J. DIFFERENTIAL GEOMETRY
6 (1971) 33-46

HOLOMORPHIC MAPPINGS OF POLYDISCS INTO COMPACT COMPLEX MANIFOLDS

K. KODAIRA

In this paper we prove an inequality in the manner of the Nevanlinna theory expressing certain properties of holomorphic mappings of n-dimensional polydiscs into compact complex manifolds of the same dimension and discuss some of its applications.

1. Let W be a compact complex manifold of dimension n. For a point w in W, we denote a local coordinate of w by $(w^1, w^2, \cdots, w^n)$. Take a complex line bundle L over W. By a theorem of de Rham, the Chern class $c(L)$ of L can be regarded as a d-cohomology *class* of d-closed 2-forms on W. We say that a real (1, 1)-form

$$\gamma = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(w) dw^\alpha \wedge d\bar{w}^\beta , \qquad i = \sqrt{-1} ,$$

on W is *positive semidefinite* (*or positive definite*) if the Hermitian matrix $(g_{\alpha\beta}(w))_{\alpha,\beta=1,\cdots,n}$ is positive semidefinite (or positive definite) at every point $w \in W$. Denote the canonical bundle of W by K. In this section we assume the existence of a complex line bundle L over W together with a positive integer m satisfying the following condition: *The Chern class* $c(L)$ *contains a positive semidefinite d-closed real* (1, 1)*-form and*

$$\dim H^0(W, \mathcal{O}(K^m \otimes L^{-1})) > 0 , \tag{1}$$

where $\mathcal{O}(K^m \otimes L^{-1})$ denotes the sheaf over W of germs of holomorphic sections of $K^m \otimes L^{-1}$.

Cover W by a *finite* number of small neighborhoods U_j, $j = 1, 2, \cdots$, and fix a local coordinate: $w \to (w_j^1, \cdots, w_j^n)$ on each U_j. Take a 1-cocycle $\{l_{jk}\}$ determining the line bundle L composed of nonvanishing holomorphic functions $l_{jk} = l_{jk}(w)$ defined, respectively, on $U_j \cap U_k$. We then find a 0-cochain $\{a_j\}$ composed of C^∞-differentiable functions $a_j = a_j(w) > 0$ defined, respectively, on U_j satisfying

$$a_j(w)^m = |l_{jk}(w)|^2 a_k(w)^m , \qquad \text{on } U_j \cap U_k ,$$

such that

Received September 14, 1970, and, in revised form, November 5, 1970.

$$\gamma = i \sum_{\alpha,\beta=1}^{n} g_{j\alpha\beta}(w) dw_j^\alpha \wedge d\bar{w}_j^\beta = i\partial\bar{\partial} \log a_j(w)$$

is positive semidefinite. Note that the d-closed real (1, 1)-form $m\gamma$ belongs to the Chern class $c(L)$. We choose a holomorphic section

$$\varphi \in H^0(W, \mathcal{O}(K^m \otimes L^{-1})) , \qquad \varphi \neq 0 ,$$

and denote by $\varphi_j(w)$ the fibre coordinate of $\varphi(w)$ over U_j. It is clear that

$$v = a_j(w) |\varphi_j(w)|^{2/m} (i/2)^n dw_j^1 \wedge d\bar{w}_j^1 \wedge \cdots \wedge dw_j^n \wedge d\bar{w}_j^n$$

is a *volume element*, i.e., a real continuous $2n$-form which is nonnegative everywhere on W. Fix a point $p^0 \in W$ such that $\varphi(p^0) \neq 0$, and assume that $p^0 \in U_1$. We normalize the volume element v by the condition:

$$(2) \qquad a_1(p^0) |\varphi_1(p^0)|^{2/m} = 1 .$$

Let $\mathbf{C}^n$ denote the space of n complex variables, define $|z| = \max_\lambda |z_\lambda|$ for $z = (z_1, \cdots, z_\lambda, \cdots, z_n) \in \mathbf{C}^n$, and denote by Δ_r a polydisc of radius r:

$$\Delta_r = \{z \in \mathbf{C}^n \,|\, |z| < r\} .$$

Take a polydisc $\Delta_R \subseteq \mathbf{C}^n$, consider a holomorphic mapping f of Δ_R into W, and assume that the Jacobian of f does not vanish at the origin $0 \in \Delta_R$ and that

$$(3) \qquad f(0) = p^0 .$$

For simplicity we write

$$dV(z) = (i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n ,$$

and let $f^*(v)$ denote the volume element on Δ_R induced from v by the mapping f. Then we have

$$f^*(v) = \xi(z) dV(z) , \qquad \xi(z) = a_j(f(z)) |\varphi_j(f(z))|^{2/m} |J_j(z)|^2 ,$$

where

$$J_j(z) = \det (\partial w_j^\alpha / \partial z_\lambda)_{\alpha,\lambda=1,\cdots,n} , \qquad (w_j^1, \cdots, w_j^n) = f(z) .$$

By hypothesis the Jacobian $J_j(z)$ of f does not vanish identically, and therefore the equation $\xi(z) = 0$ defines a proper analytic subset of Δ_R. Hence, by applying a suitable linear transformation to $\mathbf{C}^n$ if necessary, we may assume that, for any fixed values of $z_1, \cdots, z_{\lambda-1}, z_{\lambda+1}, \cdots, z_n$, the function $\xi(z_1, \cdots, z_\lambda, \cdots, z_n)$ of z_λ does not vanish identically and that

$$(4) \qquad J_1(0) = 1 \ .$$

Set

$$\sigma_\lambda = (i/2)^{n-1} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{\lambda-1} \wedge dz_{\lambda+1} \wedge \cdots \wedge d\bar{z}_n \ ,$$

$$\sigma = \sum_{\lambda=1}^{n} \sigma_\lambda \ ,$$

$$|\partial f(z)/\partial z_\lambda|^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\beta}(f(z))(\partial w_j^\alpha/\partial z_\lambda)(\partial \bar{w}_j^\beta/\partial \bar{z}_\lambda) \ ,$$

where $(w_j^1, \cdots, w_j^n) = f(z)$. Moreover, setting $z_\lambda = r_\lambda e^{i\theta_\lambda}$, we introduce polar coordinates $(r_\lambda, \theta_\lambda)$ and let

$$dS(z) = \sum_{\lambda=1}^{n} r_\lambda d\theta_\lambda \wedge \sigma_\lambda \ .$$

We denote the bundary of the polydisc $\varDelta_r$ by $\partial\varDelta_r$.

Now we define functions $M(r), A(r)$ and $N(r)$ of r, $0 < r < R$, as follows:

$$M(r) = r^{-1} \int_{\partial \varDelta_r} \log \xi(z) dS(z) \ ,$$

$$A(r) = 4 \int_{\varDelta_r} \sum_{\lambda=1}^{n} |\partial f(z)/\partial z_\lambda|^2 dV(z) \ ,$$

$$N(r) = 4\pi m^{-1} \int_{(f^*\varphi) \cap \varDelta_r} \sigma + 4\pi \int_{(J) \cap \varDelta_r} \sigma \ ,$$

where $(f^*\varphi)$ and (J) denote, respectively, the divisors of the holomorphic functions $\varphi_j(f(z))$ and $J_j(z)$.

Theorem 1. *We have the inequality*:

$$(5) \qquad \int_0^r A(t) t^{-1} dt + \int_0^r N(t) t^{-1} dt \leq M(r) \ .$$

Proof. Let

$$\mu(z) = \log \xi(z) \ .$$

The set $\Gamma = \{z \mid \xi(z) = 0\}$ is a proper analytic subset of $\varDelta_R$, and $\mu(z)$ is C^∞-differentiable outside Γ. For brevity we write

$$z = (z_1, \zeta) \ , \qquad \zeta = (z_2, \cdots, z_n) \ .$$

We set

$$\mu_1(r, \zeta) = \int_0^{2\pi} \mu(re^{i\theta}, \zeta)d\theta \ .$$

Lemma. *$\mu_1(r, \zeta)$ is a continuous function of (r, ζ), $0 < r < R$, $|\zeta| < R$, and is a piecewise smooth function of r, $0 < r < R$, when ζ is fixed.*

To prove this lemma, take a point ζ^0, $|\zeta^0| < R$, and a real number r^0, $0 < r^0 < R$, such that $(r^0 e^{i\theta}, \zeta^0) \notin \Gamma$ for $0 \leq \theta < 2\pi$. Moreover, for each $\zeta, |\zeta| < R$, denote by $\rho_h(\zeta), h = 1, 2, 3, \cdots$, the roots of the equation:

$$\varphi_j(f(z_1, \zeta))J_j(z_1, \zeta)^m = 0 \ .$$

Then for a small positive number ε we have, for $|z_1| < r^0, |\zeta - \zeta^0| < \varepsilon$,

$$\mu(z) = 2m^{-1} \sum_h \log |z_1 - \rho_h(\zeta)| + \tau(z) \ ,$$

where the summation is extended over all roots $\rho_h(\zeta)$ with $|\rho_h(\zeta)| < r^0$, and $\tau(z)$ is a C^∞-differentiable function of z. Using the formula

$$\int_0^{2\pi} \log |re^{i\theta} - \rho| d\theta = 2\pi \max \{\log r, \log |\rho|\} \ ,$$

we hence obtain

$$(6) \qquad \mu_1(r, \zeta) = 4\pi m^{-1} \sum_h \max \{\log r, \log |\rho_h(\zeta)|\} + \tau_1(r, \zeta) \ ,$$

where $\tau_1(r, \zeta)$ is a C^∞-differentiable function of (r, ζ), $|r| < r_0$, $|\zeta - \zeta^0| < \varepsilon$. Since the roots $\rho_h(\zeta)$, arranged in an appropriate order, are continuous functions of $\zeta, |\zeta - \zeta^0| < \varepsilon$, the formula (6) proves the lemma.

Define

$$M(r_1, r_2, \cdots, r_n) = \int \mu(z_1, z_2, \cdots, z_n)d\theta_1 d\theta_2 \cdots d\theta_n \ ,$$

where the integral is extended over the domain: $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_2 < 2\pi$, $\cdots, 0 \leq \theta_n < 2\pi$. Since

$$M(r_1, r_2, \cdots, r_n) = \int \mu_1(r_1, z_2, \cdots, z_n)d\theta_2 \cdots d\theta_n \ ,$$

we infer from the above lemma that $M(r_1, r_2, \cdots, r_n)$ is a continuous function of $(r_1, r_2, \cdots, r_n) \neq (0, \cdots, 0)$, while, by (2), (3) and (4), the function $\mu(z)$ of z is C^∞-differentiable in a neighborhood of 0. Consequently $M(r_1, \cdots r_n)$ is a continuous function of $(r_1, \cdots, r_n)$, $0 \leq r_\lambda < R$.

Let ∂_1 denote the exterior differentiation with respect to the variable z_1. We then have

$$i\partial_1\bar{\partial}_1\mu(z) = i\partial_1\bar{\partial}_1 \log a_j(f(z)) = |\partial f(z)/\partial z_1|^2 idz_1 \wedge d\bar{z}_1 .$$

Define

$$B(r, \zeta) = \int_{|z_1|<r} 2i\partial_1\bar{\partial}_1\mu(z) = \int_{|z_1|<r} 2|\partial f(z)/\partial z_1|^2 idz_1 \wedge d\bar{z}_1 .$$

Setting $z_1 = x + iy$, we have

$$2i\partial_1\bar{\partial}_1\mu = d{*}d\mu, \qquad {*}d\mu = (\partial\mu/\partial x)dy - (\partial\mu/\partial y)dx .$$

Moreover the function $\mu(z_1, \zeta)$ is C^∞-differentiable in z_1 for $z_1 \neq \rho_h(\zeta)$. Hence, letting

$$\oint_\rho {*}d\mu(z) = \lim_{\varepsilon\to 0} \int_{|z_1-\rho|=\varepsilon} {*}d\mu(z_1, \zeta) ,$$

we obtain

$$B(r, \zeta) = \int_{|z_1|=r} {*}d\mu(z) - \sum_{|\rho|<r} \oint_\rho {*}d\mu(z) .$$

Note that $\oint_\rho {*}d\mu(z) = 0$ for $\rho \neq \rho_h(\zeta)$, $h = 1, 2, \cdots$. We denote by $\nu(r, \zeta, f^*\varphi)$ and $\nu(r, \zeta, J)$, respectively, the number of the roots on the disc $|z_1| < r$ of the equations $\varphi(f(z_1, \zeta)) = 0$ and $J_j(z, \zeta) = 0$. Since

$$\mu(z) = \log a_j(f(z)) + 2m^{-1} \log|\varphi_j(f(z))| + 2 \log|J_j(z)| ,$$

we have

$$\sum_{|\rho|<r} \oint_\rho {*}d\mu(z) = 4\pi m^{-1}\nu(r, \zeta, f^*\varphi) + 4\pi\nu(r, \zeta, J) .$$

Moreover we see readily that

$$\int_{|z_1|=r} {*}d\mu(z) = r\partial\mu_1(r, \zeta)/\partial r .$$

Hence, setting

$$\nu(r, \zeta) = 4\pi m^{-1}\nu(r, \zeta, f^*\varphi) + 4\pi\nu(r, \zeta, J) ,$$

we obtain

$$B(r, \zeta) + \nu(r, \zeta) = r\partial\mu_1(r, \zeta)/\partial r ,$$

and therefore

$$(7) \qquad \int_s^r B(t, \zeta) t^{-1} dt + \int_s^r \nu(t, \zeta) t^{-1} dt = \mu_1(r, \zeta) - \mu_1(s, \zeta) .$$

This proves the inequality

$$\mu_1(r, z_2, \cdots, z_n) \geq \mu_1(s, z_2, \cdots, z_n) , \qquad \text{for } r > s > 0 .$$

It follows that

$$M(r, r_2, \cdots, r_n) \geq M(s, r_2, \cdots, r_n) , \qquad \text{for } r > s .$$

Thus we infer that $M(r_1, \cdots, r_\lambda, \cdots, r_n)$ *is a monotone nondecreasing function of each variable* r_λ. Since, by (2), (3) and (4), $\xi(0)$ is equal to 1, we get

$$(8) \qquad M(r_1, r_2, \cdots, r_n) \geq 0 .$$

Define

$$A(t, u) = \int_{|\zeta| \leq u} B(t, \zeta) dV(\zeta) ,$$

$$N(t, u) = \int_{|\zeta| \leq u} \nu(t, \zeta) dV(\zeta) ,$$

$$M_1(t, u) = \int_{|\zeta| \leq u} \mu_1(t, \zeta) dV(\zeta) ,$$

where

$$dV(\zeta) = \sigma_1 = (i/2)^{n-1} dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n .$$

Since $i dz_\lambda \wedge d\bar{z}_\lambda = 2 r_\lambda dr_\lambda d\theta_\lambda$, we have

$$M_1(r, u) = \int_0^u M(r, r_2, r_3, \cdots, r_n) r_2 dr_2 r_3 dr_3 \cdots r_n dr_n ,$$

where the integral is extended over the domain: $0 \leq r_\lambda \leq u, \lambda = 2, 3, \cdots, n$. Hence, using (8), we obtain from (7) the inequality

$$(9) \qquad \int_0^r A(t, u) t^{-1} dt + \int_0^r N(t, u) t^{-1} dt \leq M_1(r, u) .$$

Set

$$M_\lambda(r) = \int_0^r M(t_2, \cdots, t_\lambda, r, t_{\lambda+1}, \cdots, t_n) t_2 dt_2 \cdots t_n dt_n ,$$

$$A_\lambda(r) = 4 \int_{\Delta_r} |\partial f(z)/\partial z_\lambda|^2 dV(z) ,$$

$$N_\lambda(r) = 4\pi m^{-1} \int_{(f^*\varphi) \cap \Delta_r} \sigma_\lambda + 4\pi \int_{(J) \cap \Delta_r} \sigma_\lambda .$$

Since $M_1(r) = M_1(r, r)$, $A_1(t) = A(t, t) \leq A(t, u)$ and $N_1(t) = N(t, t) \leq N(t, u)$ for $t \leq u$, we derive from (9) the inequality

$$\int_0^r A_1(t) t^{-1} dt + \int_0^r N_1(t) t^{-1} dt \leq M_1(t) .$$

We infer in the same manner that

$$\int_0^r A_\lambda(t) t^{-1} dt + \int_0^r N_\lambda(t) t^{-1} dt \leq M_\lambda(t) . \tag{10}$$

Since

$$rM(r) = \int_{\partial \Delta_r} \mu(z) dS(z) = \sum_{\lambda=1}^n \int_{|z| = |z_\lambda| = r} \mu(z) r_\lambda d\theta_\lambda \wedge d\sigma_\lambda ,$$

we have

$$M(r) = \sum_{\lambda=1}^n M_\lambda(r) ,$$

while it is obvious that

$$A(t) = \sum_{\lambda=1}^n A_\lambda(t) , \qquad N(t) = \sum_{\lambda=1}^n N_\lambda(t) .$$

Hence the inequality (5) follows from (10). q.e.d.

For a positive number β, we define

$$\Omega_\beta(r) = \int_{\partial \Delta_r} \xi(z)^\beta dS(z) ,$$

and set

$$S(r) = \int_{\partial \Delta_r} dS(z) = 2n\pi^n r^{2n-1} .$$

Theorem 2. *We have the inequality*

$$\int_0^r A(t)t^{-1}dt + \int_0^r N(t)t^{-1}dt \leq \beta^{-1}r^{-1}S(r)\log(\Omega_\beta(r)/S(r)) . \tag{11}$$

Proof. Since $\log x$ is a *concave* function of $x, x > 0$, we have

$$\begin{aligned} rM(r) &= \int_{\partial\Delta_r} \log \xi(z)dS(z) = \beta^{-1}\int_{\partial\Delta_r}\log \xi(z)^\beta dS(z) \\ &\leq \beta^{-1}S(r)\log\left(S(r)^{-1}\int_{\partial\Delta_r}\xi(z)^\beta dS(z)\right), \end{aligned}$$

which together with (5) gives the inequality (11). q.e.d.

We have assumed so far that the system of coordinates $(z_1, \cdots, z_\lambda, \cdots, z_n)$ is *general* in the sense that, for each λ and any fixed values of $z_1, \cdots, z_{\lambda-1}, z_{\lambda+1}, \cdots, z_n$, the function $\xi(z_1, \cdots, z_\lambda, \cdots, z_n)$ of z_λ does not vanish identically. However, this assumption is irrelevant to the inequality (11). *The inequality* (11) *holds for any system of coordinates* $(z_1, \cdots, z_n)$ *satisfying the conditions* (3) *and* (4). To prove this, suppose that the coordinates $(z_1, \cdots, z_n)$ are obtained from a fixed system of coordinates $(z_1^{(0)}, \cdots, z_n^{(0)})$ by means of a linear transformation $u = (u_{\lambda\nu})$ with $\det(u_{\lambda\nu}) = 1$:

$$z_\lambda = \sum_{\nu=1}^{n} u_{\lambda\nu}z_\nu^{(0)} .$$

There exists an everywhere dense subset G of the special linear group $SL(n, \mathbf{C})$ such that, for every $u \in G$, the corresponding system of coordinates $(z_1, \cdots, z_n)$ is general and, consequently, the inequality (11) holds. For our purpose it suffices, therefore, to verify that each term of (11) depends continuously on u. It is obvious that $\int_0^r A(t)t^{-1}dt$ and $\Omega_\beta(r)$ are continuous in u. Denoting the positive part of $\log x$ by $\log^+ x$, we have

$$\int_0^r N(t)t^{-1}dt = 4\pi m^{-1}\int_{(f^*\varphi)+m(J)} \log^+(r/|z|)\sigma ,$$

which shows that $\int_0^r N(t)t^{-1}dt$ depends continuously on u. q.e.d.

Note that

$$\int_{\Delta_r} \xi(z)^\beta dV(z) = \int_0^r \Omega_\beta(t)dt . \tag{12}$$

Since $A(t)$ and $N(t)$ are nonnegative, the inequality (11) implies that

$$\Omega_\beta(r) \geq S(r) . \tag{13}$$

Combining this with (12), we get

$$\int_{\Delta_r} \xi(z)^\beta dV(z) \geq \pi^n r^{2n} . \tag{14}$$

In particular, setting $\beta = 1$, we obtain

$$\int_{\Delta_r} f^*(v) \geq \pi^n r^{2n} . \tag{15}$$

2. A holomorphic mapping is said to be *totally degenerate* if its Jacobian vanishes identically. Let v_0 be a volume element which is positive everywhere on W. Then, for any holomorphic mapping f of Δ_r into W, the quotient $\int_{\Delta_r} f^*(v_0) \Big/ \int_W v_0$ may be regarded as a *mean degree* of the mapping $f\colon \Delta_r \to W$. Define

$$\deg (f \mid \Delta_r) = \int_{\Delta_r} f^*(v_0) \Big/ \int_W v_0 ,$$

and further set

$$P_m = \dim H^0(W, \mathcal{O}(K^m)) , \qquad \text{for} \quad m = 1, 2, 3, \cdots .$$

Theorem 3. *Let W be a compact complex manifold of dimension n. If there exists a holomorphic mapping f of $\mathbf{C}^n$ into W which is not totally degenerate, and if*

$$\liminf_{r \to +\infty} r^{-2n} \deg (f \mid \Delta_r) = 0 , \tag{16}$$

then all the plurigenera P_m of W vanish.

Proof. Suppose that one of the plurigenera, say P_m, is positive. Then, letting L be a trivial bundle, we have the inequality (1). Hence, by (15), we obtain

$$\int_{\Delta_r} f^*(v) \geq \pi^n r^{2n} ,$$

which contradicts (16), since the quotient v/v_0 is bounded on W. q.e.d.

By a surface we shall mean a compact complex manifold of dimension 2. A surface W is said to be *regular* if the first Betti number $b_1(W)$ of W vanishes. *A regular surface W is rational if and only if all the plurigenera P_m of W vanish* (see [9, Theorem 54]).

Theorem 4. *If a regular surface W contains $\mathbf{C}^2$ as its open subset, then W is a rational surface.*

Proof. Let W be a regular surface containing $\mathbf{C}^2$ and let $f: \mathbf{C}^2 \subsetneq W$ denote the inclusion map. It is obvious that $\deg (f|\Delta_r) < 1$ for each polydisc $\Delta_r \subset \mathbf{C}^2$. Thus by Theorem 3 all the plurigenera P_m of W vanish, and hence W is a rational surface. q.e.d.

Letting U be a non-empty open subset of a compact complex manifold W, we call W a *compactification* of U if the complement $W - U$ of U is an analytic subset of W. F. Hirzebruch mentioned in his list [6] of problems the classification of all compactifications of $\mathbf{C}^n$. Concerning this problem, A. Van de Ven [13] pointed out that all the known examples of compactifications of $\mathbf{C}^2$ are rational surfaces.

Theorem 5. *Every compactification of $\mathbf{C}^2$ is a rational surface.*

Proof. Let W be a compactification of $\mathbf{C}^2$. It is then obvious that $b_1(W) = b_1(\mathbf{C}^2) = 0$. Hence, by Theorem 4, W is a rational surface. q.e.d.

The condition $\mathbf{C}^2 \subsetneq W$ is much weaker than that W is a compactification of $\mathbf{C}^2$. In fact, there exists an infinite sequence of *mutually disjoint* open subsets $U_1, U_2, U_3, \cdots$ of $\mathbf{C}^2$ each of which is biholomorphically isomorphic to $\mathbf{C}^2$ (see § 4 below). Thus, if $\mathbf{C}^2 \subsetneq W$, then $U_1 \subsetneq \mathbf{C}^2 \subsetneq W$, and the existence of $U_1 \subsetneq W$ together with the vanishing of $b_1(W)$ already implies the rationality of W.

3. Letting W be a projective algebraic manifold of dimension n, we call W an algebraic manifold of *general type* if

$$\limsup_{m\to+\infty} m^{-n} \dim H^0(W, \mathcal{O}(K^m)) > 0 , \tag{17}$$

where K denotes the canonical bundle of W. Recently Iitaka [7] introduced the concept of canonical dimension. The condition (17) is equivalent to saying that the canonical dimension of W coincides with the dimension n of W. In this section we apply Theorem 1 to algebraic manifolds of general type and derive a recent result of Griffiths [5].

Let W be an algebraic manifold of general type of dimension n, X a general hyperplane section of W, and $L = [X]$ the complex line bundle over W determined by the divisor X. Then, letting K_X denote the restriction of K to X, we have the exact sequence:

$$0 \to H^0(W, \mathcal{O}(K^m \otimes L^{-1})) \to H^0(W, \mathcal{O}(K^m)) \to H^0(X, \mathcal{O}(K_X^m)) \to \cdots ,$$

while $\dim H^0(X, \mathcal{O}(K_X^m))$ is a function of m of order $O(m^{n-1})$. Hence, by (17), $\dim H^0(X, \mathcal{O}(K^m \otimes L^{-1}))$ is positive for a large integer m, and thus we have the inequality (1). Obviously we may assume that the real (1, 1)-form

$$i \sum g_{j\alpha\beta}(w) dw_j^\alpha \wedge d\bar{w}_j^\beta = i\partial\bar{\partial} \log a_j(w)$$

is *positive definite*. Therefore, setting

$$g_j(w) = \det (g_{j\alpha\beta}(w)) ,$$

we find a positive constant c such that

$$a_j(w) |\varphi_j(w)|^{2/m} \leq c^n g_j(w) , \qquad \text{for } w \in W . \tag{18}$$

Now consider a holomorphic mapping $f: \Delta_R \to W$ satisfying the conditions (3) and (4), and set

$$\Omega(r) = \Omega_{1/n}(r), \qquad T(r) = \int_{\Delta_r} \xi(z)^{1/n} dV(z) .$$

Since

$$g_j(f(w)) | J_j(z)|^2 \leq \prod_{\lambda=1}^{n} |\partial f(z)/\partial z_\lambda|^2 ,$$

we have, in consequence of (18),

$$\xi(z) \leq c^n \prod_{\lambda=1}^{n} |\partial f(z)/\partial z_\lambda|^2 , \qquad \xi(z)^{1/n} \leq n^{-1} c \sum_{\lambda=1}^{n} |\partial f(z)/\partial z_\lambda|^2 ,$$

from which follows

$$T(r) \leq (4n)^{-1} c A(r) .$$

Combining this with (11) we obtain

$$\int_0^r T(t) t^{-1} dt \leq (4r)^{-1} c S(r) \log (\Omega(r)/S(r)) . \tag{19}$$

Set

$$Q(r) = \int_0^r T(t) t^{-1} dt , \qquad \Psi(r) = 2n\pi^{-n} r^{-2n} Q(r) ,$$

and note that, by (14), $T(r) \geq \pi^n r^{2n}$, $Q(r) \geq (2n)^{-1}\pi^n r^{2n}$ and $\Psi(r) \geq 1$. The inequality (19) implies that

$$r \leq r_0 , \qquad r_0 = r_0(c, n) ,$$

where $r_0(c, n)$ is a constant depending only on c and n (see Nevanlinna [11, p. 235]). In fact, if $\Omega(r) \leq r^2 Q(r)^4$, then the inequality (19) yields

$$r^2 \Psi(r) \leq n^2 c (4 \log \Psi(r) + (6n + 3) \log r + 3n \log \pi) .$$

Since $\Psi(r) \geq 1$ and $e \log x \leq x$ for $x > 0$, this proves that

$$r \leq r_1 = \max\{1, n^2ce^{-1}(6n + 7) + 3n \log \pi\} .$$

Therefore, if $r > r_1$, then (19) implies that $\Omega(r) > r^2Q(r)^4$. It follows that either $\Omega(r) > T(r)^2$ or $T(r) > rQ(r)^2$. If $\Omega(r) > T(r)^2$, then

$$dr = \Omega(r)^{-1}dT(r) < T(r)^{-2}dT(r) .$$

If $T(r) > rQ(r)^2$, then

$$dr = T(r)^{-1}rdQ(r) < Q(r)^{-2}dQ(r) .$$

Hence we get

$$\begin{aligned} r - r_1 = \int_{r_1}^{r} dt &< -\int_{r_1}^{r} d(T(t)^{-1} + Q(t)^{-1}) \\ &< T(r_1)^{-1} + Q(r_1)^{-1} < (2n + 1)\pi^{-n} , \end{aligned}$$

which proves that

$$r \leq r_0 , \qquad r_0 = r_1 + (2n + 1)\pi^{-n} .$$

Thus we obtain the following

Theorem 6. *Let W be an algebraic manifold of general type, and p^0 a point on W such that $\varphi(p^0) \neq 0$ for an element $\varphi \in H^0(W, \mathcal{O}(K^m \otimes L^{-1}))$. Then there exists a constant r_0 with the following properties: For any holomorphic mapping $f: \Delta_R \to W$ with $f(0) = p^0$ and $J_1(0) = 1$, the inequality $R \leq r_0$ holds, where $J_1(0)$ denotes the Jacobian of f at the origin* 0.

This theorem has been proved by Griffiths [5] under the assumption that the canonical system $|K|$ is ample. We remark that his proof also applies to the case in which $|K|$ is not assumed to be ample, and establishes the above Theorem 6 (see Kobayashi and Ochiai [8, Addendum]).

4. Bieberbach [2] constructed an example of a biholomorphic mapping f of $\mathbf{C}^2$ onto a proper open subset U of $\mathbf{C}^2$. His construction is as follows. Let $\eta: z \to \eta z$ be a biholomorphic automorphism of $\mathbf{C}^2$ of which the origin 0 is a fixed point: $\eta 0 = 0$. Obviously η induces a linear transformation of the tangent space $T_0(\mathbf{C}^2)(\cong \mathbf{C}^2)$ of $\mathbf{C}^2$ at 0. Let λ and μ denote the eigenvalues of this linear transformation, and assume that $|\lambda| \leq |\mu| < 1$. Then there exists a biholomorphic mapping $f_0: z \to f_0(z)$ of a neighborhood N of 0 into $\mathbf{C}^2$ with $f_0(0) = 0$ such that $g = f_0^{-1}\eta f_0$ takes the normal form

$$g: z = (z_1, z_2) \to gz = (\lambda z_1 + \beta z_2^p, \mu z_2) ,$$

where p is a positive integer and β is a constant which vanishes unless $\lambda = \mu^p$ (see Lattès [10], Sternberg [12]). Obviously g is a contraction in the sense that

$$\lim_{m\to+\infty} g^m z = 0 , \qquad \text{for } z \in \mathbf{C}^2 .$$

For every positive integer m, we have

$$f_0(z) = \eta^{-m} f_0(g^m z) , \qquad \text{for } z \in N , \tag{20}$$

provided that $gN \subset N$. Since $\eta^{-m} f_0 g^m$ is defined on $g^{-m}N$ and $\bigcup_m g^{-m}N = \mathbf{C}^2$, it follows from (20) that f_0 can be continued analytically to a biholomorphic mapping f of $\mathbf{C}^2$ onto an open subset U of $\mathbf{C}^2$ (see Sternberg [12, p. 816]). For every integer m we have

$$f(z) = \eta^{-m} f(g^m z) , \qquad \text{for } z \in \mathbf{C}^2 .$$

It follows that

$$U = \{z \mid \lim_{m\to+\infty} \eta^m z = 0\} .$$

Now we specify η to be the automorphism

$$\eta \colon z = (z_1, z_2) \to \eta z = (z_2, \lambda^2 z_1 + (\lambda^2 - 1)(\sin z_2 - z_2)) ,$$

where λ is a constant with $0 < |\lambda| < 1$. Note that the normal form of this η is

$$g \colon z = (z_1, z_2) \to gz = (\lambda z_1, -\lambda z_2) .$$

We define a translation

$$\tau \colon z = (z_1, z_2) \to (z_1 + 2\pi, z_2 + 2\pi) .$$

Then η and τ are commutative: $\eta\tau = \tau\eta$, and therefore, for each integer k, $\tau^k 0 = (2k\pi, 2k\pi)$ is a fixed point of η and

$$\tau^k U = \{z \mid \lim_{m\to+\infty} \eta^m z = \tau^k 0\} .$$

It follows that $\tau^k U$ and $\tau^j U$ are disjoint for $k \neq j$. Thus we obtain an infinite sequence of mutually disjoint open subsets $\tau^k U, k = 0, \pm 1, \pm 2, \cdots$, each of which is biholomorphically isomorphic to $\mathbf{C}^2$.

Letting $\{\tau\}$ denote the infinite cyclic group generated by τ, we have

$$\mathbf{C}^2/\{\tau\} = \mathbf{C}^* \times \mathbf{C} .$$

Clearly we may regard $U = \bigcup_k \tau^k U/\{\tau\}$ as an open subset of $\mathbf{C}^* \times \mathbf{C}$. Thus we see the existenc of a biholomorphic mapping: $\mathbf{C}^2 \subsetneq \mathbf{C}^* \times \mathbf{C}$. Combining this with Theorem 4, we infer that if a regular surface W contains $\mathbf{C}^* \times \mathbf{C}$ as its open subset, then W is a rational surface. This result can be verified also in the same manner as in the proof of Theorem 4. In fact, if $\mathbf{C}^* \times \mathbf{C} \subset W$, then

$f\colon (z_1, z_2) \to (\exp z_1, z_2)$ is a holomorphic mapping of $\mathbf{C}^2$ into W with $\deg (f \mid \Delta_r) = O(r)$. Thus by Theorem 3 all the plurigenera of W vanish, and hence W is a rational surface.

References

[1] L. V. Ahlfors, *Geometrie der Riemannschen Flächen*, C. R. Congrès Internat. Math. Oslo, 1936, 239–248.

[2] L. Bieberbach, *Beispiel zweier ganzer Funktionen zweier komplexer Variablen welche eine schlicht volumetreue Abbildung des* R_4 *auf einen Teil seiner selbest vermitteln*, Preuss. Akad. Wiss. Sitzungsber. 1933, 476–479.

[3] S. S. Chern, *Complex analytic mappings of Riemann surfaces.* I, Amer. J. Math. **82** (1960) 323–337.

[4] ——, *The integrated form of the first main theorem for complex analytic mappings in several complex variables*, Ann. of Math. **71** (1960) 536–551.

[5] P. A. Griffiths, *Holomorphic mappings into canonical algebraic varieties*, Ann. of Math. **93** (1971) 439–458.

[6] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. **60** (1954) 213–236.

[7] S. Iitaka, *On* D-*dimensions of algebraic varieties*, Proc. Japan Acad. **46** (1970) 487–489.

[8] S. Kobayashi & T. Ochiai, *Mappings into compact complex manifolds with negative first Chern classes*, J. Math. Soc. Japan **23** (1971) 137–148.

[9] K. Kodaira, *On the structure of compact complex analytic surfaces.* IV, Amer. J. Math. **90** (1968) 1048–1066.

[10] M. S. Lattès, *Sur les formes réduites des transformations ponctuelles à deux variables*, C. R. Acad. Sci. Paris **152** (1911) 1566–1569.

[11] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, Berlin, 1936.

[12] S. Sternberg, *Local contractions and a theorem of Poincaré*, Amer. J. Math. **79** (1957) 809–824.

[13] A. Van de Ven, *Analytic compactification of complex homology cells*, Math. Ann. **147** (1962) 189–204.

UNIVERSITY OF TOKYO