HOLOMORPHIC CURVES AND METRICS OF NEGATIVE CURVATURE

By

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0. Introduction

In this paper we shall prove some of the basic results in the theory of holomorphic curves using the method of negative curvature which has recently been fruitful in the study of equidimensional holomorphic mappings. The eventual goal of the theory is to understand the position of holomorphic curves in general algebraic varieties; and it seemed to us that substantial progress on this problem necessitated finding new proofs of the classical results. To explain this a little better, it may be useful to give a historical sketch of the subject.

The classical theory deals with a non-degenerate holomorphic mapping $f:\mathbf{C}\rightarrow\mathbf{P}^n$, which we shall call a *holomorphic curve*, and in brief outline developed as follows:

(i) E. Borel (1896, [4]), showed that the image curve $f(\mathbf{C})$ can miss at most $n+1$ hyperplanes in general position, thus extending Picard's theorem $(n=1)$.

(ii) A. Bloch (1926, [3]) proved Borel's theorem in finite form, among other things computing the Kobayashi metric [18] of $\mathbf{P}^2-\{4$ lines in general position$\}$.

(iii) H. Cartan (1928, [8]) clarified and extended the work of Bloch, along the way essentially proving that $\mathbf{P}^n-\{2n+1$ hyperplanes in general position$\}$ is complete hyperbolic (cf. [11], [15], [12], and [17] for further developments).

(iv) H. and J. Weyl (1938, [21]) undertook the task of extending the quantitative form of Picard's theorem, the beautiful *defect relation* of R. Nevanlinna [19], to holomorphic curves in $\mathbf{P}^n$ (cf. [21]). Although unable to accomplish this, they did develop the important theory of associated curves and non-compact Plücker formulae.

(v) Finally, L. Ahlfors (1941, [2]) was able to prove the defect relations, not only for the holomorphic curve but for the associated curves as well. (An earlier paper by H. Cartan [9] contains a defect relation for the holomorphic curve alone.) Ahlfors' work stands as the culmination of a 60 year development in the branch of function theory dealing with Picard's theorem and the study of the value distribution of entire functions (cf. the introduction to [21]), and was followed by a lull in the subject, until being recently revived by Chern [10], Kobayashi [18], Wu [22], and others.

As one reason for this lull, we think that the great beauty of the subject was perhaps offset by the technical difficulty in the proofs of the main theorems, the theorems of Bloch and Ahlfors. (A glance at the introductions of [21], [22], and [10] reveals an awe of the difficulty of Ahlfors' proofs.) As a consequence, the proofs of these results may have loomed larger than the basic principles of the subject—the Second Main Theorem and related question of contact—thus hindering further progress on the general theory. (The Ahlfors theorem strikes us as one of the few instances where *higher co-dimension* has been dealt with *globally* in complex-analytic geometry.) In this paper we give what is hopefully a conceptually simple and technically straightforward proof of the Ahlfors defect relation, a proof based on the use of negatively curved metrics.

The general philosophy is that a metric of negative curvature forces very strong global behaviour on a holomorphic mapping. Instead of attempting to formalize this philosophy, which would probably be a mistake anyway, we have tried to illustrate how it is used operationally by showing in Section 2 how such a negatively curved metric leads to Schottky-Landau theorems and defect relations. The method here is to use a potential-theoretic integral formula (Section 1), which is related to *Jensen's theorem* and the *Gauss-Bonnet theorem* [22]. Having a metric $h(\zeta)|d\zeta|^2$ of negative curvature means that h is subharmonic, and thus the signs in potential-theoretic integral formulae all go the right way.

If the reader is thus at least somewhat convinced that having a metric of negative curvature gives a defect relation, then the proof of the Ahlfors theorem follows by simply writing down such a metric (6.3) and computing its curvature. (The reader who is familiar with the standard foundational material on holomorphic curves ([21], [22]) and the formalism of Frenet frames [10] may find a proof of the Ahlfors theorem in Sections 1(a) and 6(a).) Actually, instead of just one metric we use a collection of n metrics constructed from f and its associated curves, none of which individually has negative curvature, but where the collection as a whole has negative curvature. The general philosophy of negatively curved metrics applies to such collections as well (Section 2(c)). The form of the metrics (6.3) was suggested by the Poincaré metric (2.3) on the punctured disc. (The heuristic reasoning which led to the metrics (6.3) is given in Section 6(d).) The calculation of the curvature is most effectively carried out using the Frenet equations for the holomorphic curve, which are reviewed in Section 5(a).

As further applications of the use of negatively curved metrics, we have given:

(i) in Section 5(b) a proof of the Ahlfors inequalities [2], which are traditionally derived by integral geometry and which furnish the main tool for the previous proofs of the defect relations;

(ii) a proof of the big Picard theorem for maps of the punctured disc into $\mathbf{P}^n - \{n+2$ hyperplanes in general position$\}$ (Section 6(c)); and

(iii) a proof of the defect relations for the associated curves to a holomorphic curve (Section 7). In his paper [2], Ahlfors deduced the defect relations for the holomorphic curve from the general result on associated curves, but we feel that it is conceptually clearer to treat the special case separately, since the main ideas appear here and might become lost in the combinatorial arguments necessary in the general situation.

A final comment concerning our viewpoint on defect relations: the First Main Theorem (Section 3(a)) for a holomorphic curve $f:\mathbf{C}\to\mathbf{P}^n$ gives an inequality (cf. (3.4))

$$N(A,r) < T_0(r) + C \tag{0.1}$$

bounding the *counting function* $N(A,r)$ for the points of intersection of $f(\mathbf{C})$ with a hyperplane A by its average (cf. (3.5)) $T_0(r) = \int_A N(A,r)\, d\mu(A)$, called the *order function* of f. By a *defect relation* we shall philosophically mean a lower bound

$$\sum N(A_\nu, r) \geqq CT_0(r) + C', \tag{0.2}$$

where the sum is over a set of hyperplanes in general position, and where the notation $\|$ means that the stated inequality holds outside exceptional intervals. Playing off the upper bound (0.1) against the lower bound (0.2) gives the usual form of defect relations, but it seems conceptually clearer to us to think of them as the combination of an upper and lower bound.

1. Integral formulae

(a) **Basic integral formula.** Let $\Delta_s = \{\zeta \in \mathbf{C} : |\zeta| < s\}$ be the disc of radius s in $\mathbf{C}$. Suppose given on this disc a function $h(\zeta)$ such that:

(i) near a point $\zeta_0 \in \Delta_s$, $h(\zeta)$ has the form

$$h(\zeta) = |\zeta - \zeta_0|^{2\mu} (\log|\zeta - \zeta_0|)^{2\nu} h_0(\zeta) \tag{1.1}$$

where $h_0(\zeta)$ is positive and C^∞, and

(ii) $h(\zeta)$ is positive and C^∞ near $\zeta = 0$ (this condition is not essential, but allows more uniform formulas). We call $\mu = \mu(\zeta_0)$ the *multiplicity* of ζ_0 in (1.1), and define the divisors

$$R = \sum_{\mu \geq 0} \mu(\zeta_0) \cdot \zeta_0$$

$$D = \sum_{\mu \leq 0} -\mu(\zeta_0) \cdot \zeta_0$$

Here, for reasons to be explained below, R stands for "ramification" and D for "singular divisor".

Counting functions $n(D,r)$ and $N(D,r)$ are defined as usual by

$$\left\{ \begin{aligned} & n(D,r) = \text{degree of } (D \cap \Delta_r) = \sum_{\mu<0} -\mu(\zeta_0), \ \zeta_0 \in \Delta_r \\ & N(D,r) = \int_0^r n(D,\rho) \frac{d\rho}{\rho}, \end{aligned} \right. \tag{1.2}$$

and similarly for $n(R,r)$ and $N(R,r)$. In addition to the exterior derivative ($\zeta = re^{i\theta}$),

$$d = \partial + \bar{\partial} = \frac{\partial}{\partial r}dr + \frac{\partial}{\partial \theta}d\theta,$$

we define the real operator

$$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial) = \frac{1}{4\pi}r\frac{\partial}{\partial r}d\theta - \frac{1}{4\pi r}\frac{\partial}{\partial \theta}dr,$$

and observe that

$$dd^c = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}d\zeta \wedge d\bar{\zeta}$$

is essentially the Laplacian.

Proposition. (Basic integral formula.) *For $r < s$,*

$$\frac{1}{4\pi}\int_{|\zeta|=r} \log h \cdot d\theta + N(D,r)$$

(1.3)

$$= N(R,r) + \int_0^r \left(\int_{\Delta_\rho} dd^c \log h \right) \frac{d\rho}{\rho} + \tfrac{1}{2}\log h(0).$$

Proof. An easy argument shows that both sides of (1.3) are continuous functions of r, and so we may assume that $\log h$ is C^∞ on the circle $|\zeta| = r$. In $\Delta_{r+\epsilon}$, we may thus write

$$h(\zeta) = \pi(\zeta) \cdot h_0(\zeta)$$

where the product

$$\pi(\zeta) = \prod_m |\zeta - \zeta_m|^{2\mu_m}$$

is over all points in $(R + D) \cap \Delta_{r+\epsilon}$, and where locally around any $\zeta_0 \in \Delta_{r+\epsilon}$,

$$h_0(\zeta) = (\log|\zeta - \zeta_0|)^{2\nu} h_1(\zeta)$$

with $h_1(\zeta)$ being positive and C^∞. It will obviously suffice to prove (1.3) for $\pi(\zeta)$ and $h_0(\zeta)$ separately.

For $\pi(\zeta)$, (1.3) results from *Jensen's theorem* [19] whose fundamental role in value distribution theory may be explained as follows:

Given an entire holomorphic function $f(\zeta)$, the number of solutions $n(r,a)$ of the equation

$$f(\zeta)=a$$

in the disc $|z|<r$ is given by the *Cauchy integral formula*

$$n(r,a)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f'(\zeta)d\zeta}{f(\zeta)-a}=\frac{1}{2\pi i}\int_{|\zeta|=r}\partial\log[f(\zeta)-a].$$

The difficulty in using this equation to estimate $n(r,a)$ is that the integrand is complex while $n(r,a)$ is real. This suggests that we add to it the conjugate formula obtaining

$$\begin{aligned}
n(r,a)&=\frac{1}{4\pi i}\int_{|\zeta|=r}\partial\log[f-a]-\frac{1}{4\pi i}\int_{|\zeta|=r}\overline{\partial\log[f-a]}\\
&=\frac{1}{4\pi i}\int_{|\zeta|=r}\partial\log|f-a|^2-\bar\partial\log|f-a|^2\quad(\bar\partial\log[f-a]=0)\\
&=\int_{|\zeta|=r}d^c\log|f-a|^2\\
&=r\frac{\partial}{\partial r}\left(\frac{1}{4\pi}\int_{|\zeta|=r}\log|f-a|^2d\theta\right)
\end{aligned}$$

by the formula for d^c given above. Integrating the equation

$$n(\rho,a)=\rho\frac{\partial}{\partial\rho}\left(\frac{1}{4\pi}\int_{|\zeta|=\rho}\log|f-a|^2d\theta\right)$$

with respect to $\frac{d\rho}{\rho}$ from 0 to r gives

$$\log|f(0)-a|+N(r,a)=\frac{1}{4\pi}\int_{|\zeta|=r}\log|f-a|^2\,d\theta,$$

which is Jensen's theorem, and which implies (1.3) for $\pi(\zeta)$.

As for $h_0(\zeta)$, in case there are no local $(\log|\zeta-\zeta_0|)^{2\nu}$ factors, $\log h_0$ is C^∞ and

$$\int_0^r\Big(\int_{\Delta_\rho} dd^c\log h_0\Big)\frac{d\rho}{\rho}=\int_0^r\Big(\int_{|\zeta|=\rho} d^c\log h_0\Big)\frac{d\rho}{\rho}$$

$$=\frac{1}{4\pi}\int_0^r\Big(\rho\frac{\partial}{\partial\rho}\int_{|\zeta|=\rho}\log h_0\,d\theta\Big)\frac{d\rho}{\rho}$$

$$=\frac{1}{4\pi}\int_{|\zeta|=r}\log h_0\,d\theta-\tfrac{1}{2}\log h_0(0),$$

where we have used Stokes' theorem and the formula for d^c in polar coordinates given above.

In general, singularities of the type $\log(\log|\zeta-\zeta_0|)^{2\nu}$ are sufficiently mild that the same calculation still goes through (cf. [6, Lem. 1.4]). Q.E.D.

(b) **A variant for the punctured disc.** Let $\Delta^*=\{0<|\zeta|<1\}$ be the punctured disc. We shall be interested in possible singularities at the puncture $\zeta=0$, and shall thus always assume that the functions under consideration are defined on the larger punctured disc $\{0<|\zeta|<1+\epsilon\}$ for some $\epsilon>0$. Given $h(\zeta)$ on Δ^* which has the local form (1.1) and which has no singularities on $|\zeta|=1$, we set

$$A_r=\Big\{\frac{1}{r}<|\zeta|\leqq 1\Big\}$$

$$n(D,r)=\text{degree of }(D\cap A_r)$$

$$N(D,r)=\int_1^r n(D,\rho)\frac{d\rho}{\rho}.$$

The same proof as for (1.3), only applied now to the annuli A_ρ, gives:

Proposition 1.4. *With the above assumptions and notation,*

$$\frac{1}{4\pi}\int_{|\zeta|=1} \log h\, d\theta + \int_1^r \Big(\int_{A_\rho} dd^c \log h\Big)\frac{d\rho}{\rho} - \Big(\int_{|\zeta|=1} d^c \log h\Big)\log r + N(R,r)$$

$$= N(D,r) + \frac{1}{4\pi}\int_{|\zeta|=1/r} \log h\; d\theta.$$

Corollary. *If R is empty and $h \geqq 1$, then*

$$N(D,r) \leqq \int_1^r \Big(\int_{A_\rho} dd^c \log h\Big)\frac{d\rho}{\rho} + C\log r + C'. \tag{1.5}$$

2. Metrics of negative curvature

(a) **A Schottky-Landau theorem for one pseudo-metric.** A *pseudo-metric* ω on Δ_s is given by a differential form of type (1, 1),

$$\omega = \frac{\sqrt{-1}}{2\pi} h(\zeta)\, d\zeta \wedge d\bar{\zeta},$$

with $h(\zeta)$ being a C^∞ function such that locally

$$h(\zeta) = |\zeta - \zeta_0|^{2\mu} h_0(\zeta) \tag{2.1}$$

where h_0 is positive, μ is non-negative, and $h(0) \neq 0$. (The adjective "pseudo" means that the coefficient function $h(\zeta)$ is C^∞ but may have isolated zeroes. In case $h(\zeta)$ only satisfies (1.1), we shall refer to ω as a *singular metric*.) The *Ricci form* Ric ω is the C^∞ (1, 1) form given by

$$\operatorname{Ric}\omega = dd^c \log h = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2 \log h}{\partial\zeta\partial\bar{\zeta}} d\zeta \wedge d\bar{\zeta}.$$

(Ricci forms are discussed in a general setting in [14, Sect. 0(b)]. The main property we shall use is the relation

$$\mathrm{Ric}(e^{\mu}\cdot\omega) = dd^c\mu + \mathrm{Ric}\,\omega$$

for a function μ.) If $K(h) = -\frac{1}{h}\frac{\partial^2 \log h}{\partial\zeta\partial\bar\zeta}$ is the *Gaussian curvature* of the Hermitian metric $h(\zeta)|d\zeta|^2$, then

$$\mathrm{Ric}\,\omega = -K(h)\cdot\omega$$

so that the conditions

$$\begin{cases} \mathrm{Ric}\,\omega \geqq \omega \\ K(h) \leqq -1 \end{cases}$$

are equivalent. Since $\frac{\partial^2 \log h}{\partial\zeta\partial\bar\zeta}$ is C^∞, the points where $h=0$ should be considered as having curvature $-\infty$.

The *Poincaré metric* $\tilde\omega_s$ given on Δ_s by

$$\tilde\omega_s = \frac{\sqrt{-1}}{\pi}\frac{s^2 d\zeta d\bar\zeta}{(s^2-|\zeta|^2)^2}$$

satisfies

$$\mathrm{Ric}\,\tilde\omega_s = \tilde\omega_s. \tag{2.2}$$

Using the covering transformation

$$w \to \zeta = e^{2\pi i w}$$

from the upper half plane $Im w > 0$ to the punctured disc $\Delta^* = \{0 < |\zeta| < 1\}$, the Poincaré metric induces on Δ^* the metric

(2.3) $$\tilde{\omega}_{\Delta^*} = \frac{\sqrt{-1}}{\pi} \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta|^2 \left(\log \frac{1}{|\zeta|^2}\right)^2}$$

which still satisfies (2.2).

Proposition 2.4. (Schottky-Landau for negatively curved pseudo-metrics.) *Given on* Δ_s *a pseudo-metric* ω *which satisfies* Ric $\omega_s \geqq \omega_s$, *then*

$$s \leqq S(h(0)) < \infty .$$

First proof. By the *Ahlfors lemma* [1], [18],

$$h(\zeta) \leqq \frac{2s^2}{(s^2 - |\zeta|^2)^2}$$

the R.H.S. being the coefficient of the Poincaré metric discussed above. Taking $\zeta = 0$ gives

$$s \leqq \sqrt{\frac{2}{h(0)}}.$$

Second proof. This argument, which is based on the integral formula (1.3), is more complicated but has the advantage of applying to metrics where the coefficient function h may have singularities. In fact, it is this proof together with the construction of suitable negatively curved metrics which will give the defect relations (cf. Section 2(b)).

Define the *order function* for ω by

(2.5) $$T_\omega(r) = \int_0^r \left(\int_{\Delta_\rho} \omega \right) \frac{d\rho}{\rho}.$$

Then

$$r\frac{dT_\omega}{dr} = \int_{\Delta_r} \omega = \int_0^r \left(\frac{1}{\pi} \int_0^{2\pi} h(\rho e^{i\theta}) d\theta \right) \rho d\rho ,$$

so that differentiating once more gives

$$\frac{1}{r^2}\left[\frac{d^2T_\omega}{(d\log r)^2}\right]=\frac{1}{\pi}\int_{|\zeta|=r} h(\zeta)d\theta. \tag{2.6}$$

Referring to the basic integral formula (1.3), where we assume for simplicity that $h(0)=1$, and using

$$N(D,r)\equiv 0$$

$$N(R,r)\geqq 0$$

$$\frac{1}{2\pi}\left(\int_0^{2\pi}\log f d\theta\right)\leqq\log\left(\frac{1}{2\pi}\int_0^{2\pi} f d\theta\right)$$

(*concavity of the logarithm*—concavity of the logarithm is ubiquitous in Nevanlinna theory), we obtain

$$T_\omega(r)\leqq\tfrac{1}{2}\log\left(\frac{1}{2\pi}\int_{|\zeta|=r} h d\theta\right),$$

which by (2.6) gives

$$T_\omega(r)\leqq\tfrac{1}{2}\log\left[\frac{2}{r^2}\frac{d^2T_\omega}{(d\log r)^2}\right]. \tag{2.7}$$

Now the order function $T_\omega(r)$ is a convex function of $\log r$, and because of the log on the R.H.S. it seems reasonable that an inequality such as (2.7) cannot hold for arbitrarily large r. To make this precise, we assume for a moment that $s=+\infty$ and seek a contradiction. The following calculus lemma is taken from [19]:

Lemma. *Suppose that $f(r)$, $g(r)$, $\alpha(r)$ are positive increasing functions of r where $g'(r)$ is continuous and $f'(r)$ is piecewise continuous. Then*

(2.8) $$f'(r) \leqq g'(r)\alpha(f(r))$$

outside a union E of exceptional intervals where

$$\int_E dg \leqq \int_{r_0}^{\infty} \frac{dr}{\alpha(r)}.$$

Proof.

$$\int_E dg \leqq \int_E \frac{f'(r)dr}{\alpha(f(r))} \leqq \int_{r_0}^{\infty} \frac{dr}{\alpha(r)}.$$ Q.E.D.

We shall use the notation

$$A(r) \leqq B(r) \qquad \|$$

to mean that the stated inequality holds outside an exceptional set E where

$$\int_E \frac{dr}{r} < +\infty.$$

Taking $f(r) = T_\omega(r)$, $g(r) = \log r$, and $\alpha(r) = r^{1+\epsilon}$ in (2.8), we obtain

(2.9) $$\frac{dT_\omega(r)}{d\log r} = r\frac{dT_\omega(r)}{dr} \leqq T_\omega(r)^{1+\epsilon}. \qquad \|$$

Now taking $f(r) = \dfrac{dT_\omega(r)}{d\log r}$ and g, α as before,

(2.10) $$\frac{d^2T_\omega(r)}{d\log r^2} \leqq \left(\frac{dT_\omega(r)}{d\log r}\right)^{1+\epsilon}. \qquad \|$$

Combining (2.9) and (2.10) and using a slightly larger ϵ gives

(2.11) $$\frac{d^2T_\omega(r)}{(d\log r)^2} \leqq [T_\omega(r)]^{1+\epsilon}. \qquad \|$$

On the other hand, $h(\zeta)$ is a subharmonic function and thus by the mean value principle

$$h(0) \leqq \frac{1}{2\pi} \int_{|\zeta|=\rho} h(\zeta) d\theta.$$

Integrating this twice gives respectively

$$h(0)\rho^2 \leqq \int_{\Delta_\rho} \omega,$$

(2.12)

$$\tfrac{1}{2} h(0) r^2 \leqq T_\omega(r).$$

It is now clear that (2.7), (2.11), and (2.12) cannot all hold, and so we have a contradiction to the assumption $s = +\infty$.

By being more careful in our use of the calculus lemma, it is possible using (2.12) to prove that $s \leqq S(h(0))$ (cf. [5, pp. 289–290]). Q.E.D.

(b) **Defect relation for one singular metric.** A *singular metric* ω on **C** is given by

$$\omega = \frac{\sqrt{-1}}{2\pi} h(\zeta) d\zeta \wedge d\bar{\zeta}$$

where the coefficient function h satisfies (1.1) above. The Ricci form is again defined by Ric $\omega = dd^c \log h$.

Lemma 2.13. (i) *ω is integrable $\Leftrightarrow$ $\mu \geqq -1$ and $\nu < -\frac{1}{2}$ if $\mu = -1$;*
(ii) *if* Ric $\omega \geq \omega$, *then ω is integrable*; *and*
(iii) Ric ω *is always integrable.*

Proof. Using polar coordinates $\zeta - \zeta_0 = re^{i\theta}$ in (1.1), we see that

$$h = \frac{h_0}{r^{-2\mu}\left(\log \frac{1}{r}\right)^{-2\nu}},$$

and so (i) follows because

$$\int_0^\epsilon \frac{rdr}{r^{-2\mu}\left(\log \frac{1}{r}\right)^{-2\nu}} < \infty$$

only under the stated conditions.

To prove (ii), we look in the punctured disc $\Delta_\epsilon^* = \{0 < |\zeta - \zeta_0| < \epsilon\}$ around ζ_0 where (1.1) holds. For simplicity, change coordinates to make $\epsilon = 1$. Then by the Ahlfors lemma

$$\omega \leqq \tilde{\omega}_{\Delta^*}$$

in Δ^*, where the Poincaré metric $\tilde{\omega}_{\Delta^*}$ is given by (2.4), which is the case $\mu = -1$, $\nu = -\frac{1}{2}$ in (1.1). Now apply (i).

Finally, (iii) follows from the computation

$$-\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}} \log\left(\log\frac{1}{|\zeta|^2}\right)^2 = \frac{2}{|\zeta|^2\left(\log\frac{1}{|\zeta|^2}\right)^2},$$

which is the statement that the Gaussian curvature of $\tilde{\omega}_{\Delta^*}$ is constant negative. Q.E.D.

Let ω, ϕ be singular metrics on $\mathbf{C}$ and assume that

$$\text{(2.14)} \qquad \operatorname{Ric}\omega \geqq \phi + \omega.$$

(The use of negative curvature in the guise of an *excess relation* by the singular metric ϕ will be the most convenient.) According to the Schottky-Landau Theorem 2.4, ω must have singularities, and we can even estimate where the first one occurs. Our next result will give a lower bound on the size of the singular divisor D of ω; *it is essentially a defect relation.* Before stating it, we remark that, by (2.13), ω and ϕ are both integrable and so their order functions $T_\phi(r)$ and $T_\omega(r)$ are defined.

Proposition 2.15. (Defect relation for a single negatively curved metric.) *If* ω, ϕ *are singular metrics on* $\mathbf{C}$ *which satisfy* (2.14), *then given* $\epsilon > 0$ *we have an estimate*

(2.16) $$N(D,r) \geqq T_\phi(r) + (1-\epsilon)T_\omega(r). \qquad \|$$

Proof. Using (2.14) and the basic integral formula (1.3), we obtain for $r \geqq 1$,

$$T_\phi(r) + T_\omega(r) \leqq N(D,r) + \frac{1}{4\pi} \int_{|\zeta|=r} \log h \cdot d\theta + C$$

$$\leqq N(D,r) + \tfrac{1}{2}\log\left[\frac{d^2 T_\omega(r)}{(d \log r)^2}\right] + C,$$

the last step being (2.6). Applying (2.11) we have

$$\epsilon T_\omega(r) - \tfrac{1}{2}\log\left[\frac{d^2 T_\omega(r)}{(d \log r)^2}\right] - C \geq 0, \qquad \|$$

which when added to the previous inequality gives (2.16). Q.E.D.

(c) **The case of several metrics.** The geometric situation of a holomorphic curve in $\mathbf{P}^n$ will give us, instead of a single metric, a collection

(2.17) $$\omega_i = \frac{\sqrt{-1}}{2\pi} h_i d\zeta \wedge d\bar{\zeta} \qquad (i = 1,\cdots,n)$$

of singular metrics, none of which individually is negatively curved, but where we have an inequality of essentially the following sort

(2.18) $$\sum_{i=1}^{n} \operatorname{Ric} \omega_i \geqq \sum_{i=1}^{n} \omega_i.$$

Definition. We shall say that the *collection* $\{\omega_i\}$ *of metrics is negatively curved* if (2.18) is satisfied.

Given a negatively curved collection of metrics (2.17), we may construct a single negatively curved metric $\omega = \frac{\sqrt{-1}}{2\pi} h\, d\zeta \wedge d\bar{\zeta}$ by setting

(2.19) $$h = (h_1 \cdots h_n)^{1/n}.$$

Then, using the inequality of arithmetic and geometric means,

$$\operatorname{Ric}\omega = dd^c \log h = \frac{1}{n}\Big(\sum_i \operatorname{Ric}\omega_i\Big)$$

$$\geqq \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{n}(h_1 + \cdots + h_n)\right] d\zeta \wedge d\bar{\zeta}$$

$$\geqq \frac{\sqrt{-1}}{2\pi}(h_1 \cdots h_n)^{1/n} d\zeta \wedge d\bar{\zeta} = \omega .$$

Applying (2.4) to ω we obtain:

Proposition 2.20. (Schottky-Landau for a negatively curved collection of pseudo metrics.) *Let $\{\omega_i\}$ be a negatively curved collection of pseudometrics on Δ_s. Then*

$$s \leqq S(h_1(0), \cdots, h_n(0)) < \infty .$$

Before giving the defect relation, we observe from (iii) in (2.13) and (2.18) that ω_i is integrable, and thus the order function

$$T_i(r) = T_{\omega_i}(r) = \int_0^r \Big(\int_{\Delta_\rho} \omega_i\Big)\frac{d\rho}{\rho}$$

is defined. The singular divisor of ω_i is denoted by D_i.

Proposition 2.21. (Defect relations for a negatively curved collection of metrics.) *Let $\{\omega_i\}$, ϕ be singular metrics on* **C** *and assume that*

$$\sum_i \operatorname{Ric}\omega_i \geqq \sum_i \omega_i + \phi .$$

Then, given $\epsilon > 0$ we have

$$\sum_i N(D_i, r) \geqq (1-\epsilon)\Big\{\sum_i T_i(r)\Big\} + T_\phi(r) . \quad \|\tag{2.22}$$

Proof. Define $k = h_1 \cdots h_n$, so that

$$\begin{cases} dd^c \log k \geqq \Sigma\, \omega_i + \phi \\ D = D_1 + \cdots + D_n \end{cases}$$

where D is the singular divisor of k. From (1.3) we have, assuming for simplicity that $k(0) = 1$,

$$(2.23) \qquad (1-\epsilon)\Big(\sum_i T_i(r)\Big) + T_\phi(r) + \epsilon\Big(\sum_i T_i(r)\Big)$$

$$\leqq \sum N(D_i, r) + \frac{1}{4\pi}\int_{|\zeta|=r} \log k\, d\theta.$$

On the other hand, by (2.6),

$$\frac{1}{4\pi}\int_{|\zeta|=r} \log k\, d\theta = \frac{1}{4\pi}\sum_i \int_{|\zeta|=r} \log h_i d\theta$$

$$\leqq \sum_i \tfrac{1}{2}\log\left[\frac{2}{r^2}\frac{d^2 \log T_i(r)}{(d\log r)^2}\right],$$

which using (2.11) gives, for $r \geqq 1$,

$$(2.24) \qquad \frac{1}{4\pi}\int_{|\zeta|=r} \log k\, d\theta \leqq \sum_i \tfrac{1}{2}\log |T_i(r)|^{1+\epsilon}. \qquad \|$$

Combining (2.24) and (2.23) gives (2.22) as in the case of a single metric.

Q.E.D.

3. Holomorphic curves in algebraic varieties

(a) The order function and First Main Theorem (F.M.T.).

Let M be a compact, complex manifold. A holomorphic mapping $f: \mathbf{C} \to M$ will be called a *holomorphic curve*. A natural problem is to study the position of $f(\mathbf{C})$ in M. In particular, for obvious dimension reasons, we wish to see how the image $f(\mathbf{C})$ meets the divisors on M.

For this the terminology of *line bundles* and *Chern classes* gives a convenient formalism (cf. [14, Sect. 0] for a general discussion). Suppose that $L \to M$ is a positive line bundle with metric $|\sigma|^2$ ($\sigma \in \mathcal{O}(L)$) and corresponding positive Chern class

$$\omega = dd^c \log \frac{1}{|\sigma|^2}. \tag{3.1}$$

We denote by $|L|$ the *complete linear system* of all effective divisors $D = (\sigma)$ ($\sigma \in \mathcal{O}(M, L) = H^0(M, \mathcal{O}(L))$). By the compactness of M, $\dim \mathcal{O}(M, L) < \infty$ and $(\sigma) = (\sigma')$ if, and only if, $\sigma = \lambda\sigma'$ for some $\lambda \in \mathbf{C}^*$. Thus $|L|$ is a finite-dimensional projective space.

To measure the "growth" of a holomorphic curve $f : \mathbf{C} \to M$ relative to the given line bundle, we set $\omega_f = f^*\omega$ and define the *order function*

$$T(L, r) = \int_0^r \Big(\int_{\Delta_\rho} \omega_f \Big) \frac{d\rho}{\rho}. \tag{3.2}$$

If f is non-constant, then clearly $T(L, r) \to \infty$ as $r \to \infty$. Moreover, changing the metric in $L \to M$ changes the order function by an $O(1)$ term ([14, Sect. 5]). Thus $T(L, r)$ is essentially intrinsic.

The holomorphic curve $f : \mathbf{C} \to M$ is said to be *non-degenerate* relative to $L \to M$ if the image $f(\mathbf{C})$ does not lie in any $D \in |L|$. Assuming this to be the case, we set $D_f = f^{-1}(D)$ and define the *counting function*

$$N(D, r) = N(D_f, r),$$

the R.H.S. being given by (1.2). Choose $\sigma \in \mathcal{O}(M, L)$ with $D = (\sigma)$ and set

$$m(D, r) = \frac{1}{4\pi} \int_{|\zeta| = r} \log \frac{1}{|f^*\sigma|^2} \, d\theta \qquad (\textit{proximity form}).$$

A different choice of σ changes $m(D, r)$ by an additive constant, and it will be convenient to always assume that $|\sigma| \leqq 1$ so that $m(D, r) \geqq 0$.

Proposition 3.3. (F.M.T.)

$$N(D,r)+m(D,r)=T(L,r)+O(1,D).$$

Proof. In case $0\notin D_f$, we may take $h(\zeta)=1/|\sigma(f(\zeta))|^2$ in the basic integral formula (1.3), and use (3.1) and (3.2) to obtain the F.M.T. If $0\in D_f$, then near $\zeta=0$, $|\sigma(f(\zeta))|^2=|\zeta|^{2\mu}h_0$ where $h_0(0)>0$. We may then apply (1.3) to $h(\zeta)=\dfrac{|\zeta|^{2\mu}}{|\sigma(f(\zeta))|^2}$ to obtain the F.M.T., where the counting function must now be defined by

$$N(D,r)=\int_0^r [n(D,\rho)-n(0,\rho)]\frac{d\rho}{\rho}+n(0,r)\log r$$

since (1.2) no longer has meaning. Q.E.D.

Corollary. (Nevanlinna inequality.)

$$N(D,r)\leqq T(L,r)+O(1,D). \tag{3.4}$$

The beautiful inequality, which generalizes the estimate on the number of zeroes of an analytic function by its maximum modulus, underlies all of Nevanlinna theory. The reader is invited to read the discussion in [19] of the F.M.T. and subsequent inequality (3.4) for a very pretty explanation of the global symmetry in an entire meromorphic function.

(b) **Crofton's formula and the Liouville theorem.** Suppose now that $M=\mathbf{P}^n$ and $L\to\mathbf{P}^n$ is the hyperplane line bundle. The metric and Chern class in L will be given explicitly in Section 4 below. For the moment all we need to know is that the unitary group operating on $\mathbf{C}^{n+1}$ induces an action on $L\to\mathbf{P}^n$ leaving the metric and Chern class ω invariant. The complete linear system $|L|$ is the dual projective space $\mathbf{P}^{n*}$ of hyperplanes in $\mathbf{P}^n$, and there is a unique measure $d\mu(D)$ on $\mathbf{P}^{n*}$ which is invariant under the unitary group and which satisfies $\int_{\mathbf{P}^{n*}} d\mu(D)=1$.

Propositioır 3.5. (Crofton's formula.)

$$\int_{D\in\mathbf{P}^{n*}} N(D,r)\,d\mu(D) = T(L,r).$$

Proof. Let G be the unitary group and $d\mu(g)(g\in G)$ the invariant measure with suitable normalization. For an integrable function $\eta(D)$ defined on $\mathbf{P}^{n*}$ and fixed hyperplane D_0,

$$\int_{D\in\mathbf{P}^{n*}} \eta(D)\,d\mu(D) = \int_{g\in G} \eta(gD_0)\,d\mu(g). \tag{3.6}$$

Fixing $\sigma_0 \in \mathcal{O}(\mathbf{P}^n, L)$ which defines D_0,

$$\int_{g\in G} \log\frac{1}{|\sigma_0(gx)|^2}\,d\mu(g) = \int_{g\in G} \log\frac{1}{|g^*\sigma_0(x)|^2}\,d\mu(g)$$

is a constant C independent of $x\in\mathbf{P}^n$ since G acts transitively on $\mathbf{P}^n$. Changing σ_0 to $\lambda\sigma_0$ adds $\log\dfrac{1}{|\lambda|}$ to C, and so we may assume that $C=0$. Thus by (3.6) and the definition

$$\int_{D\in\mathbf{P}^{n*}} m(D,r)\,d\mu(D) = \int_{D\in\mathbf{P}^{n*}} \left(\frac{1}{4\pi}\int_{|\zeta|=r} \log\frac{1}{|\sigma(f(\zeta))|^2}\,d\theta\right) d\mu(D)$$

$$= \frac{1}{4\pi}\int_{|\zeta|=r}\left(\int_{g\in G} \log\frac{1}{|g^*\sigma_0(f(\zeta))|^2}\,d\mu(g)\right) d\theta = 0.$$

Similarly, $\displaystyle\int_{D\in\mathbf{P}^{n*}} O(1,D)\,d\mu(D) = 0$ since $O(1,D) = \log\dfrac{1}{|\sigma(f(0))|^2}$ for $0\notin D_f$.

Integrating the F.M.T. (3.3) over $\mathbf{P}^{n*}$ gives (3.5). Q.E.D.

Remark. Differentiating (3.5) gives

$$\int_{D \in \mathbf{P}^{n*}} n(D,r)\, d\mu(D) = \int_{\Delta_r} \omega_f, \tag{3.7}$$

which is the usual form of Crofton's formula ([20, pp. 12–13]). Chern has remarked that the Nevanlinna inequality

$$N(D,r) \leqq T(L,r) + O(1)$$

should be viewed as a non-compact version of the *Wirtinger theorem*, which states the degree ($\leftrightarrow N(D,r)$) of an algebraic curve in $\mathbf{P}^n$ is equal to its area ($\leftrightarrow T(L,r)$). (This ties in nicely with Hermann Weyl's interpretation of the Second Main Theorem as non-compact Plücker relations.) It is the combination of the Nevanlinna inequality (3.4) and Crofton formula (3.5) which seem to force such delicate and refined results as the Picard theorem and defect relations. As a first indication of this, we shall prove the following:

Corollary 3.8. (Liouville theorem.) *A non-degenerate holomorphic curve in* $\mathbf{P}^n$ *meets almost all hyperplanes* $D \in \mathbf{P}^{n*}$.

Proof. Let $E \subset \mathbf{P}^{n*}$ be the set of hyperplanes which $f(\mathbf{C})$ misses. By (3.5) and (3.4),

$$T(L,r) = \int_{D \in \mathbf{P}^{n*}} N(D,r)\, d\mu\,(D)$$

$$= \int_{D \in \mathbf{P}^{n*} - E} N(D,r)\, d\mu\,(D)$$

$$\leqq \mu(\mathbf{P}^n - E)\, T(L,r) + O(1),$$

from which it follows that E cannot contain an open set $E \subset \mathbf{P}^{n*}$. Q.E.D.

Remark. Ahlfors' proof of the defect relation was a great refinement of this argument, where the averaging was with respect to a singular density in $\mathbf{P}^{n*}$.

4. Holomorphic Curves in $\mathbf{P}^n$; the Second Main Theorem

(a) **Projective spaces and Grassmannians.** We shall represent points in $\mathbf{P}^n$ by homogeneous coordinates

$$Z = [z_0, \cdots, z_n].$$

The hyperplane line bundle $L \to \mathbf{P}^n$ has global sections $\mathcal{O}(\mathbf{P}^n, L)$ given by the linear forms $A = (a_0, \cdots, a_n)$ on $\mathbf{C}^{n+1}$ where

$$A(Z) = \langle Z, A \rangle = \sum_{i=0}^{n} z_i a_i.$$

The divisor associated to A is given by $\langle Z, A \rangle = 0$, and we denote this hyperplane also by A. The complete linear system $|L| = \mathbf{P}^{n*}$ is the dual projective space of hyperplanes in $\mathbf{P}^n$. The metric in $L \to \mathbf{P}^n$ is given explicitly by

$$|A(Z)|^2 = \frac{|\langle Z, A \rangle|^2}{|Z|^2 |A|^2} = \frac{|Z, A|^2}{|Z|^2 |A|^2}, \tag{4.1}$$

where the first equality is a definition and the second is notation. The Chern class

$$\Omega = dd^c \log \frac{1}{|A(Z)|^2} = dd^c \log |Z|^2 \tag{4.2}$$

is the Kähler form associated to the usual *Fubini-Study metric* on $\mathbf{P}^n$.

The Grassmann manifold of linear k-spaces in $\mathbf{P}^n$ is denoted by $G(k, n)$. Using the identification

$$G(k, n) = \{(k+1)\text{–planes through the origin in } \mathbf{C}^{n+1}\},$$

a point $\Lambda \in G(k,n)$ is given by choosing $k+1$ vectors $Z_0,\cdots,Z_k$ which span Λ. If we denote the standard basis for $\mathbf{C}^{n+1}$ by $e_0,\cdots,e_n$, then

$$Z_0 \wedge \cdots \wedge Z_k = \sum_{i_0<\cdots<i_k} \Lambda_{i_0,\cdots,i_k} e_{i_0} \wedge \cdots \wedge e_{i_k} \tag{4.3}$$

where the $\Lambda_{i_0,\cdots,i_k}$ are the homogeneous *Plücker coordinates* of $G(k,n)$ in $\mathbf{P}^{\binom{n+1}{k+1}-1}$. According to (4.2), the metric on $G(k,n)$ induced by the Plücker embedding is

$$\Omega_k = dd^c \log|\Lambda|^2 = dd^c \log|Z_0 \wedge \cdots \wedge Z_k|^2. \tag{4.4}$$

A holomorphic mapping $f:\Delta_r \to \mathbf{P}^n$ will be called a *holomorphic curve.* We may represent $f(\zeta)$ by a holomorphic vector $Z(\zeta) = [z_0(\zeta),\cdots,z_n(\zeta)]$. It is useful to allow that $Z(\zeta_0) = (0,\cdots,0)$ for isolated points ζ_0. If this happens, then near ζ_0 we write

$$Z(\zeta) = (\zeta - \zeta_0)^{\nu} \tilde{Z}(\zeta), \quad \tilde{Z}(\zeta_0) \neq (0,\cdots,0), \tag{4.5}$$

and f is given by $\tilde{Z}(\zeta)$. To compute the *ramification* of f at ζ_0 where $Z(\zeta_0) \neq 0$, one proceeds as follows:

By a suitable linear transformation, bring $Z(\zeta)$ to the form

$$Z(\zeta) = [1+\cdots,(\zeta-\zeta_0)^{\mu_1+1}+\cdots,\cdots,(\zeta-\zeta_0)^{\mu_n+1}+\cdots] \tag{4.6}$$

where $0 \leqq \mu_1 \leqq \cdots \leqq \mu_n$ and "$\cdots$" denotes higher order terms in $(\zeta-\zeta_0)$. *Then the ramification index of f at ζ_0 is μ_1.*

(b) **The associated curves.** Let $f:\Delta_r \to \mathbf{P}^n$ be a holomorphic curve given by $Z(\zeta) \in \mathbf{C}^{n+1}$. It is permissible to multiply $Z(\zeta)$ by a holomorphic function $\rho(\zeta)$, to change coordinates in $\mathbf{C}^{n+1}$ by a non-singular matrix if we are interested in the linear structure, and by a unitary matrix if we are using the metric structure. The holomorphic curve is *non-degenerate* if the image does not lie in a proper linear subspace of $\mathbf{P}^n$. This is equivalent to the condition

$$Z(\zeta) \wedge Z'(\zeta) \wedge \cdots \wedge Z^{(n)}(\zeta) = \det(z_i^{(k)}(\zeta)) \neq 0, \tag{4.7}$$

where $\det(z_i^{(k)}(\zeta))$ is the *Wronskian* of the coordinate functions $z_0(\zeta),\cdots,z_n(\zeta)$. Such an analytic condition for non-degeneracy is not yet available for holomorphic curves in general algebraic varieties.

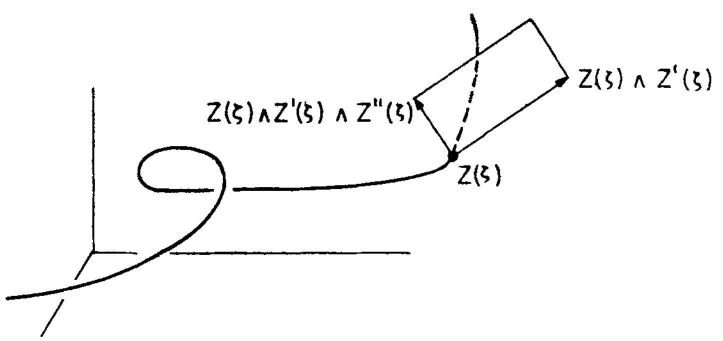
Given a holomorphic curve f, there are naturally associated holomorphic mappings

$$f_k : \Delta_r \to G(k,n) \qquad (k = 0,\cdots,n-1)$$

given by the homogeneous coordinate vectors

$$\Lambda_k(\zeta) = Z(\zeta) \wedge \cdots \wedge Z^{(k)}(\zeta) \in \mathbf{C}^{\binom{n+1}{k+1}}.$$

Thus $f_0 = f, f_1$ is the tangent line to the curve, f_2 is the osculating 2-plane, etc.



These *associated curves* are of fundamental importance in the theory. Geometrically they provide a means for interpolating between the 1-dimensional curve and n-dimensional projective space.

It is standard that a non-constant meromorphic function $f(\zeta)$ has around $\zeta = 0$ the local form

$$f(\zeta) = \zeta^\mu + \cdots$$

in a suitable linear coordinate system on $\mathbf{P}^1$. In homogeneous coordinates this is the same as

$$f(\zeta) = [1, \zeta^\mu + \cdots], \tag{4.8}$$

and we want to generalize this normal form to a non-degenerate holomorphic curve. Introduce integers $a_{-1}, a_0, \cdots, a_{n-1}$ such that

$$Z^{(a_{-1})}(0), Z^{(1+a_{-1}+a_0)}(0), \cdots, Z^{(n+a_{-1}+\cdots+a_{n-1})}(0) \tag{4.9}$$

are the lowest order linearly independent derivatives at $\zeta = 0$. The point is a *regular point* in case $a_0 = \cdots = a_{n-1} = 0$; otherwise we have a *stationary point*. Choosing the vectors (4.9) as a linear coordinate system for $\mathbf{C}^{n+1}$ and multiplying $Z(\zeta)$ by a suitable factor, we may give $f(\zeta)$ by

$$Z(\zeta) = [1, \zeta^{1+a_0} + \cdots, \frac{1}{2!}\zeta^{2+a_0+a_1} + \cdots, \cdots, \frac{1}{n!}\zeta^{n+a_0+\cdots+a_{n-1}} + \cdots]. \tag{4.10}$$

This is the analogue of (4.8). From (4.10),

$$\begin{aligned} \Lambda_k(\zeta) &= \zeta^{ka_0+(k-1)a_1+\cdots+a_{k-1}} e_0 \wedge \cdots \wedge e_k + \cdots \\ &= \zeta^{\nu_k} e_0 \wedge \cdots \wedge e_k + \cdots. \end{aligned} \tag{4.11}$$

Lemma 4.12. *The associated curve $f_k : \Delta_r \to G(k,n)$ has a ramification point of order a_k at $\zeta = 0$.*

Proof. We shall do the case $n = 3$, $k = 1$. Then

$$Z(\zeta) = [1, \zeta^{1+a} + \cdots, \zeta^{2+a+b} + \cdots, \zeta^{3+a+b+c} + \cdots],$$

$$\begin{aligned} \Lambda_1(\zeta) &= \zeta^a e_0 \wedge e_1 + \zeta^{1+a+b} e_0 \wedge e_2 + \cdots \\ &= \zeta^a \{e_0 \wedge e_1 + \zeta^{1+b} e_0 \wedge e_2 + \cdots\}. \end{aligned}$$

Using the prescription at the end of Section 4(a), $\Lambda_1(\zeta)$ has a ramification of order b at $\zeta = 0$. Q.E.D.

Remark. Referring to (4.11),

$$a_k = \nu_{k+1} + \nu_{k-1} - 2\nu_k \tag{4.13}$$

is the second difference of the ν_k's. If $a_k > 0$, we say that $\zeta = 0$ is a *stationary point of index k and order* a_k. The divisor

$$R_k = \Sigma a_k(\zeta) \cdot \zeta$$

measures the ramification of the kth associated curve, and we set (cf. (1.2))

$$N_k(r) = N(R_k, r). \tag{4.14}$$

(c) **The Second Main Theorem** (S.M.T.). Given a non-degenerate holomorphic curve $f: \mathbf{C} \to \mathbf{P}^n$, we denote by

$$\Omega_k = dd^c \log |\Lambda_k(\zeta)|^2 = \frac{\sqrt{-1}}{2\pi} h_k d\zeta \wedge d\bar{\zeta} \tag{4.15}$$

the pseudo-metric on $\mathbf{C}$ induced by the standard metric (cf. (4.4)) on $G(k, n)$, and let

$$T_k(r) = \int_0^r \left(\int_{\Delta_\rho} \Omega_k \right) \frac{d\rho}{\rho}$$

be the *order function* for the kth associated curve. Then $T_0(r)$ is the order function for f. The S.M.T. regulates the balance between the growth of the various T_k's, and represents an intrinsic form and generalization of the relation between the growth of an analytic function and of its derivative. According to a beautiful remark of H. Weyl [21], the S.M.T. should be thought of as *non-compact Plücker formulae.*

We wish to compute the Ricci form

$$\mathrm{Ric}\,\Omega_k = dd^c \log h_k$$

for Ω_k given by (4.15), and the following is the main step:

Lemma 4.16.

$$h_k = \frac{|\Lambda_{k-1}|^2 |\Lambda_{k+1}|^2}{|\Lambda_k|^4}.$$

Proof. It will suffice to check (4.16) at a regular point ζ_0; this is because h_k is obviously C^∞, and $\frac{|\Lambda_{k-1}|^2|\Lambda_{k+1}|^2}{|\Lambda_k|^4}$ is C^∞ since $\nu_{k+1}+\nu_{k-1}-2\nu_k \geqq 0$ (cf. (4.11) and (4.13)). Representing $f(\zeta)$ by a vector $Z(\zeta)$, we wish to first find $\alpha_1,\cdots,\alpha_{k+1}$ such that, if

$$\tilde{Z}(\zeta)=\left[1+\alpha_1(\zeta-\zeta_0)+\cdots+\frac{\alpha_{k+1}}{(k+1)!}(\zeta-\zeta_0)^{k+1}\right]Z(\zeta),$$

then the inner products

$$(\tilde{Z}^{(k+1)}(\zeta_0),\tilde{Z}^{(j)}(\zeta_0))=0 \qquad (j=0,\cdots,k). \tag{4.17}$$

Now $\tilde{Z}^{(j)}(\zeta_0)=Z^{(j)}(\zeta_0)+\binom{j}{1}\alpha_1 Z^{(j-1)}(\zeta_0)+\cdots+\binom{j}{j}\alpha_j Z(\zeta_0)$, so that (4.17) is equivalent to the system

$$(\tilde{Z}^{(k+1)}(\zeta_0),Z^{(j)}(\zeta_0))=0 \qquad (j=0,\cdots,k)$$

of $k+1$ linear equations in the $k+1$ unknowns $\alpha_1,\cdots,\alpha_{k+1}$ with non-singular coefficient matrix $(Z^{(i)}(\zeta_0),Z^{(j)}(\zeta_0))$ $(0\leqq i,j\leqq k)$. Solving these equations gives (4.17).

Changing notation, we assume (4.17) for $Z(\zeta)$. Then at $\zeta=\zeta_0$,

$$\begin{cases}\text{(i)} \quad (Z\wedge\cdots\wedge Z^{(k)},\ Z\wedge\cdots\wedge Z^{(k-1)}\wedge Z^{(k+1)})=0\\ \text{(ii)} \quad (Z\wedge\cdots\wedge Z^{(k-1)}\wedge Z^{(k+1)},\ Z\wedge\cdots\wedge Z^{(k-1)}\wedge Z^{(k+1)})\end{cases} \tag{4.18}$$

$$=\frac{|\Lambda_{k-1}|^2|\Lambda_{k+1}|^2}{|\Lambda_k|^2}.$$

Computing derivatives at $\zeta=\zeta_0$, we find

$$h_k(\zeta_0)=\frac{\partial^2\log(Z\wedge\cdots\wedge Z^{(k)},Z\wedge\cdots\wedge Z^{(k)})}{\partial\zeta\partial\bar{\zeta}} \qquad \text{(by (4.15))}$$

$$=\frac{\partial}{\partial\zeta}\left(\frac{(Z\wedge\cdots\wedge Z^{(k)},Z\wedge\cdots\wedge Z^{(k-1)}\wedge Z^{(k+1)})}{(Z\wedge\cdots\wedge Z^{(k)},Z\wedge\cdots\wedge Z^{(k)})}\right) \qquad (\bar{\partial}Z^{(i)}=0)$$

$$= \frac{(Z \wedge \cdots \wedge Z^{(k-1)} \wedge Z^{(k+1)}, Z \wedge \cdots \wedge Z^{(k-1)} \wedge Z^{(k+1)})}{(Z \wedge \cdots \wedge Z^{(k)}, Z \wedge \cdots \wedge Z^{(k)})} \qquad \text{(by (4.18i))}$$

$$= \frac{|\Lambda_{k-1}|^2 |\Lambda_{k+1}|^2}{|\Lambda_k|^4} \qquad \text{(by (4.18ii))}.$$

Q.E.D.

(Our proof of (4.16) is a little different from the usual argument in that we have chosen a special representative $Z(\zeta)$ for $f(\zeta)$ to make the "cross-terms" drop out, thus circumventing much of the linear algebra. This device will be used consistently; roughly speaking, the general philosophy is that terms which have no intrinsic meaning may, by suitable choice of coordinates, be made equal to zero at a given point.)

From (4.15) and (4.16) we find

$$\text{(4.19)} \qquad \operatorname{Ric} \Omega_k = dd^c \log h_k = \Omega_{k+1} + \Omega_{k-1} - 2\Omega_k.$$

Applying the basic integral formula (1.3) to (4.19) gives:

Second Main Theorem

$$\text{(4.20)} \qquad T_{k-1}(r) + T_{k+1}(r) + N_k(r) = 2T_k(r) + \frac{1}{4\pi} \int_{|\zeta|=r} \log h_k d\theta + C'.$$

Corollary 4.21. $T_{k-1}(r) + T_{k+1}(r) \leqq 2T_k(r) + C \log T_k(r) + C'$. ||

Proof. This follows from (4.20) and

$$\frac{1}{4\pi} \int_{|\zeta|=r} \log h_k d\theta \leqq \tfrac{1}{2} \log\left(\frac{1}{2\pi} \int_{|\zeta|=r} h_k d\theta\right)$$

$$= \tfrac{1}{2} \log\left[\frac{2}{r^2} \frac{d^2 T_k(r)}{(d \log r)^2}\right] \qquad \text{(by (2.6))}$$

$$\leqq C \log T_k(r) \qquad \text{(by (2.11))}.$$

||

Remark. The S.M.T. 4.20 may be written as

$$\begin{cases} T_{k-1}(r) - 2T_k(r) + T_k(r) = \epsilon(T_k(r)) \\ \epsilon(T_k(r)) \leqq C \log T_k(r) + C', \end{cases} \qquad \|$$

an estimate on the second differences of the $T_k(r)$ which should be compared with (4.13).

From (4.21) we obtain the inequalities (see [22])

$$(4.22) \qquad \begin{cases} (k+1)T_l(r) \leqq (l+1)T_k(r) + \epsilon(T(r)) & (k < l) \\ (n+1-l)T_k(r) \leqq (n+1-k)T_l(r) + \epsilon(T(r)) & (k < l) \end{cases}$$

where $T(r) = \max_k T_k(r)$. It is in this sense that the original curve and its associated curves have the same order of growth. Thus, e.g., f is *rational* $\Leftrightarrow$ $T_0(r) = O(\log r)$ $\Leftrightarrow$ $T_k(r) = O(\log r)$ for some k; f has *finite order* $\Leftrightarrow$ $T_0(r) = O(r^\lambda)$ $\Leftrightarrow$ $T_k(r) = O(r^\lambda)$ for some k, etc. For $n = 2$, the inequalities (4.22) reduce to

$$(4.23) \qquad \begin{cases} T_1(r) \leqq 2T_0(r) + \epsilon(T(r)) \\ T_0(r) \leqq 2T_1(r) + \epsilon(T(r)). \end{cases}$$

5. Holomorphic curves in $\mathbf{P}^n$; the Frenet formalism

(a) Following Chern [10], we shall use the Frenet frames associated to a non-degenerate holomorphic curve. As may be familiar from the study of ordinary differentiable curves in $\mathbf{R}^3$, this formalism most clearly exhibits the geometry of the curve, especially those aspects dealing with *contact* which the curve may have with a linear space, in terms of the parametric equations of the curve. We shall use frames on an intuitive level, letting "dZ" symbolize an "infinitesimal displacement of Z" and so forth. The rigorous basis for this symbolism is given in [7, Chap. 3].

A *frame* is a unitary basis $\{Z_0, \cdots, Z_n\}$ for $\mathbf{C}^{n+1}$. The set of frames is a manifold F_{n+1}, which may obviously be identified with the unitary group

U_{n+1}. If $Z: F_{n+1} \to \mathbf{C}^{n+1}$ is a smooth function, the infinitesimal displacement dZ may be resolved into its components relative to the frame vectors:

$$dZ = \sum_{k=0}^{n} \theta_k Z_k,$$

where the θ_k are 1-forms on F_{n+1}. Applying this principle to the frame vectors themselves gives

$$dZ_k = \sum_{l=0}^{n} \theta_{kl} Z_l, \tag{5.1}$$

where the $\theta_{kl} = (dZ_k, Z_l)$ are the basic Maurer-Cartan forms on F_{n+1}. Differentiation of $(Z_k, Z_l) = \delta_l^k$ gives

$$\theta_{kl} + \bar{\theta}_{lk} = 0, \tag{5.2}$$

and exterior differentiation of (5.1) gives

$$d\theta_{kl} = \sum_{i=0}^{n} \theta_{ki} \wedge \theta_{il}. \tag{5.3}$$

The relations (5.1)–(5.3) are the structure equations for F_{n+1}.

The map $F_{n+1} \xrightarrow{\pi} \mathbf{P}^n$ given by $\pi\{Z_0, \cdots, Z_n\} = Z_0$ is a fibration with fibre F_n. As an illustration of how one calculates with frames, we shall prove:

Lemma 5.4.

$$\pi^*\Omega = \frac{\sqrt{-1}}{2\pi}\left\{\sum_{\rho=1}^{n} \theta_{o\rho} \wedge \bar{\theta}_{o\rho}\right\},$$

Ω being the standard Kähler form on $\mathbf{P}^n$.

Proof. The mapping π is given by

$$Z_0 = \frac{Z}{|Z|} e^{i\psi} \tag{5.5}$$

where Z is a homogeneous coordinate on $\mathbf{P}^n$ and ψ is C^∞ and real.

Differentiating $|Z|^2=(Z,Z)$ gives

$$2|Z|\,d|Z|=(dZ,Z)+(Z,dZ). \tag{5.6}$$

Combining (5.5) and (5.6) we obtain

$$\begin{aligned}\theta_{oo}=(dZ_0,Z_0)&=\frac{(dZ,Z)}{|Z|^2}-\frac{d|Z|}{|Z|}+id\psi\\&=\frac{(dZ,Z)}{|Z|^2}-\frac{1}{2|Z|^2}\{(dZ,Z)+(Z,dZ)\}+id\psi\\&=\frac{1}{2|Z|^2}\{(dZ,Z)-(Z,dZ)\}+id\psi\\&=\tfrac{1}{2}(\partial-\bar{\partial})\log|Z|^2+id\psi.\end{aligned}$$

Thus

$$\frac{\sqrt{-1}}{2\pi}\theta_{oo}=-d^c\log|Z|^2-\frac{1}{2\pi}d\psi. \tag{5.7}$$

Taking the exterior derivative of (5.7) gives

$$\begin{aligned}\pi^*\Omega&=dd^c\log|Z|^2 &&\text{(by (4.2))}\\&=-\frac{\sqrt{-1}}{2\pi}d\theta_{oo} &&\text{(by (5.7))}\\&=\frac{\sqrt{-1}}{2\pi}\left\{\sum_{\rho=1}^{n}\theta_{o\rho}\wedge\bar{\theta}_{o\rho}\right\} &&\text{(by (5.3) and (5.2)).}\end{aligned}$$

Q.E.D.

Definition. Given a non-degenerate holomorphic curve, a *Frenet frame* $\{Z_0,\cdots,Z_n\}$ is a unitary frame such that $Z_0,\cdots,Z_k$ and $Z,\cdots,Z^{(k)}$ span the same linear space in $\mathbf{C}^{n+1}$; i.e.,

$$e^{i\psi_k}\frac{Z\wedge\cdots\wedge Z^{(k)}}{|Z\wedge\cdots\wedge Z^{(k)}|}=Z_0\wedge\cdots\wedge Z_k \qquad (k=0,\cdots,n). \tag{5.8}$$

Frenet frames are defined at the regular points of the curve, and there they provide a coordinate system especially adapted to the curve in a neighborhood of the point. A Frenet frame is unique up to rotations

$$Z_\alpha \to e^{2\pi i\psi_\alpha}Z_\alpha . \tag{5.9}$$

Restricting the structure equations (5.1)–(5.3) to Frenet frames, they simplify as follows:

Lemma 5.10. (Frenet equations.) (i) $\theta_{ij}=0$ *for* $i+1<j$ *and* $\theta_{i,i+1}$ *is of type* (1,0); *and*

$$\text{(ii)}\qquad \begin{cases} dZ_k = \theta_{k,k-1}Z_{k-1}+\theta_{k,k}Z_k+\theta_{k,k+1}Z_{k+1} \\ \Omega_k = \dfrac{\sqrt{-1}}{2\pi}\theta_{k,k+1}\wedge\bar\theta_{k,k+1}. \end{cases}$$

Proof. Since Z_k is a linear combination of $Z, Z', \cdots, Z^{(k)}$, dZ_k is a linear combination of $Z, Z', \cdots, Z^{(k+1)}$ which implies that $\theta_{ij}=(dZ_i, Z_j)=0$ for $i+1<j$. Similarly, since $\bar\partial Z^{(l)}=0$ it follows that $\bar\partial Z_k$ is a linear combination of $Z_0,\cdots,Z_k$, and thus $\theta''_{k,k+1}=(\bar\partial Z_k, Z_{k+1})=0$. This proves (i), and the first equation in (ii) follows from (i) and (5.1)–(5.3).

To prove the second equation there, we first derive the relation

$$\frac{\sqrt{-1}}{2\pi}(\theta_{o,o}+\cdots+\theta_{k,k})=-d^c\log|Z\wedge\cdots\wedge Z^{(k)}|^2-\frac{1}{2\pi}d\psi_k \tag{5.11}$$

using the same method as in the proof of (5.7). Taking exterior derivatives gives

$$\begin{aligned}\Omega_k &= dd^c\log|Z\wedge\cdots\wedge Z^{(k)}|^2 && \text{(by (4.15))}\\ &= -\frac{\sqrt{-1}}{2\pi}(d\theta_{o,o}+\cdots+d\theta_{k,k}) && \text{(by (5.11))}\\ &= \frac{\sqrt{-1}}{2\pi}\theta_{k,k+1}\wedge\bar\theta_{k,k+1}\end{aligned}$$

by (5.2) and using that $d\theta_{i,i} = \theta_{i,i-1} \wedge \theta_{i-1,i} + \theta_{i,i+1} \wedge \theta_{i+1,i}$ so that the sum telescopes. Q.E.D.

For future easy reference, we collect together the structure equations for the curve using Frenet frames:

Structure equations 5.12.

(i) $Z_0 \wedge \cdots \wedge Z_k = \dfrac{e^{i\psi_k} Z \wedge \cdots \wedge Z^{(k)}}{|Z \wedge \cdots \wedge Z^{(k)}|} = e^{i\psi_k} \dfrac{\Lambda_k}{|\Lambda_k|}$;

(ii) $dZ_k = \theta_{k,k-1} Z_{k-1} + \theta_{k,k} Z_k + \theta_{k,k+1} Z_{k+1}$;

(iii) $\theta_{k,k+1}$ has type (1,0);

(iv) $$\begin{cases} \theta_{ij} + \bar{\theta}_{ji} = 0 \\ d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj}\,; \end{cases}$$

(v) $\Omega_k = \dfrac{\sqrt{-1}}{2\pi} dd^c \log|\Lambda_k|^2 = \dfrac{\sqrt{-1}}{2\pi} \theta_{k,k+1} \wedge \bar{\theta}_{k,k+1}$; and

(vi) $\Omega_k = \dfrac{\sqrt{-1}}{2\pi} h_k d\zeta \wedge d\bar{\zeta}$ where $h_k = \dfrac{|\Lambda_{k-1}|^2 |\Lambda_{k+1}|^2}{|\Lambda_k|^4}$.

(b) **The Ahlfors inequalities.** As an application of the calculus of Frenet frames, we shall derive the Ahlfors inequalities [2] in the form of the conjectured equation (124) in [10]. An heuristic discussion of our method appears at the end of this section. These results will not be needed for the proofs of the defect relations in Sections 6,7 below.

We begin with the following linear algebra convention: For a k-vector B and an h-vector C in $\mathbf{C}^{n+1}$, the *interior product* $\langle B, C\rangle$ is the unique $(k-h)$-vector such that for all $(k-h)$-vectors D. We also set

$$|\langle B,C\rangle| = |B,C|.$$

Let $f:\mathbf{C}\to\mathbf{P}^n$ be a non-degenerate holomorphic curve, and $A\in(\mathbf{C}^{n+1})^*$ a unit vector defining a hyperplane $A\subset\mathbf{P}^n$. Setting

$$\begin{aligned}\phi_k = \phi_k(A) &= |Z_0,A|^2+\cdots+|Z_k,A|^2 \\ &= \frac{|Z\wedge\cdots\wedge Z^{(k)},A|^2}{|Z\wedge\cdots\wedge Z^{(k)}|^2},\end{aligned} \tag{5.13}$$

we observe that $\phi_k(\zeta_0)=0 \Leftrightarrow$ *the curve has contact of order* $k+1$ *with* A *at* ζ_0.

Proposition. (Ahlfors' inequalities.) *Given* $\epsilon>0$, *for* $\mu\geqq\mu(\epsilon)\geqq 1$,

$$\int_0^r\left(\int_{\Delta_\rho}\frac{\phi_{k+1}\Omega_k}{\phi_k\left(\log\frac{\mu}{\phi_k}\right)^2}\right)\frac{d\rho}{\rho}<\epsilon T_k(r)+C. \tag{5.14}$$

Proof. For a positive C^∞ function ϕ,

$$\frac{\sqrt{-1}}{4\pi}\partial\bar\partial\log\frac{1}{\left(\log\frac{1}{\phi}\right)^2}=\frac{\sqrt{-1}}{2\pi}\frac{\partial\bar\partial\log\phi}{\log 1/\phi}+\frac{\sqrt{-1}}{2\pi}\frac{\partial\phi\wedge\bar\partial\phi}{\phi^2\left(\log\frac{1}{\phi}\right)^2}. \tag{5.15}$$

This equation is straightforward to check. Using functions of the sort

$$\log\left(\frac{1}{\phi^2\left(\log\frac{1}{\phi}\right)^2}\right)$$

as potentials is suggested by the formula for the Poincaré metric (2.3) on the punctured disc. We wish to use (5.15) when $\phi=\phi_k/\mu$, and so we need the following two lemmas, which together with (4.16) constitute the main computations of this paper:

Lemma 5.16. $\frac{\sqrt{-1}}{2\pi}(\partial\phi_k\wedge\bar\partial\phi_k)=(\phi_{k+1}-\phi_k)(\phi_k-\phi_{k-1})\Omega_k.$

Lemma 5.17. $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\phi_k = \left(\frac{\phi_{k+1}\phi_{k-1}}{\phi_k^2}\right)\Omega_k - \Omega_k.$

Proof of (5.16). Using a rotation (5.9), we may assume that $\theta_{j,j}(\zeta_0) = 0$ at a given point ζ_0. Calculating derivatives at ζ_0 and using (ii) and (iii) in (5.12), we have

$$\begin{aligned}\partial\phi_k &= \partial(\langle Z_0 \wedge \cdots \wedge Z_k, A\rangle, \langle Z_0 \wedge \cdots \wedge Z_k, A\rangle)\\ &= (\langle Z_0 \wedge \cdots \wedge Z_{k-1} \wedge Z_{k+1}, A\rangle, \langle Z_0 \wedge \cdots \wedge Z_k, A\rangle)\,\theta_{k,k+1}\\ &= A_{k+1}\bar{A}_k\,\theta_{k,k+1}\end{aligned}$$

where $A_k = \langle Z_k, A\rangle$. Thus by (5.12v) and (5.13)

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}(\partial\phi_k \wedge \bar{\partial}\phi_k) &= |A_{k+1}|^2|A_k|^2\frac{\sqrt{-1}}{2\pi}(\theta_{k,k+1} \wedge \bar{\theta}_{k,k+1})\\ &= (\phi_{k+1} - \phi_k)(\phi_k - \phi_{k-1})\Omega_k.\end{aligned}$$

Q.E.D.

Proof of (5.17). By the definition (5.13) and (5.12v),

$$\phi_k = \frac{|\Lambda_k, A|^2}{|\Lambda_k|^2}$$

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|\Lambda_k|^2 = \Omega_k.$$

Thus we must check that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|\Lambda_k, A|^2 = \left(\frac{\phi_{k+1}\phi_{k-1}}{\phi_k^2}\right)\Omega_k, \tag{5.18}$$

which will turn out to be the S.M.T. (4.16) for the *contracted holomorphic curve*

$$\Lambda_A(\zeta) = \langle\Lambda_l(\zeta), A\rangle.$$

Indeed, we will check that

$$\langle \Lambda_k, A \rangle = \frac{(\Lambda_A)_{k-1}}{\langle Z, A \rangle^{k-1}}, \tag{5.19}$$

which upon using (5.12vi) and (4.16) gives

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|\Lambda_k, A|^2 &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|(\Lambda_A)_{k-1}|^2 \\ &= \frac{\sqrt{-1}}{2\pi}\frac{|(\Lambda_A)_{k-2}|^2|(\Lambda_A)_k|^2}{|(\Lambda_A)_{k-1}|^4}d\zeta \wedge d\bar{\zeta} \\ &= \frac{\sqrt{-1}}{2\pi}\frac{|\Lambda_{k-1}, A|^2|\Lambda_{k+1}, A|^2}{|\Lambda_k, A|^4}d\zeta \wedge d\bar{\zeta} \\ &= \frac{\phi_{k-1}\phi_{k+1}}{\phi_k^2}\Omega_k,\end{aligned}$$

thereby proving (5.18).

It will suffice to verify (5.19) at a regular point ζ_0 where $\langle Z(\zeta_0), A \rangle \neq 0$. Arguing as in the proof of Lemma 4.16, we may assume that

$$\langle Z^{(j)}(\zeta_0), A \rangle = 0 \qquad (j = 1, \cdots, n). \tag{5.20}$$

We now proceed by induction, assuming (5.19) for k and then deriving it for $k+1$. Calculating at ζ_0,

$$\begin{aligned}(\Lambda_A)_{k-1} &= \langle Z, A \rangle^{k-1}\langle \Lambda_k, A \rangle && \text{(by induction)} \\ &= \langle Z, A \rangle^{k-1}\langle Z \wedge Z' \wedge \cdots \wedge Z^{(k)}, A \rangle && \text{(by definition)} \\ &= \langle Z, A \rangle^{k} Z' \wedge \cdots \wedge Z^{(k)} && \text{(by (5.20))};\end{aligned}$$

$$\begin{aligned}(\Lambda_A)^{(k)} &= \langle Z \wedge Z^{(k+1)}, A \rangle && \text{(by (5.20))} \\ &= \langle Z, A \rangle Z^{(k+1)} && \text{(by (5.20))};\end{aligned}$$

and so finally

$$(\Lambda_A)_k = (\Lambda_A)_{k-1} \wedge (\Lambda_A)^{(k)} \qquad \text{(by definition)}$$

$$= \langle Z, A\rangle^{k+1} Z' \wedge \cdots \wedge Z^{(k+1)}$$

$$= \langle Z, A\rangle^{k} \langle \Lambda_{k+1}, A\rangle \qquad \text{(by (5.20))}.$$

Q.E.D.

Continuing with the proof of (5.14), we use (5.16) and (5.17) in (5.15) with $\phi = \phi_k / \mu$ to obtain

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left[\frac{1}{\log\frac{\mu}{\phi_k}}\right]^2 + \epsilon\Omega_k = \left[\epsilon - \frac{2}{\log\frac{\mu}{\phi_k}}\right]\Omega_k$$

$$+2\left[\frac{(\phi_{k+1}-\phi_k)(\phi_k-\phi_{k-1})}{\phi_k^2\left(\log\frac{\mu}{\phi_k}\right)^2} + \frac{\phi_{k+1}\phi_{k-1}}{\phi_k^2\log\frac{\mu}{\phi_k}}\right]\Omega_k,$$

which easily implies the main technical inequality of this paper

$$\text{(5.21)} \qquad \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left[\frac{1}{\log\frac{\mu}{\phi_k}}\right]^2 + \epsilon\Omega_k \geqq 2\frac{\phi_{k+1}}{\phi_k\left(\log\frac{\mu}{\phi_k}\right)^2}\Omega_k$$

provided $\mu \geqq \mu(\epsilon)$. Applying the basic integral formula (1.3) to the function

$$h = \left(\frac{1}{\log\frac{\mu}{\phi_k}}\right)^2,$$

we obtain from (5.21) the estimate

$$2\int_0^r\left(\int_{\Delta_\rho}\frac{\phi_{k+1}}{\phi_k\left(\log\frac{\mu}{\phi_k}\right)^2}\Omega_k\right)\frac{d\rho}{\rho} < \epsilon T_k(r) + \frac{1}{4\pi}\int_{|\zeta|=r}\log h\, d\theta + C'$$

$$< \epsilon T_k(r) + C$$

since h is bounded. Q.E.D.

Remarks. We wish to discourse a little on inequalities of the type (5.14) and their relationship to the philosophy of negative curvature. Using

$$\phi_k\left(\log\frac{\mu}{\phi_k}\right)^2 > C(1-\lambda)\phi_k^\lambda \qquad (\lambda < 1)$$

in (5.14), we arrive at an inequality

$$(5.22) \qquad \int_0^r\left(\int_{\Delta_\rho}\frac{\phi_{k+1}\Omega_k}{\phi_k^\lambda}\right)\frac{d\rho}{\rho} < \left(\frac{1}{1-\lambda}\right)T_k(r) + C$$

after suitable adjusting of constants. This is the conjectured inequality (124) in [10], which, as was carried out in [10] in the case $n = 2$, leads to the Ahlfors defect relation, a result which we shall prove by a somewhat different method in the next section. Taking $k = 0$, (5.22) becomes

$$(5.23) \qquad \int_0^r\left(\int_{\Delta_\rho}\frac{|Z_0,A|^2+|Z_1,A|^2}{|Z_0,A|^{2\lambda}}\Omega_0\right)\frac{d\rho}{\rho} < \left(\frac{1}{1-\lambda}\right)T_0(r) + C.$$

The (1,1) form

$$\frac{\Omega_0}{|Z_0,A|^{2\lambda}} = \omega_\lambda(A)$$

fails to be integrable exactly where the hyperplane A has high order contact with the holomorphic curve. Thus, by Lemma 2.13 (iii), $\omega_\lambda(A)$ *cannot be the Ricci form of a singular metric.* However, the factor $|Z_0,A|^2+|Z_1,A|^2$ vanishes when A has high contact with A, and when put in front of $\omega_\lambda(A)$ gives an integrable (1,1) form. Thus it is at least possible that

$$\frac{|Z_0,A|^2+|Z_1,A|^2}{|Z_0,A|^{2\lambda}}\Omega_0$$

be dominated by the Ricci form of a singular metric, in which case [Ahlfors] type inequalities may be expected to follow from the basic integral formula (1.3).

By extremely ingenious arguments in integral geometry (cf. the introductions to [21] and [22]), Ahlfors derived estimates of the general type as (5.22).

Our main point is that if one adopts the philosophy of trying to find metrics of negative curvature using the potential suggested by the Poincaré metric on the punctured disc as a guide, then the Ahlfors estimates and much more (cf. Sections 6,7 below) fall out quite naturally using straightforward computations based on the Frenet frames.

6. Defect relation for a holomorphic curve and some applications

(a) **Proof of the defect relation.** Let $f: \mathbf{C} \to \mathbf{P}^n$ be a non-degenerate holomorphic curve with order function

$$T_0(r) = \int_0^r \left(\int_{\Delta_\rho} \Omega_0 \right) \frac{d\rho}{\rho}$$

relative to the hyperplane line bundle. For each hyperplane $A \subset \mathbf{P}^n$, the counting function $N(A, r)$ measures the number of points of intersection of the curve with A, and satisfies the Nevanlinna inequality (3.4)

$$N(A, r) \leqq T_0(r) + C.$$

Using this, one defines the *Nevanlinna defect*

$$\delta(A) = 1 - \overline{\lim_{r \to \infty}} \left(\frac{N(A, r)}{T_0(r)} \right), \tag{6.1}$$

with the properties

$$\begin{cases} 0 \leqq \delta(A) \leqq 1 \\ \delta(A) = 1 \text{ if } f(\mathbf{C}) \text{ fails to meet } A. \end{cases}$$

In general, the defect $\delta(A) > 0$ exactly when the curve meets A less than it meets an average hyperplane (cf. (3.5)). The *Ahlfors defect relation* [2] is the estimate

$$\sum_\nu \delta(A_\nu) \leqq n + 1 \tag{6.2}$$

where A_ν is a set of hyperplanes in *general position*; i.e., no $n+1$ of the A_ν's are linearly dependent. We will prove (6.2) using the basic integral formula (1.3) and a negatively curved collection of metrics (Section 2 (c)). Referring to (2.22), the idea is to find singular metrics leading to a lower bound on the sum

$$\Sigma N(A_\nu, r),$$

which may then be played off against the upper bound in the Nevanlinna inequality.

Now (6.2) is obvious if there are less than $n+2$ hyperplanes A_ν, and so we may assume that $A_1, \cdots, A_N$ are hyperplanes in general position with $n+2 \leqq N < \infty$, and will then prove (6.2). Define for $i = 0, \cdots, n-1$,

$$\omega_i = c_i \prod_\nu \left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\left(\log \frac{\mu}{\phi_i(A_\nu)}\right)^2} \right)^{i/(n-i)} \Omega_i. \tag{6.3}$$

Proposition. *Given* $\epsilon > 0$, *for suitable choice of constants* c_i, μ *we have*

$$\sum_{i=0}^{n-1} (n-i) \operatorname{Ric} \omega_i \geqq (N-n-1)\Omega_0 + \sum_{i=0}^{n-1} \omega_i - \epsilon \left(\sum_{i=0}^{n-1} \Omega_i \right). \tag{6.4}$$

Aside from the term with the ϵ, (6.4) says that the $\{\omega_i\}$ form a *negatively curved collection of metrics* (Section 2 (c)). Moreover, the term with the ϵ may be estimated by (4.22). Thus the proof of Proposition 2.21 will still apply to yield the desired lower bound.

In fact, let us assume (6.4) and carry out in detail the proof of (6.2). Using the method of (2.21), we will first derive the lower bound

$$\sum_\nu N(A_\nu, r) \geqq (N-n-1-\epsilon) T_o(r) + C. \tag{6.5}$$ ||

Proof of (6.5). Let $N_i(A_\nu, r)$ be the counting function for the points $\zeta \in \mathbf{C}$ where $f(\mathbf{C})$ has contact of order $i+1$ with A_ν, and $N_i(r)$ the counting function for the stationary points of index i. Writing $\omega_i = \frac{\sqrt{-1}}{2\pi} h_i d\zeta \wedge d\bar{\zeta}$ and applying the basic integral formula (1.3) to h_i, we find

(6.6)
$$\frac{n-i}{4\pi} \int_{|\zeta|=r} \log h_i d\theta = (n-i) N_i(r)$$
$$+ \sum_\nu [N_{i+1}(A_\nu, r) - N_i(A_\nu, r)] + (n-i) \int_0^r \Big(\int_{\Delta_r} \operatorname{Ric} \omega_i \Big) \frac{d\rho}{\rho} + C_i .$$

Using (2.6) and (2.11), we may estimate the boundary integrals for $r \geqq 1$ by an inequality

(6.7)
$$\frac{n-i}{4\pi} \int_{|\zeta|=r} \log h_i d\theta \leqq C' \log (T_{\omega_i}(r)). \qquad \|$$

Finally, using (4.22) we have

(6.8)
$$\epsilon \Big(\sum_{i=0}^{n-1} T_i(r) \Big) \leqq C'' \epsilon T_0(r). \qquad \|$$

Now sum (6.6) for $i = 0, \cdots, n-1$ use (6.4), (6.7), and (6.8) to obtain an estimate

(6.9)
$$\sum_\nu N(A_\nu, r) \geqq \sum_i (n-i) N_i(r) + (N - n - 1 - \epsilon) T_0(r) + C \qquad \|$$

(where ϵ has replaced ϵC). Q.E.D.

Proof of (6.2).
$$\sum_\nu \delta(A_\nu) = \sum_\nu \Big(1 - \overline{\lim_{r\to\infty}} \frac{N(A_\nu, r)}{T_0(r)} \Big)$$
$$\leqq N - \overline{\lim_{r\to\infty}} \Big(\sum_\nu \frac{N(A_\nu, r)}{T_0(r)} \Big)$$
$$\leqq n + 1 + \epsilon$$

by (6.5). Q.E.D.

Proof of (6.4). We first note that

$$\sum_{i=0}^{n-1}(n-i)\operatorname{Ric}\Omega_i = \sum_{i=0}^{n-1}(n-i)(\Omega_{i-1}+\Omega_{i+1}-2\Omega_i) \tag{6.10}$$
(4.19)

$$= -(n+1)\Omega_0.$$

Since the sum telescopes (this is the $n+1$ in the Ahlfors defect relation). Next, by (5.21),

$$dd^c \log\left(\frac{1}{\left(\log\frac{\mu}{\phi_k(A_\nu)}\right)^2}\right) \geqq -\epsilon\Omega_k + \frac{2\phi_{k+1}(A_\nu)\Omega_k}{\phi_k(A_\nu)\left(\log\frac{\mu}{\phi_k(A_\nu)}\right)^2} \tag{6.11}$$

for sufficiently large μ. Summing (6.11) gives

$$\sum_{i=0}^{n-1} dd^c \log\left[\prod_\nu \frac{\phi_{i+1}(A\nu)}{\phi_i(A_\nu)(\log\mu/\phi_i(A_\nu))^2}\right]$$

$$= -\sum_\nu dd^c \log\phi_0(A_\nu) + \sum_{i=0}^{n-1}\sum_\nu dd^c \log\left|\frac{1}{(\log\mu/\phi_i(A_\nu))^2}\right|$$

$$\geqq N\Omega_0 + 2\sum_{i=0}^{n-1}\left\{\sum_\nu \frac{\phi_{i+1}(A_\nu)\Omega_i}{\phi_i(A_\nu)\log(\mu/\phi_i(A_\nu))^2}\right\} - \epsilon\sum_{i=0}^{n-1}\Omega_i,$$

the middle step being by telescoping a sum. Combining this with (6.10) gives

$$\sum_{i=0}^{n-1}(n-i)\operatorname{Ric}\omega_i \geqq (N-n-1)\Omega_0 \tag{6.12}$$

$$+2\sum_{i=0}^{n-1}\left\{\sum_\nu \frac{\phi_{i+1}(A_\nu)\Omega_i}{\phi_i(A_\nu)\left(\log\frac{\mu}{\phi_i(A_\nu)}\right)^2}\right\} - \epsilon\sum_{i=0}^{n-1}\Omega_i.$$

Using that Ric $(c\omega) = \operatorname{Ric}\omega$, (6.4) follows from (6.12) and the final:

Lemma. (Sums into products.) *For $\{A_\nu\}$ in general position,*

$$\sum_{\nu=1}^{N}\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\left(\log\frac{\mu}{\phi_i(A_\nu)}\right)^2} \geqq C_i\prod_{\nu=1}^{N}\left(\frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\left(\log\frac{\mu}{\phi_i(A\nu)}\right)^2}\right)^{1/(n-i)} \tag{6.13}$$

Proof. For B a unit $(i+1)$-vector in $\mathbf{C}^{n+1}$, $\langle B, A_\nu \rangle = 0$ for at most $n-i$ of the A_ν, since the A_ν are in general position. Using continuity and compactness of the Grassmannian,

$$|\langle B, A_\nu \rangle|^2 \geqq m > 0$$

for all but at most $n-i$ of the A_ν and all B. In particular,

$$\phi_i(A_\nu) \geqq m > 0$$

for all but at most $n-i$ of the A_ν. Set

$$\Phi_\nu = \frac{\phi_{i+1}(A_\nu)}{\phi_i(A_\nu)\left(\log \dfrac{\mu}{\phi_i(A_\nu)}\right)^2}.$$

By what was just said, there exists a constant $L > 0$ such that $\Phi_\nu \leqq L$ for all but at most $n-i$ of the A_ν's.

We may renumber so that $\Phi_\nu(\zeta) > L$ at most for $\nu = 1, \cdots, q \leqq n-i$. Then at ζ,

$$\begin{aligned}\sum_\nu \Phi_\nu &\geqq \sum_{\nu=1}^{q} \Phi_\nu \\ &\geqq C \prod_{\nu=1}^{q} \Phi_\nu^{1/q} \qquad \text{(arithmetic and geometric means)} \\ &\geqq C \prod_{\nu=1}^{N} \Phi_\nu^{1/q} \cdot L^{(q-N)/q} \\ &\geqq C' \prod_{\nu=1}^{N} \Phi_\nu^{1/(n-i)}.\end{aligned}$$

(b) **The Borel theorem.**

Proposition. (Borel [4].) *Let $f : \Delta_s \to \mathbf{P}^n - (A_1 + \cdots + A_{n+2})$ be a non-degenerate holomorphic curve which omits $n+2$ hyperplanes in general position. Then*

(6.14)

We will derive (6.14) from (6.4) and the following assertion about pseudo-metrics:

Let $\{\omega_i\}$, ϕ, ψ be pseudo-metrics on Δ_s and assume that the conditions

$$(6.15)\qquad \begin{cases} \sum_i \operatorname{Ric}\omega_i \geqq \sum \omega_i + \phi - \psi \\ T_\psi(r) \leqq \log\left(\dfrac{1}{r^2}\dfrac{d^2 T_\phi(r)}{(d\log r)^2}\right) + \epsilon T_\phi(r) \qquad (\epsilon < 1) \qquad \| \end{cases}$$

are satisfied. Then

$$(6.16)\qquad s < \infty.$$

Proof. As in the proof of (2.22), the assumptions (6.15) imply the inequality

$$(6.17)\qquad \sum_i T_{\omega_i}(r) + (1-\epsilon)T_\phi(r)$$

$$\leqq \sum_i C\log T_{\omega_i}(r) + C'\log T_\phi(r) + C'' \qquad \|$$

where C, C' are universal constants and C'' depends on $\omega_i(0)$, $\phi(0)$, $\psi(0)$. It follows immediately that $s \neq +\infty$. Q.E.D.

(c) **A big Picard theorem.** Let $\Delta^* = \{0 < |\zeta| < 1\}$ be the punctured disc and $A_1, \cdots, A_{n+2}$ hyperplanes in general position in $\mathbf{P}^n$.

Proposition 6.18. ([15]) *A non-degenerate holomorphic mapping $f: \Delta^* \to \mathbf{P}^n - (A_1 + \cdots + A_{n+2})$ extends across $\zeta = 0$.*

The proof breaks naturally into four short steps. We assume throughout that all functions are defined on the slightly larger disc $\{0 < |\zeta| < 1 + \epsilon\}$ for some $\epsilon > 0$.

Step one. We begin with a simple one-variable lemma.

Lemma 6.19. *Suppose that $\psi(\zeta)$ is a meromorphic function on Δ^* such that, for some open set $U \subset \mathbf{P}^1$ and constant $B > 0$, the equation*

$$\psi(\zeta) = a \qquad (a \in U) \tag{6.20}$$

has at most B solutions. Then $\psi(\zeta)$ extends across $\zeta = 0$.

Proof. Let $a_0 \in U$ be an interior point where (6.20) has a maximum number of solutions $\zeta_1, \cdots, \zeta_m$. If $\rho = \min_{i=1,\cdots,m} |\zeta_i|$, then in the punctured disc $0 < |\zeta| < \rho - \epsilon$ the meromorphic function $\psi(\zeta)$ omits an open neighborhood of a_0. Thus $\dfrac{1}{\psi(\zeta) - a_0}$ is bounded near $\zeta = 0$, and so the Riemann extension theorem applies. Q.E.D.

Step two. let $f : \Delta^* \to \mathbf{P}^n$ be a holomorphic mapping, $\psi_\alpha = Z_\alpha / Z_0$ the rational coordinate functions on $\mathbf{P}^n$, and

$$\Psi_\alpha = f^* \psi_\alpha$$

the corresponding meromorphic functions on Δ^*. The mapping f extends across $\zeta = 0$ if, and only if, the meromorphic functions Ψ_α extend. Referring to (6.20), the solutions to that equation for Ψ_α are given by $f^{-1}(A)$ where $A \subset \mathbf{P}^n$ is a suitable hyperplane. Setting $A_r = \{1/r < |\zeta| < 1\}$ and

$$n(A, r) = \text{degree } (f^{-1}(A) \cap A_r)$$

$$N(A, r) = \int_1^r n(A, \rho) \frac{d\rho}{\rho},$$

we obtain from Lemma 6.19 the following:

Lemma. *$f : \Delta^* \to \mathbf{P}^n$ extends across $\zeta = 0$ if, for some open set of hyperplanes $U \subset \mathbf{P}^{n*}$, we have*

$$N(A, r) \leqq B \log r + B' \tag{6.21}$$

for $A \in U$.

Indeed, it is clear that $n(A, r) \leqq B$ if, and only if, (6.21) is satisfied.

Step three. Keeping $f:\Delta^*\to\mathbf{P}^n$ as above, we let

$$T_0(r)=\int_1^r\Big(\int_{A_\rho}\Omega_0\Big)\frac{d\rho}{\rho}$$

be the order function.

Lemma. *f extends across* $\zeta=0$ *if, and only if,*

$$T_0(r)=O(\log r). \tag{6.22}$$

Proof. The set S of hyperplanes A such that $f^{-1}(A)$ meets the circle $|\zeta|=1$ is closed and lower dimensional in $\mathbf{P}^{n*}$. Taking $h=1/|\sigma_A|^2$ in (1.5) where $\sigma_A\in\mathcal{O}(\mathbf{P}^n,L)$ defines the hyperplane A, we find an estimate

$$N(A,r)\leqq T_0(r)+C\log r \tag{6.23}$$

for all $A\in U$ where U is a relatively compact open set in $\mathbf{P}^{n*}-S$. Our lemma now follows from (6.21)–(6.23). Q.E.D.

Step four. Proposition 6.18 follows now from (6.4), (6.22), and the following lemma about metrics.

Lemma 6.24. *Let* $\{\omega_i\}$, ϕ, ψ *be pseudo-metrics on* Δ^* *and assume that, for some* $\epsilon<1$,

$$\begin{cases}\sum_i \operatorname{Ric}\omega_i\geqq\sum_i\omega_i+\phi-\psi\\[2ex] T_\psi(r)\leqq\epsilon T_\phi(r)+C.\end{cases}\qquad\|$$

Then it follows that

$$T_\phi(r)=O(\log r).$$

Proof. Applying (1.4), (2.6) (which also holds on Δ^*), and (2.11) we have

$$\sum_i T_{\omega_i}(r)+T_\phi(r)\leqq A\log r+\sum_i B\log T_{\omega_i}(r)+\epsilon T_\phi(r)+C,\qquad\|$$

which simplifies to an estimate of the sort

(6.25) $$T_\phi(r) \leqq A \log r + B. \qquad \|$$

We claim that (6.25) implies

(6.26) $$\int_{A_r} \phi \leqq A$$

for all r, thus proving the lemma. If (6.26) is false for some r_0, then for $r > r_0$,

$$\begin{aligned} T_\phi(r) &= \int_{r_0}^{r} \left(\int_{A_\rho} \phi \right) \frac{d\rho}{\rho} + B \\ &\geqq \int_{r_0}^{r} \left(\int_{A_{r_0}} \phi \right) \frac{d\rho}{\rho} + B \\ &> A \log r + C, \end{aligned}$$

which contradicts (6.25). Q.E.D.

(d) **Intuitive remarks on the construction of negatively curved metrics.** If, on the basis of the arguments in Section 2, one believes that finding metrics of negative curvature will lead to Picard theorems, defect relations, etc., then the following heuristic arguments may show how metrics of the form (6.3) naturally arise. For simplicity, consider a holomorphic plane curve $f: \mathbf{C} \to \mathbf{P}^2$ in which we want to see how it meets a set of lines $A_1, \cdots, A_N$. Recalling that

$$\begin{cases} \phi_0(A) = |Z_0, A|^2 & (= 0 \Leftrightarrow f(\mathbf{C}) \text{ meets } A) \\ \phi_1(A) = |Z_0, A|^2 + |Z_1, A|^2 & (= 0 \Leftrightarrow f(\mathbf{C}) \text{ meets } A \text{ to 2nd order}), \end{cases}$$

the Ahlfors lemma and formula (2.3) for the Poincaré metric suggest looking at

(6.27)
$$\omega_1 = \left(\prod_\nu \frac{1}{\phi_0(A_\nu)\left(\log \frac{1}{\phi_0(A_\nu)}\right)^2} \right) \Omega_0 .$$

The Ricci form of ω_1 is given by (cf. (4.19))

(6.28)
$$\operatorname{Ric} \omega_1 = (N-2)\Omega_0 + \Omega_1 + \sum_\nu dd^c \log \left[\frac{1}{\log \frac{1}{\phi_0(A_\nu)}} \right]^2$$

and thus, for $N \geqq 3$ and using (5.21),

$$\operatorname{Ric} \omega = \chi \cdot \omega$$

where $\chi > 0$ *but where* $\chi(\zeta) \to 0$ *if* $f(\zeta)$ *tends tangentially toward a line* A_ν. Thus what is suggested is that we modify (6.27) by setting

$$\omega_2 = \left(\prod_\nu \frac{\phi_1(A_\nu)}{\phi_0(A_\nu)\left(\log \frac{1}{\phi_0(A_\nu)}\right)^2} \right) \Omega_0 .$$

Letting $\Omega_1(A) = dd^c \log |Z \wedge Z', A|^2$,

(6.29)
$$\operatorname{Ric} \omega_2 = (N-2)\Omega_0 - (N-1)\Omega_1 + \sum_\nu \Omega_1(A_\nu) + \sum_\nu dd^c \log \left(\frac{1}{\log \frac{1}{\phi_0(A_\nu)}} \right)^2 .$$

Ignoring the trouble arising at points of intersection of two of the lines, it follows from (5.21) that

(6.30)
$$\operatorname{Ric} \omega_2 \geqq \chi \cdot \omega_2 + \left(\sum_\nu \Omega_1(A_\nu) - (N-1)\Omega_1 \right)$$

where χ is bounded away from zero. Thus, aside from the C^∞ term $(\Sigma_\nu \Omega_1(A_\nu) - (N-1)\Omega_1)$, ω_2 has negative curvature. To take care of the C^∞ term, we use a second metric

$$\omega_3 = \left(\prod_\nu \frac{1}{\phi_1(A_\nu) \log \left(\frac{1}{\phi_1(A_\nu)} \right)^2} \right) \Omega_1, \tag{6.31}$$

whose Ricci form is given by

$$\operatorname{Ric} \omega_3 = (N-2)\Omega_1 + \Omega_o - \sum_\nu \Omega_1(A\nu) + \sum_\nu dd^c \log \left(\frac{1}{\log \frac{1}{\phi_1(A)}} \right)^2 . \tag{6.32}$$

Adding (6.29) and (6.32) gives

$$\operatorname{Ric} \omega_2 + \operatorname{Ric} \omega_3 = (N-1)\Omega_o - \Omega_1 + (\cdots) \tag{6.33}$$

where $(\cdots)$ are the $dd^c \log(1/\log)^2$ terms. Now by the S.M.T. (cf. (4.23)), we have roughly that

$$\Omega_1 \leqq (2+\varepsilon)\Omega_o,$$

and another $\varepsilon \Omega_0$ is required to make the $dd^c \log(1/\log)^2$ terms positive ((5.21)). Thus (6.33) implies roughly that

$$\operatorname{Ric} \omega_2 + \operatorname{Ric} \omega_3 \geqq (N-3-\varepsilon)\Omega_o + (\omega_2 + \omega_3). \tag{6.34}$$

Consequently, aside from the trouble arising at intersection points of lines, we have a negatively curved collection of metrics for $N \geqq 4$. *The fractional exponents in* (6.3) *are used to resolve the trouble at points of intersection.*

7. Defect relations for the associated curves

Let $f: \mathbf{C} \to \mathbf{P}_n$ be a non-degenerate holomorphic curve, and let $f_k: \mathbf{C} \to G(k,n)$, the Grassman manifold of projective k-planes in $\mathbf{P}_n$, be the kth associated curve. Since $G(k,n)$ can be imbedded in $\mathbf{P}(\Lambda^{k+1}\mathbf{C}^{n+1}) \cong \mathbf{P}^N$ where $N = \binom{n+1}{k+1} - 1$, then f_k can be viewed as a holomorphic curve in $\mathbf{P}^N$. If A^k is a unit $(k+1)$-vector in $\mathbf{C}^{n+1}$, then f_k may lie in the hyperplane in $\mathbf{P}^N$ defined by A^k, i.e.,

$$\langle \Lambda_k, A^k \rangle \equiv 0$$

and thus f_k may be degenerate. However, if A^k is decomposable, then $\langle \Lambda_k, A^k \rangle \equiv 0$ implies f is degenerate. Indeed, if $A^k = a_0 \wedge \cdots \wedge a_k$, then $\langle \Lambda_k, A^k \rangle = 0$ if and only if the holomorphic curve $\tilde{Z}: \mathbf{C} \to \mathbf{C}^{k+1}$ given by

$$\tilde{Z} = (\langle Z, a_0 \rangle, \cdots, \langle Z, a_k \rangle) \tag{7.1}$$

satisfies $\tilde{Z} \wedge \cdots \wedge \tilde{Z}^{(k)} \equiv 0$, i.e., if and only if $\tilde{Z}$ is degenerate, which implies Z is degenerate.

Thus for decomposable A^k, we may still consider the deficiency $\delta(A^k)$, for f_k considered as a curve in $\mathbf{P}^N$. In [1], Ahlfors showed that the defect relation (6.2) holds for f_k as a curve in $\mathbf{P}^N$, even though f_k may be degenerate; that is

$$\sum \delta(A^k_\nu) \leqq N + 1 \tag{7.2}$$

$$\leqq \binom{n+1}{k+1}$$

for $\{A^k_\nu\}$ a family of hyperplanes in general position in $\mathbf{P}^N$ defined by decomposable $(k+1)$-vectors A^k_ν; $\nu = 1, \cdots, q$.

The proof of (7.2) is similar to that of (6.2) in that we apply our main inequalities (5.21) to the *contracted curve* $\langle \Lambda_k, A^{k-1} \rangle$, where A^{k-1} is a decomposable k-vector in $\mathbf{C}^{n+1}$, and then construct a collection of metrics having negative curvature. Of course, the defect relations for a non-degenerate holomorphic curve (6.2) are a special case of (7.2); but the combinatorial problems which arise in treating the associated curves are trivial in the case $k = 0$. Thus we feel it is conceptually clearer to treat the holomorphic curve f and its associated curves f_k separately.

We will use the notation A^h for a decomposable unit $(h+1)$-vector in $\mathbf{C}^{n+1}$ and

$$\phi_k(A^h) = |\Lambda_k, A^h|^2 / |\Lambda_k|^2 \tag{7.3}$$

$$\Omega_k(A^h) = dd^c \log |\Lambda_k, A^h|^2 \tag{7.4}$$

$$\Lambda_{A^h} = \langle \Lambda_{h+1}, A^h \rangle \tag{7.5}$$

for $k \geqq h$. We adopt the convention that $\phi_k(A^h) = 0$ for $k < h$ and $\phi_k(A^{-1}) = 1$.

Let $A = A^h = a_0 \wedge \cdots \wedge a_h, a_j \in \mathbf{C}^{n+1}$, and put

$$A^\perp = \{z \in \mathbf{C}^{n+1}; \langle z, a_j \rangle = 0, j = 0, \cdots, h\}.$$

Then the contracted curve $\Lambda_{A^h} : \mathbf{C} \to A^\perp$ is a holomorphic curve in $\mathbf{P}(A^\perp) \cong \mathbf{P}^{n-h-1}$. The associated curves of Λ_{A^h} satisfy the relation:

$$(\Lambda_{A^h})_j = \langle \Lambda_h, A^h \rangle^j \langle \Lambda_{h+1+j}, A^h \rangle, 0 \leqq j \leqq n-h-1. \tag{7.6}$$

Proof. True for $h = 0$ by (5.19). Assume true for h. Given A^{n+1}, find A^h and A^0 such that $A^{h+1} = A^h \wedge A^0$. Then

$$\Lambda_{A^{h+1}} = \langle \Lambda_{h+2}, A^{h+1} \rangle$$

$$= \langle \langle \Lambda_{h+2}, A^h \rangle, A^0 \rangle$$

$$= \langle \Lambda_h, A^h \rangle^{-1} \langle (\Lambda_{A^h})_1, A^0 \rangle \rangle \quad \text{by the}$$

induction hypothesis. Thus

$$(\Lambda_{A^{h+1}})_j = \langle \Lambda_h, A^h \rangle^{-(j+1)} (\langle (\Lambda_{A^h})_1, A^0 \rangle)_j$$

$$= \langle \Lambda_h, A^h \rangle^{-(j+1)} \langle \Lambda_{A^h}, A^0 \rangle^j \langle (\Lambda_{A^h})_{j+1}, A^0 \rangle$$

$$= \langle \Lambda_{h+1}, A^{h+1} \rangle^j \langle \Lambda_{h+j+2}, A^{h+1} \rangle.$$

Q.E.D.

Note that (7.6) in the case $j = n - h - 1$ implies that Λ_{A^h} is a *non-degenerate* holomorphic curve.

Applying (4.16) to Λ_{A^h}, using (7.6) to compute the associated curves, we have

$$\Omega_k(A^h) = dd^c \log |(\Lambda_{A^h})_{k-h-1}|^2 \tag{7.7}$$

$$= \frac{\phi_{k-1}(A^h)\phi_{k+1}(A^h)}{\phi_k^2(A^h)} \Omega_k.$$

We wish to apply the estimate (5.21) to the curve Λ_{A^h}; in order to avoid confusion we use the notation # to denote "relative to the curve Λ_{A^h}". Assume $A^{h+1} = A^h \wedge A^0$; then

$$\#\,\phi_k(A^0) = \frac{|(\Lambda_{A^h})_k, A^0|^2}{|(\Lambda_{A^h})_k|^2} = \frac{|\langle \Lambda_{h+k+1}, A^h \rangle, A^0|^2}{|\Lambda_{h+k+1}, A^h|^2} = \frac{\phi_{h+k+1}(A^{h+1})}{\phi_{h+k+1}(A^h)} \tag{7.8}$$

and

$$\#\,\Omega_k = dd^c \log|(\Lambda_{A^h})_k|^2 = dd^c \log|\Lambda_{h+k+1}, A^h|^2 = \Omega_{h+k+1}(A^h) \quad \text{(by (7.4))}. \tag{7.9}$$

Thus (5.21) yields, for $j > h$,

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left[\frac{1}{\log\dfrac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}}\right]^2 + \epsilon\Omega_j(A^h) \\ &\geqq \frac{2\phi_{j+1}(A^{h+1})/\phi_{j+1}(A^h)\Omega_j(A^h)}{\phi_j(A^{h+1})/\phi_j(A^h)\left(\log\dfrac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}\right)^2} \\ &\geqq 2\frac{\phi_{j+1}(A^{h+1})}{\phi_j(A^{h+1})}\frac{\phi_{j-1}(A^h)\Omega_j}{\phi_j(A^h)\left(\log\dfrac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}\right)^2}, \quad \text{(by (7.7))}. \end{aligned} \tag{7.10}$$

For a given A^k choose a system of orthonormal vectors spanning A^k, and let $A^h \subset A^{h+1}$ be spanned by h, $h+1$ of these vectors, respectively. Summing (7.10) over the finite number of possible $A^h \subset A^{h+1}$ (relative to the fixed orthonormal system) we obtain

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\prod\frac{1}{\log\frac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}}\right)^2+\epsilon\,\Sigma\Omega_j(A^h) \tag{7.11}$$

$$\geqq 2\left(\sum\frac{\phi_{j+1}(A^{h+1})}{\phi_j(A^{h+1})}\frac{\phi_{j-1}(A^h)}{\phi_j(A^h)}\right)\frac{\Omega_j}{\prod\left(\log\frac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}\right)^2}.$$

But, following Weyl [21], we have

$$\left(\sum\frac{\phi_{j+1}(A^{h+1})}{\phi_j(A^{h+1})}\frac{\phi_{j-1}(A^h)}{\phi_j(A^h)}\right)\sum\frac{\phi_j(A^{h+1})}{\phi_{j+1}(A^{h+1})}\geqq(k-h)\sum\frac{\phi_{j-1}(A^h)}{\phi_j(A^h)}$$

since for fixed A^h there are $(k-h)$ A^{h+1}'s containing the A^h. Putting

$$\psi_{j,h}(A^k)=\sum\frac{\phi_j(A^h)}{\phi_{j+1}(A^h)}, \tag{7.12}$$

we have from (7.11) that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\prod\frac{1}{\log\frac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}}\right)^2+\epsilon\,\Sigma\Omega_j(A^h) \tag{7.13}$$

$$\geqq 2(k-h)\frac{\psi_{j-1,h}(A^k)}{\psi_{j,h+1}(A^k)}\frac{1}{\prod\left(\log\frac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}\right)^2}\Omega_j,$$

for $j>h$, $k>h$.

The proof of (5.14) applied to (7.13) yields the *Ahlfors inequalities*:

$$\int_0^r\left\{\int_{\Delta_\rho}\frac{\psi_{j-1,h}(A^k)\Omega_j}{\psi_{j,h+1}(A^k)\prod\left(\log\frac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})}\right)^2}\right\}\frac{d\rho}{\rho} \tag{7.14}$$

$$\leqq\epsilon\,\Sigma\,T_j(A^h)+C,\ \text{for a suitable constant } \mu,$$

where we have set

$$T_j(A^h) = \int_0^r \int_{\Delta_\rho} \Omega_j(A^h) \frac{d\rho}{\rho}. \tag{7.15}$$

Since $\phi_j(A^h) \leqq 1$, our basic integral formula (1.3) with $h(\zeta) = 1/\phi_j(A^h)$ implies:

$$T_j - T_j(A^h) + O(1) = \int_0^r \left(\int_{\Delta_\rho} dd^c \log 1/\phi_j(A^h) \right) \frac{d\rho}{\rho} + O(1) \geqq 0. \tag{7.16}$$

Thus (7.14) and (7.16) yield:

$$\int_0^r \left\{ \int_{\Delta_\rho} \frac{\psi_{j-1,h}(A^k)\Omega_j}{\psi_{j,h+1}(A^k) \prod \left(\log \dfrac{\mu\phi_j(A^h)}{\phi_j(A^{h+1})} \right)^2} \right\} \frac{d\rho}{\rho} \tag{7.17}$$

$$\leqq \epsilon T_j + C.$$

Let $\{A^k_\nu\}$ be a family of decomposable hyperplanes in $\mathbf{P}_N$, i.e., each A^k_ν is a decomposable $(k+1)$-vector in $\mathbf{C}^{n+1}$, $\nu = 1, \cdots, q$. Define $p_k(j,h)$ to be the maximum possible number of A^k_ν's such that $\phi_j(A^h)(\zeta) = 0$ for all A^h contained in A^k_ν at any point $\zeta \in \mathbf{C}$.

Lemma 7.18.

$$p_k(j,h) = \sum_{l=k-h}^{\infty} \binom{n-j}{l+1} \binom{j+1}{k-l}; j \geqq h, k \geqq h.$$

Proof. Let $B = b_0 \wedge \cdots \wedge b_j$, $b_i \in \mathbf{C}^{n+1}$, and V be the span of $\{b_0, \cdots, b_j\}$. Put

$$V^\perp = \{z \in \mathbf{C}^{n+1} | \langle b_i, z \rangle = 0, i = 0, \cdots, j\}$$

and let W be any complementary subspace to $V^\perp$ in $\mathbf{C}^{n+1}$. Then $\langle B, A^h \rangle = 0$ for all $A^h \subset A^k$ if and only if $A^k = A^l \wedge A^{k-l-1}$, A^l a decomposable $(l+1)$-vector in $V^\perp$, A^{k-l-1} a decomposable $(k-l)$-vector in $\mathbf{C}^{n+1}$, and $k-l-1 < h$.

Thus if $\langle B, A^h \rangle = 0$ for all $A^h \subset A^k$ then A^k is in the subspace

$\sum_{l=k-h}^{\infty} \wedge^{l+1}(V^{\perp}) \wedge \wedge^{k-l} W$, which has dimension $\sum_{l=k-h}^{\infty} \binom{n-j}{l+1}\binom{j+1}{k-l}$. But general position implies that the number of A_ν^k's in any proper subspace cannot exceed the dimension of the subspace. Q.E.D.

Define

$$\omega_{jh} = c_{jh} \prod_{\nu=1}^{q} \left(\frac{\psi_{j-1,h-1}(A_\nu^k)}{\psi_{j,h}(A_\nu^k) \prod \left(\log \dfrac{\mu\phi_j(A^{h-1})}{\phi_j(A^h)}\right)^2} \right)^{1/p_k(j,h)} \Omega_j \tag{7.19}$$

for $j \geqq h \geqq 0$.

Remark. The metric ω_{jh} is singular only where $\psi_{jh}(A_\nu^k) = 0$ for some ν. It is not enough only to consider when $\phi_j(A_\nu^k) = 0$, i.e., when $(\Lambda_j)^{\perp} \cap A \neq 0$ (where we now interpret Λ_j, $A = A^k$ as subspaces of $\mathbf{C}^{n+1}$ instead of as $j+1$, $k+1$ vectors). We must also know the dimensionality of the intersection, i.e., when $\dim(\Lambda_j)^{\perp} \cap A = k - h + 1$, and this occurs only when $\psi_{j,h}(A_\nu^k) = 0$ and $\psi_{j,h+1}(A_\nu^k) \neq 0$.

Proposition. *Given $\epsilon > 0$, for suitable choice of constants c_{jh} and μ, we have for $k \leqq n-1$, $q > \binom{n+1}{k+1}$,*

$$\begin{aligned} &\sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} p_k(j,h) \operatorname{Ric} \omega_{jh} \\ &\geqq \sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} \omega_{j,h} + \left[q - \binom{n+1}{k+1} \right] \Omega_k \\ &\quad - \epsilon \sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} \sum_{A^{h-1} \subset A_\nu^k} \Omega_j(A^{h-1}). \end{aligned} \tag{7.20}$$

Proof. By (7.18), $p_k(j,h) = 0$ if $k - h \geqq n - j$, thus

$$\begin{aligned} &\sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} p_k(j,h) \operatorname{Ric} \Omega_j \\ &= \sum_{h=0}^{k} \sum_{j=h}^{n-1} p_k(j,h) \operatorname{Ric} \Omega_j \\ &= \sum_{j=0}^{k-1} \sum_{h=0}^{j} p_k(j,h) \operatorname{Ric} \Omega_j + \sum_{j=k}^{n-1} \sum_{h=0}^{k} p_k(j,h) \operatorname{Ric} \Omega_j. \end{aligned} \tag{7.21}$$

In order to simplify (7.21) we use the *Vandermonde convolution*

$$\binom{s}{r} = \sum_{i=0} \binom{s-p}{r-i}\binom{p}{i} \tag{7.22}$$

which is a simple consequence of

$$(1+x)^{s-p}(1+x)^p = (1+x)^s, \quad \text{for} \quad 0 \leqq p \leqq s.$$

Then for $0 \leqq j \leqq k-1$,

$$\begin{aligned} \sum_{h=0}^{j} p_k(j,h) &= \sum_{h=0}^{j}\sum_{l=k-h}^{k}\binom{n-j}{l+1}\binom{j+1}{k-l} \\ &= \sum_{l=k-j}^{k}[l+j-k+1]\binom{n-j}{l+1}\binom{j+1}{k-l} \\ &= \sum_{l=k-j}^{k}[j+1]\binom{n-j}{l+1}\binom{j}{k-l} \\ &= [j+1]\binom{n}{k+1} \qquad \text{(by (7.22))}. \end{aligned} \tag{7.23}$$

For $k \leqq j \leqq n-1$,

$$\begin{aligned} \sum_{h=0}^{k} p_k(j,h) &= \sum_{h=0}^{k}\sum_{l=k-h}^{k}\binom{n-j}{l+1}\binom{j+1}{k-l} \\ &= \sum_{l=0}^{k}[l+1]\binom{n-j}{l+1}\binom{j+1}{k-l} \\ &= \sum_{l=0}^{k}[n-j]\binom{n-j-1}{l}\binom{j+1}{k-l} \\ &= [n-j]\binom{n}{k} \qquad \text{(by (7.22))}. \end{aligned} \tag{7.24}$$

Now

$$\sum_{j=0}^{k-1}(j+1)\operatorname{Ric}\Omega_j = \sum_{j=0}^{k-1}(j+1)\{\Omega_{j-1}-2\Omega_j+\Omega_{j+1}\} \qquad \text{(by 4.16)} \tag{7.25}$$

$$= k\Omega_k - (k+1)\Omega_{k-1},$$

since the sum telescopes; similarly,

$$\sum_{j=k}^{n-1}(n-j)\operatorname{Ric}\Omega_j = (n-k)\Omega_{k-1} - (n-k+1)\Omega_k. \tag{7.26}$$

Using (7.23)–(7.26) we can simplify (7.21) to obtain

$$\sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p_k(j,h)\operatorname{Ric}\Omega_j \tag{7.27}$$

$$= \binom{n}{k+1}\{k\Omega_k - (k+1)\Omega_{k-1}\} + \binom{n}{k}\{(n-k)\Omega_{k-1} - (n-k+1)\Omega_k\}$$

$$= -\binom{n+1}{k+1}\Omega_k.$$

In (7.12) we see that

$$\psi_{j,k}(A^k) = \phi_j(A^k)/\phi_{j+1}(A^k)$$

so that

$$\sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} dd^c \log\left(\prod_\nu \psi_{j-1,h-1}(A_\nu^k)/\psi_{jh}(A_\nu^k)\right) \tag{7.28}$$

$$= -\sum_{j=k}^{n-1} dd^c \log\left(\prod_\nu \psi_{j,k}(A_\nu^k)\right) - dd^c \log\left(\prod_\nu \phi_k(A_\nu^k)\right)$$

$$= \sum_{\nu=1}^{q}\{\Omega_k - \Omega_k(A_\nu^k)\} \qquad \text{(by (7.3) and (7.4))}$$

$$= q\Omega_k,$$

since $\langle \Lambda_k, A^k\rangle$ is a holomorphic function implies $\Omega_k(A^k)=0$.

From (7.13) we obtain

(7.29)
$$dd^c \log \prod_\nu \left(\prod \log \frac{\mu \phi_j(A^{h-1})}{\phi_j(A^h)} \right)^{-2}$$

$$\geqq c \sum_\nu \frac{\psi_{j-1,h-1}(A^h_\nu)\Omega_j}{\psi_{jh}(A^k_\nu) \prod \left(\log \frac{\mu \phi_j(A^{h-1})}{\phi_j(A^h)} \right)^2}$$

$$- \epsilon \sum_{A^{h-1} \subset A^k_\nu} \Omega_j(A^{h-1}).$$

Now (7.27)–(7.29) imply

(7.30)
$$\sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} p_k(j,h) \operatorname{Ric} \omega_{jh}$$

$$\geqq \sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} c \sum_\nu \frac{\psi_{j-1,h-1}(A^k_\nu)\Omega_j}{\psi_{jh}(A^k_\nu) \prod \left(\log \frac{\mu \phi_j(A^{h-1})}{\phi_j(A^h)} \right)^2}$$

$$+ \left[q - \binom{n+1}{k+1} \right] \Omega_k - \epsilon \sum_{h=0}^{k} \sum_{j=h}^{n-1+h-k} \sum_{A^{h-1} \subset A^k_\nu} \Omega_j(A^{h-1}).$$

Lemma. (Sums into products.) *Let* $\{A^k_\nu\}$, $\nu = 1, \cdots, q$, *be in general position*; $q > \binom{n+1}{k+1}$.

Then

(7.31)
$$\sum_\nu \frac{\psi_{j-1,h-1}(A^k_\nu)}{\psi_{jh}(A^k_\nu) \prod \left(\log \frac{\mu \phi_j(A^{h-1})}{\phi_j(A^h)} \right)^2}$$

$$\geqq c_{jh} \prod \left[\frac{\psi_{j-1,h-1}(A^k_\nu)}{\psi_{j,h}(A^k_\nu) \prod \left[\log \frac{\mu \phi_j(A^{h-1})}{\phi_j(A^h)} \right]^2} \right]^{1/p_k(j,h)}$$

Proof. By the same method as in the proof of (6.13), there is a constant m_{jh} such that

$$\psi_{jh}(A_\nu^k) \geqq m_{jh} > 0 \tag{7.32}$$

for all but at most $p_k(j,h)$ of the ν's.
Put

$$\Phi_\nu = \frac{\psi_{j-1,h-1}(A_\nu^k)}{\psi_{j,h}(A_\nu^k) \prod \left[\log \frac{\mu\phi_j(A^{k-1})}{\phi_j(A^h)}\right]^2}. \tag{7.33}$$

Since $\phi_{j-1}(A^{h-1}) \leqq \phi_j(A^{h-1})$, then $\psi_{j-1,h-1}(A^k) \leqq \binom{k+1}{h}$, the number of A^{h-1}'s in A_ν^k. Thus there exists $L > 1$ such that

$$\Phi_\nu \leqq L, \tag{7.34}$$

for all but at most $p_k(j,h)$ of the ν's. Since q, the number of A_ν^k's satisfies $q \geqq p_k(j,h)$, the proof of (6.13) can now be applied to (7.34) and (7.31) is the result. Q.E.D.

Write $\omega_{jh} = \frac{\sqrt{-1}}{2\pi} H_{jh} d\zeta d\bar{\zeta}$ and let $N_{j,h}(A^k)$, $N_j(A^k)$ denote the counting function for the zeros of $\psi_{j,h}(A^k)$, $\phi_j(A^k)$ respectively. Using (1.3) we find

$$\begin{aligned} p_k(j,h)\frac{1}{4\pi}\int_{|\zeta|=r} \log H_{jh}\, d\theta \\ = p_k(j,h)N_j(r) + \sum_\nu [N_{j-1,h-1}(A_\nu^k) - N_{j,h}(A_\nu^k)] \\ + p_k(j,h)\int_0^r \left(\int_{\Delta_\rho} \operatorname{Ric}\omega_{jh}\right)\frac{d\rho}{\rho} + O(1). \end{aligned} \tag{7.35}$$

For $r \geqq 1$,

(7.36)
$$p_k(j,h)\frac{1}{2\pi}\int_{|\zeta|=r}\log H_{jh}\,d\theta$$
$$\leqq p_k(j,h)\log\left[\int_0^r\left(\int_{\Delta_\rho}\omega_{jh}\right)\frac{d\rho}{\rho}\right] \qquad \|$$
$$\leqq \int_0^r\left(\int_{\Delta_\rho}\omega_{jh}\right)\frac{d\rho}{\rho}. \qquad \|$$

Using (7.16) and (4.22) we have

(7.37)
$$\epsilon\sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k}\sum_{A^{h-1}\subset A_\nu^k} T_j(A^{h-1}) \leqq \epsilon\sum\sum T_j \leqq \epsilon C T_k.$$

Summing (7.35) for $h=0,\cdots,k$; $j=h,\cdots,n-1+h-k$, we obtain from (7.20), (7.36) and (7.37),

(7.38)
$$\sum_\nu\sum_{j=k}^{n-1} N_{j,k}(A_\nu^k)$$
$$\geqq \sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p_k(j,h)N_j(r) + \left[q-\binom{n+1}{k+1}-\epsilon\right]T_k(r). \qquad \|$$

But $N_{j,k}(A^k) = N_j(A^k) - N_{j+1}(A^k)$, so that (7.38) yields

(7.39)
$$\sum_\nu N_k(A^k)$$
$$\geqq \sum_{h=0}^{k}\sum_{j=h}^{n-1+h-k} p_k(j,h)N_j(r) + \left[q-\binom{n+1}{k+1}-\epsilon\right]T_k(r). \qquad \|$$

The proof that (6.5) implies (6.2) can now be applied to show that (7.39) yields the desired defect relations (7.20). Q.E.D.

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