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Holography, a Covariant c Function, and the Geometry of the Renormalization Group

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We propose a covariant geometrical expression for the c function for theories which admit dual gravitational descriptions. We state a c theorem with respect to this quantity and prove it. We apply the expression to a class of geometries, from domain walls in gauged supergravities, to extremal and near extremal Dp-branes, and the AdS Schwarzschild black hole. In all cases, we find agreement with expectations.

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I. INTRODUCTION

The holographic encoding of information in gravitational theories appears to be a manifestation of a fundamental physical principle. The importance of this projection of information was realized in the context of classical general relativity through entropy bounds and black hole thermodynamics [1,2,3]. More recently, we have learned from string theory that this phenomenon appears to have intriguing connections with scaling and renormalization group flow in nongravitational theories [4,5,6,7,8,9,10,11]. A great deal remains to be unraveled about this connection, but there are already indications that this line of thought may hold the resolution of some of the paradoxical issues arising from black hole physics.

In this work, we investigate the connection between renormalization group flow and holography by proposing a covariant, geometric measure for the effective central charge for the so-called ''boundary theory'' of Maldacena's duality [12,13,14]. Central charge, or the ''c function,'' is a measure of the degrees of freedom of a theory, the number of independent species of excitations. For theories in two dimensions, Zamolodchikov [15] was able to prove a set of elegant statements describing the central role played by this quantity in the renormalization group flow. The effective central charge was shown to be a function of the couplings of the theory that is monotonically decreasing as one flows to lower energies; fixed points described by conformal field theories correspond to the extrema of this function, and the gradients over coupling space are related to the beta functions of the theory. Attempts to generalize some of these statements to higher dimensions have been met with very limited success. In the context of Maldacena's duality, we acquire geometrical tools to study this question in regimes where a theory is strongly coupled.

The basic conceptual ingredient in our proposal is a remarkably simple, yet powerful prescription proposed by Bousso [16]. The observation is that holographic statements should have a covariant nature. Consequently, Bousso proposes to use congruences of null geodesics as probes for the sampling of holographically encoded information. We believe that this principle is a general one. The proposal of Maldacena in regimes where one focuses on bulk dynamics in the nonstringy gravity sector must have a similar covariant nature.

In the next section, we motivate and construct a covariant expression for the central charge for the boundary theory. We will guide ourselves by a set of intuitively driven principles inspired by the Bousso entropy bound. We will then prove a c theorem; Bousso's criterion for the convergence of the congruence and the null convergence criterion are identified as the necessary and sufficient conditions.[1] We will then proceed to apply the prescription to certain classes of geometries: domain wall solutions in gauged supergravities, near-horizon regions of extremal and near-extremal Dp-branes for $p<5$, and the anti–de Sitter (AdS) Schwarzschild black hole in four dimensions. In the first class, we find exact agreement with [10]. For the second class, the scaling of the c function is found to match onto the expected asymptotic behaviors given by the perturbative supersymmetric Yang-Mills (SYM) theory, the matrix string, the M-theory membrane, and the M-theory five-brane theories. We also find that our expression, applied to flow along ''radial congruences,'' is insensitive to the presence of a thermodynamic horizon; as expected, the latter corresponds to a thermodynamic state in the same dual theory. We end with a discussion assessing the evidence presented.

II. COVARIANT c FUNCTION

Consider a D-dimensional spacetime with metric g_{ab} foliated by a choice of constant-time surfaces. Let this vacuum solve Einstein's equations with a negative cosmological constant. Focus on a spacelike $(D-2)$-dimensional surface $\mathcal{M}$ at some fixed time (see Fig. 1). There are generally four light sheets projected out of this surface consisting of the spacetime points visited by a congruence of null geodesics orthogonal to $\mathcal{M}$ [17]. As prescribed by Bousso [16], we pick a light sheet along which the congruence converges. If n^a denotes the tangents to these geodesics, we can construct a null vector field m^a on the light sheet such that it is orthogonal to $\mathcal{M}$ and satisfies $m^a n_a=-1$ [18]. Then $\mathcal{M}$ admits the metric

$$h_{ab}=g_{ab}+n_{(a}m_{b)}\,. \tag{1}$$

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[1]This is for cases involving shearless flow; a more general statement can also be proved for the cases with shear.

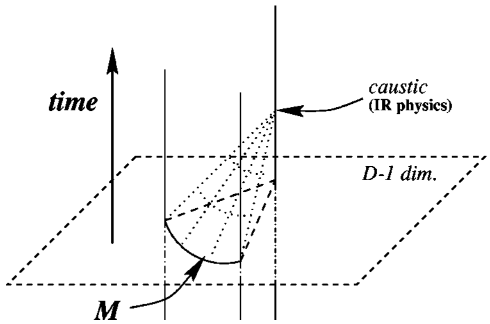

FIG. 1. An illustration of the construction of holographic duals; the $(D-2)$-dimensional surface $\mathcal{M}$ is shown, along with a caustic ending the congruence of null geodesics.

The *second fundamental form* is defined by [17,19]

$$B_{ab} \equiv \nabla_b n_a = \nabla_a n_b = B_{ba}. \tag{2}$$

The symmetry property follows from Frobenius's theorem and the fact that the vector field n^a is surface orthogonal. We define the $(D-2)$-dimensional matrix

$$\hat{B}^a_b = g^{ac} B_{cb} = h^{ac} B_{cb} \tag{3}$$

and its trace

$$\theta = \mathrm{Tr}\, \hat{B}. \tag{4}$$

The condition for the convergence of the geodesics is stated as [16]

$$\theta \leqslant 0. \tag{5}$$

The geodesics are to be extended as long as this condition is satisfied; for spacetimes curved by matter satisfying the null convergence condition,[2] these geodesics will typically end at caustics ($\theta \to -\infty$). We would like to think of a sense in which information on such a light sheet is holographically encoded on $\mathcal{M}$. This is the nature of Bousso's entropy bound, and it is also consistent with Maldacena's conjecture.[3]

For simplicity, let us require that the spacetimes we consider admit a timelike Killing vector field along which our choice of time flows. One may propose to time-flow the whole light sheet forward and backward, generating a D-dimensional region of spacetime which becomes the ''bulk,'' and the ''boundary'' is the time flow of $\mathcal{M}$. The c function we will be considering is to be accorded to the theory that is in some sense residing on this boundary and is dual to the bulk. We want to associate renormalization group flow to lower energy scales with motion along the converging congruence of null geodesics. A more precise and careful version of this statement will be postponed to future work. For now, we will use the null geodesics described above as tools to probe covariantly the dual theory at lower energy scales; the success of our proposal in the examples we will consider can be viewed as evidence to this approach.

Without claim to rigor, we next motivate the geometrical formula for the c function. The first principle we accord to is that the *local geometry* transverse to the flow encodes the information about the decrease of the effective central charge due to the coarse graining of the boundary theory. This is partly motivated by the work of [20], where the energy momentum tensor of the boundary theory for AdS spacetimes was written in terms of local quantities, essentially the extrinsic curvature of a foliation and its trace. From the same line of thought, we expect the c function to be proportional to G_D^{-1}, the inverse of the D-dimensional gravitational coupling. The next tool is dimensional analysis; we need a local *covariant* object with dimensions length^{D-2} to construct a dimensionless c function; motivated by the Bousso construction, we allow ourselves to use only covariant data from the congruence. Intuitively, we also require *invariance* under boundary diffeomorphisms; the most natural way to assure this is through an integration over the boundary using the proper measure constructed from h_{ab}. Finally, we require that the formula give the proper scaling for the central charge in AdS spaces, i.e., a constant proportional to l^{D-2}/G_D [21], where the cosmological constant scales as $-1/l^2$. For AdS spaces with metric in the Poincaré coordinates,

$$ds^2 = \frac{l^2}{z^2}(dz^2 + d\vec{x}^2_{D-2} - dt^2), \tag{6}$$

and a choice of constant z foliations at fixed time for $\mathcal{M}$, we have $\theta \sim z/l^2$ and $\sqrt{h} \sim (l/z)^{D-2}$. Putting everything together, we are left with combinations of two simple expressions $\det \hat{B}$ and θ, differing from each other only if the flow of the congruence has shear (see definition below); we pick a form that appears the most natural:

$$c(\tau) \simeq \frac{c_0}{G_D \int \sqrt{h} \det \hat{B}|_\tau}. \tag{7}$$

Another obvious option is to replace $(\det \hat{B})$ by θ^{D-2}. The integral is evaluated at some proper time τ along the geodesics; τ is to be related to the energy scale of renormalization group flow through the UV-IR relation. There are two problems with expression (7): dependence on the scale parameter for the proper time and it is divergent for many relevant geometries with noncompact boundaries $\mathcal{M}$. Both problems can be regulated by proposing that the local expression we have can yield only a fractional decrease in the central charge as a result of renormalization. We then propose the cure

[2]The null convergence condition is discussed in detail in [17]; it is the statement that the energy momentum tensor satisfies the condition $T_{ab}k^a k^b \geqslant 0$ for all null k^a. This follows from the strong or weak energy conditions that are believed to be satisfied by all known forms of matter.

[3]The past and future history of the light sheet is causally related to it, so that the spacelike notion of holography we may naively accord to Maldacena's duality is a subset of this covariant statement.

$$\frac{c_{\tau_2}}{c_{\tau_1}}=\frac{G_D\int\sqrt{h}\,\det\hat{B}|_{\tau_1}}{G_D\int\sqrt{h}\,\det\hat{B}|_{\tau_2}}. \tag{8}$$

One fixes the central charge at some proper time τ_1 from other data, and the formula predicts the central charge at lower energy scales deep in the bulk at τ_2. The expression is manifestly covariant, independent of the arbitrary scaling of the proper time, in principle convergent, and invariant under boundary diffeomorphisms. Typically, one would expect to work in a gravitational theory with negative cosmological constant so as to have an asymptotic AdS vacuum configuration corresponding to a UV fixed point. c_{τ_1} gets fixed by the conformal algebra in this region, and c_{τ_2} predicts the central charge of the deformed conformal theory at lower energy scales. The product of c_{τ_1} with the numerator of the right-hand side is a numerical coefficient times regulator factors canceling with the denominator of the right-hand side. For pure AdS spaces, Eq. (8) gives one by construction.

In practical calculations, we will have symmetries that allow us to write a slightly simplified formula. For the rest of this work, we assume that the flow under consideration is shearless; the matrix $\hat{B}$ can be decomposed generally as [17]

$$\hat{B}=\frac{\theta}{D-2}\mathbf{1}+\hat{\sigma}, \tag{9}$$

where the symmetric matrix $\hat{\sigma}$ is referred to as shear. For all cases under considerations, we have $\hat{\sigma}=0$, so that $\hat{B}$ is proportional to a $(D-2)$-dimensional identity matrix. We then write

$$c(\tau)=\frac{c_0}{G_D\int\sqrt{h}\,\det\hat{B}|_{\tau}}\rightarrow\frac{c_0'}{G_D\int\sqrt{h}\,\theta^{D-2}|_{\tau}}, \tag{10}$$

where c_0 and c_0' are products of numerical coefficients, and a factor regulating the size of $\mathcal{M}$ in the cases where $\mathcal{M}$ is noncompact. As mentioned above, for shearless flow, we are unable to discriminate between combinations of the two expressions that we were led to in the arguments above, i.e., $\det\hat{B}\sim\theta^{D-2}$. We are, however, intuitively driven to propose that the general form for the central charge should be given by Eq. (8). We expect that the flow of the central charge should be sensitive to the phenomenon of shear.

We will next prove a c theorem for Eq. (10). A more general version with respect to Eq. (8) can be proved as well (the convergence criterion gets slightly generalized and we would require that the congruence be principal null with respect to the Weyl tensor). However, our understanding of the physical role of shear from the renormalization group perspective is primitive; we defer the more general statement to future work where we hope to explore explicitly examples with shear.

III. c THEOREM

Theorem. Consider a congruence of null geodesics emanating from a $(D-2)$-dimensional surface $\mathcal{M}$ as defined above; then, Eq. (10) is monotonically decreasing for increasing τ if the null convergence condition is satisfied and if everywhere along the flow $\theta\leqslant 0$.

The proof is straightforward. Differentiating the logarithmic of Eq. (10), the derivative slices through the integral (the congruence is orthogonal to $\mathcal{M}$) and we get

$$\frac{d}{d\tau}\ln c=-\frac{1}{\int\sqrt{h}\,\theta^{D-2}}\int\frac{d}{d\tau}(\sqrt{h}\,\theta^{D-2}). \tag{11}$$

The monotonicity follows immediately from Raychaudhuri's equation

$$\frac{d\theta}{d\tau}=-\frac{1}{D-2}\theta^2-R_{ab}n^a n^b \tag{12}$$

and from

$$\frac{d}{d\tau}\sqrt{h}=\delta_n\sqrt{h}=\sqrt{h}\,\theta. \tag{13}$$

We then have

$$\frac{d}{d\tau}\ln c=-\frac{(D-2)}{\int\sqrt{h}|\theta|^{D-2}}\int\sqrt{h}|\theta|^{D-3}R_{ab}n^a n^b\leqslant 0, \tag{14}$$

where we have used $\theta\leqslant 0$. Here $R_{ab}n^a n^b\geqslant 0$ follows from the null convergence condition and Einstein's equations in the presence of a cosmological constant since $n^a n_a=0$. The null convergence condition follows from either the weak or strong energy conditions.

Note that Eq. (14) vanishes for $D=2$, i.e., when the boundary is described by quantum mechanics. This is consistent with the fact that the renormalization group flow prescription, in the spirit defined in field theoretical settings, does not exist in this regime.

Let us simplify the formulas further to make a few observations. Often the metric for spacetimes of interest depends on a single "radial" coordinate, and we choose the surface $\mathcal{M}$ to be at constant value with respect to this coordinate. Spacetime is therefore parametrized by time, a radial coordinate, and $D-2$ spatial coordinates residing on $\mathcal{M}$. The integral in Eq. (10) can then be evaluated, canceling the potential divergence in the numerator, so we write the finite expressions

$$c(\tau)=\frac{c_0}{G_D\sqrt{h}|\theta|^{D-2}|_{\tau}}, \tag{15}$$

$$\frac{d}{d\tau}\ln c=-(D-2)\frac{R_{ab}n^a n^b}{|\theta|}, \tag{16}$$

where c_0 is a finite numerical coefficient. Implied is the statement that for larger values of proper time, we penetrate deeper in the bulk and therefore flow to lower energies. This statement will be made more precise in the next section. When $|\theta|\rightarrow\infty$, putting an end to the sampling of the bulk points and assuming $R_{ab}n^a n^b$ is finite, Eq. (16) indicates that we have reached an infrared fixed point. Depending on whether the combination $\sqrt{h}|\theta|^{D-2}$ is finite or infinite, we have a finite or zero central charge. This implies that all renormalization group flows to lower energies lead to IR fixed points. This is certainly a desirable statement; we have correlated caustics with IR fixed points.[4] The other criterion for the theorem, the null convergence condition, was also a condition advocated by the work of [10] for the statement of monotonicity. Our covariant approach indicates that this observation is a general one.

IV. COVARIANT UV-IR RELATION

We need to prescribe a relation between the proper time and the energy cutoff in the renormalization group flow. We have very little to guide ourselves with in this regard. The UV-IR relation, as sketched in, for example, [22], is a rough scaling relationship, and it is still associated with various paradoxes. A fundamental formulation of the relation between scale on the boundary and bulk physics is yet unknown. In the spirit of our previous discussion, we should try to write a covariant UV-IR relation. One can write trivially the statement of [22], i.e., $g_{tt}dt^2 \sim g_{rr}dr^2$, covariantly:

$$\frac{1}{\mu(\tau)} \simeq \int_{\tau_{\rm UV}}^{\tau} d\tau' n^a \nabla_a t. \tag{17}$$

Here t is the function foliating the equal time surfaces, $\tau_{\rm UV}$ is the proper time chosen in the UV, and $\mu(\tau)$ is the energy cutoff associated with the proper time τ. This approach is particularly naive, and moreover it is not unique. One can readily write other covariant expressions, particularly ones that make reference to the second fundamental form and appear to be more natural choices. Given the correlation between IR fixed points and caustics we advocated earlier, it is desirable to state a UV-IR relation such that caustics naturally correspond to the zero-energy limit of the cutoff. For the standard scenarios analyzed in the literature in the context of Eq. (17), this is certainly the case. However, our understanding of the underlying physics in this regard in a general context is limited; we will then adopt for now Eq. (17) as a rough guide for the purpose of tracking the scaling relation between energy scale and a coordinate in the bulk. A more fundamental geometrical understanding of this issue and the coarse graining prescription is needed for a more rigorous map between the renormalization group and geometry.

V. TEST AND EXAMPLES

A. Gauged supergravity

We first consider the class of geometries describing domain walls in gauged supergravities. These solutions interpolate between two asymptotic AdS regions. For $D=5$, they are believed to correspond to compactifications of type-IIB vacua on manifolds that get deformed from the spherical geometry of the Freund-Rubin scenario as one flows from the UV to the IR. For $D=4$ or $D=7$, they correspond to M-theory compactifications. A wide class of solutions were summarized in [11], and the metrics have the generic form

$$ds^2 = e^{2A(r)}(d\vec{x}^2_{(D-2)} - dt^2) + dr^2. \tag{18}$$

We pick the $(D-2)$-dimensional manifold $\mathcal{M}$ as a surface of constant r, and we measure time/energy by the coordinate t. The congruence of ingoing null geodesics can easily be found using the timelike Killing vector field ∂_i; one finds

$$n^a = \gamma e^{-2A}\partial_t - \gamma e^{-A}\partial_r, \tag{19}$$

where γ is an arbitrary parameter, chosen to be positive, which scales the proper time; this is an arbitrariness characteristic of null geodesics. The Christoffel variables are found to be

$$\Gamma^r_{ii} = -A' e^{2A}\eta_{ii}, \quad \Gamma^i_{ri} = A', \tag{20}$$

where $\eta_{tt} = -1$ and $\eta_{ii} = +1$ for $i \in \mathcal{M}$. We construct the second fundamental form

$$B_{bc} = \gamma \begin{pmatrix} A' e^{-A} & A' & \cdots 0 \cdots \\ A' & A' e^{A} & \cdots 0 \cdots \\ \vdots & \vdots & \\ 0 & 0 & -A' e^{A}\mathbf{1} \\ \vdots & \vdots & \end{pmatrix}. \tag{21}$$

The data for the c function become

$$\hat{B} = -\gamma A' e^{-A}\mathbf{1}_{D-2}, \quad \sqrt{h} = e^{(D-2)A}, \tag{22}$$

yielding

$$c = \frac{c_0}{G_D A'^{D-2}}. \tag{23}$$

This is the expression that was proposed for the c function in [10,11] along with compelling evidence in its favor. The form of Eq. (8) assures that the parameter c_0 here is such that, in the asymptotic AdS UV region, the central charge is given by that of the UV fixed point.

B. D*p*-branes

We next consider the near-horizon geometries of Dp-branes. It will be more convenient to first study a class of metrics of the form

$$ds^2 = e^{2A(z)}[f(z)^{-1}dz^2 + d\vec{x}^2_{D-2} - f(z)dt^2]. \tag{24}$$

We will coordinate transform the Dp-brane geometries into this form later. We choose a constant z and t surface for our manifold $\mathcal{M}$. The congruence of null geodesics is given by the vector field

[4]If $R_{ab}n^a n^b$ is to diverge, we expect stringy physics to set in to regulate the conclusion.

$$n^a = \gamma f^{-1} e^{-2A}\partial_t + \gamma e^{-2A}\partial_z . \tag{25}$$

The Christoffel variables are

$$\Gamma^z_{zz} = \frac{1}{2}\left(-\frac{f'}{f} + 2A'\right), \quad \Gamma^z_{ii} = -fA',$$

$$\Gamma^z_{tt} = \frac{f}{2}(f' + 2fA'),$$

$$\Gamma^i_{zi} = A', \quad \Gamma^t_{zt} = \frac{1}{2}\left(\frac{f'}{f} + 2A'\right). \tag{26}$$

The data for the c function become

$$\hat{B} = -\gamma A' e^{-2A}, \quad \sqrt{h} = e^{(D-2)A}. \tag{27}$$

We then get the expression

$$c = \frac{c_0}{G_D e^{-(D-2)A} A'^{D-2}} . \tag{28}$$

The function $f(z)$ disappeared from the expression for the c function. The role of this function in the metric is to excite the geometry above extremality, i.e., to create a thermodynamic horizon. Correspondingly, in the dual description, we excite a finite-temperature vacuum in the same theory. The c function should not change when the vacuum reflects a thermodynamic state in the same theory with the same degrees of freedom. This insensitivity of our expression to thermodynamic horizons is the second nontrivial piece of evidence in its favor. We will come back to this issue later in the context of the AdS Schwarzschild black hole; for now, let us proceed to Dp-branes.

A Dp-brane metric in the Maldacena scaling regime is given by [23,24]

$$ds^2_{\mathrm{str}} = \left(\frac{r}{q}\right)^{(7-p)/2} (d\vec{x}^2_{(p)} - f dt^2) + \left(\frac{q}{r}\right)^{(7-p)/2} (f^{-1} dr^2 + r^2 d\Omega^2_{8-p}). \tag{29}$$

with the dilation being

$$e^{\phi} = \left(\frac{q}{r}\right)^{(7-p)(3-p)/4} \tag{30}$$

and

$$q^{7-p} \simeq g_{\mathrm{str}} N, \quad f = 1 - \left(\frac{r_0}{r}\right)^{7-p}, \tag{31}$$

where (*after* taking the decoupling limit) we have chosen units such that $\alpha' = 1$. Energy in the SYM dual is measured with respect to the coordinate time t. Given that we lack numerical accuracy in the relation between energy scale and radial extent r at present (for that matter, we also lack rigorous conceptual understanding in this context), we will now start being careless with numerical coefficients and aim at determining only the scaling of the c function with respect to the physical parameters of the SYM theory. We apply the coordinate transformation

$$r = z^{2/(p-5)} \quad \text{with} \quad p < 5. \tag{32}$$

After rescaling the metric to the Einstein frame, as well as absorbing certain constants in the transverse coordinates

$$g^{\mathrm{Ein}}_{\mu\nu} = e^{-\phi/2} g^{\mathrm{str}}_{\mu\nu}, \quad (p-5)\frac{q^{(p-7)/2}}{2}(\vec{x},t) \to (\vec{x},t), \tag{33}$$

we get

$$ds^2_{\mathrm{Ein}} = q^{(p+1)(7-p)/8} z^{(p-3)^2/4(p-5)} \left(\frac{2}{p-5}\right)^2 \times \left(\frac{1}{z^2}(f^{-1}dz^2 - f dt^2 + d\vec{x}^2_p) + \left(\frac{p-5}{2}\right)^2 d\Omega^2_{8-p}\right). \tag{34}$$

Note that we have effectively rescaled the SYM energy. This metric is of the form (24) except for the transverse $(8-p)$-sphere factor. Even though it is straightforward to extend our formalism to this extended space with the transverse sphere, it is easier to track the scaling of the physical parameters by imagining that we have compactified the geometry on this sphere, with the effect that the gravitational coupling in the lower $D = p+2$ dimensions scales as

$$G_{(p+2)} \simeq \frac{g^2_{\mathrm{str}}}{\mathrm{Vol}(\Omega_{8-p})} = g^2_{\mathrm{str}} z^{p-8} e^{(p-8)A(z)}, \tag{35}$$

where $A(z)$ refers to the corresponding function identified from matching Eq. (24) with Eq. (34). We then have

$$A' \simeq \frac{1}{z}. \tag{36}$$

The c function becomes

$$c = c_0 \frac{z^{8-p} e^{8A}}{g^2_{\mathrm{str}} A'^p} = c_0' g_{\mathrm{str}}^{(p-3)/2} N^{(p+1)/2} z^{(p-3)^2/(p-5)}, \tag{37}$$

where c_0 and c_0' are numerical coefficients. Applying the UV-IR relation given by Eq. (17) (necessarily at zero temperature; see comments clarifying the relevance of this statement in the Schwarzschild black hole section, Sec. V C), we get [22]

$$\mu(z) \sim \frac{1}{\sqrt{g_{\mathrm{str}} N z}}, \tag{38}$$

where we interpret μ as the renormalization energy cutoff scale. We have eliminated the proper time in favor of the coordinate z using the trajectory of the geodesics. We have also undone the rescaling of the time variable in Eq. (33), so that μ is energy scale as measured in the SYM theory. Putting things together and defining the effective large N dimensionless coupling as

$$g_{\text{eff}}^2(\mu) \equiv g_Y^2 N \mu^{p-3}, \tag{39}$$

where $g_Y^2 = g_{\text{str}}$, we arrive at an expression for the c function for the $(p+1)$-dimensional SYM theory:

$$c_{\text{D}p}(\mu) \simeq g_{\text{str}}^{(p-3)/(5-p)} N^{(p-7)/(p-5)} \mu^{(p-3)^2/(5-p)}$$
$$\simeq g_{\text{eff}}^{2(p-3)/(5-p)}(\mu) N^2 \simeq c_{\text{SYM}}(g_{\text{eff}}). \tag{40}$$

The first thing to note is that, when the energy scale is $\mu \sim \mu_{\text{YM}}$ such that the curvature scale in the region of space where the integral of Eq. (10) is evaluated becomes of order the string scale, we have $g_{\text{eff}}^2(\mu_{\text{YM}}) \sim 1$, and therefore, for all p,

$$c_{\text{D}p}(\mu_{\text{YM}}) \simeq N^2; \tag{41}$$

i.e., the gravitational description breaks down at the Horowitz-Polchinski correspondence point [25] and the central charge scales as in the perturbative SYM regime. This happens in the UV for $p<3$ and in the IR for $p=4$. For $p=3$, we note that the c function is constant and of order N^2 as expected for the conformal $(3+1)d$ $\mathcal{N}=4$ SYM theory.

Now let us analyze the different scenarios more closely: for $p=1$, we get

$$c_{\text{D1}}(\mu) \simeq \frac{N^2}{g_{\text{eff}}(\mu)} \simeq \frac{N^{3/2}}{g_Y} \mu. \tag{42}$$

This result was obtained in [26] by different methods; the authors there could use the correlation function with insertions of two energy-momentum tensors to read off the central charge. This c function, as noted by them, interpolates between the $(1+1)$-dimensional SYM and the matrix string theory regimes; the latter arises in the IR at energy scales $\mu_{\text{MS}} \sim g_Y/N^{1/2}$, which is again a Horowitz-Polchinski correspondence point in a dual geometry [27]; Eq. (42) yields, at this energy scale,

$$c_{\text{D1}}(\mu_{\text{MS}}) \sim N \tag{43}$$

as expected for the matrix string theory [28,29].

Next, consider $p=2$; we have

$$c_{\text{D2}}(\mu) \simeq \frac{N^2}{g_{\text{eff}}^{2/3}(\mu)} \simeq \frac{N^{5/3}}{g_Y^{2/3}} \mu^{1/3}. \tag{44}$$

Moving from the 2+1 SYM theory to the IR, at energy scales $\mu_{\text{M2}} \sim g_Y^2 N^{-1/2}$, as shown in [27], the membrane theory is encountered. We find the corresponding central charge is

$$c_{\text{D2}}(\mu_{\text{M}_2}) \approx N^{3/2}. \tag{45}$$

This is indeed the proposed behavior for the membrane theory [30]. Beyond this point, the geometry becomes AdS_4, and it appears we have reached a fixed point of flow.[5]

For $p=4$, we flow from the SYM in the IR to the (2,0) theory on a circle sitting in the UV. We have

$$c_{\text{D4}}(\mu) \sim g_Y^2 \mu N^3. \tag{46}$$

As we flow to the UV, we will start probing the size of the 11th dimension. This happens at $\mu_{\text{M5}} \sim 1/R_{11} \sim g_Y^{-2}$. The central charge then becomes

$$c_{\text{D4}}(\mu_{\text{M5}}) \sim N^3. \tag{47}$$

This is indeed the characteristic scaling for the central charge of the M5-branes [21]. The geometry becomes beyond this scale $\text{AdS}_7 \times \text{S}^4$, i.e., the near-horizon geometry of M5-branes.

The reader may have noticed that $p=5$ was a special case in our analysis. The coordinate transformation applied for these examples breaks down in this setting. In this case, one probes the delicate Neveu-Schwarz, 5-brane (NS5-brane) geometry; a more careful analysis of the geodesic flow is in order. The results are bound to have more of a predictive nature than of a test of our proposal; we will postpone this task to the future.

Equation (40) appears then to correctly reflect the renormalization group flow in $(p+1)$-dimensional SYM theories for $p<5$. Furthermore, the expression is insensitive to excitations of the vacuum above extremality to finite temperatures. We will say more on this issue in the next section. One may argue that the matching onto the M2- and M5-brane central charges is not terribly impressive since the geometries become AdS at these energy scales, and our expression is tuned to give the right answer for AdS spaces. We note, however, that, at the Horowitz-Polchinski correspondence points, this issue cannot be raised. We have two powers of the dimensionless quantities g_{eff} and N to check against; if one gets fixed by the AdS region, the other is free. Note that in the case for D1-branes, the Horowitz-Polchinski correspondence criteria bound both sides of the flow. And these matched the proper asymptotics known from other reliable methods. Turning around the argument, this becomes a test of our hypothesis that the c function is expressible in terms of local geometrical quantities, a statement which is intuitively in tune with the prescription of the renormalization group flow. When we reach a fixed point, i.e., an AdS region, it is irrelevant how we got there; there may be different routes to flowing to a fixed points from different neighboring conformal field theories. The outcome must be the same; the

[5]One may expect naively that the energy scale μ_{M2} must be $1/R_{11}$; however, the geometry of the lifted D2-branes is given by that of smeared M2-branes, whose near-horizon geometry is *not* AdS_4; the energy scale $\mu_{\text{M2}} \sim g_Y^2 N^{-1/2}$ was identified in [27] as the scale where the localized membrane theory sets in; it is this geometry which is AdS_4. Correspondingly, we find the characteristic $N^{3/2}$ scaling for membranes at this scale of the flow.

central charge is fixed by the end point, i.e., by the cosmological scale and gravitational coupling of the AdS region. This viewpoint, along with the nontrivial matchings with the perturbative SYM regimes for all p and the matrix string theory, constitutes compelling evidence in favor of a local expression for the central charge.

C. AdS Schwarzschild black holes

We briefly explore here the four-dimensional AdS Schwarzschiled black hole case to illustrate a previous point in a simpler setting. Let us consider the metric [31]

$$ds^2 = -g(r)dt^2 + f(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{48}$$

For the AdS black hole, we have

$$g(r) = 1 - \frac{r_0}{r} + \frac{r^2}{l^2} = f(r)^{-1}, \tag{49}$$

where l is related to the cosmological constant. One then finds

$$c \sim f(r)g(r). \tag{50}$$

The inverse relation between f and g characteristic of horizon excitations yields a constant c function. As in the near-extremal Dp-brane cases, we find here as well that our assessment of the central charge in the theory does not get affected by a thermodynamic horizon. The central charge for the AdS Schwarzschild black hole geometry is a constant equal to the value set by the asymptotic AdS region.

Note that this example also demonstrates that the disappearance of the function f from the central charge in the cases of near-extremal Dp-branes was *not* a result of focusing on the space $\mathcal{M}$ transverse to the z-t plane; it is the relative relationship between f and g that the central charge probes. One may get troubled from the fact that we used the *zero-temperature* UV-IR relation when we wrote the central charge as a function of the cutoff energy; i.e., we used the extremal metric. The complaint would be that the insensitivity of the central charge, written as a function of energy scale, to the presence of the black horizon was partly put by hand. The point is that the relation between energy scale in the boundary theory and extent in the bulk should be an *independent* statement; a covariant formulation of the UV-IR statement should be insensitive to the presence of a thermodynamic horizon in the geometry independently from any other statement. It would be unphysical if the presence of a background thermal bath affected our assessment of the relation between the location of an excitation in the bulk and its energy as measured in the boundary theory. This would have been needed if the cancelation of the horizon factors in the central charge expression was not to occur. This feature of our expression is then positive evidence in its favor.

In what sense then is the presence of a black horizon special? It seems that we are drawn to the conclusion that the central charge for the AdS Schwarzschild black hole geometry is set by the asymptotic AdS region; the black hole is simply a thermal state in the conformal field theory dual to the AdS vacuum [32,33]. The answer has to do with the causal aspect of the horizon. Consider a surface $\mathcal{M}$ sitting *at* the horizon. As Bousso notes, there are now three classes of null congruences which are candidates for sampling the bulk. One set will sample the inside of the horizon, but the other two sets, the trapped geodesics, will sample the surface area. These ones saturate the sampling criterion $\theta \leqslant 0$; i.e., they satisfy $\theta = 0$. This indicates that the *gravitational* dynamics *at* the horizon in some sense is a holographic dual to the gravitational dynamics *within* the horizon, both descriptions being duals to a conformal field theory. This viewpoint, in its current fetal state, presents an intriguing marriage between our understanding of the special causal aspects of a black horizon, its thermodynamic character, and renormalization group flow.

In principle, one can arrange matter configurations so as to curve spacetime as in Eq. (48) with arbitrary f and g. Asymptotically, we must recover AdS space with the cosmological constant set by our gravitational action. Our analysis suggests that, if this setup is stable, unlike the black hole scenario, it would correspond to perturbing the boundary theory away from conformality, generating nontrivial renormalization group flow.

The generalization of this example to higher dimensions is straightforward; we expect no change in the conclusions.

VI. DISCUSSION

Let us recap the proposal and critically assess the evidence we have presented in its favor. We used a principle of covariant holography and a set of intuitively driven, yet nonrigorous, arguments to define holographic duals. In this context, we proposed a c function for the boundary theory; it is a geometrical, local, covariant expression holographic in nature:

$$c \sim \frac{1}{\int_{\mathcal{M}} \det \hat{B}}. \tag{51}$$

The inverse central charge is simply written as the integral of the determinant of the second characteristic form on $\mathcal{M}$. The expression was explicitly tuned to yield a constant for AdS spaces. Unfortunately, we do not have a more physical understanding of the form Eq. (51). If this proposal is indeed correct, it is a statement about understanding the holographic encoding of information in the language of the renormalization group. The evidence we presented in favor of Eq. (51) was as follows.

(i) We could prove a c theorem: this essentially followed from Raychaudhuri's equation. One of the two criteria for a monotonically decreasing c function, the condition $\theta < 0$, correlates with Bousso's criterion for sampling the bulk space for information; the second criterion, the null convergence condition, was advocated independently in the example of [10].

(ii) For domain wall solutions, our result agrees with [10,11]. This may be regarded as merely a test for the possibility to formulate a c function through the formalism of congruences of null geodesics.

(iii) For the Dp-brane geometries, our expression appears to interpolate correctly between known asymptotics. This constitutes a nontrivial test for the principle that a covariant and local expression for the c function exists. As such, however, any other covariant local expression is a candidate as well.

(iv) Exciting a thermodynamic horizon in the bulk space does not change the central charge. This constitutes a nontrivial test for the form we have proposed, beyond the test for covariance.

The proposal (51) may be incomplete, with additional corrections needed as we move further into the bulk [20]. The successes we demonstrated may have been accidents due to certain symmetries in the cases considered. Any covariant term, invariant under boundary diffeomorphisms and vanishing in the maximally symmetric AdS case, is *a priori* allowed. Less interestingly, terms involving arbitrary powers of det $\hat{B}$ and θ such that the dimensions are right are allowed; we can write combinations involving objects like Tr $\hat{B}\cdots\hat{B}$. Such possibilities are endless, as well as being uninteresting; unfortunately, without examples that probe the effect of shear, one cannot distinguish between them. We propose the expression (51) for the c function as it appears to be the simplest and most natural form amongst these possibilities. On the other hand, more general terms can be multiplied by powers of the gravitational coupling to make them dimensionless; their origin would then probably be stringy. One may in principle add terms constructed from pullbacks of the curvature tensors; adding Weyl-tensor-dependent terms would not affect our conclusions in the cases of extremal Dp-branes and domain walls; it would, however, change the conclusion for near extremal Dp-branes, which would be undesirable. It is possible that our approach may be a first order approximation to the underlying physics, and perhaps probing the geometry by geodesics can go so far; one may need to study the full quantum field theory in a given background geometry (or for that matter the full string theory) to decode renormalization group data from gravitational physics. On the other hand, the principle of covariant holography accords an attractive special physical role to null geodesics.[6] We certainly expect corrections of string theoretical origin as the geometrical description starts to break down. However, within energy scales where the low energy gravity sector is a good approximation, a fundamental quantity like the c function may be expected to have a simple geometrical representation such as Eq. (51). This is in the spirit of the frugal statement that relates the entropy of a black hole with the area of its horizon. We believe that we have presented enough evidence to make the proposal worthy of further investigation.

One of the most attractive aspects of Eq. (51) is the fact that it is in practice easy to computationally handle. It can readily be applied to a myriad of geometries, tested, as well as used to understand the nature of certain ill-understood dual theories (such as five-branes). There are also more stringent tests that the expression can be subjected to: in particular, an understanding of the relation between the first order Callan-Symanzik equations and the second order Einstein equations is of direct relevance to this proposal. Work in this direction is in progress.

We stated in the beginning of the first section the condition that the gravitational vacuum under consideration should solve Einstein's equations in the presence of a negative cosmological constant. From the string theory side, we know that there exists an energy regime that screens out regions of spacetime that are not candidates for holography. This typically leads to focusing on the near-horizon geometries of Dp-branes, which are conformal to AdS spaces. On the side of the boundary theory, fixed points play a fundamental role in defining renormalization group flows. These special points indeed correspond to AdS spaces. It is in this light that we are motivated to state that holography, in general, and the formalism we presented, in particular, need to be thought of in the context of a gravitational theory with a negative cosmological constant. This line of thought rules out extending these ideas to flat Minkowski space. It would be interesting, however, to explore this approach in scenarios where the spacetime does not admit a timelike Killing vector field.

An important issue that we have not been able to address properly is a covariant formulation of the UV-IR correspondence. This issue is related to an understanding of the process of coarse graining as seen by the gravity side. A possible picture for this was presented in [9]. On the other hand, it is tempting to believe that the gravitational vacuum that solves Einstein's equations reflects the state of the dual theory at all energy scales; locally, foliations are snapshots of the theory at different energy scales. The metric and its first derivative on $\mathcal{M}$ (essentially the content of the congruence data) encode all the necessary information about the theory at a given scale. From the difficulty we are having in formulating a covariant UV-IR correspondence, it appears that this line of thought may be only part of the whole picture.

Finally, certain simplifying assumptions were made in the text to arrive at leaner conclusions and to focus on the relevant physics. The assumption of shearless flow, however, may hold rich physics. Even though it may appear straightforward to generalize the approach to this case, there are subtleties which we do not understand in this context. Flow with shear is in particular a characteristic of boundary theories coupled to background gravity. Our understanding of the effect of this using the renormalization prescription is limited.

If a fundamental relation between renormalization group flow and geometry exists, it should be possible to find a geometrical interpretation for every object in the renormalization group prescription. We hope to have stimulated further investigation in this direction.

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[6]Null geodesics were also used in a similar approach in the work of [34].

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