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Holomorphic convexity and Carleman approximation by entire functions on Stein manifolds

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Abstract We give necessary and sufficient conditions for totally real sets in Stein manifolds to admit Carleman approximation of class $\mathcal{C}^k$, $k \geq 1$, by entire functions.

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1 Introduction

In 1927, Carleman [2] proved a remarkable extension of Weierstrass' approximation theorem: If $f, \epsilon \in \mathcal{C}(\mathbb{R})$ are continuous functions on the real line in the complex plane, ϵ strictly positive, then there exists an entire function $g \in \mathcal{O}(\mathbb{C})$ such that $|g(x) - f(x)| < \epsilon(x)$ for all $x \in \mathbb{R}$. This theorem has been generalized to one-dimensional sets in $\mathbb{C}^N$ by Alexander [1], who proved in 1979 the same result for smoothly embedded curves in $\mathbb{C}^N$, and more recently, in 2002, by Gauthier and Zeron [5], who gave a proof in the case of locally rectifiable curves with trivial topology in $\mathbb{C}^N$.

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We treat the case of higher dimensional totally real manifolds and, more generally, totally real sets. A totally real set M in a Stein manifold X is said to admit Carleman approximation if there for each $f, \epsilon \in \mathcal{C}(M)$, ϵ strictly positive, exists an entire function $g \in \mathcal{O}(X)$ such that $|g(x) - f(x)| < \epsilon(x)$ for all $x \in M$. If M is a totally real manifold of class $\mathcal{C}^k$, it is also possible to consider $\mathcal{C}^k$ Carleman approximation of $f \in \mathcal{C}^k(M)$ by $g \in \mathcal{O}(X)$; if $X = \mathbb{C}^N$ then this is obtained by requiring all partial derivatives of $g - f$ along M of order $\leq k$ to be smaller than $\epsilon(x)$ at each $x \in M$. We show in Sect. 2.2 how to define $\mathcal{C}^k$ Carleman approximation on totally real sets of class $\mathcal{C}^k$ in Stein manifolds, and the main object of the paper is to give necessary and sufficient conditions for $\mathcal{C}^k$ Carleman approximation on totally real sets of class $\mathcal{C}^k$, $k \geq 1$, to be possible.

Totally real affine linear subspaces of $\mathbb{C}^N$ always allow Carleman approximation, as was shown by Hoischen [7] and Scheinberg [14]. For more general sets, extra conditions must be imposed. Firstly, we need the totally real set in question to be polynomially convex, or holomorphically convex if the ambient space is a Stein manifold. However, a recent construction by the second author [18] shows that polynomial convexity alone is not sufficient: There exists a smoothly embedded polynomially convex totally real surface in $\mathbb{C}^3$ which does not allow Carleman approximation. We will therefore in addition require that the set has what we call bounded exhaustion hulls, or E-hulls, in the ambient space; see Definition 2.1. This condition is present in proofs of Carleman approximation in, e.g., [2,16], and we are able to show that it is a necessary condition for $\mathcal{C}^k$ Carleman approximation, $k \geq 1$.

We give some examples to illustrate these notions. If $M \subset \mathbb{C}^N$ is a locally rectifiable curve with trivial topology, then it will be polynomially convex and have bounded E-hulls, as is shown in [1,5], building upon fundamental work by Stolzenberg [16]. Another example satisfying both conditions is given by any Lipschitz graph over $\mathbb{R}^N$ with Lipschitz constant $\alpha < 1$; see Proposition 4.2. A third example is given in the one-dimensional case $M \subset G \subset \mathbb{C}$; the condition of bounded E-hulls is then equivalent to requiring the complement of M in G to be locally connected at infinity. This is also a necessary condition in Nersesjan's approximation theorem [12] (see Remarks 2.2 and 2.8). The example in [18] mentioned above is polynomially convex, but does not have bounded E-hulls. There are also simple examples of $M \subset \mathbb{C}$ which are polynomially convex, but does not have bounded E-hulls; see [4]. These examples are not totally real sets, however.

It is shown in [8] that for a totally real manifold $M \subset \mathbb{C}^N$ of real dimension at most $N-1$, the condition of polynomial convexity and the condition of bounded E-hulls are both generic, so that any sufficiently small $\mathcal{C}^1$ perturbation of M will be polynomially convex and have bounded E-hulls.

Our main result can then be stated as follows:

Theorem 1.1 *Let X be a Stein manifold, and let $M \subset X$ be a totally real set of class $\mathcal{C}^k$ with $k \geq 1$.*

(a) *M admits $\mathcal{C}^k$ Carleman approximation if and only if M is holomorphically convex and has bounded E-hulls in X.*
(b) *If any of the equivalent conditions in* (a) *are satisfied, then M admits $\mathcal{C}^k$ Carleman approximation with interpolation.*

The notion of $\mathcal{C}^k$ Carleman approximation is made more precise by Definitions 2.3 and 2.6, which use the same setup as in [11]. One useful consequence of Theorem 1.1 (see Corollary 3.2) is that the set M has a Runge and Stein neighborhood basis.

In light of Theorem 1.1, it is natural to expect that polynomial convexity and the property of having bounded E-hulls are both necessary conditions for admitting Carleman approximation of *continuous* functions. The methods of this article do not seem to provide a proof of that.

The fact that the conditions in (a) are sufficient to obtain $\mathcal{C}^k$ Carleman approximation was proved by the first author in the unpublished work [10].

2 Preliminaries

If X is a complex manifold, we let $\mathcal{O}(X)$ denote the holomorphic functions on X. If $K \subset X$ is a compact set, we let $\mathcal{O}(K)$ denote the continuous functions on K which are restrictions of holomorphic functions on some open neighborhood of K. The neighborhood may depend on the function. We will always assume that the manifold X is equipped with some riemannian metric giving rise to a distance $|x - x'|$ for $x, x' \in X$.

2.1 Holomorphic convexity and exhaustion hulls

If M is a compact subset of a complex manifold X, we define, as usual, the *holomorphically convex hull* of M to be

$$\widehat{M}_{\mathcal{O}(X)} = \{x \in X; |f(x)| \leq \|f\|_M, \forall f \in \mathcal{O}(X)\}.$$

If $X = \mathbb{C}^N$ for some N, we drop the subscript $\mathcal{O}(X)$. The hull then coincides with the polynomial hull of M.

If M is a closed, noncompact set, we define the hull of M by

$$\widehat{M}_{\mathcal{O}(X)} = \bigcup_{k=1}^{\infty} \widehat{M^k}_{\mathcal{O}(X)},$$

where $\{M^k\}$ is a normal exhaustion of M. Note that the definition of $\widehat{M}_{\mathcal{O}(X)}$ is independent of the exhaustion. We call M holomorphically convex if $\widehat{M}_{\mathcal{O}(X)} = M$. If $X = \mathbb{C}^N$ and $\widehat{M}_{\mathcal{O}(\mathbb{C}^N)} = M$, we call M polynomially convex; in other words, this means that M can be exhausted by polynomially convex compact subsets.

For a closed set $M \subset X$, we let $h(M)$ denote the set

$$h(M) = \overline{\widehat{M}_{\mathcal{O}(X)} \setminus M}.$$

Definition 2.1 Let $E = \{E^k\}$ be a normal exhaustion of X. We say that M has *bounded exhaustion hulls (or E-hulls) in* X if the set $h(E^k \cup M)$ is compact in X for all choices of k. Note that this property is independent of the exhaustion E.

Remark 2.2 If $G \subset \mathbb{C}$ is a domain and $M \subset G$ is a closed subset, then M is holomorphically convex iff $G^* \setminus M$ is connected, and M has bounded E-hulls iff $G^* \setminus M$ is locally connected at infinity iff $G \setminus M$ has no bounded component, where $G^* = G \cup \{\infty\}$ is the one-point compactification of G (see [4]).

2.2 Pointwise seminorms

Let $M \subset X$ be some set and let $k \in \mathbb{Z}_+$. For each point $x \in M$ we introduce an equivalence relation on germs of $\mathcal{C}^k$-smooth complex-valued functions at x, namely $f_x \sim g_x$ if and only if $f - g$ vanishes to order k at x. The set of equivalence classes, denoted by $\mathcal{J}_x^k$, forms in a natural way a finite dimensional complex vector space called the k-jet space at the point x. The collection of all k-jet spaces for all points $x \in M$ forms in a natural way a complex vector bundle $\mathcal{J}^k(X, M)$ over M, where transition functions can be expressed in terms of transition functions (and their derivatives) on X.

To any $\mathcal{C}^k$-smooth function f on X, we associate a continuous section $\mathcal{J}^k(f)$ of $\mathcal{J}^k(X, M)$ by $\mathcal{J}^k(f)(x) := [f_x]$. Let $|\cdot|$ be a fiberwise norm on $\mathcal{J}^k(X, M)$ that varies continuously with x, *i.e.*, for any local section $s \in \Gamma(\mathcal{J}^k(X, M)|_{M \cap U})$, where U is an open subset of X, the funtion $x \mapsto |s(x)|$ is continuous. Finally, we define the pointwise seminorms

$$|f|_{k,x} = |\mathcal{J}^k(f)(x)|$$

whenever $f \in \mathcal{C}^k(U)$ and $x \in M \cap U$, where U is an open subset of X.

Definition 2.3 Let X be a complex manifold, let $M \subset X$ be a closed set, and let $|\cdot|_{k,x}$ be a pointwise seminorm on M. Let $\mathcal{F}$ be a family of complex-valued functions contained in $\mathcal{C}^k(X)$. We say that M admits $\mathcal{C}^k$ Carleman approximation of functions in $\mathcal{F}$ if there for every function $f \in \mathcal{F}$ and every strictly positive function $\epsilon \in \mathcal{C}(M)$ exists an entire function $g \in \mathcal{O}(X)$ with

$$|g - f|_{k,x} < \epsilon(x)$$

for all $x \in M$.

Remark 2.4 If we make another choice of pointwise seminorm $|\cdot|'_{k,x}$, we have

$$C(x)^{-1}|f|_{k,x} \leq |f|'_{k,x} \leq C(x)|f|_{k,x}$$

for some positive continuous function C on M. In particular, the validity of $\mathcal{C}^k$ Carleman approximation is independent of the choice of the norm.

Definition 2.5 We recall that a manifold M contained in a complex manifold X is said to be totally real if at all points $p \in M$ the tangent space T_pM contains no complex line. We say that a set $M \subset X$ is a totally real set of class $\mathcal{C}^k$, $k \geq 1$, if M is closed and locally contained in a totally real $\mathcal{C}^k$-manifold.

It is shown in [6] that M is a totally real set of class $\mathcal{C}^k$ if and only if we can write $M = \rho^{-1}(0)$ for some non-negative real $\mathcal{C}^{k+1}$-function ρ which is strictly plurisubharmonic on some neighborhood of M.

Definition 2.6 Let X be a complex manifold and let M be a totally real set of class $\mathcal{C}^k$ in X. Let $f \in \mathcal{C}^k(X)$ for some $k \geq 1$. If $[\overline{\partial}(D^\alpha f)](x) = 0$ for all $x \in M$ and all $|\alpha| \leq k-1$, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N}}$ for some holomorphic coordinate system $z = (z_1, \dots, z_N)$ near x, then we say that f is $\overline{\partial}$-flat to order k along M, and we write $f \in \mathcal{H}_k(X, M)$. We declare $f \in \mathcal{H}_0(X, M)$ for any continuous $f \in \mathcal{C}(X)$.

Note that $\mathcal{H}_k(X, M)$ is closed under multiplication, and hence becomes an algebra. We will in this paper be interested in $\mathcal{C}^k$ Carleman approximation of functions in $\mathcal{H}_k(X, M)$.

When M is a totally real manifold of class $\mathcal{C}^k$, it is possible to consider $\mathcal{C}^k$-approximation by entire functions of functions defined only on M. If M is a totally real set, then it is possible to cover it by totally real manifolds M_j and use $\mathcal{C}^k$-functions f_j on M_j which fit together suitably to obtain $\mathcal{C}^k$-objects on M to approximate. The following proposition can then be used to show that M will admit Carleman approximation with this definition if and only if it does so as defined in Definition 2.3.

Proposition 2.7 *Let X be a complex manifold, and let $M \subset X$ be a totally real set of class $\mathcal{C}^k$. Let $f \in \mathcal{C}^k(X)$ be any function. Then there exists a function $g \in \mathcal{C}^k(X)$ such that $g(x) = f(x)$ for all $x \in M$ and such that g is $\overline{\partial}$-flat to order k along M.*

Proof There are a locally finite open cover $\{U_i\}_{i=1}^\infty$ of M and totally real manifolds $M_i \subset U_i$ of maximal real dimension N such that $M \cap U_i \subset M_i$ for each i. Consider the restriction $f|_{M_i}$. Since M_i is of maximal dimension, the Cauchy-Riemann equations determine at each point $x \in M_i$ a unique $\overline{\partial}$-flat k-jet agreeing with $j_x^k(f)$ along M_i. By Whitney's extension theorem [17], $f|_{M_i}$ can be extended to a function $f_i \in \mathcal{C}^k(U_i)$ which is $\overline{\partial}$-flat along M_i. Let $\phi_i \in \mathcal{C}^k(X)$ be functions with $\text{Supp}(\phi_i) \subset U_i$ and such that $\sum \phi_i = 1$ for all $x \in M$. Again, by Whitney's extension theorem, there are $\tilde{\phi}_i \in \mathcal{H}_k(U_i, M_i)$ which extend $\phi_i|_{M_i}$. It is possible to extend by the zero function wherever $\phi_i|_{M_i}$ is locally zero, and away from M_i the extension can be arbitrary. We can thus obtain $\text{Supp}(\tilde{\phi}_i) \subset U_i$, and defining $\tilde{\phi}_i \equiv 0$ on $X \setminus U_i$ we get $\tilde{\phi}_i \in \mathcal{H}_k(X, M)$. Then $g = \sum \tilde{\phi}_i f_i \in \mathcal{H}_k(X, M)$ is well defined, and at each $x \in M$ we have $g(x) = \sum \tilde{\phi}_i(x) f_i(x) = \sum \phi_i(x) f(x) = f(x)$. □

Remark 2.8 In the one-dimensional case, Nersesjan's theorem [12] (see also [4]) characterizes sets which admit Carleman approximation of functions in $A(M)$: If $G \subset \mathbb{C}$ is a domain and $M \subset G$ is a closed and proper subset, then M admits Carleman approximation of functions in $A(M)$ iff (i) $G^* \setminus M$ is connected, (ii) $G^* \setminus M$ is locally connected at ∞, and (iii) for any compact $K \subset G$ there is a neighborhood V of ∞ in G^* such that no component of M° intersects both K and V. Here $G^* = G \cup \{\infty\}$, as in Remark 2.2. Note that (iii) is vacuously satisfied whenever M has empty interior, and (i) and (ii) will then characterize sets which admit Carleman approximation of functions in $\mathcal{C}(M)$.

3 The sufficient condition

The goal of this section is to prove the sufficiency in Theorem 1.1. We prove the more general result:

Theorem 3.1 *Let X be a Stein manifold and let $M \subset X$ be a totally real set which is holomorphically convex and has bounded E-hulls in X. Then the following holds: For any compact set $K \subset X$ with $K \cup M$ holomorphically convex, $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_i\}_{i=1}^{\infty}$ discrete sequences of points in X with $A \subset M$ and $B \subset X \setminus (K \cup M)$, $C = \{c_i\}_{i=1}^{m} \subset K$ a finite set of points, $\{q_i\}_{i=1}^{\infty}$ a collection of germs of holomorphic functions at the points b_i, $\{d_i\}_{i=0}^{\infty} \subset \mathbb{N}$, $f \in \mathcal{C}(K \cup M) \cap \mathcal{O}(K)$, and $\epsilon \in \mathcal{C}(K \cup M)$ a strictly positive function, there exists a $g \in \mathcal{O}(X)$ such that*

(i) $|g(x) - f(x)| < \epsilon(x)$ *for all* $x \in K \cup M$,
(ii) $g(x) = f(x)$ *for all* $x \in A$,
(iii) $g(x) - q_i(x) = O(|x - b_i|^{d_i+1})$ *as* $x \to b_i$ *for all* $i \in \mathbb{N}$, *and*
(iv) $g(x) - f(x) = O(|x - c_i|^{d_0+1})$ *as* $x \to c_i$ *for* $i = 1, \ldots, m$.

If, in addition, M is a totally real set of class $\mathcal{C}^k$ and $f \in \mathcal{H}_k(X, M)$, we may additionally achieve that

(i)' $|g - f|_{k,x} < \epsilon(x)$ *for all* $x \in M$, *and*
(ii)' $|g - f|_{k,x} = 0$ *for all* $x \in A$.

Before we attend to the proof of this theorem, we give a useful corollary.

Corollary 3.2 *Let X and M be as in the previous theorem, and let $K \subset X$ be a compact set such that $K \cup M$ is holomorphically convex. Then $K \cup M$ has a Runge and Stein neighborhood basis.*

Proof Let U be an arbitrary neighborhood around $K \cup M$. As in the compact case, we will define an analytic polyhedron $\Omega \subset U$, but we will need infinitely many defining functions. Let $\{X_j\}_{j=1}^{\infty}$ be a compact exhaustion of X such that $X_j \cup M$ is holomorphically convex for each $j \in \mathbb{N}$, and where $X_1 = K$.

If $\partial U = \emptyset$, we define $\Omega = U$. Otherwise, for each point $q \in \partial U$, choose j maximal such that $q \notin X_j$. By Theorem 3.1 there exists a function $f_q \in \mathcal{O}(X)$ such that $f_q(q) = 2$ and such that $|f_q(x)| < 1$ for all $x \in X_j \cup M$. Let $\{q_i\}_{i=1}^{\infty} \subset \partial U$ be a discrete sequence of points in X such that the set

$$\{x \in X; |f_{q_i}(x)| > \frac{3}{2} \text{ for some } i \in \mathbb{N}\}$$

covers ∂U. Define

$$\Omega = \{x \in U; |f_{q_i}(x)| < 1 \text{ for all } i \in \mathbb{N}\}.$$

The set Ω is open. To see this, let $p \in \Omega$ be any point, and let $V \subset\subset X$ be an open set containing p. Then $V \subset X_j$ for some j, and since $\{q_i\}$ is discrete in X, we obtain for all sufficiently large i that $|f_{q_i}(x)| < 1$ for all $x \in V$. Hence $\Omega \cap V$ is a finite intersection of open sets.

Let $C \subset \Omega$ be compact. We claim that $\widehat{C}_{\mathcal{O}(X)} \cap \Omega$ is compact in Ω. Assume that this is not the case. Since $\widehat{C}_{\mathcal{O}(X)}$ is compact in X, there is a sequence of points $x_j \in \widehat{C}_{\mathcal{O}(X)}$ converging to a point $x \in \partial\Omega$. Then either $x \in \partial U$ or $x \in U$. If $x \in \partial U$, there exists an $i \in \mathbb{N}$ such that $x \in \{|f_{q_i}| > \frac{3}{2}\}$, which is a contradiction since $\|f_{q_i}\|_C < 1$. If $x \in U$, there exists an $i \in \mathbb{N}$ such that $|f_{q_i}(x)| \geq 1$, which is a contradiction for the same reason.

We then have

$$\widehat{C}_{\mathcal{O}(\Omega)} \subset \widehat{C}_{\mathcal{O}(X)} \cap \Omega \subset\subset \Omega,$$

and this shows that Ω is Runge and Stein. □

We will prove Theorem 3.1 by an induction procedure, where we approximate on larger and larger compact sets. First we need a version of the Oka-Weil approximation theorem, which we will call the Oka-Weil theorem with jet interpolation. Finite jet interpolation is possible on manifolds more general than Stein manifolds, and we include a brief discussion.

Definition 3.3 Let X be a complex manifold. Given a finite set of points $A = \{a_1, \ldots, a_m\}$ and an integer $d \in \mathbb{N}$, we let $\mathcal{J}_A^d$ denote the vector space of d-jets at the points a_i.

Definition 3.4 We say that a complex manifold X admits *finite jet interpolation with bounds* if the following holds: For any compact set $K \subset X$, any finite set of points $A = \{a_1, \ldots, a_m\} \subset X$ without repetition, any norm $|\cdot|$ on $\mathcal{J}_A^d$, and any integer $d \in \mathbb{N}$, there exists a constant C such that for all $j \in \mathcal{J}_A^d$ there exists a function $f \in \mathcal{O}(X)$ with $j_{a_i}^d(f) = j_i$ for $i = 1, \ldots, m$ and $\|f\|_K \leq C|j|$.

Definition 3.5 Let X be a complex manifold. Given a compact set $K \subset X$ and a function $g \in \mathcal{O}(K)$, we say that g admits uniform approximation on K if there exists a sequence $\{f_j\}_{j=1}^{\infty} \subset \mathcal{O}(X)$ such that $f_j \to g$ uniformly on K. If, additionally, we for any finite set of points $A = \{a_1, \ldots, a_m\} \subset K^\circ$ and any integer $d \in \mathbb{N}$ may also achieve that $j_{a_i}^d(f_j - g) = 0$ for $i = 1, \ldots, m$ and for all $j \in \mathbb{N}$, we say that g admits uniform approximation with jet interpolation on K.

Lemma 3.6 *Let X be a complex manifold that admits finite jet interpolation with bounds, let $K \subset X$ be a compact set, and let $g \in \mathcal{O}(K)$. If g admits uniform approximation on K then g admits uniform approximation with jet interpolation on K.*

Proof Let $f_j \in \mathcal{O}(X)$, $j \in \mathbb{N}$, be functions such that $f_j \to g$ uniformly on K, and let $A = \{a_1, \ldots, a_m\} \subset K^\circ$. Let $h_j \in \mathcal{O}(X)$ be functions such that $j_{a_i}^d(h_j) = j_{a_i}^d(f_j - g)$ for $i = 1, \ldots, m$ and $\|h_j\|_K \leq C|\mathcal{J}^d(f_j - g)|$ for all $j \in \mathbb{N}$, where $\mathcal{J}^d(f_j - g)$ is the element in $\mathcal{J}_A^d$ induced by $f_j - g$. By the Cauchy inequalities, $|\mathcal{J}^d(f_j - g)| \to 0$ as $j \to \infty$. It follows that $f_j - h_j \to g$ uniformly on K and interpolates the jets of g on A. □

Proposition 3.7 *Let X be a complex manifold. Then X admits finite jet interpolation with bounds if and only if $\mathcal{O}(X)$ separates points and local coordinates are given by entire functions.*

Proof One of the implications is clear. The other implication will follow from:

Lemma 3.8 *Let X be a complex manifold such that $\mathcal{O}(X)$ separates point and such that local coordinates are given by globally defined functions. Let $A = \{a_1, \ldots, a_m\} \subset X$ be distinct points and let $d \in \mathbb{N}$. Then there exists a function $f \in \mathcal{O}(X)$ such that*

(i) $f(x) = O(\|x - a_i\|^{d+1})$ *as* $x \to a_i$ *for* $i = 1, \ldots, m-1$, *and*
(ii) $f(x) = 1 + O(\|x - a_m\|^{d+1})$ *as* $x \to a_m$.

Before we prove this, we show how the proposition follows. Let f_i, $i = 1, \ldots, m$, be functions as in the lemma, but with a_i in place of a_m, so that f_i is tangent to 1 to order d at a_i and vanishes to order d at the other points in A. Let a compact set $K \subset X$ be given, and choose the constant C_1 such that $\|f_i\|_K \le C_1$ for $i = 1, \ldots, m$. Let $z^i = \{z^i_1, \ldots, z^i_N\}$ be local coordinates at a_i given by entire functions with $z^i(a_i) = 0$ for $i = 1, \ldots, m$. Since all the z^i_js are bounded on K and any d-jet at a_i can be expressed as a polynomial in the z^i_js, it is clear that there exists a constant C_2 such that for any d-jet j^d_i at a_i there is an entire function g_i with $j^d_{a_i}(g_i) = j^d_i$ and $\|g_i\|_K \le C_2|j^d_i|$. The function $g = \sum_{i=1}^m f_i \cdot g_i$ now interpolates the given jet to order d, and $\|g\|_K \le mC_1C_2|j^d|$.

We proceed to prove the lemma. Note first that it is enough to prove it in the case that $m = 2$. Given that, one constructs functions f_i such that f_i is 1 to order d at a_m and zero to order d at a_i for $i = 1, \ldots, m-1$ and then defines $f := \Pi_{i=1}^m f_i$.

Let $z_1, .., z_N$ be local coordinates near a_1 and let $w_1, \ldots, w_N$ be local coordinates near a_2, all given by entire functions and such that $z(a_1) = w(a_2) = 0$. Since $\mathcal{O}(X)$ separates points, we may assume that $z_j(a_2) \neq 0$ for $j = 1, \ldots, N$. By choosing polynomials in the z_js, we can create entire functions that vanish to any given order at a_1. In particular, there exists an entire function $g(x) = P(z_1(x), \ldots, z_N(x))$ such that g vanishes to order d at a_1 and such that $g(a_2) = 1$. Expanding g at a_2 gives that

$$g(x) = 1 + P_s(w_1(x), \ldots, w_N(x)) + h.o.t.,$$

where P_s is a homogenous polynomial of degree s. Consider the function $g(x) \cdot (1 - P_s(w_1(x), \ldots, w_N(x)))$. This function will be tangent to 1 to some order greater than s. Proceed like this until a function which is tangent to a sufficiently high degree is obtained. □

Theorem 3.9 *Let X be a Stein manifold and let $K \subset X$ be a holomorphically convex compact set. Then K admits uniform approximation with jet interpolation of any function $f \in \mathcal{O}(K)$.*

Proof It is well known that K admits uniform approximation of any function $f \in \mathcal{O}(K)$, and so this follows from Lemma 3.6 and Proposition 3.7. □

We will build up to the proof of Theorem 3.1 through two simpler approximation results; first we approximate functions supported on small subsets of $M \setminus K$, and then we approximate functions whose support does not intersect K. Along the way we give some useful corollaries.

Proposition 3.10 *Let X be a complex manifold, $K \subset X$ a compact set, $M \subset X$ a totally real set of class $\mathcal{C}^k$, $M_0 \subset M$ compact, and assume that $K \cup M_0$ is a Stein compactum. Then for any point $p \in M_0 \setminus K$, any open neighborhood V of p, and any set of points $\{a_i\}_{i=1}^m \subset (K \cup M_0) \setminus V$, there exist a neighborhood $U' \subset V$ of p and a Stein neighborhood Ω of $K \cup M_0$ such that the following hold: For any $f \in \mathcal{H}_k(X, M)$ with* $\mathrm{Supp}(f) \subset U'$ *and any $d \in \mathbb{N}$, there exists a sequence $\{g_j\}_{j=1}^\infty \subset \mathcal{O}(\Omega)$ satisfying*

(i) $|g_j - f|_{k,x} \to 0$ *uniformly on* $K \cup M_0$ *as* $j \to \infty$,
(ii) $|g_j - f|_{k,p} = 0$, *and*
(iii) $g_j(x) = O(|x - a_i|^{d+1})$ *as* $x \to a_i$ *for* $i = 1, \ldots, m$.

Proof Since M is locally contained in a totally real manifold of maximal (real) dimension N, there exists an open neighborhood V of p with $V \cap K = \emptyset$ and a closed, totally real submanifold M_V of V of dimension N such that $V \cap M \subset M_V$. We may assume that $f \in \mathcal{H}_k(U, M_V)$. By [9], p. 522, there are neighborhoods

$$U' \subset\subset U'' \subset\subset U \subset\subset V$$

of p and a neighborhood $W \subset U$ of $M \cap \partial U''$ such that if $f \in \mathcal{H}_k(U, M_V \cap U)$ has compact support in U', there is a sequence of holomorphic functions $\{h_j\}_{j=1}^{\infty} \subset \mathcal{O}(U)$ such that $|h_j - f|_{k,x} \to 0$ uniformly on $M_V \cap U$ as $j \to \infty$ and such that $\|h_j\|_W \to 0$ uniformly (see the remark after this proof). By the Oka-Weil theorem with jet interpolation, we may assume that $|h_j - f|_{k,p} = 0$ for all j.

Let $\{\Omega_j\}_{j=1}^{\infty}$ be a Stein neighborhood basis of $K \cup M_0$, and define $U_j^1 = \Omega_j \cap U''$ and $U_j^2 = (\Omega_j \setminus U'') \cup (W \cap \Omega_j)$. If j is large enough, we have that U_j^2 is an open set, and clearly $\Omega_j = U_j^1 \cup U_j^2$ and $U_j^1 \cap U_j^2 \subset W$. Fix a j large enough so that this holds, and drop the subscript j.

We solve a Cousin problem on Ω with respect to the cover U^1, U^2. By the solution of Cousin I with estimates (see, *e.g.*, [13], p. 304), there are sequences of functions $g_j^i \in \mathcal{O}(U^i)$ such that $h_j = g_j^1 - g_j^2$ on $U^1 \cap U^2$ and such that $g_j^i \to 0$ uniformly on compact subsets of U^i as $j \to \infty$. By Oka-Weil with jet interpolation, we may assume that all g_j^1 vanish to order k at p, and that all g_j^2 vanish to order d at the points a_i. Keeping in mind the Cauchy inequalities, we see that the sequence defined by $g_j = h_j - g_j^1$ on U^1 and $g_j = -g_j^2$ on U^2 satisfies the conclusions of the proposition. □

Remark 3.11 In [9] the approximation result is stated as follows. Given a $\mathcal{C}^k$-smooth function f *on* M_V with support in $M_V \cap U'$ there exist functions h_j holomorphic on U such that the h_js approximate f in $\mathcal{C}^k$-norm on M_V. Since in our case the Cauchy-Riemann equations for f along M_V are satisfied to order k, and since M_V is of maximal dimension, it follows that $|h_j - f|_{k,x} \to 0$ as $j \to \infty$.

The following corollary will be used in Sect. 4.

Corollary 3.12 *Any point* $p \in M_0 \setminus K$ *is a peak point for the uniform closure of* $\mathcal{O}(\Omega)|_{K \cup M_0}$. *If* $X = \mathbb{C}^N$ *and* $K \cup M_0$ *is polynomially convex, then any* $p \in M_0 \setminus K$ *is a peak point for* $\mathcal{P}(K \cup M_0)$.

Proof This is obvious. □

Proposition 3.13 *Let* X *be a Stein manifold,* $K \subset X$ *a compact set,* $M \subset X$ *a totally real set,* $M_0 \subset M$ *compact, and assume that* $K \cup M_0$ *is holomorphically convex. Then for any* $f \in \mathcal{C}(X)$ *with* $\mathrm{Supp}(f) \cap K = \emptyset$, $\{b_i\}_{i=1}^{n} \subset X \setminus (K \cup M)$ *and* $\{c_i\}_{i=1}^{m} \subset K$ *finite sets of points,* q_i *germs of holomorphic functions at the points* b_i*, and* $d \in \mathbb{N}$*, there exists a sequence* $\{h_j\}_{j=1}^{\infty} \subset \mathcal{O}(X)$ *such that*

(i) $\|h_j - f\|_{K\cup M_0} \to 0$ *as* $j \to \infty$,
(ii) $h_j(x) - q_i(x) = O(|x - b_i|^{d+1})$ *as* $x \to b_i$ *for* $i = 1, \ldots, n$, *and*
(iii) $h_j(x) = O(|x - c_i|^{d+1})$ *as* $x \to c_i$ *for* $i = 1, \ldots, m$.

If, in addition, M *is a totally real set of class* $\mathcal{C}^k$, $\{a_i\}_{i=1}^s \subset M_0 \setminus K$ *is a finite set of points, and* $f \in \mathcal{H}_k(X, M)$, *then we can also obtain*

(iv) $|h_j - f|_{k,x} \to 0$ *uniformly on* M_0 *as* $j \to \infty$, *and*
(v) $|h_j - f|_{k,a_i} = 0$ *for* $i = 1, \ldots, s$.

Proof Let U be a neighborhood of K which does not meet $\mathrm{Supp}(f)$ and which does not contain any a_i, and let $M_0' = \overline{M_0 \setminus U}$. At each point $p \in M_0'$, let V_p be an open neighborhood such that $\overline{V_p} \cap K = \emptyset$ and such that V_p contains none of the a_i, except possibly if $p = a_i$ for some i. Choose neighborhoods $U_p' \subset V_p$ as in Proposition 3.10. Let $\{U_{p_i}'\}_{i=1}^t$ be a finite cover of M_0' such that $p_i = a_i$ for $i = 1, \ldots, s$, and let M_i denote the open set $M \cap U_{p_i}'$ for $i = 1, \ldots, t$. Let $\phi^i \in \mathcal{C}_0^k(U_{p_i}')$ be functions for $i = 1, \ldots, t$ such that $\sum_{i=1}^t \phi^i(x) = 1$ for all $x \in M_0'$. As in the proof of Proposition 2.7, we may assume that each ϕ^i is $\bar\partial$-flat to order k along M. Then

$$f|_{M_0} = \sum_{i=1}^t \phi^i \cdot f,$$

where each $\phi^i \cdot f \in \mathcal{H}_k(X, M)$. For each i, choose a Stein neighborhood Ω_i of $K \cup M_0$ such that $b_j \notin \overline{\Omega}_i$ for all j and such that there is a sequence $\{g_j^i\}_{j=1}^\infty \subset \mathcal{O}(\Omega_i)$ approximating $\phi^i \cdot f$ in accordance with Proposition 3.10. Let $\Omega = \cap_{i=1}^t \Omega_i$ and let $\tilde g_j = \sum_{i=1}^t g_j^i$. Then $\{\tilde g_j\}_{j=1}^\infty$ satisfies claims (i), (iii), (iv), and (v) of the present proposition, but with Ω in place of X.

For $i = 1, \ldots, n$, let W_i be a neighborhood of the point b_i such that q_i has a representative which is holomorphic on W_i, $W_i \cap \Omega = \emptyset$, and such that $W_i \cap W_j = \emptyset$ whenever $i \neq j$. Define a sequence of functions $\{g_j\}_{j=1}^\infty \subset \mathcal{O}(\Omega \cup (\cup_{i=1}^n W_i))$ by $g_j = \tilde g_j$ on Ω and $g_j = q_i$ on W_i for $i = 1, \ldots, n$. Since $K \cup M_0 \cup \{b_1, \ldots, b_n\}$ is holomorphically convex, we can apply the Oka-Weil theorem with jet interpolation to approximate the g_j by entire functions. We thus obtain $\{h_j\}_{j=1}^\infty \subset \mathcal{O}(X)$ with all the required properties of the present proposition. □

The following corollary will be used in Sect. 4.

Corollary 3.14 *With* X, M_0, *and* K *as in the previous proposition, if* $f \in \mathcal{C}(K \cup M_0)$ *satisfies* $f|_K \equiv 1$, *then* f *is in the uniform closure of* $\mathcal{O}(X)|_{K\cup M_0}$.

Proof The function $1 - f$ can be uniformly approximated on $K \cup M_0$ by continuous functions which vanish on some (varying) neighborhood of K. Now apply Proposition 3.13. □

Proof of Theorem 3.1. Without loss of generality, $\epsilon(x) < 1$ for all $x \in K \cup M$. Let $\{K_j\}_{j=0}^\infty$ be a normal exhaustion of X such that $K_j \cup M$ is holomorphically convex for each j. We may assume that $K_0 = K$ and that $f \in \mathcal{O}(K_2)$. For $j = 1, 2, \ldots$, let

$\chi_j \in \mathcal{C}_0^\infty(X)$ be a cutoff function such that $\chi_j \equiv 1$ near K_j and such that $\mathrm{Supp}(\chi_j) \subset K_{j+1}^\circ$. As in the proof of Proposition 2.7, we may assume that χ_j is $\bar\partial$-flat along M for each j. For $j = 1, 2, \ldots$, let $C_j \in \mathbb{R}$ be a constant such that $|\chi_j \cdot F|_{k,x} \leq C_j \cdot |F|_{k,x}$ for all $x \in M$ and all $F \in \mathcal{C}^k(X)$. Choose the constants such that $1 \leq C_j \leq C_{j+1}$ for all j.

We will construct a sequence of approximating functions by induction, and the following is our inductive hypothesis I_j for $j \geq 1$: We have constructed functions $g_s \in \mathcal{O}(X)$ for $s = 0, \ldots, j$ such that

(a) $|g_j - f|_{k,x} < \sum_{s=1}^{j} 2^{-s-1}\epsilon(x)$ for all $x \in K \cup (M \cap K_{j+1})$,
(b) $|g_j - f|_{k,x} < \frac{2^{-j-1}}{C_j}\epsilon(x)$ for all $x \in M \cap \overline{K_{j+1} \setminus K_j}$,
(c) $\|g_j - g_{j-1}\|_{K_{j-1}} < 2^{-j}$,
(d) $|g_j - f|_{k,x} = 0$ for all $x \in A \cap K_{j+1}$, and
(e) $g_j(x) - q_i(x) = O(|x - b_i|^{d_i})$ as $x \to b_i$ for all $b_i \in B \cap K_j$.
(f) $g_j(x) - f(x) = O(|x - c_i|^{d_0})$ as $x \to c_i$ for all $c_i \in C$.

By the assumption that $f \in \mathcal{O}(K_2)$, we get a function $g_1 \in \mathcal{O}(X)$ satisfying conditions (a) and (c)–(f) by applying the Oka-Weil theorem with jet interpolation. Let $g_0 = g_1$, so that I_1 is satisfied.

Assume that I_j holds for some $j \geq 1$. Let $d' = \max\{d_i; b_i \in B \cap K_{j+1}\}$. The support of the function $f_j = (1 - \chi_j) \cdot (f - g_j)$ does not intersect K_j, so we may apply Proposition 3.13 to get a function $h_j \in \mathcal{O}(X)$ satisfying

(g) $|h_j - f_j|_{k,x} < \frac{2^{-j-2}}{C_{j+1}}\epsilon(x)$ for all $x \in K_j \cup (M \cap K_{j+2})$,
(h) $|h_j - f_j|_{k,x} = 0$ for all $x \in A \cap K_{j+2}$,
(i) $h_j(x) = O(|x - b_i|^{d'+1})$ as $x \to b_i$ for all $b_i \in B \cap K_j$,
(j) $h_j(x) - (g_j - q_i)(x) = O(|x - b_i|^{d'+1})$ as $x \to b_i$ for all $b_i \in B \cap (K_{j+1} \setminus K_j)$, and
(k) $h_j(x) = O(|x - c_i|^{d_0+1})$ as $x \to c_i$ for all $c_i \in C$.

It follows from (g) that $\|h_j\|_{K_j} < 2^{-j-1}$. Let $g_{j+1} = g_j + h_j$; then

$$|g_{j+1} - f|_{k,x} \leq |h_j - f_j|_{k,x} + |\chi_j(g_j - f)|_{k,x}$$

for all $x \in K \cup M$, and it is straightforward to verify that g_{j+1} satisfies the conditions in I_{j+1}.

Let $\{g_j\}_{j=1}^\infty \subset \mathcal{O}(X)$ be a sequence constructed by the inductive procedure. It is straightforward to verify that g_j converges to a limit $g \in \mathcal{O}(X)$ satisfying all the claims of the theorem. □

4 The necessary condition

Having verified the sufficiency in Theorem 1.1, it remains to show the following:

Theorem 4.1 *Let X be a Stein manifold and let $M \subset X$ be a totally real set that admits $\mathcal{C}^1$ Carleman approximation of functions in $\mathcal{H}_1(X, M)$. Then M is holomorphically convex and has bounded E-hulls in X.*

We first note that it is enough to prove this theorem in the case that $X = \mathbb{C}^N$ for some $N \in \mathbb{N}$. It is well known that if M admits uniform approximation of continuous functions on compact sets, then M is holomorphically convex. By the embedding theorem of Remmert, the pair (X, M) embeds holomorphically as closed submanifolds of $\mathbb{C}^N$, and by Cartan's Theorem B, the image M_0 of M admits Carleman approximation. If M_0 has bounded E-hulls in $\mathbb{C}^N$, then clearly M has bounded E-hulls in X.

We start by establishing a sufficient condition on certain closed sets for being polynomially convex and having bounded E-hulls in $\mathbb{C}^N$, recalling that polynomial convexity of closed sets is defined in terms of normal exhaustions. Write $\mathbb{C}^N$ as a decomposition $\mathbb{C}^N = (\mathbb{R}^k \times \mathbb{R}^{N-k}) \oplus \mathrm{i}\mathbb{R}^N$, $1 \leq k \leq N$. A graph Z over a set $S \subset \mathbb{R}^k$ is a set

$$Z = \{z = (x, y) + \mathrm{i}w \in \mathbb{C}^N; \, y = \phi(x), \, w = \psi(x), \, x \in S\},$$

where $\phi : S \to \mathbb{R}^{N-k}$ and $\psi : S \to \mathbb{R}^N$ are continuous functions.

Proposition 4.2 *Let $Z \subset \mathbb{C}^N$ be a graph over a closed set $S \subset \mathbb{R}^k$, as above, and assume that there is some $\beta < 1$ such that $\psi : S \to \mathbb{R}^N$ satisfies the Lipschitz condition $\|\psi(x) - \psi(x')\| \leq \beta\|x - x'\|$ for all $x, x' \in S$. Then Z is polynomially convex and has bounded E-hulls in $\mathbb{C}^N$.*

Proof We first observe that there is no loss in generality in assuming $k = N$, as Z also is a Lipschitz graph with the same β over the set $\{(x, \phi(x)); \, x \in S\} \subset \mathbb{R}^N$. We therefore assume that $Z = \{x + \mathrm{i}\psi(x); \, x \in S\}$ with $S \subset \mathbb{R}^N$. By Kirszbraun's theorem (see, e.g., [3]), ψ extends to a Lipschitz function $\tilde{\psi} : \mathbb{R}^N \to \mathbb{R}^N$ with the same Lipschitz constant β as ψ. Let

$$\tilde{Z} := \{x + \mathrm{i}\tilde{\psi}(x); \, x \in \mathbb{R}^N\} \supset Z.$$

To see that Z is polynomially convex, let $w_0 \in \mathbb{C}^N \setminus Z$, and let $z_0 \in \tilde{Z}$ be such that $\mathrm{Re}(z_0) = \mathrm{Re}(w_0)$, where we do not exclude the possibility that $z_0 = w_0$. Define $g(z) = \sum_{j=1}^N (z_j - (z_0)_j)^2$; then $\mathrm{Re}(g(w_0)) \leq 0$ and $\mathrm{Re}(g(z)) \geq 0$ for all $z \in \tilde{Z}$. If we set $f = \exp(-g)$, then $|f(w_0)| > \sup_{z \in Z} |f(z)|$. Approximating f by polynomials, we get that Z can be exhausted by polynomially convex compact sets.

Next we consider E-hulls, and we may assume that the origin is contained in Z. Let $K_R = \{z = x + \mathrm{i}y \in \mathbb{C}^N; \, \|x\| \leq R, \|y\| \leq R\}$; it suffices to show that $h(K_R \cup Z)$ is bounded for each $R > 0$. Choose $R' > \frac{2R}{1-\beta}$, and let $P_{R'} = \{x + \mathrm{i}y \in \mathbb{C}^N; \, \|x\| \leq R', \|y\| \leq 2R'\}$. Choose any $w_0 = u_0 + \mathrm{i}v_0 \in \mathbb{C}^N \setminus (P_{R'} \cup Z)$.

If $w_0 \notin \tilde{Z}$, then there is $z_0 = x_0 + \mathrm{i}y_0 \in \tilde{Z}$ such that $\|x_0\| \geq R'$ and $\|x_0 - u_0\| < \|y_0 - v_0\|$. Define $g(z) = \sum_{j=1}^N (z_j - (z_0)_j)^2$; then $\mathrm{Re}(g(w_0)) < 0$, if $z \in Z$ then $\mathrm{Re}(g(z)) \geq 0$ by the Lipschitz condition, and if $z \in K_R$ then

$$\mathrm{Re}(g(z)) = \|x - x_0\|^2 - \|y - y_0\|^2 \geq (\|x_0\| - R)^2 - (\beta\|x_0\| + R)^2 > 0$$

by the choices made above.

If $w_0 \in \tilde{Z}$, then let $z_0 = w_0$ and define $g(z)$ as above. We get that $g(w_0) = 0$ and that $\mathrm{Re}(g(z)) > 0$ for all $z \in K_R \cup Z$.

In any case, it follows that $w_0 \notin h(K_R \cup Z)$, and hence that $h(K_R \cup Z) \subset P_{R'}$. □

Proposition 4.3 *Let $K \subset \mathbb{C}^N$ be compact, let $F : \mathbb{C}^N \to \mathbb{C}^M$ be an entire function, and let $Y = F(K)$. For a point $y \in Y$, let F_y denote the fiber $F^{-1}(y)$, and let K_y denote the restricted fiber $F_y \cap K$. If $y \in Y$ is a peak point for the algebra $\mathcal{P}(Y)$, then*

$$\widehat{K} \cap F_y = \widehat{K}_y.$$

Proof Since F_y is an analytic set, we have that $\widehat{K}_y \subset F_y$, so clearly $\widehat{K} \cap F_y \supset \widehat{K}_y$. For the other inclusion, let $x \in F_y$, $x \notin \widehat{K}_y$. Choose a polynomial P with $P(x) = 1 > \|P\|_{K_y}$. There is a neighborhood V of y such that $|P| < 1$ on $F^{-1}(V) \cap K$, and there is a function $Q \in \mathcal{P}(Y)$ such that $Q(y) = 1$ and $|Q| < 1$ on $Y \setminus \{y\}$. For a large enough integer $m \in \mathbb{N}$, we define $f = P \cdot (Q \circ F)^m$ and get that $f(x) = 1 > \|f\|_K$. Since Q is approximable by polynomials, we get that $x \notin \widehat{K}$. □

Proposition 4.4 *Let M be a totally real set of class $\mathcal{C}^k$ in $\mathbb{C}^N$, $k \geq 1$. If M admits $\mathcal{C}^1$ Carleman approximation, then there exists a holomorphic map $F : \mathbb{C}^N \to \mathbb{C}^{2N}$ such that $F|_M$ is proper, and $F(M)$ is a totally real set of class $\mathcal{C}^k$ which is polynomially convex and has bounded E-hulls in $\mathbb{C}^{2N}$.*

Proof Let $R : \mathbb{C}^N \to \mathbb{R}^{2N}$ denote the real coordinate map

$$z = (z_1, \ldots, z_N) \mapsto (x_1, \ldots, x_{2N}),$$

where $z_j = x_{2j-1} + \mathrm{i}x_{2j}$ for $j = 1, \ldots, N$, and let M_0 denote the set $M_0 = R(M)$. We regard M_0 as a subset of $\mathbb{R}^{2N} \oplus \{0\} \subset \mathbb{R}^{2N} \oplus \mathrm{i}\mathbb{R}^{2N} = \mathbb{C}^{2N}$.

By Proposition 2.7, there is a function $\tilde{R} \in \mathcal{C}^1(\mathbb{C}^N)$ such that $\tilde{R}(z) = R(z)$ for all $z \in M$ and such that $\tilde{R}$ is $\overline{\partial}$-flat along M. The map $\tilde{R}$ can be approximated arbitrarily well in $\mathcal{C}^1$-norm on M by a holomorphic map $F = f + \mathrm{i}g : \mathbb{C}^N \to \mathbb{C}^{2N}$, *i.e.*, for any strictly positive, continuous function δ on M, we may find F such that

(i) $|f - \tilde{R}|_{1,x} < \delta(x)$ and
(ii) $|g|_{1,x} < \delta(x)$

for all $x \in M$. If δ is chosen small enough, then $f|_M$ is an embedding. Defining $\psi := g \circ f^{-1} : S \to \mathbb{R}^{2N}$, we get that $Z = (f + \mathrm{i}g)(M)$ is a graph $\{(x, \psi(x))\}$ over $S := f(M)$. Since $\|\tilde{R}(x) - \tilde{R}(y)\| = \|x - y\|$ for all $x, y \in M$, where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^{2N}$, we may choose δ small enough such that

(iii) $\|f(x) - f(y)\| \geq \frac{2}{3}\|x - y\|$ for all $x, y \in M$

and such that

(iv) $\|g(x) - g(y)\| \leq \frac{1}{2}\|x - y\|$ for all $x, y \in M$.

For $x', y' \in S$, we write $x' = f(x)$, $y' = f(y)$. It follows from (iii) and (iv) that $\|\psi(x') - \psi(y')\| = \|g(x) - g(y)\| \leq \frac{3}{4}\|x' - y'\|$. By Proposition 4.2, we get that Z is polynomially convex and has bounded E-hulls in $\mathbb{C}^{2N}$. □

We may now tie the previous results together into a proof of Theorem 4.1.

Proof of Theorem 4.1. As already noted, it is enough to prove the theorem in the case where $X = \mathbb{C}^N$. It is a well known fact that M has to be polynomially convex for approximation to hold (see, e.g., [15]). Let $K \subset \mathbb{C}^N$ be a compact set; we have to show that $h(K \cup M)$ is bounded.

Let $F : \mathbb{C}^N \to \mathbb{C}^{2N}$ be in accordance with Proposition 4.4, and let M_0 denote the image $M_0 = F(M)$. Since M_0 has bounded E-hulls in $\mathbb{C}^{2N}$, there exists $R > 0$ such that

$$h(F(K) \cup M_0) \subset R\overline{\mathbb{B}},$$

where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^{2N}$.

Let $X \subset K \cup M$ be a compact set, and let $Y = F(X)$. We first show that $h(X) \subset F^{-1}(R\overline{\mathbb{B}})$. Let $x \in \mathbb{C}^N \setminus X$ be such that $\|F(x)\| > R$. If $F(x) \notin Y$, then there exist a polynomial P such that $|P(F(x))| > \|P\|_Y = \|P(F)\|_X$, and hence $x \notin \widehat{X}$. If $F(x) = y \in Y$, then y is a peak point for $\mathcal{P}(Y)$ by Proposition 3.12, and by Lemma 4.3 it follows that $\widehat{X} \cap F_y = \widehat{X}_y$. Since X_y consists of only one point, it follows that $x \notin \widehat{X}$, and hence we must have $h(X) \subset F^{-1}(R\overline{\mathbb{B}})$.

Since $F|_M$ is proper, we have that $M \cap F^{-1}(R\overline{\mathbb{B}})$ is compact, and hence also $(K \cup M) \cap F^{-1}(R\overline{\mathbb{B}})$ is compact. To finish the proof, we show that $h(X) = h(X \cap F^{-1}(R\overline{\mathbb{B}})) \subset h((K \cup M) \cap F^{-1}(R\overline{\mathbb{B}}))$, where the last set is independent of X.

Let $C = [X \cap F^{-1}(R\overline{\mathbb{B}})]\widehat{}$, and let $x \in \mathbb{C}^N \setminus C$ with $|F(x)| \leq R$. If $F(x) \notin \widehat{Y}$, then clearly $x \notin \widehat{X}$. If $F(x) \in \widehat{Y}$, use Corollary 3.14 to obtain an $f \in \mathcal{P}(\widehat{Y})$ such that $f \equiv 1$ on $\widehat{Y} \cap R\overline{\mathbb{B}}$ and $|f| < 1$ on $\widehat{Y} \setminus R\overline{\mathbb{B}} = Y \setminus R\overline{\mathbb{B}}$. Let P be a polynomial such that $P(x) = 1 > \|P\|_C$; then the function $g := P \cdot (f \circ F)^m$ will satisfy $g(x) = 1 > \|g\|_X$ if m is large enough. Since g can be approximated by polynomials, the result follows. □

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