.16}$$

This result is insensitive to the length scale introduced in the definitions of the anomalies, as long as the same scale is used in the two definitions.

Expressing the dilaton (2.7) in terms of the electric field we find yet another form of the central charge

$$c = \frac{3}{2} k E^2 L^4 . \tag{3.17}$$

As it stands, this result is twice as large as the result found in [11]. However, there the anomaly is attributed to two contributions, from  $T_{++}$  and  $T_{--}$  related to the two boundaries of global  $\text{AdS}_2$ . We introduce a single energy-momentum tensor  $T_{tt}$ , as seems appropriate when the boundary theory has just one spacetime dimension. In general spacetimes,  $T_{tt}$  would be a density but in one spacetime dimension there are no spatial dimensions, and so the “density” is the same as the energy. Such an energy-momentum tensor cannot be divided into left- and right-moving parts. Thus our computation agree with [11] even though our interpretations differ.

#### 4. 3D reduction and connection with 2D

Asymptotically  $\text{AdS}_2$  backgrounds have a non-trivial  $\text{SL}(2, \mathbb{R})$  group acting on the boundary that can be interpreted as one of the two  $\text{SL}(2, \mathbb{R})$  groups associated to  $\text{AdS}_3$ . To do so, we compactify pure gravity in 3D with a negative cosmological constant on a circle and find the map to the Maxwell-dilaton gravity (1.1). This dimensional reduction also shows that the  $\text{AdS}_2$  boundary stress tensor and central charge found in this paper are consistent with the corresponding quantities in  $\text{AdS}_3$ .

##### 4.1 Three dimensional gravity

Our starting point is pure three dimensional gravity described by an action

$$I = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left( \mathcal{R} + \frac{2}{\ell^2} \right) + \frac{1}{8\pi G_3} \int d^2y \sqrt{-\gamma} \left( \mathcal{K} - \frac{1}{\ell} \right) , \tag{4.1}$$

that is a sum over bulk and boundary actions like in the schematic equation (2.13). The 3D stress-tensor defined as

$$\delta I = \frac{1}{2} \int d^2y \sqrt{-\gamma} T_{3\text{D}}^{ab} \delta \gamma_{ab} , \tag{4.2}$$

becomes [12]

$$T_{ab}^{3\text{D}} = -\frac{1}{8\pi G_3} \left( \mathcal{K}_{ab} - \mathcal{K} \gamma_{ab} + \frac{1}{\ell} \gamma_{ab} \right) . \tag{4.3}$$

For asymptotically  $\text{AdS}_3$  spaces we can always choose Fefferman-Graham coordinates, where the bulk metric takes the form

$$ds^2 = d\eta^2 + \gamma_{ab} dy^a dy^b , \quad \gamma_{ab} = e^{2\eta/\ell} \gamma_{ab}^{(0)} + \gamma_{ab}^{(2)} + \dots . \tag{4.4}$$

The functions  $\gamma_{ab}^{(i)}$  depend only on the boundary coordinate  $y^a$  with  $a, b = 1, 2$ . The boundary is located at  $\eta \rightarrow \infty$ , and  $\gamma_{ab}^{(0)}$  is the 2D boundary metric defined up to conformal

transformations. The energy momentum tensor (4.3) evaluated in the coordinates (4.4) is

$$T_{ab}^{3D} = \frac{1}{8\pi G_3 \ell} \left( \gamma_{ab}^{(2)} - \gamma_{(0)}^{cd} \gamma_{cd}^{(2)} \gamma_{ab}^{(0)} \right). \quad (4.5)$$

In the case of pure gravity (4.1) we can be more explicit and write the exact solution as [32]

$$ds^2 = d\eta^2 + \left( \frac{\ell^2}{4} e^{2\eta/\ell} + 4g_+ g_- e^{-2\eta/\ell} \right) dx^+ dx^- + \ell \left( g_+ (dx^+)^2 + g_- (dx^-)^2 \right). \quad (4.6)$$

We assumed a flat boundary metric  $\gamma_{ab}^{(0)}$  parameterized by light-cone coordinates  $x^\pm$ . The function  $g_+$  ( $g_-$ ) depends exclusively on  $x^+$  ( $x^-$ ). For this family of solutions the energy-momentum tensor (4.5) becomes

$$T_{++}^{3D} = \frac{1}{8\pi G_3} g_+, \quad T_{--}^{3D} = \frac{1}{8\pi G_3} g_-. \quad (4.7)$$

## 4.2 Kaluza-Klein reduction

Dimensional reduction is implemented by writing the 3D metric as

$$ds^2 = e^{-2\psi} \ell^2 (dz + \tilde{A}_\mu dx^\mu)^2 + \tilde{g}_{\mu\nu} dx^\mu dx^\nu. \quad (4.8)$$

The 2D metric  $\tilde{g}_{\mu\nu}$ , the scalar field  $\psi$ , and the gauge field  $\tilde{A}_\mu$  all depend only on  $x^\mu$  ( $\mu = 1, 2$ ). The coordinate  $z$  has period  $2\pi$ . The 3D Ricci scalar expressed in terms of 2D fields reads

$$\mathcal{R} = \tilde{R} - 2e^\psi \tilde{\nabla}^2 e^{-\psi} - \frac{\ell^2}{4} e^{-2\psi} \tilde{F}^2. \quad (4.9)$$

The 2D scalar curvature  $\tilde{R}$ , and the covariant derivatives  $\tilde{\nabla}_\mu$ , are constructed from  $\tilde{g}_{\mu\nu}$ . Inserting (4.9) in the 3D bulk action in (4.1) gives the 2D bulk action

$$\tilde{I}_{\text{bulk}} = \frac{\ell}{8G_3} \int d^2x \sqrt{-\tilde{g}} e^{-\psi} \left( \tilde{R} + \frac{2}{\ell^2} - \frac{\ell^2}{4} e^{-2\psi} \tilde{F}^2 \right). \quad (4.10)$$

The action (4.10) is on-shell equivalent to the action (2.22) for the constant dilaton solutions (2.10). To find the precise dictionary we first compare the equations of motion. Variation of the action (4.10) with respect to the scalar  $\psi$  and metric  $\tilde{g}_{\mu\nu}$  gives

$$\tilde{R} + \frac{2}{\ell^2} - \frac{3\ell^2}{4} e^{-2\psi} \tilde{F}^2 = 0, \quad (4.11a)$$

$$\tilde{g}_{\mu\nu} \left( \frac{1}{\ell^2} - \frac{\ell^2}{8} e^{-2\psi} \tilde{F}^2 \right) + \frac{\ell^2}{2} e^{-2\psi} \tilde{F}_{\mu\alpha} \tilde{F}_\nu{}^\alpha = 0, \quad (4.11b)$$

which implies<sup>4</sup>

$$e^{-2\psi} \tilde{F}^2 = -\frac{8}{\ell^4}, \quad (4.12a)$$

$$\tilde{R} = -\frac{8}{\ell^2}. \quad (4.12b)$$

---

<sup>4</sup>A check on the algebra: inserting (4.12) into the formula (4.9) for the 3D Ricci scalar yields  $\mathcal{R} = -6/\ell^2$ , concurrent with our definition of the 3D AdS radius.

The analogous equations derived from the 2D action (2.5) take the same form, but with the identifications

$$\tilde{g}_{\mu\nu} = a^2 g_{\mu\nu}, \tag{4.13a}$$

$$\ell = aL, \tag{4.13b}$$

$$e^{-\psi} \tilde{F}_{\mu\nu} = \frac{1}{2} e^{\phi} F_{\mu\nu}, \tag{4.13c}$$

with  $a$  an arbitrary constant.

In order to match the overall normalization on-shell we evaluate the bulk action (4.10) using the on-shell relations (4.12a) and (4.12b)

$$\tilde{I}_{\text{bulk}} = -\frac{\ell}{2G_3} \int d^2x \frac{\sqrt{-\tilde{g}}}{\ell^2} e^{-\psi}. \tag{4.14}$$

and compare with the analogous expression

$$I_{\text{bulk}} = \frac{4\alpha}{\pi} \int d^2x \frac{\sqrt{-g}}{L^2} e^{-2\phi}. \tag{4.15}$$

computed directly from the 2D action (2.22). Equating the on-shell actions  $I_{\text{bulk}} = \tilde{I}_{\text{bulk}}$  and simplifying using (4.13a), (4.13b) we find

$$\alpha = -\frac{\pi \ell}{8G_3} e^{2\phi - \psi}. \tag{4.16}$$

We see again that the unusual sign  $\alpha < 0$  is the physically correct one. According to (2.2) we can write the 3D/2D identification as

$$\frac{1}{G_2} = \frac{\pi \ell e^{-\psi}}{G_3}. \tag{4.17}$$

So far we determined the 3D/2D on-shell dictionary by comparing equations of motions and the bulk action. In appendix 9 we verify that the same identification (4.16) also guarantees that the boundary actions agree. Additionally, we show that the 3D/2D dictionary identifies the 3D solutions (4.6) with the general 2D solutions (2.10). These checks give confidence in our 3D/2D map.

In summary, our final result for the dictionary between the 2D theory and the KK reduction of the 3D theory is given by the identifications (4.13) and the relation (4.16) between normalization constants. We emphasize that the map is on-shell; it is between solutions and their properties. The full off-shell theories do not agree, as is evident from the sign in (4.16). The restriction to on-shell configurations will not play any role in this paper but it may be important in other applications.

### 4.3 Conserved currents and central charge

Applying the 3D/2D dictionary from the previous subsection (and elaborations in appendix 9), we now compare the linear response functions and the central charge computed by reduction from 3D to those computed directly in 2D.

The starting point is the 3D energy momentum tensor (4.2). The KK-reduction formula (9.1) decomposes the variation of the boundary metric  $\gamma_{ab}$  as

$$\delta\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta\tilde{h}_{tt} + \ell^2 e^{-2\psi} \begin{pmatrix} 2\tilde{A}_t & 1 \\ 1 & 0 \end{pmatrix} \delta\tilde{A}_t, \quad (4.18)$$

and the determinant  $\sqrt{-\gamma} = \ell e^{-\psi} \sqrt{-\tilde{h}}$  so that the 3D stress tensor (4.2) becomes

$$\begin{aligned} \delta I &= \int dx \sqrt{-\tilde{h}} \left[ \frac{1}{2} (2\pi\ell e^{-\psi}) T_{3d}^{tt} \delta\tilde{h}_{tt} + (\pi\ell^3 e^{-3\psi}) (T_{3d}^{tt}\tilde{A}_t + T_{3d}^{zt}) \right] 2\delta\tilde{A}_t \\ &= \int dx \sqrt{-\tilde{h}} (h^{tt})^2 \left[ \frac{1}{2} (2\pi L e^{-\psi} T_{tt}^{3d}) \delta h_{tt} + \left( \frac{\pi}{2} L^3 e^{-\psi+2\phi} A_t T_{tt}^{3d} \right) \delta A_t \right], \end{aligned} \quad (4.19)$$

where we used the 3D-2D dictionary (4.13) and wrote the variation of the boundary fields as

$$\delta\tilde{h}_{tt} = a^2 \delta h_{tt}, \quad \delta\tilde{A}_t = \frac{1}{2} e^{\psi+\phi} \delta A_t. \quad (4.20)$$

Indices of the stress tensor in (4.19) are lowered and raised with  $\tilde{h}_{tt}$  and  $\tilde{h}^{tt}$ , respectively. Comparing (4.19) with the 2D definition of stress tensor and current (3.1) we find

$$T_{tt}^{2D} = 2\pi L e^{-\psi} T_{tt}^{3D}, \quad (4.21a)$$

$$J_t = \frac{\pi}{2} L^3 h^{tt} e^{-\psi+2\phi} A_t T_{tt}^{3D}, \quad (4.21b)$$

for the relation between 3D and 2D quantities.

The next step is to rewrite the 3D energy momentum tensor (4.7) in a notation more appropriate for comparison with 2D. We first rescale coordinates according to (9.8) and then transform into 2D variables using (9.10b), (9.11a). The result is

$$T_{tt}^{3D} = -\frac{1}{8\pi G_3 \ell} h_1, \quad T_{zz}^{3D} = \frac{1}{8\pi G_3} \ell e^{-2\psi}. \quad (4.22)$$

Inserting these expressions in (4.21), along with the asymptotic values of the background fields in the solution (2.10), we find

$$T_{tt}^{2D} = \frac{2\alpha}{L\pi} e^{-2\phi} h_1, \quad (4.23a)$$

$$J_t = -\frac{2\alpha}{\pi} e^{-\phi} e^{-2\eta/L} h_1, \quad (4.23b)$$

after simplifications using our 3D-2D dictionary (4.13) and the rescaling mentioned just before (9.11). The current (4.23b) vanishes on the boundary  $\eta \rightarrow \infty$  but the subleading term given here is significant for some applications. The expressions (4.23) are our results for the 2D linear response functions, computed by reduction from 3D. They should be compared with the analogous functions (3.2) defined directly in 2D, with those latter expressions evaluated on the solution (2.10). These results agree precisely.

Using the relations between the conserved currents, we now proceed to compare the central charges in 2D and 3D. Under the diffeomorphisms which preserve the three dimensional boundary, the 3D stress tensor transforms as [12]

$$\delta T_{tt}^{3D} = 2T_{tt}^{3D} \partial_t \xi(t) + \xi(t) \partial_t T_{tt}^{3D} - \frac{c}{24\pi} \partial_t^3 \xi(t), \quad (4.24)$$



with the central term given by the standard Brown-Henneaux central charge

$$c_{3D} = \frac{3\ell}{2G_3} . \tag{4.25}$$

From the relation (4.21a) between 2D and 3D stress tensor, and by comparing the transformations (3.9) and (4.24), the central charges are related as

$$c_{2D} = 2\pi e^{-\psi} c_{3D} . \tag{4.26}$$

Inserting (4.25) and using (4.16) we find

$$c_{2D} = 2\pi e^{-\psi} \left( \frac{3\ell}{2G_3} \right) = -24\alpha e^{-2\phi} . \tag{4.27}$$

This is the result for the 2D central charge, obtained by reduction from 3D. It agrees precisely with the central charge (3.10) obtained directly in 2D.

In summary, in this section we have given an explicit map between 3D and 2D. We have shown that it correctly maps the equations of motion and the on-shell actions, it maps 3D solutions to those found directly in 2D, it maps the linear response functions correctly between the two pictures, and it maps the central charge correctly.

## 5. Black hole thermodynamics

In this section we apply our results to discuss the entropy of 2D black holes. We start by computing the temperature and mass of the black hole and the relation of these quantities to the 2D stress tensor. By using the renormalized on-shell action and the first law of thermodynamics, we obtain the Bekenstein-Hawking entropy. Finally, we discuss the identification of the black hole entropy with the ground state entropy of the dual CFT.

### 5.1 Stress tensor for AdS<sub>2</sub> black holes

For  $h_0 = 1$  and constant  $h_1$ , the solution (2.10) becomes

$$ds^2 = d\eta^2 - \frac{1}{4} e^{4\eta/L} \left( 1 + h_1 e^{-4\eta/L} \right)^2 dt^2 , \tag{5.1a}$$

$$A_t = \frac{1}{L} e^{-\phi} e^{2\eta/L} \left( 1 - h_1 e^{-4\eta/L} \right) . \tag{5.1b}$$

Solutions with positive  $h_1$  correspond to global AdS<sub>2</sub> with radius  $\ell_A = L/2$ , while solutions with negative  $h_1$  describe black hole geometries.

An AdS<sub>2</sub> black hole with horizon at  $\eta = \eta_0$  corresponds to  $h_1$  given by

$$h_1 = -e^{4\eta_0/L} . \tag{5.2}$$

Regularity of the Euclidean metrics near the horizon determines the imaginary periodicity  $t \sim t + i\beta$  as

$$\beta = \pi L e^{-2\eta_0/L} . \tag{5.3}$$

We identify the temperature of the 2D black hole as  $T = \beta^{-1}$ .

Our general AdS<sub>2</sub> stress tensor (3.2), (4.23a) is

$$T_{tt} = \frac{2\alpha}{\pi L} e^{-2\phi} h_1 = -\frac{h_1}{4\pi G_2 L} . \quad (5.4)$$

The stress tensor for global AdS<sub>2</sub> ( $h_1 > 0$ ) is negative. This is reasonable, because the Casimir energy of AdS<sub>3</sub> is negative as well. Importantly, the black hole solutions ( $h_1 < 0$ ) are assigned positive energy, as they should be. The assignment  $\alpha < 0$  is needed in (5.4) to reach this result, giving further confidence in our determination of that sign.

We can rewrite the stress tensor (5.4) as

$$T_{tt} = \frac{\pi L T^2}{4G_2} = c \frac{\pi L T^2}{12} , \quad (5.5)$$

where we used the central charge (3.11). We interpret this form of the energy as a remnant of the 3D origin of the theory, as the right movers of a 2D CFT.

The mass is generally identified as the local charge of the current generated by the Killing vector  $\partial_t$ . This amounts to the prescription

$$M = \sqrt{-g^{tt}} T_{tt} = 2e^{-2\eta/L} T_{tt} \rightarrow 0 , \quad (5.6)$$

for the mass measured asymptotically as  $\eta \rightarrow \infty$ . The solutions (5.1) are therefore all assigned vanishing mass, due to the redshift as the boundary is approached. We will see in the following that this result is needed to uphold the Bekenstein-Hawking area law.

## 5.2 On-shell action and Bekenstein-Hawking entropy

The boundary terms in (2.22) were constructed so that the variational principle is well-defined, but they are also supposed to cancel divergences and render the on-shell action finite. It is instructive to compute its value.

The on-shell bulk action (4.15) becomes

$$\begin{aligned} I_{\text{bulk}} &= \frac{2\alpha}{\pi L^2} e^{-2\phi} \int dt d\eta \left( e^{2\eta/L} + h_1 e^{-2\eta/L} \right) \\ &= \frac{\alpha\beta}{\pi L} e^{-2\phi} \left( e^{2\eta/L} - h_1 e^{-2\eta/L} \right) \Big|_{\eta_0}^{\infty} , \end{aligned} \quad (5.7)$$

for the 2D black hole (5.1). The boundary terms in (2.22) were evaluated in (9.6) with the result

$$I_{\text{boundary}} = -\frac{2\alpha\beta}{\pi L} e^{-2\phi} \sqrt{-h_{tt}} = -\frac{\alpha\beta}{\pi L} e^{-2\phi} e^{2\eta/L} . \quad (5.8)$$

The divergence at the boundary  $\eta \rightarrow \infty$  cancels the corresponding divergence in the bulk action (5.7). The renormalized on-shell action becomes finite with the value

$$I = I_{\text{bulk}} + I_{\text{boundary}} = -\frac{2\alpha\beta}{\pi L} e^{-2\phi} e^{2\eta_0/L} = -2\alpha e^{-2\phi} = \frac{1}{4G_2} . \quad (5.9)$$

The third equality used (5.3) and the last one used (2.2).

We computed the on-shell action in Lorentzian signature to conform with the conventions elsewhere in the paper. The Euclidean action has the opposite sign  $I_E = -I$ , and it is that action which is related to the free energy in the standard manner

$$\beta F = I_E = \beta M - S, \tag{5.10}$$

when we consider the canonical ensemble.<sup>5</sup> We found vanishing  $M$  in (5.6) and so the black hole entropy becomes

$$S = -I_E = I = \frac{1}{4G_2}. \tag{5.11}$$

This is the standard Bekenstein-Hawking result.

### 5.3 Black hole entropy from Cardy's formula

One of the motivations for determining the central charge of  $\text{AdS}_2$  is that it may provide a short-cut to the black hole entropy. We will just make preliminary comments on this application.

A 2D chiral CFT with  $c_0$  degrees of freedom living on a circle with radius  $R$  has entropy given by the Cardy formula

$$S = 2\pi \sqrt{\frac{c_0}{6}(2\pi RH)}. \tag{5.12}$$

Here  $H$  denotes the energy of the system. This formula generally applies when  $2\pi RH \gg \frac{c_0}{24}$  but, for the CFTs relevant for black holes, we expect it to hold also for  $2\pi RH \sim \frac{c_0}{24}$  [33]. Since the Casimir energy for such a theory is  $2\pi RH = \frac{c_0}{24}$  we recover the universal ground state entropy

$$S = 2\pi \cdot \frac{c_0}{12}. \tag{5.13}$$

Relating the number of degrees of freedom  $c_0$  to our result for the central charge (3.11) as  $c_0 = c/(2\pi)$  we find the ground state entropy

$$S = \frac{c}{12} = \frac{1}{4G_2}, \tag{5.14}$$

in agreement with the Bekenstein-Hawking entropy.

The relation  $c_0 = c/(2\pi)$  is not self-evident. We have defined the central charge by the transformation property (3.9) with stress tensor normalized as in (3.1). This gives the same normalization of central charge as in [11]. As we have already emphasized, the length scale  $L$  introduced in (3.9) to render the central charge dimensionless is rather arbitrary. We could have introduced  $2\pi L$  instead, corresponding to

$$(\delta_\epsilon + \delta_\Lambda)T_{tt} = 2T_{tt} \partial_t \xi + \xi \partial_t T_{tt} - \frac{c_0}{12} L \partial_t^3 \xi(t). \tag{5.15}$$

It is apparently this definition that leads to  $c_0$ , the measure of degrees of freedom.

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<sup>5</sup>Strictly, the on-shell action is related to a thermodynamic potential that is a function of the temperature  $T$  and the electrostatic potential  $\Phi$ . However, the boundary term for the gauge field leads to a net charge  $Q = 0$ , and so the thermodynamic potential reduces to the standard Helmholtz free energy.

The situation is illuminated by our 3D-2D dictionary. We can implement this by using the 3D origin of the 2D coordinate  $t$  (9.8) or, simpler, the 3D origin of the 2D central charge (4.26). In 3D the dimensionless central charge that counts the degrees of freedom is introduced without need of an arbitrary scale. The relation (4.26) to the 2D central charge therefore motivates the factor  $2\pi$  in  $c_0 = c/(2\pi)$ . Furthermore, there is a conformal rescaling of the central charge due to an induced dilaton  $e^{-\psi}$ . To get a feel for this consider the canonical 4D BPS black holes [34, 35], supported by four mutually BPS charges  $n_1, n_2, n_3, n_0$  of which  $n_0$  is the KK-momentum along the circle. The conformal rescaling brings the 3D central charge  $c_{3D} = 6n_1n_2n_3$  into the more symmetrical value

$$c_0 = 6\sqrt{n_1n_2n_3n_0} \tag{5.16}$$

It would be interesting to understand this value directly from the 2D point of view.

It is natural to consider a more general problem. The 2D black holes (5.1) are lifted by our 2D/3D map in section 4 to the general BTZ black holes in three dimensions. The BTZ black holes are dual to a 2D CFT, with both right and left movers. The 2D description keeps only one chirality and so it is challenging to understand how the general entropy can be accounted for directly in 2D. Our result equating the ground state entropy of the chiral 2D CFT with the Bekenstein-Hawking entropy of any AdS<sub>2</sub> black hole indicates that this is in fact possible, but the details remain puzzling.

## 6. Backgrounds with non-constant dilaton

In this section we generalize our considerations to backgrounds with non-constant dilaton. We find that the counterterms determined for constant dilaton give a well-defined variational principle also in the case of a non-constant dilaton. We discuss some properties of the general solutions. In particular we identify an extremal solution that reduces to the constant dilaton solution (2.10) in a near horizon limit. For recent work on non-constant dilaton solutions in 2D Maxwell-Dilaton gravity see [36].

### 6.1 General solution with non-constant dilaton

We start by finding the solutions to the equations of motion. The spacetime and gauge curvature are determined by solving  $\mathcal{E}_\phi = 0$  and  $\mathcal{E}_\mu = 0$  in (2.4), which gives

$$R = -\frac{8}{L^2}, \quad F_{\mu\nu} = 2E \epsilon_{\mu\nu} . \tag{6.1}$$

In the case of non-constant dilaton we may use the dilaton as one of the coordinates

$$e^{-2\phi} = \frac{r}{L} . \tag{6.2a}$$

This statement is true everywhere except on bifurcation points of bifurcate Killing horizons. We do not exhaustively discuss global issues here and therefore disregard this subtlety. For dimensional reasons we have included a factor  $1/L$  on the right hand side of the

definition (6.2a). Using the residual gauge freedom we employ again a gauge where the line element is diagonal and the gauge field has only a time component,

$$ds^2 = g_{rr} dr^2 + g_{tt} dt^2, \quad A_\mu dx^\mu = A_t dt. \quad (6.2b)$$

Solving  $\mathcal{E}_\phi = 0$  yields  $g_{tt} = -1/g_{rr}$ , and the last equations of motion  $\mathcal{E}_{\mu\nu} = 0$  gives

$$g_{tt} = -\frac{4r^2}{L^2} + 2L^3 E^2 r + 4M, \quad (6.2c)$$

and

$$A_t = 2Er. \quad (6.2d)$$

The electric field  $E$  and ‘mass’  $M$  are constants of motion. The former has dimension of inverse length squared, the latter is dimensionless in our notation.

There is a Killing vector  $k = \partial_t$  that leaves the metric, gauge field and dilaton invariant. There are two other Killing vectors that leave invariant the metric, but not the dilaton. This is the breaking of  $SL(2, \mathbb{R})$  to  $U(1)$  mentioned before (2.5).

The Killing horizons are determined by the zeroes of the Killing norm. The norm squared is given by  $k^\mu k^\nu g_{\mu\nu} = g_{tt}$ , and therefore by solving  $g_{tt} = 0$  the horizons are located at

$$r_h = L \left[ \frac{E^2 L^4}{4} \pm \sqrt{\left(\frac{E^2 L^4}{4}\right)^2 + M} \right]. \quad (6.3)$$

For positive  $M$ , there is exactly one positive solution to (6.3). If  $M$  is negative two Killing horizons exist, provided the inequality  $E^2 > 4\sqrt{-M}/L^4$  holds. If the inequality is saturated,

$$M_{\text{ext}} = -\frac{L^8 E^4}{16}, \quad (6.4)$$

then the Killing horizon becomes extremal and the value of the dilaton (6.2a) on the extremal horizon,  $r_h/L = L^4 E^2/4$ , coincides with the constant dilaton result (2.7). This is consistent with the universality of extremal black hole spacetimes [1].

The geometric properties of the solution (6.2) are developed further in [23, 24] and references therein. The thermodynamic properties are a special case of those discussed in [28].<sup>6</sup>

## 6.2 Asymptotic geometry and counterterms

In order to compare the asymptotic geometry with our previous results we introduce

$$e^{2\eta/L} = \frac{4r}{L}, \quad (6.5)$$

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<sup>6</sup>The solution (6.2) is the special case  $U(X) = 0$ ,  $V(X) = -\frac{4X}{L^2} + L^2 E^2$  where the functions  $U$ ,  $V$  are introduced in the definition of the action (1.1) of [28] and  $X = e^{-2\phi}$  is the dilaton field.

and write the solutions (6.2) in the form

$$g_{\mu\nu}dx^\mu dx^\nu = d\eta^2 - \frac{1}{4}e^{4\eta/L}dt^2 + \dots, \quad (6.6a)$$

$$A_\mu dx^\mu = \frac{1}{2}LEe^{2\eta/L}dt, \quad (6.6b)$$

$$\phi = -\frac{\eta}{L}. \quad (6.6c)$$

At this order the solutions agree with the constant dilaton background (2.10), except the dilaton diverges linearly with  $\eta$  rather than approaching a constant. Therefore, the solution (6.2) may be called ‘linear dilaton solution’.

Asymptotically linear dilaton solutions have Fefferman-Graham like expressions analogous to (2.11), except that we must allow a logarithmic modification in the expansion of the dilaton

$$h_{tt} = e^{4\eta/L} h_{tt}^{(0)} + h_{tt}^{(1)} + \dots, \quad (6.7a)$$

$$A_t = e^{2\eta/L} A_t^{(0)} + A_t^{(1)} + \dots, \quad (6.7b)$$

$$\phi = \eta \phi^{(\log)} + \phi^{(0)} + \dots. \quad (6.7c)$$

As for the full solution, the asymptotic geometry and field strength of the linear dilaton solutions respect the asymptotic  $SL(2, \mathbb{R})$  symmetry, but the asymptotic dilaton respects only the Killing vector  $\partial_t$ . The explicit linear dilaton solution (6.2) is obviously of the asymptotically linear dilaton form (6.7). Its boundary values are

$$h_{tt}^{(0)} = -\frac{1}{4}, \quad A_t^{(0)} = \frac{1}{2}LE, \quad \phi^{(\log)} = -\frac{1}{L}. \quad (6.8)$$

We want to set up a consistent boundary value problem and variational principle, as in section 2. There we wrote down the most general local counter terms and determined their coefficients, essentially by demanding the vanishing of the momenta (2.18) at the boundary. A non-constant dilaton could give rise to some additional local boundary terms, but all candidate terms vanish too rapidly to affect the variational principle — they are irrelevant terms in the boundary theory. Since there are no new counter terms and the coefficients of the existing ones are fixed by considering constant dilaton configurations, it must be that the same counter terms suffice also in the more general case.

We can verify this argument by explicit computation. For our choice of counter terms, the momenta (2.18b), (2.18c) vanish no matter the dilaton profile. For consistency we have verified that (2.18a) also does not lead to new conditions. We conclude that the logarithmic modification in the dilaton sector inherent to (6.8) does not destroy the consistency of the full action (2.22). Moreover, the discussion of gauge invariance in section 2.3 also carries through. This is so, because the new boundary term (2.23) does not depend on the dilaton field.

In summary, we find that the full action (2.22) encompasses the boundary value problems (2.12) and (6.8), has a well-defined variational principle, and is extremized by the constant dilaton solutions (2.10) as well as by the linear dilaton solutions (6.2). Our discussion therefore exhausts all solutions to the equations of motion (2.4).

## 7. Discussion

We conclude this paper with a few comments on questions that are left for future work:

- **Universal Central Charge:** Our result for the central charge can be written as (3.11)

$$c = \frac{3}{G_2}. \tag{7.1}$$

This form of the central charge does not depend on the detailed matter in the theory, i.e. the Maxwell field and the charge of the solution under that field. This raises the possibility that the central charge (7.1) could be universal, i.e. independent of the matter in the theory. It would therefore be interesting to study more general theories and establish in which cases (7.1) applies.

- **Mass Terms for Gauge Fields:** One of the subtleties we encountered in this paper was the presence of the mass term

$$I_{\text{new}} \sim \int_{\partial\mathcal{M}} m A^2, \tag{7.2}$$

for the boundary gauge field. Related boundary terms are known from Chern-Simons theory in three dimensions [37, 38], but apparently not in higher dimensions. It would be interesting to find situations where such boundary terms do appear in higher dimensions, after all. A challenge is that typical boundary conditions have the gauge field falling off so fast at infinity that these boundary terms are not relevant, but there may be settings with gauge fields that fall off more slowly.

- **Unitarity:** Our computations have several unusual signs. The most prominent one is that the overall constant in the action (1.1) must be negative

$$\alpha < 0. \tag{7.3}$$

With this assignment the various terms in the action would appear to have the “wrong” sign, raising concerns about the unitarity. The sign we use is required to get positive central charge, positive 3D Newton constant, positive energy of the 2D black holes, and positive black hole entropy. This suggests that  $\alpha < 0$  is in fact the physical sign. Nevertheless, a more direct understanding of unitarity would be desirable.

We hope to return to these questions in future work.

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## 8. Conventions and notations

The 2D Newton’s constant is determined by requiring that the normalization of the gravitational action is given by

$$I = -\frac{1}{16\pi G_2} \int_{\mathcal{M}} d^2x \sqrt{-g} R + \dots, \tag{8.1}$$

where the unusual minus sign comes from requiring positivity of several physical quantities, as explained in the body of the paper. Comparing (2.1) with (8.1) gives the relation (2.2) for constant dilaton backgrounds.

For sake of compatibility with other literature we choose a somewhat unusual normalization of the AdS radius  $L$  so that in 2D  $R_{AdS} = -8/L^2$ , and for electric field  $E$  we use  $F_{\mu\nu} = 2E \epsilon_{\mu\nu}$ . For the same reason our gauge field has inverse length dimension. As usual, the quantity  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength for the gauge field  $A_\mu$ , its square is defined as  $F^2 = F^{\mu\nu} F_{\mu\nu} = -8E^2$ , and the dilaton field  $\phi$  is defined by its coupling to the Ricci scalar of the form  $e^{-2\phi} R$ .

Minkowskian signature  $-, +, \dots$  is used throughout this paper. Curvature is defined such that the Ricci-scalar is negative for AdS. The symbol  $\mathcal{M}$  denotes a 2D manifold with coordinates  $x^\mu$ , whereas  $\partial\mathcal{M}$  denotes its timelike boundary with coordinate  $x^a$  and induced metric  $h_{ab}$ . We denote the 2D epsilon-tensor by

$$\epsilon_{\mu\nu} = \sqrt{-g} \varepsilon_{\mu\nu}, \tag{8.2}$$

and fix the sign of the epsilon-symbol as  $\varepsilon^{t\eta} = -\varepsilon_{t\eta} = 1$ .

In our 2D study we use exclusively the Fefferman-Graham type of coordinate system

$$ds^2 = d\eta^2 + g_{tt} dt^2, \tag{8.3}$$

in which the single component of the induced metric on  $\partial\mathcal{M}$  is given by  $h_{tt} = g_{tt}$  with ‘determinant’  $h = h_{tt}$ . In the same coordinate system the outward pointing unit vector normal to  $\partial\mathcal{M}$  is given by  $n^\mu = \delta_\eta^\mu$ , and the trace of the extrinsic curvature is given by

$$K = \frac{1}{2} h^{tt} \partial_\eta h_{tt}. \tag{8.4}$$



Our conventions in 3D are as follows. Again we use exclusively the Fefferman-Graham type of coordinate system

$$ds^2 = d\eta^2 + \gamma_{ab} dx^a dx^b, \quad (8.5)$$

in which the induced metric on the boundary is given by the 2D metric  $\gamma_{ab}$ . In the same coordinate system the extrinsic curvature is given by

$$\mathcal{K}_{ab} = \frac{1}{2} \partial_\eta \gamma_{ab}, \quad (8.6)$$

with trace  $\mathcal{K} = \gamma^{ab} \mathcal{K}_{ab}$ . The 3D AdS radius  $\ell$  is normalized in a standard way,  $\mathcal{R}_{\text{AdS}} = -6/\ell^2$ . Without loss of generality we assume that the AdS radii are positive:  $L, \ell > 0$ .

## 9. Dictionary between 2D and 3D

We have derived in section 4.2 the relation (4.16) between the normalization constants in 2D and 3D. As a consistency check on our 3D interpretation of the 2D theory we show in section 9.1 that the boundary terms also reduce correctly. Also, since the 2D Maxwell-dilaton theory is on-shell equivalent to the KK-reduction of 3D gravity, the 3D solutions respecting the appropriate isometry must agree with a 2D solution. We construct the explicit map in section 9.2.

### 9.1 Kaluza-Klein reduction: the boundary terms

Applying the KK-reduction (4.8) to a 3D metric in the Fefferman-Graham form (4.4) we can write

$$ds^2 = e^{-2\psi} \ell^2 (dz + \tilde{A}_t dt)^2 + \tilde{h}_{tt} dt^2 + d\eta^2. \quad (9.1)$$

Here we identify  $\tilde{h}_{tt}$  as the metric of the 1D boundary of the 2D metric  $\tilde{g}_{\mu\nu} dx^\mu dx^\nu$ . Surfaces of (infinite) constant  $\eta$  define the boundary in both 3D and in 2D, and so we can use  $\eta$  as the radial coordinate in both cases. The 3D trace of extrinsic curvature becomes

$$\mathcal{K} = \tilde{K} - \partial_\eta \psi, \quad (9.2)$$

with  $\tilde{K}$  the extrinsic curvature of the one dimensional boundary  $\tilde{h}_{tt}$ . The boundary term of the 3D theory in (4.1) therefore reduces to the boundary term

$$\tilde{I}_{\text{boundary}} = \frac{\ell}{4G_3} \int dt \sqrt{-\tilde{h}} e^{-\psi} \left( \tilde{K} - \frac{1}{\ell} \right), \quad (9.3)$$

of the 2D theory. The term proportional to the gradient in  $\psi$  canceled an identical term arising when integrating the bulk term (4.9) by parts.

In order to show that our 2D theory is equivalent on-shell to the KK-reduction of the 3D theory we must match (9.3) with the boundary term

$$I_{\text{boundary}} = \frac{\alpha}{\pi} \int dt \sqrt{-h} e^{-2\phi} \left( K - \frac{2}{L} + \frac{L}{2} e^{2\phi} A^a A_a \right) \quad (9.4)$$

determined directly in 2D. Evaluating (9.4) on the asymptotic AdS<sub>2</sub> backgrounds (2.10) we have

$$K = \frac{2}{L}, \quad h^{ab} A_a A_b = -e^{-2\phi} \frac{4}{L^2}. \quad (9.5)$$

and so

$$\begin{aligned} I_{\text{boundary}} &= -\frac{\alpha}{\pi} \int dt \sqrt{-h} e^{-2\phi} \frac{2}{L} \\ &= \frac{\ell}{4G_3} \int dt \sqrt{-\tilde{h}} e^{-\psi} \frac{1}{\ell}, \end{aligned} \quad (9.6)$$

where in the last line we used our 3D-2D dictionary (4.13a), (4.16).

Asymptotically AdS<sub>2</sub> solutions of the theory defined by the bulk action (4.10) have extrinsic curvature  $\tilde{K} = \frac{2}{\ell}$  and therefore the on-shell value of the KK reduced boundary action (9.3) exactly agrees with the on-shell value of the 2D boundary action (9.6), i.e.,  $\tilde{I}_{\text{boundary}} = I_{\text{boundary}}$ . This is what we wanted to show.

## 9.2 Asymptotically AdS solutions

Our starting point is the general 3D solution (4.6). For compactifications along  $z = x^+$  we consider  $g_+ = \text{constant}$ , and rewrite the solution in the form (9.1) as

$$\begin{aligned} ds^2 &= \left(\frac{g_+}{\ell}\right) \ell^2 \left[ dx^+ + \frac{1}{2\ell} e^{2\eta/\ell} \sqrt{\frac{\ell}{g_+}} \left(1 + \frac{16}{\ell^2} g_+ g_-(t) e^{-4\eta/\ell}\right) dt \right]^2 \\ &\quad - \frac{1}{4} e^{4\eta/\ell} \left(1 - \frac{16}{\ell^2} g_+ g_-(t) e^{-4\eta/\ell}\right)^2 dt^2 + d\eta^2, \end{aligned} \quad (9.7)$$

with

$$t = \frac{\ell}{4} \sqrt{\frac{\ell}{g_+}} x^-. \quad (9.8)$$

Comparing with the Ansatz (4.8) we read off the 2D metric

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -\frac{1}{4} e^{4\eta/\ell} \left(1 - \frac{16}{\ell^2} g_+ g_-(t) e^{-4\eta/\ell}\right)^2 dt^2 + d\eta^2, \quad (9.9)$$

and the matter fields

$$\tilde{A} = \frac{1}{2\ell} e^{2\eta/\ell} e^\psi \left(1 + \frac{16}{\ell^2} g_+ g_-(t) e^{-4\eta/\ell}\right) dt, \quad (9.10a)$$

$$e^{-2\psi} = \frac{g_+}{\ell}. \quad (9.10b)$$

The solution (9.9)-(9.10) should be equivalent to the asymptotically AdS<sub>2</sub> solutions (2.10) found directly in 2D. After the coordinate transformation  $(\eta, t) \rightarrow \frac{1}{a}(\eta, t)$  in (2.10) this expectation is correct, and we use the dictionary (4.13), (4.16) to find the relations  $h_0(t) = 1$ ,  $a(t) = 0$  and

$$h_1 = -\frac{16}{\ell^2} g_+ g_-(t), \quad (9.11a)$$

$$\alpha = -\frac{\pi\ell}{8G_3} e^{2\phi} \sqrt{\frac{g_+}{\ell}}, \quad (9.11b)$$

between the parameters of the solutions.

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