Article

Holomorphic anomaly and quantum mechanics

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Abstract

We show that the all-orders WKB periods of one-dimensional quantum mechanical oscillators are governed by the refined holomorphic anomaly equations of topological string theory. We analyze in detail the double-well potential and the cubic and quartic oscillators, and we calculate the WKB expansion of their quantum free energies by using the direct integration of the anomaly equations. We reproduce in this way all known results about the quantum periods of these models, which we express in terms of modular forms on the WKB curve. As an application of our results, we study the large order behavior of the WKB expansion in the case of the double well, which displays the double factorial growth typical of string theory.

Reference

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Holomorphic Anomaly and Quantum Mechanics

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ABSTRACT: We show that the all-orders WKB periods of one-dimensional quantum mechanical oscillators are governed by the refined holomorphic anomaly equations of topological string theory. We analyze in detail the double-well potential and the cubic and quartic oscillators, and we calculate the WKB expansion of their quantum free energies by using the direct integration of the anomaly equations. We reproduce in this way all known results about the quantum periods of these models, which we express in terms of modular forms on the WKB curve. As an application of our results, we study the large order behavior of the WKB expansion in the case of the double well, which displays the double factorial growth typical of string theory.

Contents

1 Introduction

In recent years, and through different perspectives [1–3], it has become clear that topological strings on local Calabi–Yau geometries are closely related to quantum mechanical problems and to integrable systems. This has led to a fruitful interplay between topological string theory (and its close cousin, $\mathcal{N} = 2$ gauge theory in 4d) and spectral problems, in which the results in one field have shed unexpected light on the other. For example, the so-called Nekrasov–Shatashvili (NS) limit of $\mathcal{N} = 2$ gauge theories is equivalent to the all-orders WKB quantization of certain quantum integrable systems. By using the Nekrasov instanton partition function [4], one can then obtain new, exact quantization conditions for these systems. These include simple quantum mechanical models, like the modified Mathieu equation (see for example [5] for a detailed study of this case). Studies of the connections between Quantum Mechanics and refined topological string/gauge theory have been made in for example [6–13].

In this paper, we suggest a more general implication of topological string theory for Quantum Mechanics. We claim that the refined holomorphic anomaly equations of [14, 15], characterizing the refined topological string free energies, govern the all-order WKB periods of generic one-dimensional quantum mechanical systems.

Let us recall that the holomorphic anomaly equations were first introduced in [16] to describe the non-holomorphic dependence of standard topological string amplitudes. It turns out that these equations determine these amplitudes to a large extent. In some local CY geometries, the holomorphic anomaly equations, combined with modularity and appropriate boundary conditions, can be used to compute recursively (and efficiently) the topological string free energies at all genera [17, 18]. Since topological strings are closely related to many other physical systems, the holomorphic anomaly equations have provided very powerful tools to study a variety of problems. For example, the large N expansion of matrix models is also governed by the holomorphic anomaly [17, 19, 20]. This has made possible to obtain the all-genus free energy of some matrix models and/or gauge theories (see [21, 22] for some relevant examples.)

The holomorphic anomaly equations of [16] can be extended to refined topological string theory and to $\mathcal{N} = 2$ gauge theories [14, 15]. As we noted above, the NS limit of certain gauge theories is closely related to quantum mechanical problems, like the modified Mathieu

equation. Therefore, one can use the holomorphic anomaly equations to analyze the all-orders WKB expansion of these problems. In this paper we argue that this connection is more general, i.e. we argue that, given a general quantum mechanical problem in one dimension, not necessarily related to supersymmetric gauge theories or topological strings, its all-orders WKB expansion is captured by the refined holomorphic anomaly equations in the NS limit. These equations can be in fact formulated for any algebraic curve obtained from the classical potential: they only require the special geometry determined by the classical periods, and the first quantum correction to the free energy, which can be expressed in terms of the discriminant of the curve [14, 15].

In order to test our claim, we consider the most famous one-dimensional models in Quantum Mechanics: the cubic oscillator, the symmetric double-well, and the quartic oscillator. The modern, non-perturbative treatment of these models started with the work of Voros [23, 24] and Zinn–Justin [25] in the early 1980s. In those papers, a series of "exact" quantization conditions were found, involving the Borel-resummed all-orders WKB periods of [26], as well as explicit non-perturbative corrections. These exact quantization conditions can be also derived in the framework of the uniform WKB method [27–31]. In this paper, we explicitly show that the refined holomorphic anomaly equations, combined with the direct integration method of [17, 32], determine the non-perturbative quantum period appearing in the quantization conditions, as a function of the perturbative quantum period. In other words, the refined holomorphic anomaly equations govern the all-orders WKB periods for one-dimensional oscillators, at least when the underlying geometry has genus one[1]. As a spinoff, we calculate the quantum free energies of these models explicitly and recursively in terms of modular forms on the WKB curve. This provides a new and efficient method to calculate the all-orders WKB expansion of the non-perturbative A function appearing in the quantization conditions of [25, 34–36], in terms of modular forms.

As a simple application of our results, we study the large order behavior of the WKB expansion for the energy of the symmetric double well. The standard perturbative series in the coupling constant for the ground state energy has been analyzed in much detail in for example [34, 35, 37]. This is a numerical series whose coefficients grow like $n!$, and its large order behavior is controlled by the two-instanton correction. The WKB expansion, in contrast, is a series in $\hbar^2$. Its coefficients are functions of the modulus ν (which encodes the quantum number), and they grow like $(2n)!$. This is of course the typical growth of the genus expansion in string theory and of the $1/N$ expansion [38–40]. The large order behavior of this series is still controlled by the two-instanton correction, and we derive and test a precise formula for the growth of the coefficients up to, and including, the term of order $1/n$.

This paper is organized as follows. In section 2 we review some basic facts about the all-orders WKB method and the refined holomorphic anomaly equations, and we present our main conjecture connecting both approaches. We also give some details about the direct integration of the anomaly equations. Section 3 illustrates our general considerations with a detailed analysis of the cubic oscillator and the double well. The quartic oscillator can be also obtained by an S-duality transformation of the double-well. In section 4 we present the analysis of the large order behavior of the WKB expansion in the example of the double well. Finally, in section 5 we present our conclusions and list some open problems for future research.

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[1]In the case of the double well, the connection with the refined holomorphic anomaly could have been anticipated by using the relation to the cubic matrix model found in [9], together with the results of [33]. However, our claim is more general and does not depend on a matrix model intermediary. For example, the cubic oscillator does not have any known realization as a β-deformed matrix model, yet it satisfies the refined holomorphic anomaly equation, as we show in this paper.

2   The all-orders WKB method and the holomorphic anomaly

In this section we review the all-orders WKB method of Dunham [26] (see for example [41] for a clear textbook presentation). We start with the Schrödinger equation for a one-dimensional particle in a potential $V(x)$,

$$\hbar^2 \psi''(x) + p^2(x)\psi(x) = 0, \qquad p(x) = \sqrt{2(\xi - V(x))}, \tag{2.1}$$

where ξ denotes the energy of the particle, and we have set the mass $m = 1$. The WKB ansatz for the wavefunction is

$$\psi(x) = \exp\left[\frac{i}{\hbar}\int^x Q(x')dx'\right]. \tag{2.2}$$

The Schrödinger equation for $\psi(x)$ becomes a Riccati equation for $Q(x)$,

$$Q^2(x) - i\hbar\frac{dQ(x)}{dx} = p^2(x), \tag{2.3}$$

which can be solved in power series in $\hbar$:

$$Q(x) = \sum_{k=0}^{\infty} Q_k(x)\hbar^k. \tag{2.4}$$

The functions $Q_k(x)$ can be computed recursively as

$$Q_0(x) = p(x),$$
$$Q_{n+1}(x) = \frac{1}{2Q_0(x)}\left(i\frac{dQ_n(x)}{dx} - \sum_{k=1}^{n} Q_k(x)Q_{n+1-k}(x)\right). \tag{2.5}$$

If we split the formal power series in (2.4) into even and odd powers of $\hbar$,

$$Q(x) = Q_{\text{odd}}(x) + P(x), \tag{2.6}$$

one finds that $Q_{\text{odd}}(x)$ is a total derivative,

$$Q_{\text{odd}}(x) = \frac{i\hbar}{2}\frac{P'(x)}{P(x)} = \frac{i\hbar}{2}\frac{d}{dx}\log P(x). \tag{2.7}$$

It follows that only Q_1 contributes to period integrals of $Q(x)$. In addition, the wavefunction (2.2) can be written as

$$\psi(x) = \frac{1}{\sqrt{P(x)}}\exp\left[\frac{i}{\hbar}\int^x P(x')dx'\right]. \tag{2.8}$$

Let us now consider the Riemann surface Σ defined by

$$y^2 = p^2(x). \tag{2.9}$$

The turning points of the WKB problem are the points where $p^2(x) = 0$, and correspond to the branch points of the curve (2.9). We will assume that turning points are simple. We will also restrict ourselves to curves of genus one, in which there are only two different, independent one-cycles A and B encircling turning points. The A-cycle corresponds to an allowed region for

the classical motion, while the B-cycle corresponds to a forbidden region. We can now consider the periods of the one-form $y(x)\mathrm{d}x$ along these cycles,

$$t = \frac{1}{2\pi} \oint_A y(x)\mathrm{d}x, \qquad t_D = -\mathrm{i} \oint_B y(x)\mathrm{d}x, \tag{2.10}$$

with appropriate choices of branch cuts for the function $y(x)$. We will refer to these as *classical periods*. They depend on the energy ξ and on the parameters of the potential. We note that the A-cycle corresponds to the "perturbative" cycle, while the B-cycle corresponds to the "tunneling cycle", and it typically encodes non-perturbative corrections. As in (local) mirror symmetry, it is useful to introduce a prepotential or classical free energy $F_0(t)$ by the equation

$$t_D = \frac{\partial F_0}{\partial t}. \tag{2.11}$$

We should regard t as a flat coordinate parametrizing the complex structure of the curve (2.9). The inverse function expressing ξ as a function of the classical A-period, which we will denote as $\xi_0(\nu)$, will be called the *classical mirror map*. This function is nothing but the classical Birkhoff series expressing the energy ξ as a function of the classical action ν (see [42, 43]).

The all-orders WKB method allows one to define "quantum" versions of the classical periods, by simply using the formal power series for $P(x)$,

$$P(x) = \sum_{n \geq 0} Q_{2n}(x)\hbar^{2n}. \tag{2.12}$$

In this way, we can promote the classical periods to "quantum" periods,

$$\nu = \frac{1}{2\pi\mathrm{i}} \oint_A P(x)\mathrm{d}x, \qquad \frac{\partial F}{\partial \nu} = -\mathrm{i} \oint_B P(x)\mathrm{d}x. \tag{2.13}$$

Note that both of them are defined by formal power series expansions in $\hbar^2$,

$$\nu = \sum_{n \geq 0} t^{(n)}\hbar^{2n}, \qquad \frac{\partial F}{\partial \nu} = \sum_{n \geq 0} t_D^{(n)}\hbar^{2n}, \tag{2.14}$$

and the leading order term of this expansion, which is obtained as $\hbar \to 0$, gives back the classical periods.

The quantum A-period defines what we will call, following [7], the *quantum mirror map*:

$$\xi(\nu) = \sum_{n \geq 0} \xi_n(\nu)\hbar^{2n}. \tag{2.15}$$

Note that the first term of (2.15), $\xi_0(\nu)$, is the classical mirror map. In the context of one-dimensional quantum mechanical oscillators, with a coupling constant g, the quantum mirror map was studied in detail in [42, 43], as a formal power series of quantum corrections to the classical Birkhoff series $\xi_0(\nu)$. It is closely related to the stationary perturbation series for the energy. Indeed, let us set

$$\nu = \hbar\left(m + \frac{1}{2}\right), \tag{2.16}$$

where m is the quantum number for the energy level. If we now expand each $\xi_n(\nu)$ as a power series in the coupling constant g, and we rearrange the terms in (2.15) in powers of g, we recover from $\xi(\nu)$ the Rayleigh–Schrödinger series.

The quantum B-period defines the *quantum free energy* as function of the full quantum period ν,

$$F(\nu) = \sum_{n \geq 0} F_n(\nu) \hbar^{2n}. \tag{2.17}$$

This quantity is the analog of the NS free energy in supersymmetric gauge theories and topological strings. It is then natural to conjecture that it is governed by the same set of equations, namely, the refined holomorphic anomaly equations of [14, 15] (see [44, 45] for further developments.) Let us briefly review these equations.

We consider the B-model refined topological string on a local Calabi–Yau manifold described by a Riemann surface Σ. The basic perturbative quantities are the free energies $F^{(g_1,g_2)}(t_i)$, with $g_1, g_2 \geq 0$ (see for example [15, 46] for a summary and a list of references). The standard topological string free energy corresponds to $g_1 = 0$, while the NS limit corresponds to $g_2 = 0$. We will denote

$$F^{(n,0)} = F_n. \tag{2.18}$$

Note that F_g usually denotes the standard topological string free energies $F^{(0,g)}$, but these will not play any rôle in this paper. The refined free energies can be promoted to non-holomorphic functions of the moduli t_i and their complex conjugates $\bar{t}_i$, $F^{(g_1,g_2)}(t_i, \bar{t}_i)$. The refined holomorphic anomaly equations govern the anti-holomorphic dependence of these functions, and they read,

$$\frac{\partial F^{(g_1,g_2)}}{\partial \bar{t}_k} = \frac{1}{2\gamma} \overline{C}_{\bar{k}}^{lm} \left(D_l D_m F^{(g_1,g_2-1)} + \sum_{0 < r_1 + r_2 < g_1 + g_2} D_l F^{(r_1,r_2)} D_m F^{(g_1-r_1,g_2-r_2)} \right). \tag{2.19}$$

We have introduced a constant γ to take into account different normalizations for the free energies. These equations are valid for $g_1 + g_2 \geq 2$. They involve the metric $G_{k\bar{m}}$ on the moduli space of complex structures, as well as the corresponding covariant derivatives D_i. C_{ijk} is the Yukawa coupling

$$C_{ijk} = \frac{\partial^3 F_0}{\partial t_i \partial t_j \partial t_k}, \tag{2.20}$$

and

$$\overline{C}_{\bar{k}}^{lm} = G^{l\bar{p}} G^{m\bar{n}} \overline{C}_{\bar{p}\bar{n}\bar{k}}. \tag{2.21}$$

If we denote by τ the period matrix of Σ, the metric is given by

$$G_{i\bar{j}} = -\mathrm{i}\pi \left(\tau - \overline{\tau} \right)_{ij}. \tag{2.22}$$

In the NS limit, the first term in the r.h.s. of (2.19) drops out, and we obtain the simplified equations,

$$\frac{\partial F_n}{\partial \bar{t}_k} = \frac{1}{2\gamma} \overline{C}_{\bar{k}}^{lm} \sum_{r=1}^{n-1} D_l F_r D_m F_{n-r}, \qquad n \geq 2. \tag{2.23}$$

The equations (2.23) have to be supplemented with an explicit expression for F_1. It turns out that [14, 15]

$$F_1 = -\frac{1}{\alpha} \log \Delta, \tag{2.24}$$

where Δ is the discriminant of the curve Σ, and α is a real number (in the examples considered in this paper, $\alpha = 24$). Using (2.24) as the initial datum, as well as the special geometry of the moduli space, the holomorphic anomaly equation (2.23) determines the functions F_n recursively,

up to a purely holomorphic dependence on the moduli which is usually called the *holomorphic ambiguity*.

It is very important to note that, in the holomorphic anomaly equations, the functions F_n are obtained as functions of the moduli of the curve, and in order to express them in terms of the period ν one uses the *classical* mirror map $\xi_0(\nu)$, i.e. the inverse of the *classical* period $t = t(\xi)$.

It was noted in [17, 32] and subsequent works that the holomorphic anomaly equation can be solved in a very efficient way by using the method of "direct integration." In this method, one exploits the symmetries of the problem and re-expresses all quantities in terms of modular forms of the period matrix τ. The method is most efficient for elliptic curves, although it can be successfully used as well for curves of higher genus [47]. For genus one curves, with a single geometric modulus, there is a single Yukawa coupling

$$Y = C_{ttt}. \tag{2.25}$$

The non-holomorphic dependence occurs through the so-called propagator S^{tt}, which is defined by

$$\overline{C}_{\bar{t}}^{tt} = \bar{\partial}_{\bar{t}} S^{tt}. \tag{2.26}$$

In all known models with curves of genus one, the propagator can be written as

$$S^{tt} = -\frac{1}{3\beta} \widehat{E}_2(\tau, \bar{\tau}), \tag{2.27}$$

where β is an appropriate proportionality constant, $\widehat{E}_2(\tau, \bar{\tau})$ is the non-holomorphic modular form

$$\widehat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\,\tau}, \tag{2.28}$$

and $E_2(\tau)$ is the weight two Eisenstein series. The elliptic modulus τ is related to the prepotential by the equation,

$$\tau = \frac{\beta}{2\pi \mathrm{i}} \frac{\partial^2 F_0}{\partial t^2}. \tag{2.29}$$

The covariant derivative D_t can then be written as

$$D_t = \beta Y D_\tau, \tag{2.30}$$

where D_τ is the Maass derivative acting on (almost-holomorphic) modular forms of weight k:

$$D_\tau = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{k}{4\pi \mathrm{Im}\tau}. \tag{2.31}$$

Putting all these ingredients together, one can write the anomaly equation in the form

$$\frac{\partial F_n}{\partial \widehat{E}_2} = -\frac{\beta}{6\gamma} Y^2 \sum_{r=1}^{n-1} D_\tau F_r D_\tau F_{n-r}, \qquad n \geq 2. \tag{2.32}$$

This can be integrated as

$$F_n(\tau, \bar{\tau}) = \sum_{\ell=1}^{2n-3} c_\ell^{(n)}(\tau) \widehat{E}_2^\ell(\tau, \bar{\tau}) + f_n(\tau). \tag{2.33}$$

The free energies $F_n(\tau, \bar{\tau})$ are non-holomorphic modular forms of weight zero w.r.t. the appropriate monodromy group of the elliptic curve Σ. Therefore, in (2.33), $c_\ell^{(n)}(\tau)$ are modular forms of weight -2ℓ, which are completely fixed by the holomorphic anomaly equations. The original holomorphic amplitudes are obtained by simply taking $\bar{\tau} \to \infty$, keeping τ fixed. This amounts to replacing

$$\widehat{E}_2(\tau, \bar{\tau}) \to E_2(\tau) \tag{2.34}$$

in (2.33).

The function $f_n(\tau)$ appearing in (2.33) is the holomorphic ambiguity. In order to fix it, one first finds an appropriate parametrization for it in terms of modular forms. This parametrization is determined by the monodromy group of Σ, and by the structure of the Yukawa coupling. Once this parametrization has been found, fixing the ambiguity reduces to determining a finite set of coefficients at each order n. To calculate these, we have to impose boundary conditions at special points in the moduli space. In the examples involving quantum mechanical models, we will look at singular points of the curve, i.e. at points where the discriminant of the curve vanishes. One of these points, which we will call the *conifold* point, corresponds to $\nu = 0$. Expanding around $\nu = 0$ is equivalent to performing a perturbative expansion in the coupling constant. Near this point, the holomorphic free energies $F_n(\nu)$ are the sum of a singular part and of a regular part. We will write,

$$F_n(\nu) = F_n^{\mathrm{s}}(\nu) + F_n^{\mathrm{r}}(\nu). \tag{2.35}$$

The singular part has the following structure. For $n = 0, 1$, it is of the form,

$$\begin{aligned}
F_0^{\mathrm{s}}(\nu) &= k_0 \frac{\nu^2}{2}\left(\log\left(\frac{\nu}{\nu_0}\right) - \frac{3}{2}\right), \\
F_1^{\mathrm{s}}(\nu) &= -\frac{k_1}{24}\log(\nu),
\end{aligned} \tag{2.36}$$

while for $n \geq 2$ there is a pole or order $2n - 2$,

$$F_n^{\mathrm{s}}(\nu) = k_n \frac{(1 - 2^{1-2n})(2n-3)!B_{2n}}{(2n)!}\nu^{2-2n}, \qquad n \geq 2. \tag{2.37}$$

In (2.36) and (2.37), k_n is of the form $r_1 r_2^n$, where $r_{1,2}$ are constants, and takes into account possible overall normalizations of the free energies and the parameter ν. We will fix these normalizations, and the value of k_n, in a case by case basis in the examples below. In the first line of (2.36), ν_0 is a constant which depends on the model. In (2.37), B_{2n} are Bernoulli numbers. On the other hand,

$$F_n^{\mathrm{r}}(\nu) = \mathcal{O}(1), \qquad n \geq 0, \tag{2.38}$$

is regular at $\nu = 0$. The behavior (2.37), (2.38) is called the *gap condition* at the conifold point. It was first discovered in [17], in the study of standard topological string amplitudes and of gravitational corrections to Seiberg–Witten theory. It was generalized in [15, 44, 46] to the refined case. The behavior (2.37) imposes $2n - 2$ conditions on the holomorphic ambiguity, but it does not fix it completely. Additional conditions are obtained by looking at other singular points of the curve. There is typically a dual conifold point, where a dual period ν_D, obtained by a symplectic trasnsformation of ν, vanishes. One then looks at the quantum free energies in the symplectic frame appropriate for the dual conifold point, F_n^D, and studies their behavior near $\nu_D = 0$. It turns out that the dual free energies satisfy a dual gap condition, of the form

$$F_n^D(\nu_D) = k_n^D \frac{(1 - 2^{1-2n})(2n-3)!B_{2n}}{(2n)!}\nu_D^{2-2n} + \mathcal{O}(1), \tag{2.39}$$

where k_n^D is another normalization constant. Very often, the combination of (2.37) and (2.39) fixes completely the holomorphic ambiguity. This will be also the case in models arising in one-dimensional quantum oscillators, where the gap condition (2.37) turns out to be equivalent to a well-known regularity property of the all-orders WKB periods.

In the next section we will set up and solve the anomaly equations (2.23) for various curves Σ arising in one-dimensional Quantum Mechanics, of the form (2.9). This gives the quantum free energies $F_n(\nu)$, which we will write as quasi-modular forms on the curve Σ. In order to determine the two quantum periods appearing in the WKB method, we need, in addition to the quantum free energy, the relation between ν and ξ. It turns out that the quantum mirror map (2.15) can be obtained from the quantum free energy by using what we will call the *PNP relations* between the perturbative and the tunneling cycles. The PNP relations for one-dimensional quantum mechanical oscillators were first observed by Álvarez and Casares in [29] for the cubic oscillator, generalizing a result in [48] for the Stark effect. They were established empirically for the quartic oscillator and the symmetric double well in [30] and [31], respectively, and extended recently to the periodic cosine potential and other one-dimensional models in [49]. Originally, the PNP relations for one-dimensional oscillators were formulated as follows. Let us consider the function

$$A(\nu) = \frac{\mathrm{d}F^{\mathrm{r}}}{\mathrm{d}\nu} = \sum_{n \geq 0} \frac{\mathrm{d}F_n^{\mathrm{r}}}{\mathrm{d}\nu}, \tag{2.40}$$

where we set $\hbar = 1$. Then, the derivative of the quantum mirror map and the A function are related as,

$$\frac{\mathrm{d}\xi}{\mathrm{d}\nu} = -\frac{g^2}{\sigma}\left(C\nu + g^2 \frac{\partial A}{\partial g^2}\right), \tag{2.41}$$

where σ, C are constants, and g is the coupling constant of the Quantum Mechanical problem. Since this equation relates the tunneling period to the perturbative period, it provides a link between perturbative and non-perturbative physics. Relations like (2.41) are also well-known for the NS limit of supersymmetric gauge theories, where they allow to compute the quantum mirror map from the quantum free energy. Derivations of the relationship (2.41) in some particular models can be found in [11, 50]. As we will see in examples, by using the full quantum free energy, one can integrate the PNP relation to obtain,

$$\xi(\nu) = -\varphi - \frac{1}{\sigma}g^4 \frac{\partial F(\nu)}{\partial g^2}, \tag{2.42}$$

where φ is an integration constant, independent of ν and g. Using this relation, together with our explicit expressions for $F_n(\nu)$, we find explicit formulae for the $\xi_n(\nu)$ in (2.15) with $n \geq 1$ in terms of modular forms.

We then conjecture that the all-orders WKB periods of one-dimensional quantum systems satisfy the holomorphic anomaly equations (2.23), and in the next sections we will test this conjecture in some non-trivial examples. It would be of course interesting to prove this conjecture rigorously. One possible strategy would be to adapt to our context recent results of Alba Grassi for topological strings in [51], one of which is the following. Let us assume that the quantum periods transform as the classical periods under a symplectic transformation. Then, one can introduce a non-holomorphic dependence on the quantum free energy, in such a way that the resulting non-holomorphic objects satisfy (2.23). Although these results were obtained in the context of topological string theory, they can be adapted to the quantum mechanical context of this paper. An important step in such a proof would then be to justify rigorously, in the
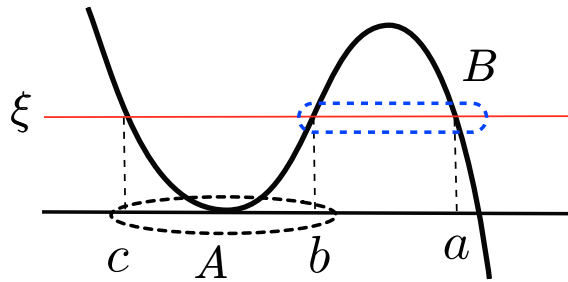
Figure 1. The cubic oscillator.

quantum mechanical case, the assumptions of [51] concerning the transformation properties of the quantum periods. This would follow, in turn, if one could express the quantum periods as differential operators acting on the classical periods. This can be shown to be the case in many examples and for low orders of the $\hbar$ expansion (see e. g. the calculations in related examples in [45]), but a general proof is lacking.

3 Examples

3.1 The cubic oscillator

The cubic oscillator is a one-dimensional quantum particle in the cubic potential

$$V(x) = \frac{x^2}{2} - gx^3. \tag{3.1}$$

This is a very much studied example in Quantum Mechanics (see for example [41] for a textbook presentation). This potential does not support bound states, but by imposing Gamow–Siegert boundary conditions, one finds an infinite tower of resonances which can be computed numerically by using complex dilatation techniques [52, 53]. The resonant energies can be also obtained by Borel resummation of the stationary perturbative series in the coupling g [29, 53–55]. In the all-orders WKB approach, one can derive an exact quantization condition involving two different periods. To write down this condition, we denote the turning points of the potential by

$$c < b < a, \tag{3.2}$$

as shown in Fig. 1. The A-cycle goes around the points c and b, while the B-cycle goes around the points b and a. The exact quantization condition can be obtained in a simple way by using the Voros–Silverstone connection formulae [23, 24, 56] (this is essentially the method followed in [57]). Alternatively, it can be derived from the uniform WKB method [28, 29]. Since the resulting series is not Borel summable, there are two different quantization conditions, depending on the choice of lateral resummation. One choice gives a perturbative quantization condition,

$$\oint_A P(x)\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{Z}_{\geq 0}, \tag{3.3}$$

while the other choice gives

$$1 + \exp\left(\frac{\mathrm{i}}{\hbar}\oint_A P(x)\mathrm{d}x\right) + \exp\left(\frac{\mathrm{i}}{\hbar}\oint_B P(x)\mathrm{d}x\right) = 0. \tag{3.4}$$

This quantization condition reduces to the previous one when the "tunneling" integral over the B cycle is neglected. The tunneling cycle gives a non-perturbative, exponentially small correction to the perturbative result. The perturbative and the tunneling periods, regarded as quantum A and B periods, define a quantum free energy $F(\nu)$ as

$$\frac{\partial F(\nu)}{\partial \nu} = -\mathrm{i} \oint_B P(x)\mathrm{d}x. \tag{3.5}$$

The Voros multiplier associated to the tunneling period, and defined as

$$f(\nu) = \exp\left(\frac{\mathrm{i}}{\hbar} \oint_B P(x)\mathrm{d}x\right), \tag{3.6}$$

has the following structure (we set $\hbar = 1$)

$$f(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + \nu\right)} \left(\frac{8}{g^2}\right)^{\nu} \mathrm{e}^{-A(\nu)}. \tag{3.7}$$

Here, $A(\nu)$ is the function defined in (2.40) and is given by the power series in g,

$$A(\nu) = \frac{2}{15g^2} + \sum_{k=1}^{\infty} c^{(k)}(\nu) g^{2k}. \tag{3.8}$$

This structure can be obtained by using the uniform WKB method, as in [28, 29], or the techniques of [57]. The coefficients $c^{(k)}$ in (3.8) turn out to be polynomials in ν with rational coefficients. They can be calculated explicitly by using the asymptotic matching of WKB expansions involving Airy and parabolic cylinder functions, but their calculation is involved and time-consuming. At the very first orders, one finds

$$\begin{aligned} c^{(1)}(\nu) &= \frac{141\nu^2}{8} + \frac{77}{32}, \\ c^{(2)}(\nu) &= \frac{7717\nu^3}{32} + \frac{13937\nu}{128}, \\ c^{(3)}(\nu) &= \frac{2663129\nu^4}{512} + \frac{5153379\nu^2}{1024} + \frac{43147783}{122880}. \end{aligned} \tag{3.9}$$

A more comprehensive list can be found in [58]. One advantage of the method developed in [28, 29] is that the tunneling period is expressed directly in terms of the quantum period ν. The tunneling period was also studied in [36], following the ideas developed previously in [25, 34, 35]. In [36], the function $A(\nu)$ defined in (3.8) was computed as a function of the energy ξ, rather than the quantum period[2]. The calculation of $A(\nu)$ in [36] is based on a direct evaluation of the period integrals, order by order in $\hbar$, and it is quite involved[3]. We will see in this section that the holomorphic anomaly equation gives an extremely powerful method to compute the function $A(\nu)$ at arbitrary order, in terms of modular forms.

---

[2]The A function of the cubic oscillator obtained in [36] was reexpressed in terms of ν in [59], and the result can be seen to agree with the results obtained previously by Álvarez and Casares in [28, 29].

[3]The convoluted integration method developed in [34–36] to calculate the periods and their quantum corrections is not really needed in the case of curves of genus one. All the integrals appearing in the cubic oscillator and other genus one examples can be computed in closed form in terms of elliptic integrals of the first and second kind.

The elliptic curve underlying the cubic oscillator is given by

$$y^2 = 2\xi - x^2 + 2gx^3. \tag{3.10}$$

It is convenient to rescale the variables so as to absorb the g dependence:

$$x \to x/g, \quad y \to y/g, \quad \xi \to \xi/g^2, \tag{3.11}$$

and the new curve reads

$$y^2 = 2\xi - x^2 + 2x^3. \tag{3.12}$$

The classical periods can be written as

$$t = \frac{1}{\pi} \int_c^b y(x) \mathrm{d}x, \qquad t_D = -2\mathrm{i} \int_b^a y(x) \mathrm{d}x. \tag{3.13}$$

One can easily check that they satisfy the Picard–Fuchs equation

$$\left[\frac{15}{2} + \Delta(\xi) \frac{\partial^2}{\partial \xi^2} \right] \Pi = 0, \tag{3.14}$$

where

$$\Delta(\xi) = \xi \left(54\xi - 1 \right) \tag{3.15}$$

is the discriminant of the elliptic curve (3.10) for $g = 1$. From the Picard–Fuchs equation, we find that the period t has an expansion around $\xi = 0$ of the form,

$$t = \sum_{n=1}^{\infty} c_n \xi^n, \tag{3.16}$$

where the coefficients satisfy the recursion relation,

$$c_n = \frac{3(6n-11)(6n-7)}{2(n-1)n} c_{n-1}, \qquad n \geq 2, \qquad c_1 = 1. \tag{3.17}$$

It is also possible to write t and t_D in terms of elliptic integrals of the first and second kind:

$$t = \frac{2\sqrt{2}}{\pi} \frac{(b-c)^2 \sqrt{a-c}}{15k^4} \left((k')^2(k^2-2)K(k) + 2(k^4 + (k')^2)E(k) \right),$$

$$t_D = 4\sqrt{2} \frac{(a-b)^2(b-c)}{\sqrt{a-c}} \frac{2(1 + k^2(k^2-1))E(k') - k^2(1+k^2)K(k')}{15k^2(k^2-1)^2}, \tag{3.18}$$

where the elliptic modulus k and its complementary k' are given by

$$k^2 = \frac{b-c}{a-c}, \qquad (k')^2 = 1 - k^2 = \frac{a-b}{a-c}. \tag{3.19}$$

An equivalent expression for the classical A period in terms of hypergeometric functions was obtained in [43]. In addition, the derivatives of the periods can be written in closed form as,

$$\frac{\partial t}{\partial \xi} = \frac{1}{\pi} \sqrt{\frac{2}{a-c}} K(k), \qquad \frac{\partial t_D}{\partial \xi} = 2\sqrt{\frac{2}{a-c}} K'(k), \tag{3.20}$$

where as usual $K'(k) = K(k')$. Finally, the tau parameter is

$$\tau = \mathrm{i}\frac{K'(k)}{K(k)} = \frac{1}{2\pi\mathrm{i}}\frac{\partial t_D}{\partial t}. \tag{3.21}$$

The classical periods have the following expansion around $\xi = 0$:

$$
\begin{aligned}
t =& \xi + \frac{15\xi^2}{4} + \frac{1155\xi^3}{16} + \frac{255255\xi^4}{128} + \cdots, \\
t_D =& \frac{2}{15} + \xi\left(\log\left(\frac{\xi}{8}\right) - 1\right) + \frac{3\xi^2}{8}\left(10\log\left(\frac{\xi}{8}\right) + 47\right) + \cdots.
\end{aligned}
\tag{3.22}
$$

Since an expansion in ξ is an expansion in g, this makes contact with the perturbative series in g. Using these results and the definition (2.11), we can compute the prepotential $F_0(t)$ in power series in t, around $t = 0$, and one finds,

$$F_0(t) = \frac{t^2}{2}\left(\log\left(\frac{t}{8}\right) - \frac{3}{2}\right) + \frac{2t}{15} + \frac{47t^3}{8} + \frac{7717t^4}{128} + \frac{2663129t^5}{2560} + \cdots. \tag{3.23}$$

Note that this has precisely the structure of a prepotential near a conifold point.

By looking at the discriminant of the curve, we find that there are two singular points. The first one is the conifold point

$$\xi = 0. \tag{3.24}$$

The second one is the dual conifold point,

$$\xi = \frac{1}{54}. \tag{3.25}$$

This is the critical value in which the energy ξ has the same magnitude as the height of the cubic barrier. It also corresponds to the value of the modulus $k^2 = 1$ in (3.19), so it sets the radius of convergence for the expansion (3.16). To formulate the dual problem, we introduce a dual energy

$$\xi_D = \frac{1}{54} - \xi, \tag{3.26}$$

such that $\xi_D = 0$ is the dual conifold point. From the Picard–Fuchs equation, one finds that exchanging ξ with ξ_D corresponds to exchanging t with t_D. In particular, the dual period t_D has a regular expansion in terms of ξ_D:

$$\frac{t_D}{2\pi} := \tilde{t}_D = \xi_D + \frac{15\xi_D^2}{4} + \frac{1155\xi_D^3}{16} + \frac{255255\xi_D^4}{128} + \mathcal{O}\left(\xi_D^5\right), \tag{3.27}$$

which is identical to the expansion of t in terms of ξ. Physically, the dual situation corresponds to inverting the cubic potential: the top of the barrier defines now the origin of energy, as in (3.26), and the tunneling and perturbative cycles are exchanged. In terms of the elliptic modulus of the curve, this is precisely an S-duality transformation,

$$\tau_D = -\frac{1}{\tau}. \tag{3.28}$$

The dual theory will play a rôle in the discussion of the boundary conditions for the holomorphic ambiguity.

As we explained in the previous section, quantum corrections to t, t_D can be computed explicitly by using the all-orders WKB method. For example, in order to compute the free energy at next-to-leading order, we note that

$$\frac{\mathrm{d}F_1}{\mathrm{d}\nu} = t_D^{(1)}\left(\xi_0(\nu)\right) - \pi\mathrm{i}\tau\left(\xi_0(\nu)\right)t^{(1)}\left(\xi_0(\nu)\right). \tag{3.29}$$

After integration, one finds that, up to an additive constant,

$$F_1(\nu) = -\frac{1}{24}\log(\nu) + \frac{77\nu}{32} + \frac{13937\nu^2}{256} + \frac{1717793\nu^3}{1024} + \frac{240109947\nu^4}{4096} + \mathcal{O}\left(\nu^5\right). \tag{3.30}$$

This turns out to agree with the expansion of

$$F_1(\nu) = -\frac{1}{24}\log\Delta(\xi), \tag{3.31}$$

where $\Delta(\xi)$ is the discriminant (3.15). This is precisely what one expects for the subleading correction to NS free energy in supersymmetric gauge theories and topological strings [14, 15]. This suggests that the higher order corrections to the quantum free energy can be computed with the holomorphic anomaly.

In order to study the holomorphic anomaly equations and its direct integration for the curve (3.10), we need to write the theory in terms of modular forms. The shortest path to this consists in mapping the curve (3.10) to the Seiberg–Witten (SW) form [60]

$$y^2 = (x^2 - 1)(x - u). \tag{3.32}$$

This is achieved by a simple linear change of variables. We choose the map in such a way that the conifold point $\xi = 0$ of (3.10) corresponds to $u = 1$ (the monopole point of the SW curve). One finds that ξ is related to u by

$$\xi = \frac{1}{108g^2}\left\{1 + \frac{u(u^2 - 9)}{(u^2 + 3)^{3/2}}\right\}. \tag{3.33}$$

We now recall that u can be written in terms of theta functions of the modulus τ in (3.21), as follows (see for example [15]),

$$u(\tau) = \frac{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)}{\vartheta_4^4(\tau)}. \tag{3.34}$$

It will be useful to express the modular forms in terms of the generators

$$\begin{aligned} K_2(q) &:= \vartheta_3^4(\tau) + \vartheta_4^4(\tau), \\ K_4(q) &:= \vartheta_2^8(\tau), \end{aligned} \tag{3.35}$$

as in [21]. These are appropriate for the monodromy group $\Gamma(2)$ characterizing the SW curve (3.32). By using all these ingredients, it is easy to derive the following explicit formula for the Yukawa coupling,

$$Y = \frac{16\sqrt{2}\left(K_2^2 + 3K_4\right)^{9/4}}{\left(K_2^2 - K_4\right)^2 K_4}. \tag{3.36}$$

We can now write the holomorphic anomaly equation in the normalization appropriate for our theory. It reads

$$\frac{\partial F_n}{\partial \widehat{E}_2} = -\frac{Y^2}{24} \sum_{r=1}^{n-1} D_\tau F_r \, D_\tau F_{n-r}. \tag{3.37}$$

As usual in the direct integration approach, the most delicate point is to find a good ansazt for the holomorphic ambiguity. As in [21], we will work in the ring of non-holomorphic modular forms generated by K_2, K_4 and $\widehat{E}_2$. Note that this ring is closed under the action of the Maass derivative, since

$$D_\tau K_2 = \frac{1}{12} \left(-K_2^2 + 2\widehat{E}_2 K_2 + 3K_4 \right),$$
$$D_\tau K_4 = \frac{1}{3} \left(K_2 + \widehat{E}_2 \right) K_4, \tag{3.38}$$
$$D_\tau \widehat{E}_2 = \frac{1}{48} \left(-K_2^2 - 3K_4 + 4\widehat{E}_2^2 \right).$$

The appropriate parametrization for the ambiguity turns out to be

$$f_n(\tau) = \frac{\left(K_2^2 + 3K_4 \right)^{\frac{3(n+1)}{2}}}{\left(K_2^2 - K_4 \right)^{4(n-1)} K_4^{2(n-1)}} \sum_{i=0}^{\left\lfloor \frac{9n-15}{2} \right\rfloor} a_i K_2^{9n-15-2i} K_4^i. \tag{3.39}$$

The coefficients a_i are fixed by requiring the expansion of $F_n(\nu)$ near $\nu = 0$ to be of the form (2.35), where the singular part is given by (2.36), (2.37), with $k_n = 1$ for all $n \geq 0$. This is the standard gap condition of [15, 44], specialized to the NS limit of the free energies. In these quantum-mechanical models, this boundary condition can be justified as follows. The singular part of the quantum B-period

$$\frac{\mathrm{d}F^{\mathrm{s}}}{\mathrm{d}\nu} = \sum_{n \geq 0} \frac{\mathrm{d}F_n^{\mathrm{s}}}{\mathrm{d}\nu}, \tag{3.40}$$

where we have set $\hbar = 1$, is precisely the asymptotic expansion of the function

$$\log \left[\frac{\Gamma\left(\frac{1}{2} + \nu \right)}{\sqrt{2\pi} 8^\nu} \right] \tag{3.41}$$

at $\nu \to \infty$ (note that the constant appearing in the first line of (2.36) has the value $\nu_0 = 8$ in this model). But the term inside the brackets is the (inverse) prefactor in (3.7). Therefore, the structure of the B-period, as determined in [28, 57], *implies* the standard gap condition. Note in particular that the function $A(\nu)$ is given by the regular part of the quantum B-period

The boundary condition (2.36) does not fix all the coefficients in the holomorphic ambiguity. We will require in addition that the *dual* function $F_n^D(\tau)$, obtained by an S-duality transformation (3.28) of $F_n(\tau)$, satisfies the same gap condition when expanded in $\nu_D = \tilde{t}_D$. In this problem, it turns out that the general dual expansion involves half-integer powers of $\tilde{t}_D$, which must also be required to vanish. This provides $6n - 6$ conditions to fulfill, so the resulting system for the coefficients a_i is overdetermined. We have found experimentally that it has a unique solution for $2 \leq n \leq 30$. For example, for F_2, we find in this way, after taking the holomorphic limit,

$$F_2 = -\frac{K_2^2 \left(K_2^2 - 9K_4 \right)^2 \left(K_2^2 + 3K_4 \right)^{5/2}}{108 \left(K_2^2 - K_4 \right)^4 K_4^2} E_2 - \frac{79 K_2 \left(K_2^2 - 9K_4 \right) \left(K_2^2 + 3K_4 \right)^{9/2}}{1080 \left(K_2^2 - K_4 \right)^4 K_4^2}. \tag{3.42}$$

The first term is fully determined by the anomaly equation, while the second one is the holomorphic ambiguity. By expanding around $\nu = 0$, we obtain

$$F_2(\nu) = -\frac{7}{5760\nu^2} + \frac{101479}{30720} + \frac{43147783\nu}{122880} + \frac{1769452671\nu^2}{65536} + \frac{1185129116647\nu^3}{655360} + \mathcal{O}\left(\nu^4\right). \quad (3.43)$$

The dependence on g is easily restored by the rescaling,

$$\nu \to g^2\nu, \qquad F_n \to (g^2)^{2n-2}F_n. \quad (3.44)$$

It is easy to generate expressions for the very first F_n's in terms of E_2, K_2 and K_4, although their explicit form is too long to be copied here. For their expansions around $\nu = 0$, we find for example,

$$F_3(\nu) = \frac{31}{161280\nu^4} + \frac{791845439}{655360} + \frac{724731745353\nu}{2621440} + \frac{157755456235861\nu^2}{4194304} + \mathcal{O}\left(\nu^3\right),$$
$$F_4(\nu) = -\frac{127}{1290240\nu^6} + \frac{1430873478800591}{1006632960} + \frac{1367769982181464281\nu}{2684354560} + \mathcal{O}\left(\nu^2\right). \quad (3.45)$$

These expansions reproduce the known results for the coefficients (3.9) as listed for example in [58] (see also [59] for similar data). This validates our claim that the all-orders WKB periods of the cubic oscillator are governed by the refined holomorphic anomaly equations. Note in particular that, by using the explicit expressions for $F_n(\tau)$ in terms of modular forms, the equations (2.40) and (3.38), and the explicit expression for the Yukawa coupling (3.36), one can write the function $A(\nu)$ (together with the singular part) as an infinite formal power series in $\hbar$ whose coefficients are modular forms.

Interestingly, the above ansatz for the holomorphic ambiguity generates a non-trivial constant term μ_n in the regular part of the free energies $F_n^{\mathrm{r}}(\nu)$, which has the form

$$F_n^{\mathrm{r}}(\nu) = \mu_n + \mathcal{O}(\nu). \quad (3.46)$$

The presence of this constant term is at first glance surprising, since the WKB method only determines the *derivative* of $F_n(\nu)$ w.r.t. ν, so the constant μ_n is not fixed by quantum special geometry. What is its meaning?

The PNP relation for the cubic oscillator, first found by Álvarez and Casares in [29], reads

$$\frac{\mathrm{d}\xi}{\mathrm{d}\nu} = -\frac{15g^2}{2}\left(\nu + g^2\frac{\partial A}{\partial g^2}\right). \quad (3.47)$$

By using (2.40) we see that, up to an integration constant independent of ν,

$$\xi(\nu) = -\frac{15g^2\nu^2}{4} - \frac{15}{2}g^4\frac{\partial F^{\mathrm{r}}(\nu)}{\partial g^2}. \quad (3.48)$$

The quantum mirror map at $\nu = 0$ is given by

$$\xi(0) = -\frac{7g^2}{16} - \frac{101479g^6}{2048} - \frac{2375536317g^{10}}{65536} + \cdots, \quad (3.49)$$

as calculated in for example [59]. It is then natural to choose a constant term μ_n in F_n, $n \geq 2$, in such a way that (3.48) is satisfied also for the ν-independent part, i.e. such that

$$\xi(0) + \frac{7g^2}{16} = -15g^4\sum_{n\geq 2}(n-1)\mu_n(g^2)^{2n-3}. \quad (3.50)$$
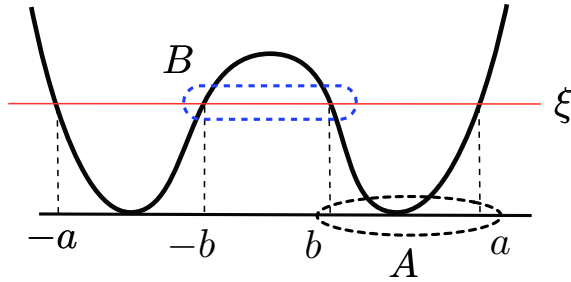
Figure 2. The symmetric double well.

This is precisely the value of μ_n which is obtained with the ansazt (3.39) for the holomorphic ambiguity, as one can verify from the explicit solution of the F_n. It is uniquely determined by the ansatz and the gap conditions. Therefore, the anomaly equation makes it possible to define an integrated form for the PNP relation, involving the quantum free energy, and with the right choice of integration constant. In fact, after including in addition the right constant for $n = 1$, and using the full free energy, one can write down the simpler relation,

$$\xi(\nu) = -\frac{3g^2}{4} - \frac{15g^4}{2}\frac{\partial F}{\partial g^2},\tag{3.51}$$

where we reintroduce g with (3.44) before taking the derivative. As opposed to (3.48), this includes the ν-independent term.

Let us make one final remark. One can verify from the explicit solutions for the quantum free energies that

$$F_n^D(\tau) = F_n(\tau),\tag{3.52}$$

i.e. the cubic oscillator is a self-dual model. Physically, this reflects the fact that inverting the cubic oscillator we obtain a completely equivalent system.

3.2 The double well and the quartic oscillator

The symmetric double well describes a quantum-mechanical particle in the potential

$$V(x) = \frac{x^2}{2}(1 + gx)^2.\tag{3.53}$$

This can be made symmetric w.r.t. $x = 0$ by shifting $x \to x - 1/(2g)$. In this symmetric realization, it has the form shown in Fig. 2. The symmetric double well is a textbook example for the importance of non-perturbative effects in Quantum Mechanics (see for example [40]). In perturbation theory, the model has two degenerate minima, but exponentially small quantum effects, due to instanton tunneling, lift the degeneracy. The exact quantization condition for this model was first conjectured in [25] by using multi-instanton calculus[4]. It can be derived in the context of the all-orders WKB method by using the exact connection formula of Voros–Silverstone (a related derivation appears in [57]). Another derivation, based on the uniform WKB method, was presented by Álvarez in [31]. The resulting condition reads,

$$1 + \exp\left(\pm\frac{\mathrm{i}}{\hbar}\oint_A P(x)\mathrm{d}x\right) = \epsilon\mathrm{i}\exp\left(\frac{\mathrm{i}}{2\hbar}\oint_B P(x)\mathrm{d}x\right).\tag{3.54}$$

---

[4]We should however note that a structurally identical, exact quantization condition for the pure quartic oscillator was already written down explicitly by Voros in [23].

Here, $\epsilon = \pm 1$ refers to the parity of the state, while the $\pm$ sign in the l.h.s. refers to the choice of lateral Borel resummation. The function appearing in the r.h.s.,

$$f(\nu) = \exp\left(\frac{\mathrm{i}}{2\hbar} \oint_B P(x)\mathrm{d}x\right), \tag{3.55}$$

has the structure:

$$f(\nu) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + \nu\right)}\left(\frac{2}{g^2}\right)^{\nu} \mathrm{e}^{-A(\nu)/2}, \tag{3.56}$$

where $A(\nu)$, defined in (2.40), is also the function appearing in the quantization condition of [25, 34] (we have set $\hbar = 1$ for simplicity.) This structural result can be obtained in the framework of [57]. It was also derived by Álvarez in [31] by using the uniform WKB method[5]. As in the cubic case, the function $A(\nu)$ has the expansion

$$A(\nu) = \frac{1}{3g^2} + \sum_{k \geq 1} c^{(k)}(\nu)g^{2k}, \tag{3.57}$$

where the coefficients $c^{(k)}(\nu)$ are polynomials in ν with rational coefficients. They can be computed by performing the WKB integrals, as in [34, 35], or by using asymptotic matching in the uniform WKB method of [31] (as in the case of the cubic oscillator, this method has the advantage of producing the function $A(\nu)$ directly as a function of the quantum period, while the standard WKB technique, as applied in [34, 35], gives $A(\nu)$ as a function of ξ). One obtains, at the very first orders (see also [49])

$$\begin{aligned}
c^{(1)}(\nu) &= \frac{19}{12} + 17\nu^2, \\
c^{(2)}(\nu) &= \frac{153\nu}{4} + 125\nu^3, \\
c^{(3)}(\nu) &= \frac{22709}{576} + \frac{23405\nu^2}{24} + \frac{17815\nu^4}{12}.
\end{aligned} \tag{3.58}$$

Let us now study this model using special geometry and the refined holomorphic anomaly equations. The elliptic curve for the double well can be taken to be

$$y^2 = (x^2 - a^2)(b^2 - x^2), \tag{3.59}$$

where we have rescaled the variables as in (3.11) to set $g = 1$. We have

$$a^2 = \frac{1}{4}\left(1 + \sqrt{32\xi}\right), \qquad b^2 = \frac{1}{4}\left(1 - \sqrt{32\xi}\right). \tag{3.60}$$

The classical periods are given by the integrals,

$$t = \frac{1}{\pi}\int_b^a y(x)\mathrm{d}x, \qquad t_D = -2\mathrm{i}\int_{-b}^b y(x)\mathrm{d}x. \tag{3.61}$$

---

[5]In order to make contact with [31], we note that: i) the function $f(\nu)$, as defined in [31], includes the factor of i appearing in the r.h.s. of (3.54), ii) our variable ν has to be identified with his $\nu/\sqrt{2}$, and iii) his g has to be set to $\sqrt{2}g^2$.

The Picard–Fuchs equation reads in this case

$$\left[6 + \xi \left(32\xi - 1 \right) \frac{\partial^2}{\partial \xi^2} \right] \Pi = 0. \tag{3.62}$$

The classical t-period can be expanded around $\xi = 0$ as in (3.16), where $c_1 = 1$ and the c_n satisfy now the recursion

$$c_n = \frac{2(4n - 7)(4n - 5)}{(n - 1)n} c_{n-1}, \qquad n \geq 2. \tag{3.63}$$

As in the cubic case, it is also possible to write t and t_D in terms of elliptic integrals of the first and second kind (see also section 4.1 of [21], where this calculation was performed in the context of the cubic Hermitian matrix model.) The results are as follows:

$$t = \frac{ab^2}{3\pi k'^2} \left[(1 + k'^2)E(k) - 2k'^2 K(k) \right],$$
$$t_D = \frac{4ab^2}{3(k')^2} \left[(1 + k'^2)E(k') - k^2 K(k') \right], \tag{3.64}$$

where the elliptic modulus is

$$k^2 = 1 - \frac{b^2}{a^2}. \tag{3.65}$$

By using the above results, one derives the following expansion of the classical periods around $\xi = 0$:

$$t = \xi + 3\xi^2 + 35\xi^3 + \frac{1155\xi^4}{2} + \frac{45045\xi^5}{4} + \cdots$$
$$t_D = \frac{1}{3} + 2\xi \left(\log \left(\frac{\xi}{2} \right) - 1 \right) + \xi^2 \left(6 \log \left(\frac{\xi}{2} \right) + 17 \right) + \cdots. \tag{3.66}$$

The prepotential $F_0(t)$ is then given by

$$F_0(t) = \frac{t}{3} + t^2 \left(\log \left(\frac{t}{2} \right) - \frac{3}{2} \right) + \frac{17t^3}{3} + \frac{125t^4}{4} + \frac{3563t^5}{12} + \frac{29183t^6}{8} + \cdots. \tag{3.67}$$

The τ parameter of the curve will be given again by iK'/K, where the elliptic modulus is (3.65), but in this case we have

$$\tau = \frac{1}{4\pi i} \frac{\partial t_D}{\partial t}. \tag{3.68}$$

The discriminant of the curve (3.59) is given by

$$\Delta(\xi) = \xi^2 (1 - 32\xi). \tag{3.69}$$

This leads to two singular points. The first one is the standard conifold point at $\xi = 0$, while the dual conifold point is

$$\xi = \frac{1}{32}. \tag{3.70}$$

This is the value of ξ in which the energy equals the height of the barrier. As in the cubic theory, we define a dual energy as

$$\xi_D = \frac{1}{32} - \xi, \tag{3.71}$$

and $\xi_D = 0$ is the dual conifold point. The classical period t_D can be expanded around $\xi_D = 0$ as

$$t_D = \sqrt{2}\pi \tilde{t}_D, \qquad \tilde{t}_D = 2\xi_D + 6\xi_D^2 + 70\xi_D^3 + \cdots \qquad (3.72)$$

Note that $\tilde{t}_D/2$ has the same expansion in terms of ξ_D, than t in terms of ξ. Physically, the dual situation corresponds to inverting the double-well and exchanging the perturbative and the tunneling cycles, up to an overall factor. The resulting model is then the inverted quartic oscillator. As we will see, the quantum free energy of this inverted quartic oscillator can be obtained by considering the S-dual theory to the double-well (i.e. by implementing the S-dual transformation (3.28)) and identifying $\tilde{t}_D$ with the quantum A-period. The dual theory will also play a rôle in the discussion of the boundary conditions for the holomorphic ambiguity.

We can now study the quantum corrections to t, t_D obtained with the all-orders WKB method. One finds that the first correction to the quantum free energy is given again by the formula (3.31). Higher order corrections can be obtained by direct integration of the holomorphic anomaly equations. In this case, life is even simpler since we can borrow the results from [21]. The Yukawa coupling was obtained in terms of modular forms in [21] and it reads,

$$Y = \frac{128\sqrt{2}K_2^{5/2}}{K_4 \left(K_2^2 - K_4\right)}. \qquad (3.73)$$

The holomorphic anomaly equation is now given by

$$\frac{\partial F_n}{\partial \widehat{E}_2} = -\frac{Y^2}{192} \sum_{r=1}^{n-1} D_\tau F_r \, D_\tau F_{n-r}, \qquad (3.74)$$

where we have chosen the appropriate normalization. The holomorphic ambiguity turns out to be parametrized as,

$$f_n(\tau) = \left(\frac{1}{\left(K_2^2 - K_4\right)K_4}\right)^{2(n-1)} K_2^{\rho_n} \sum_{i=0}^{4(n-1)-\rho_n/2} a_i K_2^{8(n-1)-\rho_n-2i} K_4^i, \qquad (3.75)$$

where

$$\rho_n = \begin{cases} 3n - 3, & \text{if } n \text{ is odd}, \\ 3n - 2, & \text{if } n \text{ is even}. \end{cases} \qquad (3.76)$$

The coefficients a_i are fixed by the boundary conditions. Since the log of (3.55) gives half of the quantum B-period, the expansion of $F_n(\nu)$ near $\nu = 0$ should be of the form (2.37), with $k_n = 2$ for all $n \geq 0$ (in this example, $\nu_0 = 2$). Let us now define the dual free energy by

$$F_n^D(\tau) = (-2)^{1-g} F_n(\tau)\Big|_{\tau \to -1/\tau}. \qquad (3.77)$$

We will require this function to have a pole of order $2n - 2$ when expanded as a function of $\tilde{t}_D$ around $\tilde{t}_D = 0$. Putting everything together, we find $4n - 5$ constraints and only $4n - 3 - \rho_n/2$ unknowns, so the system is overdetermined. However, it has a solution in all cases we have checked, up to $n = 40$. The coefficient of the singularity in the dual expansion is fixed by these

conditions and it is equal to (2.39), with $k_n^D = 1$. Using this procedure we find for example,

$$F_2 = -\frac{2\left(2K_2^2 - 3K_4\right)^2 K_2^3}{27\left(K_2^2 - K_4\right)^2 K_4^2}\widehat{E}_2 - \frac{2\left(158K_2^5 - 330K_4K_2^3 + 135K_4^2 K_2\right)K_2^3}{135\left(K_2^2 - K_4\right)^2 K_4^2},$$

$$F_3 = \frac{32\left(2K_2^2 - 3K_4\right)^3 K_2^7}{2187\left(K_2^2 - K_4\right)^4 K_4^4}\widehat{E}_2^3 + \frac{16\left(2K_2^2 - 3K_4\right)^2\left(22K_2^4 - 39K_4K_2^2 + 27K_4^2\right)K_2^6}{729\left(K_2^2 - K_4\right)^4 K_4^4}\widehat{E}_2^2 + \quad (3.78)$$

$$+ \frac{8\left(2K_2^2 - 3K_4\right)\left(7604K_2^6 - 23088K_4K_2^4 + 23625K_4^2 K_2^2 - 6345K_4^3\right)K_2^7}{3645\left(K_2^2 - K_4\right)^4 K_4^4}\widehat{E}_2 + f_3(\tau),$$

where the holomorphic ambiguity at order $n = 3$ reads

$$f_3(\tau) = \frac{4K_2^6}{76545\left(K_2^2 - K_4\right)^4 K_4^4}\left(7384904K_2^{10} - 31999716K_4K_2^8 + 53857062K_4^2 K_2^6 - \right.$$

$$\left. - 43355655K_4^3 K_2^4 + 16924950K_4^4 K_2^2 - 1927233K_4^5\right). \quad (3.79)$$

By expanding around $\nu = 0$, we obtain

$$F_2(\nu) = -\frac{7}{2880\nu^2} + \frac{131}{192} + \frac{22709\nu}{576} + \frac{217663\nu^2}{128} + \frac{61936297\nu^3}{960} + \frac{581912191\nu^4}{256} + \mathcal{O}\left(\nu^5\right),$$

$$F_3(\nu) = \frac{31}{80640\nu^4} + \frac{10483}{256} + \frac{20182631\nu}{3840} + \frac{628163783\nu^2}{1536} + \frac{178000040363\nu^3}{7168} + O\left(\nu^4\right). \quad (3.80)$$

The dependence on g is restored again by (3.44). The explicit expressions for F_4 and the higher n free energies become too long to be copied here, but we can write the first terms of their expansion around $\nu = 0$. We have, for example,

$$F_4(\nu) = -\frac{127}{645120\nu^6} + \frac{193438987}{24576} + \frac{553607616973\nu}{344064} + \frac{12106775869037\nu^2}{65536} + \mathcal{O}\left(\nu^3\right). \quad (3.81)$$

As in the cubic model, the $A(\nu)$ function appearing in (3.57) is related to the regular part of the quantum free energy by the equation (2.40) (the regular part is defined as in the cubic oscillator, by removing logarithmic terms and poles in the expansion of the F_n). The expansions of the F_n obtained above agree with previous results in quantum mechanics, as listed in for example [31] (see also [49]).

The PNP relation for the double well, found by Álvarez in [31], can be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}\nu} = -3g^2\left(2\nu + g^2\frac{\partial A}{\partial g^2}\right). \quad (3.82)$$

After integration, we find that

$$\xi(\nu) = -3g^2\nu^2 - 3g^4\frac{\partial F^{\mathrm{r}}(\nu)}{\partial g^2}, \quad (3.83)$$

up to a ν independent term. The quantum mirror map at $\nu = 0$ is given by

$$\xi(0) = -\frac{g^2}{4} - \frac{131g^6}{32} - \frac{31449g^{10}}{64} - \frac{580316961g^{14}}{4096} - \cdots. \quad (3.84)$$

This can be easily calculated for example with the BenderWu package of [61]. The constant term μ_n in F_n should be such that (3.83) is satisfied for the ν-independent part, i.e. such that

$$\xi(0) + \frac{g^2}{4} = -3g^4 \sum_{n \geq 2} (2n - 2)\mu_n (g^2)^{2n-3}. \tag{3.85}$$

As in the cubic oscillator, this is precisely the value of μ_n obtained with the holomorphic anomaly equations. One can use the full quantum free energy to write the PNP relation as in (3.51),

$$\xi(\nu) = -\frac{g^2}{2} - 3g^4 \frac{\partial F}{\partial g^2}, \tag{3.86}$$

where again we reintroduce g with (3.44) before the derivative.

Finally, we note that results for the quantum free energies of the (unstable) quartic oscillator with potential

$$V(x) = \frac{x^2}{2} - gx^4 \tag{3.87}$$

can be obtained from the results for the double well, after an S-duality transformation. The quantum free energies of the quartic oscillator are precisely the dual free energies in (3.77), while $\tilde{t}_D$ is the quantum A-period. One obtains, for example,

$$
\begin{aligned}
F_2^D(\tilde{t}_D) &= -\frac{7}{5760\tilde{t}_D^2} + \frac{513}{512} + \frac{305141\tilde{t}_D}{9216} + \frac{3105983\tilde{t}_D^2}{4096} + \frac{912774217\tilde{t}_D^3}{61440} + \cdots, \\
F_3^D(\tilde{t}_D) &= \frac{31}{161280\tilde{t}_D^4} + \frac{485523}{8192} + \frac{1056412343\tilde{t}_D}{245760} + \frac{34978331399\tilde{t}_D^2}{196608} + \cdots
\end{aligned}
\tag{3.88}
$$

This and other results agree with the calculations using uniform WKB in [28, 58]. They also agree with the calculation of the A function in [36], after expressing it in terms of the quantum period as in [59].

4 Application: the large order behavior of the WKB expansion

In the all-orders WKB method, the energy is given in terms of the quantum number m by the formal power series (2.15) in $\hbar^2$, where ν is given by (2.16). After reexpanding this series in powers of g, we recover the perturbative Rayleigh–Schrödinger series for the m-th energy level. The large order behavior of this perturbative series is governed by instanton corrections, as first noted by Bender and Wu in their seminal paper [62] (see [40] for a textbook presentation). However, one can ask a slightly different question: what is the large order behavior of (2.15), i.e. the behavior of the coefficients $\xi_n(\nu)$ when n is large? This is a series in $\hbar^2$, not in the coupling constant, and in addition its coefficients are non-trivial functions of ν. As we will see, its behavior is different from the one of the standard perturbative series. Since the direct integration of the holomorphic anomaly equation, combined with the PNP relations, easily generates many terms in the expansion (2.15), one can test in detail the predictions for its large order behavior.

We will now determine the large order behavior of (2.15) by using standard tools in resurgent analysis (see for example [39, 40] for a simple introduction, and [63, 64] for very similar applications). We first note that this series can be extended to a *trans-series* involving an exponentially small parameter, of the form

$$\Xi(\nu) = \sum_{\ell=0}^{\infty} e^{-\ell \mathcal{A}(\nu)/\hbar} \xi^{(\ell)}(\nu), \tag{4.1}$$

where

$$\xi^{(\ell)}(\nu) = \sum_{r \geq 0} \xi_r^{(\ell)}(\nu) \hbar^r. \tag{4.2}$$

Here, $\mathcal{A}(\nu)$ is the instanton action, and $\xi^{(0)}(\nu)$ is the original asymptotic series $\xi(\nu)$ in (2.15). Note that $\xi_r^{(0)}(\nu) = 0$ for odd r. The trans-series (4.1) is obtained by solving the appropriate quantization condition. We will focus on the case of the double well, so we need the trans-series solution to (3.54). We choose the $-$ sign in this equation, and we write the result in the form

$$\mathrm{i}f(\widehat{\nu}) = 1 + \mathrm{e}^{-2\pi \mathrm{i}\widehat{\nu}/\hbar}. \tag{4.3}$$

We have denoted the argument of this equation by $\widehat{\nu}$ to emphasize the fact that it now has non-perturbative corrections and is no longer given by (2.16). We have also reabsorbed the parity ϵ inside the function $f(\widehat{\nu})$. We now sketch the derivation of the trans-series $\Xi(\nu)$, following a similar derivation in [31]. The solution to (4.3) is of the form

$$\widehat{\nu} = \nu + \Delta\nu, \tag{4.4}$$

where ν is given by (2.16) and $\Delta\nu$ is non-perturbative in $\hbar$. It satisfies the equation

$$\Delta\nu = \frac{\mathrm{i}\hbar}{2\pi} \log\left(1 - \mathrm{i}f\left(\nu + \Delta\nu\right)\right). \tag{4.5}$$

We note that, from the defining expression in (3.55), one has,

$$f(\nu) = \exp\left(-\frac{\mathcal{A}(\nu)}{\hbar}\right) \left\{1 - \hbar\frac{\partial F_1}{\partial \nu} + \cdots\right\}, \tag{4.6}$$

where

$$\mathcal{A}(\nu) = \frac{1}{2}\frac{\partial F_0}{\partial \nu} = \frac{t_D(\nu)}{2} \tag{4.7}$$

is the instanton action, and $t_D(\nu)$ is given in (3.64). We now solve for $\Delta\nu$ as an instanton expansion,

$$\Delta\nu = \sum_{k \geq 1} \Delta\nu^{(k)}, \tag{4.8}$$

where

$$\Delta\nu^{(k)} \propto \mathrm{e}^{-k\mathcal{A}(\nu)/\hbar}. \tag{4.9}$$

From (4.5) one obtains,

$$\Delta\nu^{(1)} = \frac{\hbar}{2\pi}f(\nu),$$
$$\Delta\nu^{(2)} = \frac{\mathrm{i}\hbar}{4\pi}f^2(\nu) + \left(\frac{\hbar}{2\pi}\right)^2 f'(\nu)f(\nu). \tag{4.10}$$
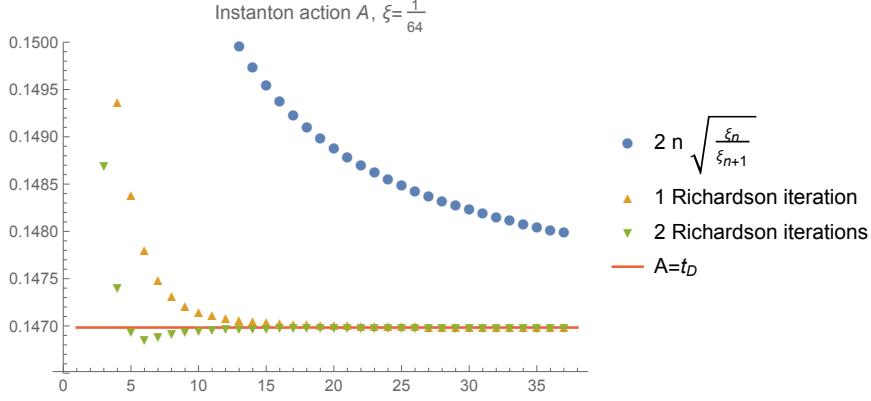
We finally obtain, up to instanton number two,

$$\Xi(\nu) = \xi(\widehat{\nu}) = \xi(\nu) + \frac{\partial \xi}{\partial \nu}\Delta\nu^{(1)} + \frac{\partial \xi}{\partial \nu}\Delta\nu^{(2)} + \frac{1}{2}\frac{\partial^2 \xi}{\partial \nu^2}\left(\Delta\nu^{(1)}\right)^2 + \cdots, \tag{4.11}$$

and we conclude that

$$\mathrm{e}^{-\mathcal{A}(\nu)/\hbar}\xi^{(1)}(\nu) = \frac{\hbar}{2\pi}f(\nu)\frac{\partial \xi}{\partial \nu},$$
$$\mathrm{e}^{-2\mathcal{A}(\nu)/\hbar}\xi^{(2)}(\nu) = \frac{\mathrm{i}\hbar}{4\pi}f^2(\nu)\frac{\partial \xi}{\partial \nu} + \frac{\hbar^2}{8\pi^2}\frac{\partial}{\partial \nu}\left(f^2(\nu)\frac{\partial \xi}{\partial \nu}\right). \tag{4.12}$$

Figure 3. The sequence (4.16) for the double well and $\xi = 1/64$, together with two Richardson transforms.

As is well-known, the large order behavior is controlled by the imaginary part of the trans-series, and the first contribution appears at the two-instanton level [37]. The imaginary part of the first two coefficients in $\xi^{(2)}(\nu)$ is given by,

$$\operatorname{Im}\xi_0^{(2)}(\nu) = \frac{\hbar}{4\pi}\frac{\partial\xi_0}{\partial\nu}, \qquad \operatorname{Im}\xi_1^{(2)}(\nu) = -\frac{\hbar}{4\pi}\frac{\partial\xi_0}{\partial\nu}\frac{\partial F_1}{\partial\nu}. \qquad (4.13)$$

The higher order coefficients $\operatorname{Im}\xi_r^{(2)}(\nu)$ with $r \geq 2$ can be computed in a straightforward way. Standard arguments (see for example [40, 63]) tell us that the large order behavior of the WKB series $\xi_n(\nu)$ is of the form[6],

$$\xi_n(\nu) \sim \frac{1}{\pi}(A(\nu))^{-2n-b}\Gamma(2n+b)\,\mu_1(\nu)\left[1 + \frac{\mu_2(\nu)A(\nu)}{2n} + \cdots\right], \quad n \gg 1, \qquad (4.14)$$

where the values of $A(\nu)$, b and the $\mu_n(\nu)$ can be obtained from the leading behavior of the imaginary part of the trans-series, i.e. from $\mathrm{e}^{-2\mathcal{A}(\nu)}\xi^{(2)}(\nu)$. One finds,

$$A(\nu) = 2\mathcal{A}(\nu) = t_D(\nu), \qquad b = -1, \qquad \mu_1(\nu) = -\frac{1}{2\pi}\frac{\partial\xi_0}{\partial\nu}, \qquad \mu_2(\nu) = -\frac{\partial F_1}{\partial\nu}. \qquad (4.15)$$

The factor of 2 and the minus sign in μ_1 are due to the fact that the relevant non-perturbative quantity is the difference between the two lateral resummations, and this introduces a factor of -2.

It is interesting to note that the WKB expansion has the $(2n)!$ growth typical of a string theory or of the $1/N$ expansion [38, 39]. This can be regarded as yet another consequence of the fact that this expansion is governed by the refined holomorphic anomaly equations of topological string theory, as we argued in this paper. In contrast, the standard perturbative expansion in the coupling constant grows like $n!$. In addition, the WKB series and its asymptotics depend on ν, which plays the rôle of a 't Hooft parameter (as already suggested in [9–11]).

The resurgent prediction (4.14) for the large order behavior can be tested in detail by studying the sequence $\xi_n(\nu)$ for high enough n. From this original sequence, one can consider various

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[6]In this section $A(\nu)$ denotes the instanton action, and it should not be confused with the function defined in (2.40).
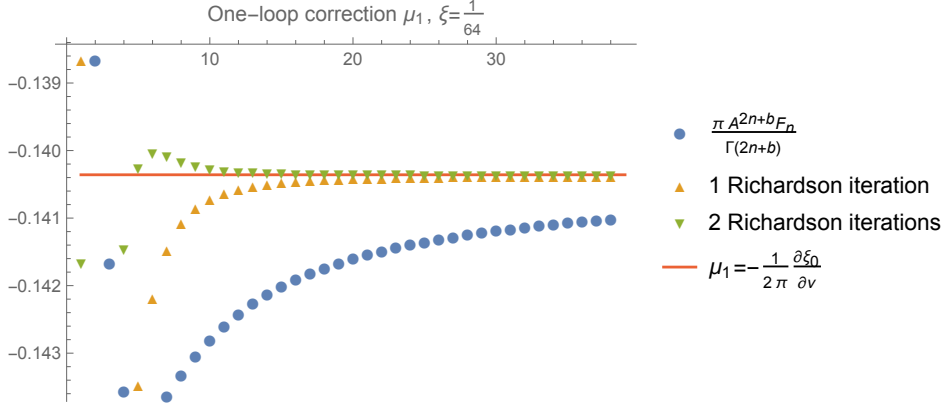
Figure 4. The sequence (4.18) for the double well and $\xi = 1/64$, together with two Richardson transforms.

auxiliary sequences to extract the coefficients. For example, to extract A, we consider the sequence

$$Q_n = 2n\sqrt{\frac{\xi_{n+1}}{\xi_n}} = A + \mathcal{O}\left(\frac{1}{n}\right). \tag{4.16}$$

The parameter b follows from the sequence

$$n\left(A^2\frac{\xi_{n+1}}{4n^2\xi_n} - 1\right) - \frac{1}{2} = b + \mathcal{O}\left(\frac{1}{n}\right). \tag{4.17}$$

The coefficients μ_1 and μ_2 are obtained from the sequences

$$\frac{\pi A^{2n+b}\xi_n}{\Gamma(2n+b)} = \mu_1\left(1 + \frac{\mu_2 A}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \tag{4.18}$$

and

$$\frac{2n}{A}\left(\frac{\pi A^{2n+b}\xi_n}{\mu_1\Gamma(2n+b)} - 1\right) = \mu_2 + \mathcal{O}\left(\frac{1}{n}\right), \tag{4.19}$$

respectively. The convergence of these sequences can be accelerated with Richardson transforms, as explained in for example [63]. We obtain in this way very good numerical approximations to the expected values (4.15), which are themselves functions of ν (or, equivalently, of the modulus ξ, which is identified with $\xi_0(\nu)$).

Let us give an example of the procedure. By combining the expression (3.86) and the results of direct integration, we generated the first forty terms in the sequence $\xi_n(\nu)$. For the value $\xi = 1/64$ (which corresponds to $\nu = 0.01654...$), we show in Fig. 3 the sequence (4.16), together with two Richardson transformations, as well as the prediction t_D. Numerically, the best estimate for the instanton action is given by

$$A^{\mathrm{num}} = 0.14698331391354... \tag{4.20}$$

The prediction from resurgent analysis is

$$t_D = 0.1469833139135404... \tag{4.21}$$
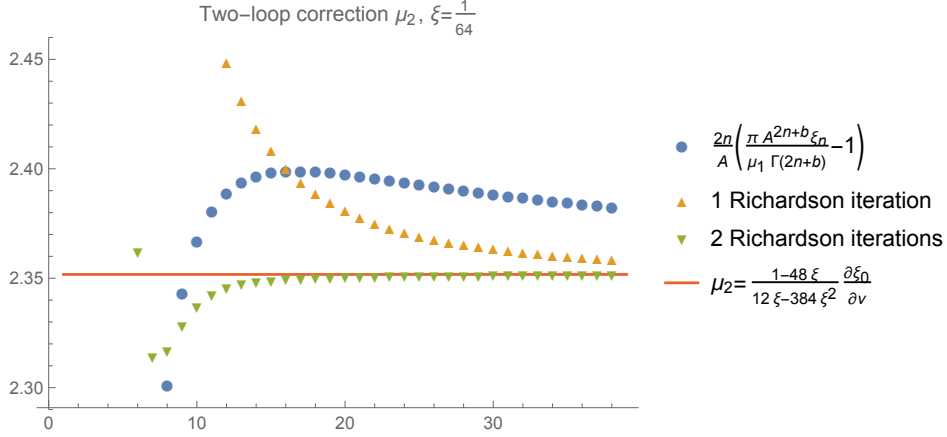
Figure 5. The sequence (4.19) and two Richardson transforms for the double well and $\xi = 1/64$.

Similarly, we show in Fig. 4 and Fig. 5 the sequences (4.18) and (4.19), respectively, for $\xi = 1/64$, together with their Richardson transforms, and we compare them to the theoretical predictions. Numerically, the best estimates are

$$\mu_1^{\mathrm{num}} = -0.1403587257682..., \qquad \mu_2^{\mathrm{num}} = 2.3517330226..., \tag{4.22}$$

while the expected values are

$$\mu_1 = -0.140358725768206..., \qquad \mu_2 = 2.351733022617... \tag{4.23}$$

In calculating μ_2, we have used that

$$\frac{\partial F_1}{\partial \nu} = -\frac{1}{24\Delta} \frac{\mathrm{d}\Delta}{\mathrm{d}\xi} \frac{\partial \xi_0}{\partial \nu}, \tag{4.24}$$

where $\Delta(\xi)$ is the discriminant (3.69). The numerical estimates obtained above are excellent, if we take into account that we have used only forty terms in the series. They are comparable to similar results in the instanton analysis of large N matrix models [63, 64]. We have checked the large order predictions for many other values of ξ (including complex ones), and in all cases we find an excellent agreement with the numerical results.

Finally, let us note that the WKB series for the energy (2.15) in the double well inherits the well-known lack of Borel summability of the perturbative series, since the instanton action t_D is positive for any $0 \leq \xi < 1/32$ inside the radius of convergence. It would be interesting to study how to recover the exact result for the energies by using the Borel–Écalle resummation of the WKB (trans-)series, instead of the resummation of the perturbative series performed in [34, 35].

5 Conclusions and outlook

In this paper we have argued that the refined holomorphic anomaly equations of topological string theory govern the all-orders WKB periods of general one-dimensional quantum mechanical systems. In the case of quantum mechanical models related to $\mathcal{N} = 2$ supersymmetric gauge theories or to topological string theory in the NS limit, like the (modified) Mathieu equation,

this is expected. However, we claim that this connection is more general and holds for oscillators which do not have a known realization in terms of gauge theories, strings or matrix models, like the cubic oscillator.

Our results seem to indicate that the holomorphic anomaly equations, in the NS limit, are just a consequence of the all orders WKB method of [26]. It would be very interesting to prove that, indeed, the equations (2.23) follow from the recursion (2.5). Note that a similar situation occurs for matrix models in the $1/N$ expansion. In principle, the genus g free energies of matrix models are holomorphic, but in the context of the topological recursion [65], one can introduce a non-holomorphic dependence through the Bergmann kernel of the curve, which is a basic building block of the recursion. It is then relatively simple to show that the resulting non-holomorphic free energies, computed by a non-holomorphic extension of the topological recursion, satisfy the standard holomorphic anomaly equation of [16], as shown in [20]. In the case of the all-orders WKB periods, the results derived in [51] for topological strings provide a strategy to derive (2.23). One important assumption needed in a derivation along the lines of [51] is that the quantum periods can be obtained from the classical periods by acting with differential operators depending on the energy (which is the modulus of this problem). This assumption is known to hold in many cases. It might be possible to derive it systematically from the recursion (2.5).

Our results also require a clarification concerning the literature on quantum curves (see for example [66] for a review). As it is well known, given a spectral curve, the topological recursion of [65] leads to a formal WKB-like wavefunction ψ_{top}. It is often claimed that ψ_{top} is annihilated by an appropriate or "natural" quantization of the underlying spectral curve, so that it provides a formal WKB solution to the operator problem defined by the quantum version of the curve. In the cases considered in this paper, for example, the "natural" quantization simply involves the promotion of y, x to canonically conjugate Heisenberg operators

$$y \to \mathsf{y}, \qquad x \to \mathsf{x}, \qquad [\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar, \tag{5.1}$$

so that we recover the standard Schrödinger operator (2.1). It has been recently proved that this claim is true when the spectral curve has genus zero [67]. However, the situation for curves of higher genus is more subtle. One hint that the topological recursion cannot lead straightforwardly to the WKB result is that the F_g's produced by the topological recursion, as applied for example to the curves studied in this paper, are simply *not* the functions F_n produced by the WKB method[7]. This is only to be expected: the former corresponds to the unrefined topological string theory, while the latter corresponds to the NS limit of the refined string. In line with this observation, a recent study of the Weierstrass curve, which has genus one, indicates that ψ_{top} is not annihilated by the quantum curve obtained by the quantization scheme (5.1) [68], in other words, it is not a formal WKB wavefunction associated to the natural quantization of the curve.

The connection unveiled in this paper between the standard WKB method and the holomorphic anomaly equations is potentially interesting to further clarify the quantum mechanical meaning of the general equations (2.19). It was famously suggested by Witten in [70] that the standard topological string partition function should be interpreted as a wavefunction. The holomorphic anomaly equation then implements the independence of this wavefunction w.r.t. the choice of polarization. If our point of view is correct, there might be a precise connection between the holomorphic anomaly equations and a Schrödinger-like equation governing the partition function.

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[7]In the case of the double-well, this is easily seen by comparing the results in [21] to the results in section 3.2 above.

There are many concrete extensions and generalizations of our approach to similar problems and settings. One could study the holomorphic anomaly equations for other one-dimensional models, like the Fokker–Planck potential or the cosine potential. Although we have focused for simplicity on models leading to curves of genus one, we expect that higher genus curves will be also governed by the refined holomorphic anomaly equations. In the case of curves of genus two, one could use already the technology developed in [47]. Notice as well that, by combining the results found in this paper with the results of [71], it is possible to compute the full $1/N$ expansion for the ground state energy of Matrix Quantum Mechanics, in terms of modular forms. This is an appealing laboratory to explore the structure and properties of the $1/N$ expansion at all orders. The analysis of the large order behavior in section 4 can be clearly extended to the cubic and the quartic oscillators, and as we already mentioned in that section, it would be interesting to understand the Borel–Écalle resummation of the WKB expansion in all the examples considered in this paper. Finally, although in the examples considered in this paper we can compute the relevant trans-series from the explicit quantization conditions, it would be interesting to study the general trans-series by extending the refined holomorphic anomaly equations to the non-perturbative sector, as it was done in [72] for the standard topological string.

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