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HOLOMORPHIC LEFSCHETZ FIXED POINT FORMULA

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1. Let X be an n-dimensional complex analytic manifold and $\varphi : X \to X$ a holomorphic map. Let Ω be the sheaf of germs of holomorphic functions on X and $H^i(X, \Omega)$ the ith cohomology group of X with coefficients in the sheaf Ω. The map φ defines endomorphisms, $H^i(\varphi)$ of $H^i(X, \Omega)$, $i \geqq 0$. Let $L(\varphi)$ be the Lefschetz number defined by

$$L(\varphi) = \sum_{i=0}^{n} (-1)^i \operatorname{trace} H^i(\varphi).$$

We are concerned with the problem of computing $L(\varphi)$.

REMARK. Let G be a compact Lie group acting on X as a group of holomorphic diffeomorphisms and $\varphi \in G$. The problem in this case has been solved by Atiyah and Singer, see [**2**]. Also in the case φ has isolated fixed points, the problem was solved in the nondegenerate case (see §2 for definition) by Atiyah and Bott in [**1**] and by Toledo and Tong in [**6**] and [**7**] in the degenerate case.

2. **The statement of main theorem.** Let X_φ be the fixed point set of the map φ, $X_\varphi = \{x \in X \text{ s.t. } \varphi(x) = x\}$. We start by stating the conditions under which we have been able to compute the Lefschetz number $L(\varphi)$.

(C_1) X_φ is a complex analytic submanifold of X and moreover with this complex analytic structure, X_φ is a Kähler manifold.

Let us write X_φ as a finite union of closed connected submanifolds of X:

$$X_\varphi = \bigcup_{i=1}^{N} Y_i. \tag{1}$$

Let $\lambda_1^i, \ldots, \lambda_{m_i}^i$ be the eigenvalues of the endomorphism $(\varphi_*)_z$ of $T_z(X)$, $z \in Y_i$, with multiplicities $n_1^i, \ldots, n_{m_i}^i$; eigenvalues λ_j^i are independent of $z \in Y_i$ because of the holomorphic nature of the situation. If 1 is an eigenvalue of the map φ_* we take $\lambda_1^i = 1$.

The vector bundles $T(X)|_{Y_i}$ decompose as a direct sum of holomorphic vector subbundles E_j^i $(1 \leqq j \leqq m_i)$ whose fibres $(E_j^i)_z$ are defined by:

$$(E_j^i)_z = \{v \in T_z(X) \text{ s.t. } (\varphi_* - \lambda_j^i I)^{n_j^i} v = 0\}.$$

We now state our other conditions.

(C_2) The fixed points are nondegenerate: 1 is an eigenvalue of

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$\varphi_*: T_z(X) \to T_z(X)$ iff the dimension r_i of Y_i is greater than zero and in case $r_i > 0$, $n_1^i = r_i$.

(C_3) There exists a hermitian metric h in $T(X)$ such that

(a) $h(v_z, w_z) = 0$ if $v_z \in T_z(Y_i)$, $w_z \in \sum_{\lambda_j^i \neq 1} (E_j^i)_z$, $z \in Y_i$.

(b) If Ω is the canonical 2-form associated to h, then, $(d\Omega)_z = (\nabla\, d\Omega)_z = 0$, $z \in X_\varphi$, ∇ is the hermitian connection defined by h.

(C_4) The vector bundles E_j^i decompose as

$$E_j^i = \sum_{k=1}^{N_{ij}} E_{jk}^i \tag{2}$$

such that each E_{jk}^i is a holomorphic subbundle and $E_{jN_{ij}}^i = 0$ and $\varphi_* - \lambda_j^i I$ maps E_{jk}^i into E_{jk+1}^i, $k \geqq 1$.

It is not very natural to impose conditions (C_3) and (C_4). We however have simple conditions which always guarantee the conditions (C_3) and (C_4):

(1) Let X be a Kähler manifold and φ preserves the metric. Then the conditions (C_1) to (C_4) are all satisfied.

(2) The condition (C_3) is satisfied if for positive integers i ($1 \leqq i \leqq N$) such that $r_i > 1$ (r_i = dimension of Y_i) the eigenvalues λ_j^i satisfy the following inequality: $\lambda_j^i \lambda_{j'}^i \neq 1$ for $j, j' \geqq 2$.

(3) Suppose that X is a Kähler manifold and $H^{0,1}(Y_i, (\Sigma_j E_j^i)^*) = 0$ for $1 \leqq i < N$ such that $r_i > 1$, where given a vector bundle ξ, ξ^* denotes the dual bundle. Then the condition (C_3) holds.

(4) If the maps $(\varphi_* - \lambda_j^i I)^k : E_j^i \to E_j^i$, $1 \leqq k \leqq n_j^i$, $1 \leqq i \leqq N$ such that $r_i > 1$, are of constant rank, then the condition (C_4) holds.

We note that if each r_i is either $n - 1$ or is at most one, then the condition (C_4) is satisfied and (C_3) is also satisfied if $\varphi_*: T_z(X) \to T_z(X)$ does not have eigenvalue -1 for $z \in Y_i$ such that $r_i = n - 1$.

We now proceed to state our theorem. Let $C_1, C_2, \ldots, C_{n_j^i}$ be Chern classes of E_j^i and consider the formal factorization:

$$1 + \sum t^k C_k = \prod_{k=1}^{n_j^i} (1 + t x_k).$$

The formal power series

$$\mathscr{U}_j^i = \prod_k \left(\frac{1 - \lambda_j^i \exp(-x_k)}{1 - \lambda_j^i} \right)^{-1}, \qquad \lambda_j^i \neq 1, \tag{3}$$

is symmetric in x_i's and hence can be expressed as a polynomial in C_k's.

THEOREM 1. *If the conditions* (C_1) *to* (C_4) *are satisfied, then*

$$L(\varphi) = \sum_{i=1}^{N} \left(\prod_{\lambda_j^i \neq 1} (1 - \lambda_j^i)^{n_j^i} \right)^{-1} \times \left\{ \left(\prod_{\lambda_j^i \neq 1} \mathscr{U}_j^i \right) \mathscr{T}(Y_i) \right\} [Y_i], \tag{4}$$

where the class $\mathscr{U}^i_j$ *is defined by* (3), $\mathscr{T}(Y_i)$ *is the Todd class of* $T(Y_i)$ *and given a class* $\alpha \in H^*(Y_i, C)$, $\{\alpha\}[Y_i]$ *denotes the evaluation of the* $2r_i$*th component of* α $(r_i$ = *complex dimension of* $Y_i)$ *on the fundamental cycle of* Y_i *determined by its natural orientation.*

3. **Outline of the proof.** We first observe that under the conditions (C_1) to (C_4) there exists a hermitian metric h in $T(X)$, the tangent bundle of X, such that the condition (C_3) is satisfied and furthermore $h(v_z, w_z) = 0$ if $z \in Y_i$, $v_z \in (E^i_{jk})_z$, $w_z \in (E^i_{j'k'})_z$, the pair $(j, k) \neq (j', k')$, where the bundles E^i_{jk} are the ones occurring in the decomposition (2) of condition (C_4).

Let $\Lambda^{0,q}$ be the bundle of differential forms of type $(0, q)$ with the metric induced from h, $d_{\bar{z}}$ be the canonical operator (exterior differentiation with respect to $\bar{z}$) from $C^\infty(\Lambda^{0,q})$ to $C^\infty(\Lambda^{0,q+1})$, $0 \leqq q \leqq n$, and $d^*_{\bar{z}}$ be its adjoint. Let $\Delta^q_{\bar{z}} = -(d_{\bar{z}}d^*_{\bar{z}} + d^*_{\bar{z}}d_{\bar{z}}) : C^\infty(\Lambda^{0,q}) \to C^\infty(\Lambda^{0,q})$ be the Laplace operator and $e^q(t, z', z)$ be the fundamental solution of the heat operator $\partial/\partial t - \Delta^q_{\bar{z}}$.

Now there exists an $\varepsilon > 0$ such that the disc bundle N_ε over the fixed point manifold X_φ defined by

$$N_\varepsilon = \{v \in T_x(X) \text{ s.t. } x \in X_\varphi \quad \text{and} \quad \|v\| < \varepsilon\},$$

(where $\|\ \|$ is defined by the metric) is diffeomorphic to a neighborhood of X_φ in X. The form $(\varphi^*_{z'}\, e^q(t, z', z))_{z'=z} {}^* 1$ defines under this diffeomorphism a form on N_ε, which we shall denote by $E^q(t, z)$.

There is a natural map $\pi_* : C^\infty(\Lambda T^*(N_\varepsilon)) \to C^\infty(\Lambda T^*(X_\varphi))$ such that $\int \psi_1 \wedge \pi_*(\psi_2) = \int \pi^*(\psi_1) \wedge \psi_2$, $\psi_2 \in C^\infty(\Lambda T^*(N_\varepsilon))$, $\psi_1 \in C^\infty(\Lambda T^*(X_\varphi))$, $\pi : N_\varepsilon \to X$ being the projection.

Let $\psi^q_t = \pi_*(E^q(t, z))$. We have the following proposition:

PROPOSITION 2. $H(\varphi) = \sum_{q=0}^n (-1)^q \int_{X_\varphi} \psi^q_t$, *as* $t \downarrow 0$, *the forms* ψ^q_t *turn out to be independent of* $\varepsilon > 0$ *(as* $t \downarrow 0$*).*

Moreover we have the following theorem:

THEOREM 3. (*Local form of Lefschetz fixed point formula.*) *We have at each point* $z \in Y_i$, $1 \leqq i \leqq N$,

$$\sum_{q=0}^n (-1)^q \psi^q_t(z) = \left(\prod_{\lambda^i_j \neq 1} (1 - \lambda^i_j)^{n^i_j} \right)^{-1}$$

$$\times\ 2r_i\text{th component of} \left[\left(\prod\nolimits_{\lambda^i_j \neq 1} \mathscr{U}^i_j \right) \mathscr{T}(Y^i) \right](z) + O(t),$$

$$\text{as } t \downarrow 0,$$

where $\mathscr{U}^i_j$'s *are the characteristic classes defined in* §2 *and here represented as a differential form by Andre Weil's homomorphism, the connections used in* $T(Y_i)$, E^i_j *are the hermitian connections defined by the hermitian metric.*

Theorem 1 is an immediate consequence of Proposition (2) and Theorem (3). Theorem (3) is of course stronger than Theorem (1). Our proof of Theorem (3) depends on the method developed in [**4**] and [**5**].

REMARK. The results have natural extension to the situation when one considers the Lefschetz number associated to the data: a holomorphic vector bundle ξ over X, a holomorphic map φ of X into itself and a vector bundle analytic homomorphism $\tilde{\varphi}$ of $\varphi^*(\xi)$ into ξ.

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REFERENCES

1. M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes*. I and II, Ann. of Math. (2) **86** (1967), 374–407, **88** (1968), 451–491. MR **35** #3701; **38** #731.

2. M. F. Atiyah and I. M. Singer, *The index of elliptic operators*. I and III, Ann. of Math. (2) **87** (1968), 484–530; **87** (1968), 546–604. MR **38** #5243; **38** #5245.

3. T. Kotake, *The fixed point theorem of Atiyah-Bott via parabolic operators*, Comm. Pure Appl. Math. **22** (1969), 789–806.

4. V. K. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Differential Geometry **5** (1971), 233–249.

5. ———, *An analytic proof of Riemann-Roch-Hirzebruch theorem for Kähler manifolds*, J. Differential Geometry **5** (1971), 251–283.

6. D. Toledo, *On the Atiyah-Bott formula for isolated fixed points*, J. Differential Geometry (to appear).

7. Yue Lin L. Tong, *deRham's integrals and Lefschetz fixed point formula for d″ cohomology*, Bull. Amer. Math. Soc. **78** (1972), 420–422.

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