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# Holomorphic line bundles on a domain of a two-dimensional Stein manifold

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**Abstract.** Let $D$ be an open subset of a two-dimensional Stein manifold $S$. Then $D$ is Stein if and only if every holomorphic line bundle $L$ on $D$ is the line bundle associated to some (not necessarily effective) Cartier divisor $\mathfrak{d}$ on $D$.

**1. Introduction.** It is well known that every holomorphic line bundle $L$ on a projective algebraic manifold $P$ is associated to some (not necessarily effective) Cartier divisor $\mathfrak{d}$ on $P$ (see Griffiths–Harris [5, p. 161]). A similar fact holds for reduced Stein spaces. For every holomorphic line bundle $L$ on a reduced Stein space $S$ there exists a holomorphic global section $s$ of $L$ such that $s$ does not vanish identically on any irreducible component of $S$. Therefore $L$ is the line bundle associated to the effective Cartier divisor $\operatorname{div}(s)$ on $S$ (see Gunning [6, pp. 120–122]).

On the other hand a holomorphic line bundle $L$ on the punctured didisk $X := \Delta^2 \setminus \{(0,0)\}$, which is not Stein, is holomorphically trivial if and only if $L$ is the line bundle associated to some Cartier divisor on $X$ (see Remark 1.4 of Ballico [1]).

In this paper we prove that an open subset $D$ of a two-dimensional Stein manifold $S$ is Stein if and only if every holomorphic line bundle $L$ on $D$ is the line bundle associated to some (not necessarily effective) Cartier divisor $\mathfrak{d}$ on $D$ (Theorem 3). By using the method of Kajiwara–Kazama [10] the proof is deduced from the above-mentioned property of the punctured didisk.

If $S$ is a Stein manifold of dimension more than two, then there can exist an open subset $D$ of $S$ such that $D$ is not Stein and every holomorphic line bundle $L$ on $D$ is the line bundle associated to some effective Cartier divisor $\mathfrak{d}$ on $D$.

**2. Lemmas.** Let $X$ be a reduced complex space. Let $\mathfrak{d}$ be a Cartier divisor on $X$ defined by a meromorphic Cousin-II distribution $\{(U_i, m_i)\}_{i\in I}$

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on $X$. We denote by $[\mathfrak{d}]$ the holomorphic line bundle on $X$ defined by the cochain $\{m_i/m_j\} \in Z^1(\{U_i\}_{i\in I}, \mathscr{O}^*)$. We say that $[\mathfrak{d}]$ is the *holomorphic line bundle associated to* $\mathfrak{d}$.

Let $\Delta := \{t \in \mathbb{C} \mid |t| < 1\}$. A holomorphic line bundle $L$ on the punctured didisk $X := \Delta^2 \setminus \{(0,0)\}$ is holomorphically trivial if and only if there exists a Cartier divisor $\mathfrak{d}$ on $X$ such that $L = [\mathfrak{d}]$ (see Remark 1.4 of Ballico [1]). More generally we have the following lemma.

LEMMA 1. *Let $S$ be a two-dimensional Stein manifold with $H^2(S,\mathbb{Z}) = 0$. Let $A$ be a non-empty discrete subset of $S$. Let $L$ be a holomorphic line bundle on the open set $S \setminus A$. Then $L$ is holomorphically trivial if and only if there exists a Cartier divisor $\mathfrak{d}$ on $S \setminus A$ such that $L = [\mathfrak{d}]$.*

*Proof.* Assume that there exists a Cartier divisor $\mathfrak{d}$ on $D := S \setminus A$ such that $L = [\mathfrak{d}]$. Since $D$ is Cousin-II (see Theorem 8.25 of Hitotumatu [7, p. 174]), $L = [\mathfrak{d}]$ is holomorphically trivial. The converse is clear. ■

Let $S$ be a reduced complex space and $D$ an open subset of $S$. Let $p$ be a boundary point of $D$ in $S$. Then we say that $D$ is *locally Stein* at $p$ if there exists a neighborhood $U$ of $p$ in $S$ such that $U \cap D$ is Stein. By using the method of the proof of Lemma 11 of Kajiwara–Kazama [10] we prove the following lemma.

LEMMA 2. *Let $S$ be a purely two-dimensional reduced Stein space and $D$ an open subset of $S$. Assume that for every holomorphic line bundle $L$ on $D$ there exists a Cartier divisor $\mathfrak{d}$ on $D$ such that $L = [\mathfrak{d}]$. Then $D$ is locally Stein at every point $p \in \partial D \setminus \mathrm{Sing}(S)$.*

*Proof.* Assume that there exists a point $p \in \partial D \setminus \mathrm{Sing}(S)$ such that $D$ is not locally Stein at $p$. Since $S$ is Stein, there exist a holomorphic map $\psi : S \to \mathbb{C}^2$ and a neighborhood $W$ of $p$ such that $W$ is non-singular, $\psi(W)$ is an open subset of $\mathbb{C}^2$ and $\psi|_W : W \to \psi(W)$ is biholomorphic (see Grauert–Remmert [4, p. 151]). Take a Stein open subset $V$ of $\mathbb{C}^2$ such that $\psi(p) \in V \Subset \psi(W)$. Then $U := \psi^{-1}(V) \cap W$ is a Stein neighborhood of $p$ and $\psi(U) = V$. Since $D$ is not locally Stein at $p$, the open set $\psi(D \cap U)$ is not Stein. By Lemma 1 of Kajiwara–Kazama [10] (see also the proof of Lemma 11 of [10]) there exist $H$, $P$, $\varepsilon$, $\varphi$ and $(b_1, b_2)$ with the following properties:

$$H = \{(w_1,w_2) \in \mathbb{C}^2 \mid |w_1| < 1,\ |w_2| < 1\}$$
$$\cup \{(w_1,w_2) \in \mathbb{C}^2 \mid 1 - 2\varepsilon < |w_1| < 1 + 2\varepsilon,\ |w_2| < 1 + 2\varepsilon\},$$
$$P = \{(w_1,w_2) \in \mathbb{C}^2 \mid |w_1| < 1 + 2\varepsilon,\ |w_2| < 1 + 2\varepsilon\}, \quad 0 < \varepsilon < 1/2,$$
$$\varphi : \mathbb{C}^2 \to \mathbb{C}^2 \text{ is a biholomorphic map}, \quad \varphi(H) \subset \psi(D \cap U),$$
$$|b_1| \le 1 - 2\varepsilon, \quad |b_2| = 1, \quad \varphi(b_1,b_2) \in \partial(\psi(D \cap U)).$$

Let $\theta = (\theta_1, \theta_2) := \varphi^{-1} \circ \psi : S \to \mathbb{C}^2$. Let $T := \{|\theta_1| < 1+2\varepsilon\}$, $T_0 := \{|\theta_2| < 1+2\varepsilon\} \cap T \cap U$ and $T_1 := \{|\theta_2| > 1+\varepsilon\} \cup (T \setminus \overline{U})$. Then $T$ is a Stein open subset of $S$ and $\{T_0, T_1\}$ is an open covering of $T$. The function $1/(\theta_2 - b_2)$ is holomorphic on $T_0 \cap T_1$. Since $H^1(\{T_0, T_1\}, \mathscr{O}) = 0$, there exist $v_i \in \mathscr{O}(T_i)$, $i = 1, 2$, such that $1/(\theta_2 - b_2) = v_1 - v_0$ on $T_0 \cap T_1$. We define a meromorphic function $v$ on $T$ by the equalities $v = v_0 + 1/(\theta_2 - b_2)$ on $T_0$ and $v = v_1$ on $T_1$. Then $v$ is holomorphic on $(\{\theta_2 \neq b_2\} \cap T_0) \cup T_1$. Let $D_1 := \{\theta_1 \neq b_1\} \cap D$ and $D_2 := ((\{\theta_2 \neq b_2\} \cap T) \cup (T \setminus \overline{U})) \cap D$. Then $\{D_1, D_2\}$ is an open covering of $D$ and the function $v/(\theta_1 - b_1)$ is holomorphic on $D_1 \cap D_2$. By assumption there exist invertible meromorphic functions $g_i$ on $D_i$, $i = 1, 2$, such that $\exp(v/(\theta_1 - b_1)) = g_1/g_2$ on $D_1 \cap D_2$. The function $g_1' := \exp(-v_0/(\theta_1 - b_1))\, g_1$ is meromorphic on $T_0 \cap D_1$ and $\exp(1/((\theta_1 - b_1)(\theta_2 - b_2))) = g_1'/g_2$ on $T_0 \cap D_1 \cap D_2$. Let $H_i := \{(w_1, w_2) \in H \mid w_i \neq b_i\}$ and $P_i := \{(w_1, w_2) \in P \mid w_i \neq b_i\}$ for $i = 1, 2$. Since $P$ is the envelope of holomorphy of $H$, the open set $P_i$ is the envelope of holomorphy of $H_i$ for each $i = 1, 2$ by Satz 7 of Grauert–Remmert [3] (see Theorem 2.5.9 of Jarnicki–Pflug [8, p. 182]). Since $H_i \subset \theta(U)$ and $\theta^{-1}(H_i) \cap U \subset T_0 \cap D_i$ for $i = 1, 2$, the function $f_1 := g_1' \circ (\theta|_U)^{-1}$ is meromorphic on $H_1$ and $f_2 := g_2 \circ (\theta|_U)^{-1}$ is meromorphic on $H_2$. By Proposition 3 of Kajiwara–Sakai [11] there exists a meromorphic function $\tilde{f}_i$ on $P_i$ such that $\tilde{f}_i = f_i$ on $H_i$ for each $i = 1, 2$. Since $f_i$ is invertible on $H_i$, $\tilde{f}_i$ is also invertible on $P_i$ by the theorem of identity. We have $\exp(1/((w_1 - b_1)(w_2 - b_2))) = \tilde{f}_1/\tilde{f}_2$ on $P_1 \cap P_2$. This contradicts Lemma 1 because the function $\exp(1/((w_1 - b_1)(w_2 - b_2))) \in \mathscr{O}^*(P_1 \cap P_2)$ defines a non-trivial holomorphic line bundle on $P_1 \cup P_2 = P \setminus \{(b_1, b_2)\}$ (see Lemma 1 of Kajiwara [9] or Serre [12, p. 372]). ∎

**3. Theorem.** We have the following theorem which characterizes a Stein open subset of a two-dimensional Stein manifold.

THEOREM 3. *Let $S$ be a two-dimensional Stein manifold and $D$ an open subset of $S$. Then the following four conditions are equivalent.*

(1) *$D$ is Stein.*
(2) *For every holomorphic line bundle $L$ on $D$ there exists an effective Cartier divisor $\mathfrak{d}$ on $D$ such that $L = [\mathfrak{d}]$.*
(3) *For every holomorphic line bundle $L$ on $D$ there exists a Cartier divisor $\mathfrak{d}$ on $D$ such that $L = [\mathfrak{d}]$.*
(4) *The image of the natural homomorphism $H^1(D, \mathscr{O}^*) \to H^1(D, \mathscr{M}^*)$ vanishes.*

*Proof.* (3)⇒(1). By Lemma 2 the open set $D$ is locally Stein at every boundary point $p$ of $D$ in $S$. It follows that $D$ is Stein by the theorem of Docquier–Grauert [2].

(1)⇒(2). Every holomorphic line bundle $L$ on an arbitrary reduced Stein space is associated to some Cartier divisor $[\mathfrak{d}]$. For the proof of this fact we refer to Gunning [6, pp. 120–122].

(2)⇒(3)⇔(4). Clear. ∎

For an open subset $D$ of a Stein manifold $S$ such that $\dim S \geq 3$ the theorem above does not hold. As an example, we take a non-empty analytic subset $A$ of $S$ such that $\operatorname{codim} A \geq 3$. The open subset $D := S \setminus A$ of $S$ is not Stein. Let $L$ be an arbitrary holomorphic line bundle on $D$. There exists a holomorphic line bundle $\tilde{L}$ on $S$ such that $\tilde{L}|_D = L$ (see Shiffman [13, p. 340]). Since $S$ is Stein, there exists an effective divisor $\tilde{\mathfrak{d}}$ on $S$ such that $\tilde{L} = [\tilde{\mathfrak{d}}]$. Then we have $L = [\tilde{\mathfrak{d}}|_D]$ and condition (2) of Theorem 3 is satisfied.

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