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HOLOMORPHIC MAPPINGS INTO TAUT COMPLEX ANALYTIC SPACES

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1. Introduction. We prove the following theorems which are generalizations of Satz 5.4. and Folgerung 5.8. of the paper by Kaup [5].

THEOREM 1. *Let X be a compact connected complex analytic space and Y a taut complex analytic space. Then the set*

$$\{f \in \mathrm{Hol}(X, Y); f(x_0) = y_0\}$$

is finite for any points x_0 of X and y_0 of Y.

THEOREM 2. *Let X be a compact connected complex analytic space and Y a compact taut complex analytic space. Then the set*

$$\{f \in \mathrm{Hol}(X, Y); f(X) = Y \text{ and } f^{-1}(y) \text{ is connected for every } y \in Y\}$$

is finite.

In this note, complex analytic spaces are always reduced and $\mathrm{Hol}(X, Y)$ stands for the set of all holomorphic mappings of a complex analytic space X into a complex analytic space Y.

Let X and Y be complex analytic spaces. Then the mapping

$$\Phi: X \times \mathrm{Hol}(X, Y) \to X \times Y$$

defined by the formula $\Phi(x, f) = (x, f(x)) \in X \times Y$ for each $(x, f) \in X \times \mathrm{Hol}(X, Y)$ is called the canonical mapping.

In this note we call an arbitrary complex analytic space X a *taut* complex analytic space, if, for every connected complex analytic space Y, the canonical mapping $\Phi: Y \times \mathrm{Hol}(Y, X) \to Y \times X$ is proper for the space $\mathrm{Hol}(Y, X)$ equipped with the compact-open topology. Namely, a complex analytic space X is said to be taut in our terminology if and only if X is hyperbolic in the terminology of Kaup [5].

Note that if X is a complex analytic space countable at infinity then, for an arbitrary connected complex analytic space Y, the compact-open topology of $\mathrm{Hol}(Y, X)$ coincides with the topology of uniform convergence on every compact set of Y. Wu [8] defined a taut complex manifold M as a connected complex analytic manifold M countable at infinity such that $\mathrm{Hol}(N, M)$ is normal for every connected complex analytic manifold N, i.e., any sequence in $\mathrm{Hol}(N, M)$ contains a subsequence which is either uniformly convergent on every compact set of N or compactly divergent on N. As far as complex analytic spaces countable at infinity are concerned, our definition of tautness is equivalent to that of Wu.

It is well known that a compact connected complex analytic space X is taut if and only if X is hyperbolic in the sense of Kobayashi [7].

For the proof of Theorems 1 and 2, we need the complex analytic structure constructed by Kaup [6] on the space of holomorphic mappings.

Let X be a compact complex analytic space and Y a complex analytic space. Kaup [6] showed that $\mathrm{Hol}(X, Y)$ equipped with the compact-open topology admits the structure of a complex analytic space which have the following properties:

(1) The canonical mapping

$$\Phi: X \times \mathrm{Hol}(X, Y) \to Y$$

defined by the formula $\Phi(x, f) = f(x)$ for each $(x, f) \in X \times \mathrm{Hol}(X, Y)$ is holomorphic.

(2) If $\phi: X \times T \to Y$ is holomorphic for a complex analytic space T, then $\tilde{\phi}: T \to \mathrm{Hol}(X, Y)$ defined by $\tilde{\phi}(t) = \phi(\cdot, t) \in \mathrm{Hol}(X, Y)$ for each $t \in T$ is holomorphic.

We have the following Lemma 1 (Theorem 1b of Kaup [6]).

LEMMA 1. *Let X and X' be compact complex analytic spaces and let $\alpha: X \to X'$ be a holomorphic surjection. Then, for a complex analytic space Y,*

$$\alpha^*: \mathrm{Hol}(X', Y) \to \mathrm{Hol}(X, Y)$$

defined by $\alpha^*(h) = h \circ \alpha$ for each $h \in \mathrm{Hol}(X', Y)$ is a biholomorphic mapping onto the complex analytic subvariety $\alpha^*\mathrm{Hol}(X', Y)$ of $\mathrm{Hol}(X, Y)$.

2. **Lemmas.** In this section, we fix a compact connected complex analytic space X and a complex analytic space Y.

LEMMA 2. *Let H be a compact complex analytic subvariety of* $\mathrm{Hol}(X, Y)$. *Then the set $\{f \in H;\ f(x_0) = y_0\}$ is finite for any points x_0 of X and y_0 of Y.*

PROOF. Assume first that X is irreducible. Consider the holomorphic mapping $\Phi: X \times H \to Y$ induced by the canonical mapping $\Phi: X \times \mathrm{Hol}(X, Y) \to Y$. Then we see easily that $H' = \{f \in H;\ f(x_0) = y_0\}$ is a complex analytic subvariety of H. Thus we have the holomorphic mapping

$\Phi: X \times H' \to Y$ by the restriction of Φ. Since H' is compact, we can take open neighborhoods U of x_0 in X and V of y_0 in Y such that

(1) $f(U) \subset V$ for every $f \in H'$, and

(2) V is biholomorphic onto a complex analytic subvariety of a domain of $\boldsymbol{C}^n$ (the Cartesian product of the complex line $\boldsymbol{C}$).

Then $\Phi: X \times H' \to Y$ induces the holomorphic mapping $\Phi: U \times H' \to V$. Since every holomorphic function defined on a compact connected complex analytic space is constant, we see that $\Phi(x, \cdot): H' \to Y$ is constant on each connected component of H' for every $x \in U$. This means that if $f, g \in H'$ are contained in the same connected component of H' then $f = g$ on U. Since X is irreducible, we see that each connected component of H' consists of one element of $\mathrm{Hol}(X, Y)$ and then H' is finite. It is now easy to complete the proof in the general case, since X is connected and has finitely many irreducible components.

LEMMA 3. *Let H be a compact connected complex analytic subvariety of* $\mathrm{Hol}(X, Y)$. *Then there exists a compact complex analytic space X' and a holomorphic surjection $\alpha: X \to X'$ which have the following properties*:

(1) *For the holomorphic mapping* $\alpha^*: \mathrm{Hol}(X', Y) \to \mathrm{Hol}(X, Y)$ *(see Lemma 1), we have a compact connected complex analytic subvariety H' of* $\mathrm{Hol}(X', Y)$ *such that* $\alpha^* H' = H$.

(2) *$h: X' \to Y$ has finite fibers over Y for any $h \in H'$.*

PROOF. Consider the following equivalence relation R on X:

$$xRy \text{ in } X \text{ if and only if } h(x) = h(y) \text{ in } Y \text{ for all } h \in H .$$

Let X' be the quotient space X/R and $\alpha: X \to X'$ the canonical projection. Then, by a theorem of H. Cartan [1], X' admits the structure of a (quotient) complex analytic space which have the following properties:

(1) $\alpha: X \to X'$ is holomorphic.

(2) Given each $h \in H$, there exists a unique holomorphic mapping $h': X' \to Y$ such that $h = h' \circ \alpha$ on X.

By Lemma 1, $\alpha^*: \mathrm{Hol}(X', Y) \to \mathrm{Hol}(X, Y)$ is a biholomorphic mapping onto the complex analytic subvariety $\alpha^* \mathrm{Hol}(X', Y)$ of $\mathrm{Hol}(X, Y)$. Since $H \subset \alpha^* \mathrm{Hol}(X', Y)$ by (2), we have a complex analytic subvariety H' of $\mathrm{Hol}(X', Y)$ such that $\alpha^*: H' \to H$ is biholomorphic. Then part (1) of Lemma 3 is obvious. Now consider the holomorphic mapping $\phi: H' \times X' \to Y$, the restriction of the canonical mapping $\phi: \mathrm{Hol}(X', Y) \times X' \to Y$. By the universality of the complex analytic space $\mathrm{Hol}(H', Y)$, we have the holomorphic mapping $\tilde{\phi}: X' \to \mathrm{Hol}(H', Y)$ defined by $\tilde{\phi}(x) =$

$\phi(\cdot, x) \in \mathrm{Hol}(H', Y)$ for each $x \in X'$. On the other hand, H' separates points of X', i.e., $\tilde{\phi}: X' \to \mathrm{Hol}(H', Y)$ is injective. By Lemma 2, we see that the set $\{x \in X'; h(x) = y\}$ is finite for any $h \in H'$ and $y \in Y$. Hence, $h: X' \to Y$ has finite fibers over Y for every $h \in H'$.

3. Proofs of main theorems.

Proof of Theorem 1. Since Y is taut, the set $\{f \in \mathrm{Hol}(X, Y); f(x_0) = y_0\}$ is a compact complex analytic subvariety of $\mathrm{Hol}(X, Y)$ for any fixed points x_0 of X and y_0 of Y (see Kaup [5]). Then the theorem follows from Lemma 2.

Proof of Theorem 2. Put

$$S = \{f \in \mathrm{Hol}(X, Y); f(X) = Y \text{ and } f^{-1}(y) \text{ is connected for every } y \in Y\}$$

in $\mathrm{Hol}(X, Y)$. Take a connected component H of $\mathrm{Hol}(X, Y)$ such that $H \cap S$ is non-empty. Note that, since Y is compact and taut, $\mathrm{Hol}(X, Y)$ is a compact complex analytic space hence H is a compact connected complex analytic subvariety of $\mathrm{Hol}(X, Y)$. Then, by Lemma 3, there exist a compact complex analytic space X' and a holomorphic surjection $\alpha: X \to X'$ such that, for each $h \in H$, we have a unique holomorphic mapping $h': X' \to Y$ with finite fibers over Y so that $h = h' \circ \alpha$ on X. Now, take an arbitrary $f \in H \cap S$ and the holomorphic mapping $f': X' \to Y$ with $f = f' \circ \alpha$ on X. Since $f^{-1}(y)$ is connected for every $y \in Y$, $f'^{-1}(y)$ is connected for every $y \in Y$. Thus $f': X' \to Y$ is a holomorphic bijection, hence a holomorphic homeomorphism, because X' is compact. X' and Y thus have the same normalization, namely, there exists a compact normal complex analytic space N (not necessarily connected) which is the normalization of X' and Y by $N \underset{\mu}{\to} X'$ and $N \underset{\nu}{\to} Y$, respectively. Then there exists a unique biholomorphic mapping $\tilde{f}: N \to N$ such that $\nu \circ \tilde{f} = f' \circ \mu$ on N (cf. Holmann [4]). It is well known that the normalization of a taut complex analytic space is also taut (cf. Kaup [5]). Hence N is a taut compact complex analytic space and there are at most finitely many biholomorphic mappings of N onto N (cf. [5]). As is easily seen, the correspondence $f \to \tilde{f}$ of $H \cap S$ into $\mathrm{Hol}(N, N)$ is injective. Thus we see that $H \cap S$ is finite. Then S is finite, because the number of connected components of $\mathrm{Hol}(X, Y)$ is finite.

4. A corollary.

Corollary 1. *Let X be a compact connected complex analytic space and Y a taut complex analytic space. Then, for any irreducible component H of the complex analytic space* $\mathrm{Hol}(X, Y)$, *we have* $\dim_{\mathbf{C}} H \leqq$

$\dim_C Y$.

PROOF. Since Y is taut, the canonical holomorphic mapping $\Phi: X \times \mathrm{Hol}(X, Y) \to X \times Y$ is proper. Furthermore, this canonical mapping Φ is discrete by Theorem 1.

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