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HOLOMORPHIC MAPS FROM $\mathbf{C}^n$ TO $\mathbf{C}^n$

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ABSTRACT. We study holomorphic mappings from $\mathbf{C}^n$ to $\mathbf{C}^n$, and especially their action on countable sets. Several classes of countable sets are considered. Some new examples of Fatou-Bieberbach maps are given, and a nondegenerate map is constructed so that the volume of the image of $\mathbf{C}^n$ is finite. An Appendix is devoted to the question of linearization of contractions.

Introduction. In Part I of this paper we investigate relations between various classes of countable subsets of $\mathbf{C}^n$ on the one hand, and holomorphic maps of $\mathbf{C}^n$ (one-to-one or not) on the other. Part II contains some results concerning the ranges of entire maps. Both parts depend to a large extent on the use of the same tool, namely those automorphisms of $\mathbf{C}^n$ that we call shears.

Throughout the paper, n will be a positive integer, usually ≥ 2 unless the contrary is stated. The *automorphisms* of $\mathbf{C}^n$ are the biholomorphic maps from $\mathbf{C}^n$ onto $\mathbf{C}^n$. They form a group under composition, denoted by $\mathrm{Aut}(\mathbf{C}^n)$. When $n = 1$, this group is quite easy to describe: its members are the functions that send z to $az + b$ ($a, b \in \mathbf{C}$, $a \neq 0$). But $\mathrm{Aut}(\mathbf{C}^n)$ is a huge and complicated group for every $n > 1$.

If $F\colon \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic map, we write $F'(z)$ for its Fréchet derivative at z. The Jacobian of F at z, written $(JF)(z)$, is the determinant of the linear operator $F'(z)$. We call F *nondegenerate* if $JF \not\equiv 0$.

Here are some of our results:

(1) The Mittag-Leffler interpolation problem (find an entire function $f\colon \mathbf{C} \to \mathbf{C}$ that has prescribed values on a given discrete set) can be solved for holomorphic maps $F\colon \mathbf{C}^n \to \mathbf{C}^n$, when $n > 1$, so that an additional requirement is satisfied, namely: $(JF)(z) = 1$ at every $z \in \mathbf{C}^n$. In other words, the interpolating map can be so chosen that it is locally volume-preserving (Theorem 1.1).

When $n = 1$ there is a similar (but considerably more difficult) theorem which we do not include here: an interpolating f can be found whose derivative is nowhere 0. We thank R. C. Gunning for showing us how to prove this one-variable result by the techniques of an earlier paper [**6**].

(2) Given any two countable *dense* subsets X and Y of $\mathbf{C}^n$ ($n > 1$), there is an $F \in \mathrm{Aut}(\mathbf{C}^n)$ so that $F(X) = Y$. The proof (in §2) actually produces such an F with $JF \equiv 1$.

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(3) The situation is very different for *discrete* subsets of $\mathbf{C}^n$. (A set $E \subset \mathbf{C}^n$ is discrete if no point of $\mathbf{C}^n$ is a limit point of E.) Every infinite discrete set $E \subset \mathbf{C}^n$ $(n > 1)$ can be mapped onto an arithmetic progression by a one-to-one holomorphic map F from $\mathbf{C}^n$ into $\mathbf{C}^n$ with $JF \equiv 1$ (Theorem 3.7), but in general not by any automorphism of $\mathbf{C}^n$ (Theorem 4.8).

We call a set $E \subset \mathbf{C}^n$ *tame* if some $F \subset \mathrm{Aut}(\mathbf{C}^n)$ maps E onto an arithmetic progression. (The term "tame" was suggested by its use in geometric topology where, for example, an arc L in R^3 is called tame if some homeomorphism of the ambient space R^3 maps L onto a straight line interval.)

In §3 we show that some apparently rather weak conditions imply tameness, and that every infinite discrete $E \subset \mathbf{C}^n$ is the union of two tame ones. In [**7**], Hermes and Peschl asked, in fact, whether every infinite discrete $E \subset \mathbf{C}^n$ is tame. (We thank Eric Bedford for telling us about this paper.) The above-mentioned Theorem 4.8 shows that the answer is no. In §5 we show more: The infinite discrete subsets of $\mathbf{C}^n$ do not form just one equivalence class modulo $\mathrm{Aut}(\mathbf{C}^n)$, but continuum many (§5.3).

(4) The preceding result follows from the existence of discrete sets $D \subset \mathbf{C}^n$ that are *rigid* relative to $\mathrm{Aut}(\mathbf{C}^n)$: the identity map is the only automorphism of $\mathbf{C}^n$ that maps D onto D. On the other hand, tame sets are what one may call *permutable* by $\mathrm{Aut}(\mathbf{C}^n)$: Every permutation of a tame set $E \subset \mathbf{C}^n$ is the restriction to E of some $F \in \mathrm{Aut}(\mathbf{C}^n)$ (§3.2).

(5) Every tame set E in $\mathbf{C}^n$ is *avoidable* by biholomorphic maps: there is a biholomorphic $F\colon \mathbf{C}^n \to \mathbf{C}^n \backslash E$. On the other hand, we show in §4 that there exist discrete sets $D \subset \mathbf{C}^n$ which intersect the range of every nondegenerate holomorphic $F\colon \mathbf{C}^n \to \mathbf{C}^n$, whether F is one-to-one or not.

In §6 we construct tame sets in $\mathbf{C}^n$ that intersect $F(\mathbf{C}^n)$ for every F with $JF \equiv 1$. From this we deduce the existence of regions $\Omega \subset \mathbf{C}^n$ (for every $n > 1$) so that $\Omega = F(\mathbf{C}^n)$ for some biholomorphic F, but not for any biholomorphic F with constant Jacobian.

Nishimura [**12**, **13**] was apparently the first to prove the existence of such regions Ω, but only in $\mathbf{C}^2$. He used an entirely different method, depending on the subharmonicity of $\log|F|$. It is not clear whether his method can be extended to $\mathbf{C}^n$ when $n > 2$.

(6) In §7 we construct holomorphic maps $F\colon \mathbf{C}^n \to \mathbf{C}^n$, for all $n > 1$, with $JF \equiv 1$, which are bounded on the complement of a set of finite volume. It follows, of course, that $\mathrm{vol}\, F(\mathbf{C}^n) < \infty$. The existence of such maps settles several questions raised by Graham and Wu [**4**, pp. 627–628, 651], and disproves a conjecture made in [**2**, p. 168].

(7) If K is a strictly convex compact set in $\mathbf{C}^n$ (or K is a point) and E is any countable set in $\mathbf{C}^n \backslash K$, then there is a biholomorphic $F\colon \mathbf{C}^n \to \mathbf{C}^n$ so that $E \subset F(\mathbf{C}^n) \subset \mathbf{C}^n \backslash K$ (Theorem 8.5). In particular, E could be dense in $\mathbf{C}^n \backslash K$. When K is a point, this yields regions $\Omega \neq \mathbf{C}^n$ that are dense in $\mathbf{C}^n$ and are biholomorphic images of $\mathbf{C}^n$.

(8) In the final section we consider an older topic, namely the Fatou-Bieberbach method of constructing biholomorphic images of $\mathbf{C}^n$ in $\mathbf{C}^n$, starting with an automorphism that has an attracting fixed point. Here is the basic theorem:

(∗) *Suppose that* $F \in \operatorname{Aut}(\mathbf{C}^n)$ *fixes a point* $p \in \mathbf{C}^n$ *and that all eigenvalues* $\lambda_1, \dots, \lambda_n$ *of* $F'(p)$ *satisfy* $|\lambda_i| < 1$. *Let* Ω *be the set of all* $z \in \mathbf{C}^n$ *for which* $\lim_{k\to\infty} F^k(z) = p$ *where* $F^k = F \circ F^{k-1}$, $F^1 = F$. *Then there exists a biholomorphic map* Ψ *from* Ω *onto* $\mathbf{C}^n$.

A large part of the long paper [**2**] by Dixon and Esterle depends on this theorem. The desired Ψ is obtained as a solution of the functional equation

$$N^{-1} \circ \Psi \circ F = \Psi \qquad (**)$$

where N is a "normal form" for F. (Except in special cases, $N = F'(p)$.) On p. 142 they refer to Reich's papers [**15**, **16**] for the solution of (∗∗). Reich [**16**, p. 235] claims to prove that

$$\Psi = \lim_{k\to\infty} N^{-k} \circ F^k$$

solves (∗∗).

However, this sequence $\{N^{-k} \circ F^k\}$ need not converge, not even in some small neighborhood of p, and not even in the formal power series sense. We give a very simple counterexample in §9.2.

In §9 we prove (∗) under the simplifying assumption that $|\lambda_i|^2 < |\lambda_j|$ for all eigenvalues λ_i, λ_j of $F'(p)$. In that case the above-mentioned sequence does converge, and our proof is quite direct and even simpler than the one found by one of us a few years ago (see [**2**, pp. 144–145]). We use this special case to exhibit several new examples of regions that are biholomorphically equivalent to $\mathbf{C}^n$.

However, the general case of (∗) deserves a correct proof. Even though it may be possible to fix the one in [**16**], we give one in an Appendix which is quite self-contained and is actually much shorter and simpler than the work in [**15** and **16**]. It relies on an analysis of what we call *lower-triangular mappings*. These may have some independent interest.

We shall use very customary notations: $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbf{C}^n$, and $\pi_1, \dots, \pi_n$ are the coordinate projections, i.e., if $z = \sum z_i e_i$, then $\pi_i(z) = z_i$.

Also, $|z| = (\sum |z_i|^2)^{1/2}$, $B = \{z\colon |z| < 1\}$ is the open unit ball of $\mathbf{C}^n$, $S = \{z\colon |z| = 1\}$ is its boundary. Occasionally, when needed because more than one dimension is involved, we shall write B_n in place of B.

As mentioned earlier, much of our work will use *shears*. These automorphisms of $\mathbf{C}^n$ are obtained by choosing some j $(1 \le j \le n)$ and adding a holomorphic function of the other $n-1$ variables to z_j. For instance, any map $F(z_1, \dots, z_n) = (w_1, \dots, w_n)$ of the form

$$w_1 = z_1 + f(z_2, \dots, z_n), \qquad w_i = z_i \quad \text{for } 2 \le i \le n$$

is a shear *in the direction* of e_1. We will often want to do this without reference to any coordinate system. In fact, most of the shears σ that we will use will have the simple form $\sigma(z) = z + f(\Lambda z)u$ where $\Lambda\colon \mathbf{C}^n \to \mathbf{C}$ is a linear functional, $\Lambda u = 0$, and $f\colon \mathbf{C} \to \mathbf{C}$ is holomorphic.

It is easy to see that σ^{-1} has the same form, with $-f$ in place of f, and that $J\sigma \equiv 1$.

Other automorphisms of $\mathbf{C}^n$, which we will only use a few times, have the form

$$w_j = z_j \exp\{c_j f(z_1^{a_1} \cdots z_n^{a_n})\} \qquad (1 \le j \le n),$$

where $a_1, \dots, a_n$ are nonnegative integers, $c_j \in \mathbf{C}$, $\sum c_j a_j = 0$, and $f\colon \mathbf{C} \to \mathbf{C}$ is holomorphic. These satisfy

$$w_1^{a_1} \cdots w_n^{a_n} = z_1^{a_1} \cdots z_n^{a_n}$$

and their Jacobian is $w_1 w_2 \cdots w_n / z_1 z_2 \cdots z_n$.

PART I. COUNTABLE SUBSETS OF $\mathbf{C}^n$

1. An immersion-interpolation theorem. The classical interpolation theorem of Mittag-Leffler states that if $\{a_i\}$ is a discrete sequence in $\mathbf{C}$, then to every choice of $\{b_i\}$ in $\mathbf{C}$ corresponds an entire function f so that $f(a_i) = b_i$ for $i = 1, 2, 3, \dots$. We show in this section that the corresponding interpolation problem for holomorphic maps from $\mathbf{C}^n$ into $\mathbf{C}^n$ can be solved (when $n > 1$) so that the interpolating map satisfies the additional requirement that its Jacobian be a nonzero constant.

1.1. THEOREM. *Assume that $n > 1$, that $\{p_j\}$ is a discrete sequence in $\mathbf{C}^n$ (without repetition), and that $\{w_j\}$ is an arbitrary sequence in $\mathbf{C}^n$.*

Then there exists a holomorphic map $\Phi\colon \mathbf{C}^n \to \mathbf{C}^n$ so that

(a) *$\Phi(p_j) = w_j$ for $j = 1, 2, 3, \dots$ and*

(b) *$(J\Phi)(z) = 1$ for every $z \in \mathbf{C}^n$.*

Conclusion (b) implies, in particular, that Φ is a local homeomorphism, i.e., an *immersion* of $\mathbf{C}^n$ into $\mathbf{C}^n$.

Our proof will use a sequence of shears. The following lemma describes the basic move.

1.2. LEMMA. *Suppose that $\varepsilon > 0$ and that*

(i) *$a_1, \dots, a_m$ are points in a compact convex set $K \subset \mathbf{C}^n$,*

(ii) *p and q are points in a hyperplane $\Pi \subset \mathbf{C}^n$ (of complex dimension $n-1$) which does not intersect K.*

Then there is a shear τ which moves p to q, fixes every a_i, and moves no point of K by as much as ε.

PROOF. Assumption (ii) implies that there is a linear functional $\Lambda\colon \mathbf{C}^n \to \mathbf{C}$ so that $\Lambda p = \Lambda q$ and Λp lies outside $\Lambda(K)$. Since $\Lambda p = \Lambda q$, there is a unit vector $u \in \mathbf{C}^n$ so that $\Lambda u = 0$, $q = p + cu$ for some $c \in \mathbf{C}$. Since $\Lambda p \notin \Lambda(K)$, and $\Lambda(K)$ is a compact convex set in $\mathbf{C}$, there is a polynomial $g\colon \mathbf{C} \to \mathbf{C}$ that satisfies

$$g(\Lambda p) = c, \qquad g(\Lambda a_i) = 0 \qquad (1 \le i \le m)$$

and $|g| < \varepsilon$ on $\Lambda(K)$. Define

$$\tau(z) = z + g(\Lambda z) u \qquad (z \in \mathbf{C}^n).$$

This τ has the desired properties.

1.3. COROLLARY. *If $a_1, \dots, a_m$, K, ε are as above, and p, q are points in $\mathbf{C}^n \backslash K$, then some composition of two shears moves p to q, fixes every a_i, and moves no point of K by as much as ε.*

PROOF. There are hyperplanes Π' and Π'', through p and q, respectively, which do not intersect K and which are not parallel. Pick $w \in \Pi' \cap \Pi''$ and apply Lemma 1.2 twice (with $\varepsilon/2$ in place of ε) to move p to w and then w to q.

1.4. PROOF OF THEOREM 1.1. We first choose the origin of $\mathbf{C}^n$ so that $0 < |p_1| < |p_2| < |p_3| < \cdots$ and then choose coordinate axes so that the hyperplane $\{z_1 = 0\}$ contains none of the points w_j. We will Φ as a composition

$$\Phi = E \circ F \tag{1}$$

in which F is a limit of a certain sequence of compositions of shears (thus $JF \equiv 1$) and

$$E(z_1, z_2, \ldots, z_n) = (e^{z_1}, z_2 e^{-z_1}, z_3, \ldots, z_n). \tag{2}$$

It is clear that $JE \equiv 1$. Thus $J\Phi \equiv 1$.

The most significant property of E, however, is the following: Every $w \in \mathbf{C}^n$ whose first coordinate $\pi_1(w)$ is different from 0 lies in the range of E; moreover, one can choose $v \in \mathbf{C}^n$ so that $w = E(v)$ and so that $|\pi_1(v)|$ is larger than any prescribed number.

We start the construction of F by setting $F_0(z) = z$. Suppose $k \geq 1$ and $F_{k-1} \in \mathrm{Aut}(\mathbf{C}^n)$ has been chosen. We can then choose $v_k \in \mathbf{C}^n$ so that $E(v_k) = w_k$ and $|\pi_1(v_k)|$ is so large that v_k lies outside the compact set $F_{k-1}(r_k \bar{B})$, where $r_k = |p_k|$. Thus there exists q_k so that $v_k = F_{k-1}(q_k)$ and $|q_k| > r_k$.

We now choose δ_k, $0 < \delta_k < r_k - r_{k-1}$, so that

$$|F_{k-1}(z') - F_{k-1}(z'')| < 2^{-k} \tag{3}$$

for all $z', z'' \in r_k \bar{B}$ with $|z' - z''| < \delta_k$. Corollary 1.3 furnishes G_k, a composition of two shears, so that

$$G_k(p_k) = q_k, \qquad G_k(p_i) = p_i \quad (1 \leq i \leq k-1) \tag{4}$$

and

$$|G_k(z) - z| < \delta_k \qquad (z \in r_{k-1}\bar{B}). \tag{5}$$

Define $F_k = F_{k-1} \circ G_k$. Then

$$F_k(p_k) = v_k, \qquad F_k(p_i) = F_{k-1}(p_i) \quad (1 \leq i \leq k-1) \tag{6}$$

and (3) and (5) show that

$$|F_k(z) - F_{k-1}(z)| < 2^{-k} \qquad (|z| \leq r_{k-1}). \tag{7}$$

It follows that $F = \lim_{k\to\infty} F_k$ exists, uniformly on compact subsets of $\mathbf{C}^n$ (because $r_k \to \infty$ as $k \to \infty$), that $F\colon \mathbf{C}^n \to \mathbf{C}^n$ is one-to-one and has $JF \equiv 1$, and that $F(p_k) = v_k$ for $k = 1, 2, 3, \ldots$, because $F_j(p_k) = v_k$ for all $j \geq k$. Since $E(v_k) = w_k$, the proof is complete.

1.5. REMARK. In the last paragraph we asserted that F is one-to-one. This property of F was actually not needed for the proof of Theorem 1.1, but we will refer to it in §3.7. Our assertion is based on the following well-known and easily proved fact which we will use repeatedly:

If $F = \lim_{k\to\infty} F_k$, uniformly on compact subsets of $\mathbf{C}^n$, and each F_k is holomorphic and one-to-one on $\mathbf{C}^n$, then either $JF \equiv 0$ (i.e., F is degenerate) or F is one-to-one on $\mathbf{C}^n$.

References, and more elaborate results of this kind, may be found in [**2**, pp. 140–141].

2. Dense sets. Theorem 2.2 will show that all countable dense subsets of $\mathbf{C}^n$ "look alike" to the group $\mathrm{Aut}(\mathbf{C}^n)$ when $n > 1$.

2.1. LEMMA. *Suppose $E, K, D \subset \mathbf{C}^n$, E is finite, K is compact, D is dense, and $n > 1$. If $a \in \mathbf{C}^n \backslash E$ and $\varepsilon > 0$, then there is a shear σ so that*
(i) *$\sigma(p) = p$ for every $p \in E$,*
(ii) *$\sigma(a) \in D$, and*
(iii) *$|\sigma(z) - z| + |\sigma^{-1}(z) - z| < \varepsilon$ for every $z \in K$.*

Note that we do not assume that $a \notin K$.

PROOF. Choose coordinates in $\mathbf{C}^n$ so that $a = 0$. Since $a \notin E$, and E is finite, there is a hyperplane Π through 0 which contains no point of E. Let u be a unit vector orthogonal to Π. Then $\langle p, u\rangle \neq 0$ for every $p \in E$. Since D is dense in $\mathbf{C}^n$, there is a sequence $\{w_i\}$ in D that converges to 0 and is "tangent" to Π. More explicitly, there are unit vectors u_i so that $w_i \perp u_i$ and $u_i \to u$ as $i \to \infty$. Hence there exists $c > 0$ so that $|\langle p, u_i\rangle| \geq c$ for all $p \in E$ as soon as i is large enough, say $i > i_0$. Define

$$g_i(\lambda) = \prod_{p \in E} \left\{1 - \frac{\lambda}{\langle p, u_i\rangle}\right\} \tag{1}$$

for $\lambda \in \mathbf{C}$, $i > i_0$, and put

$$\sigma_i(z) = z + g_i(\langle z, u_i\rangle) w_i \tag{2}$$

for $z \in \mathbf{C}^n$, $i > i_0$. Since $w_i \perp u_i$, each σ_i is a shear.

It is clear that $\sigma_i(p) = p$ for every $p \in E$ and that $\sigma_i(0) = w_i \in D$.

The denominators $\langle p, u_i\rangle$ in (1) are bounded from 0. Hence $\{g_i(\langle z, u_i\rangle)\}$ is uniformly bounded on K. Since $w_i \to 0$ as $i \to \infty$, it follows that $\sigma_i(z) \to z$ uniformly on K. The same holds for σ_i^{-1}, since σ_i^{-1} is obtained by simply replacing $+$ by $-$ in (2).

We conclude that σ_i satisfies (i), (ii), and (iii) as soon as i is large enough.

2.2. THEOREM. *If X and Y are countable dense subsets of $\mathbf{C}^n$, $n > 1$, then there is an $F \in \mathrm{Aut}(\mathbf{C}^n)$ so that $F(X) = Y$ and $JF \equiv 1$.*

PROOF. Enumerate $X = \{x_i\}$, $Y = \{y_i\}$, both without repetition.

We will construct a sequence of automorphisms F_j, starting with the identity map for F_0. Make the induction hypothesis that $j \geq 0$ and that F_j maps $2j$ points $p_1, \ldots, p_{2j}$ of X to $2j$ points $q_1, \ldots, q_{2j}$ of Y.

We will now construct F_{j+1}.

Let K be a large compact ball in $\mathbf{C}^n$ that contains $(j\bar{B}) \cup F_j(j\bar{B})$ in its interior, say at distance > 2 from the boundary of K. Choose ε_j, $0 < \varepsilon_j < 2^{-j-1}$, so that

$$|F_j^{-1}(z') - F_j^{-1}(z'')| < 2^{-j} \tag{1}$$

for all $z', z'' \in K$ with $|z' - z''| < 2\varepsilon_j$.

Let p_{2j+1} be the *first* x_i in $X \backslash \{p_1, \ldots, p_{2j}\}$. Apply Lemma 2.1 to the finite set $\{q_1, \ldots, q_{2j}\}$ and the dense set $Y \backslash \{q_1, \ldots, q_{2j}\}$ to find a shear σ_j so that

$$\begin{cases} \sigma_j(q_i) = q_i \quad \text{for } 1 \leq i \leq 2j, \\ \sigma_j(F_j(p_{2j+1})) \in Y \backslash \{q_1, \ldots, q_{2j}\}, \\ |\sigma_j(z) - z| + |\sigma_j^{-1}(z) - z| < \varepsilon_j \quad \text{for every } z \in K. \end{cases} \tag{2}$$

Put $q_{2j+1} = \sigma_j(F_j(p_{2j+1}))$, and let q_{2j+2} be the *first* y_i in $Y\backslash\{q_1, \ldots, q_{2j+1}\}$. Apply Lemma 2.1 to the finite set $\{q_1, \ldots, q_{2j+1}\}$ and the dense set

$$\sigma_j(F_j(X\backslash\{p_1, \ldots, p_{2j+1}\}))$$

to find a shear τ_j and a point $p_{2j+2} \in X$ so that

$$\begin{cases} \tau_j(q_i) = q_i \quad \text{for } 1 \le i \le 2j+1, \\ \tau_j(q_{2j+2}) = \sigma_j(F_j(p_{2j+2})), \\ |\tau_j(z) - z| + |\tau_j^{-1}(z) - z| < \varepsilon_j \quad \text{for every } z \in K. \end{cases} \tag{3}$$

Now define

$$F_{j+1} = \tau_j^{-1} \circ \sigma_j \circ F_j. \tag{4}$$

Then $F_{j+1}(p_i) = q_i$ for $1 \le i \le 2j+2$, which is our induction hypothesis, with $j+1$ in place of j.

Assume $z \in j\bar{B}$. Since $\tau_j^{-1} \circ \sigma_j$ moves no point of $F_j(j\bar{B})$ by as much as $2\varepsilon_j$, we have

$$|F_{j+1}(z) - F_j(z)| < 2\varepsilon_j < 2^{-j}. \tag{5}$$

Since $\sigma_j^{-1} \circ \tau_j$ moves no point of $j\bar{B}$ by as much as $2\varepsilon_j$, we see, because of (1), that

$$|F_j^{-1}(\sigma_j^{-1}(\tau_j(z))) - F_j^{-1}(z)| < 2^{-j};$$

in other words

$$|F_{j+1}^{-1}(z) - F_j^{-1}(z)| < 2^{-j}. \tag{6}$$

The conclusion to be drawn from (5) and (6) is that both $\{F_j\}$ and $\{F_j^{-1}\}$ converge, uniformly on compact subsets of $\mathbf{C}^n$, and that $F = \lim F_j$ satisfies the theorem; note that $\{p_j\}$ and $\{q_j\}$ are reorderings of X and Y, respectively, hence $F(X) = Y$.

3. Tame sets. It will be convenient to have the following fact available before we define what we mean by a tame set.

3.1. PROPOSITION. *Suppose $n > 1$. If $\{\alpha_i\}$ and $\{\beta_i\}$ are discrete sequences in $\mathbf{C}$, $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ when $i \neq j$, then there are three shears in $\mathbf{C}^n$ whose composition τ satisfies*

$$\tau(\alpha_i e_1) = \beta_i e_1 \qquad (i = 1, 2, 3, \ldots).$$

Here e_1 is the first element in the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbf{C}^n$. Of course, any other nonzero vector could be substituted for e_1.

PROOF. The Mittag-Leffler interpolation theorem furnishes entire functions $f, g \colon \mathbf{C} \to \mathbf{C}$ that satisfy $f(\alpha_i) = \beta_i - \alpha_i$, $g(\beta_i) = -\alpha_i$, for $i = 1, 2, 3, \ldots$. Define

$$\sigma_1(z) = z + z_1 e_2, \qquad \sigma_2(z) = z + f(z_2)e_1, \qquad \sigma_3(z) = z + g(z_1)e_2$$

for $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$, and put $\tau = \sigma_3 \circ \sigma_2 \circ \sigma_1$. The action of τ on $\alpha_i e_1$ is described by

$$\alpha_i e_1 \xrightarrow{\sigma_1} \alpha_i e_1 + \alpha_i e_2 \xrightarrow{\sigma_2} \beta_i e_1 + \alpha_i e_2 \xrightarrow{\sigma_3} \beta_i e_1.$$

3.2. REMARK. The case in which $\{\beta_i\}$ is a rearrangement of $\{\alpha_i\}$ shows that every permutation of $\{\alpha_i e_1\}$ extends to an automorphism of $\mathbf{C}^n$. This is one illustration (Theorem 2.2 is another one) of the fact that $\mathrm{Aut}(\mathbf{C}^n)$ is a huge group, for every $n > 1$.

3.3. DEFINITION. Let $N = \{e_1, 2e_1, 3e_1, \dots\}$. *We call a set* $E \subset \mathbf{C}^n$ *tame in* $\mathbf{C}^n$ if $F(E) = N$ for some $F \in \mathrm{Aut}(\mathbf{C}^n)$, and we say that E is *very tame in* $\mathbf{C}^n$ if such an F can be found that also has $JF \equiv 1$.

3.4. REMARKS. (a) If L' and L'' are complex lines in $\mathbf{C}^n$ $(n > 1)$ then some affine map with Jacobian 1 carries L' to L''. Proposition 3.1 shows therefore that every infinite discrete set $E \subset \mathbf{C}^n$ that lies in a complex line is tame (in fact, very tame) in $\mathbf{C}^n$. Our "tame" sets are thus the same as those that were called "planierbar" by Hermes and Peschl [**7**]. They did not distinguish between tame and very tame. In §6 we shall see that the very tame sets actually form a proper subclass of the tame ones.

(b) Remark 3.2 shows that the concept of a "tame sequence" $\{p_j\}$ in $\mathbf{C}^n$, as being one for which some $F \in \mathrm{Aut}(\mathbf{C}^n)$ gives $F(p_j) = je_1$, $j = 1, 2, 3, \dots$, does not differ in any essential way from that of a tame set.

(c) Instead of requiring $JF \equiv 1$ in the definition of "very tame" we could equally well have imposed the (apparently) less restrictive requirement that JF be a nonzero constant. But this would in fact define the same class of sets. For if $JF \equiv c \neq 0$ and $F(E) = N$ then $J(c^{-1/n}F) \equiv 1$ and $(c^{-1/n}F)(E) = c^{-1/n}N$, a very tame set, by Proposition 3.1.

We shall now consider the following situation: k and m are positive integers, $n = k + m$, and $\mathbf{C}^n = \mathbf{C}^k \oplus \mathbf{C}^m$, where $\mathbf{C}^k$ is spanned by $\{e_1, \dots, e_k\}$, $\mathbf{C}^m$ is spanned by $\{e_{k+1}, \dots, e_n\}$. Thus every $z \in \mathbf{C}^n$ has a unique decomposition $z = z' + z''$, with $z' \in \mathbf{C}^k$, $z'' \in \mathbf{C}^m$. We define π' and π'' by $\pi'(z) = z'$, $\pi''(z) = z''$.

The following theorem says, roughly speaking, that sets with a discrete projection and finite fibers are very tame.

3.5. THEOREM. *Suppose* $E \subset \mathbf{C}^n$ *is infinite,* $\pi'(E)$ *is discrete in* $\mathbf{C}^k$*, and to each* $p \in \pi'(E)$ *correspond only finitely many* $q \in \mathbf{C}^m$ *so that* $p + q \in E$*. Then* E *is very tame in* $\mathbf{C}^n$.

(The case $k = 1$ is in [**7**].)

PROOF. Let $\{p_1, p_2, p_3, \dots\}$ be an enumeration of $\pi'(E)$. We can successively find $w_1, w_2, w_3, \dots$ in $\mathbf{C}^m$ so that

$$|q + w_j| > j + |z'' + w_i| \tag{1}$$

for all points $p_j + q \in E$ and all $p_i + z'' \in E$ that have $i < j$. There is a holomorphic $F\colon \mathbf{C}^k \to \mathbf{C}^m$ so that $F(p_i) = w_i$ for $i = 1, 2, 3, \dots$. (When $k = 1$, use the Mittag-Leffler theorem; when $k > 1$, the required interpolation theorem is also very well known and can, in fact, be deduced from our Theorem 1.1.) We use F to define a "shear" σ_1:

$$\sigma_1(z) = z' + (z'' + F(z')) \qquad (z \in \mathbf{C}^n). \tag{2}$$

Our choice of $\{w_j\}$ shows that π'' is one-to-one on $E_1 = \sigma_1(E)$ and that $\pi''(E_1)$ is discrete in $\mathbf{C}^m$.

Hence there is a function φ, defined on $\pi''(E_1)$, so that $z_1 + \varphi(z'')$ runs through the positive integers (in one-to-one fashion) as z runs through E_1, and there is a holomorphic function $g: \mathbf{C}^m \to \mathbf{C}$ so that $g(z'') = \varphi(z'')$ on $\pi''(E_1)$. The shear

$$\sigma_2(z) = z + g(z'')e_1 \tag{3}$$

thus carries E_1 onto a set $E_2 = \sigma_2(E_1)$ so that π_1, restricted to E_2, is a one-to-one map onto the positive integers.

Finally, there are holomorphic functions $h_j: \mathbf{C} \to \mathbf{C}$ ($j = 2, 3, \ldots, n$) so that $h_j(r)$ is the jth coordinate of that point of E_2 whose first coordinate is r (where $r = 1, 2, 3, \ldots$). The shear

$$\sigma_3(z) = z - \sum_{j=2}^{n} h_j(z_j)e_j \tag{4}$$

then takes E_2 onto $N = \{e_1, 2e_1, 3e_1, \ldots\}$.

Thus $(\sigma_3 \circ \sigma_2 \circ \sigma_1)(E) = N$, and $J(\sigma_3 \circ \sigma_2 \circ \sigma_1) \equiv 1$.

3.6. COROLLARIES. (a) *Every discrete infinite set $E \subset \mathbf{C}^{n-1}$ is very tame in $\mathbf{C}^n$.*

(b) *The union of a finite set and a [very] tame set is [very] tame.*

(c) *Every discrete infinite set $E \subset \mathbf{C}^n$ ($n > 1$) is the union of two that are very tame in $\mathbf{C}^n$.*

(Hermes and Peschl have (c), but with n in place of two.)

To prove (a), apply Theorem 3.5 with $k = n - 1$.

To prove (b), it is enough to consider the union of N and a finite set, and apply Theorem 3.5 with $k = 1$.

To prove (c), let $n = k + m$ as above, and put

$$E_1 = \{z' + z'' \in E: |z''| \le |z'|\}, \qquad E_2 = \{z' + z'' \in E: |z''| > |z'|\}.$$

Theorem 3.5 applies to E_1 as it stands (over every compact set in $\mathbf{C}^k$ there are at most finitely many points of E_1), and it applies to E_2 with the roles of k and m reversed. Thus both E_1 and E_2 are very tame in $\mathbf{C}^n$.

(Note: We have ignored the possibility that E_1 or E_2 might be finite. In that case, E itself is very tame, by (b).)

Corollary (c) says that every infinite discrete $E \subset \mathbf{C}^n$ is, in a certain sense, close to being tame in $\mathbf{C}^n$. Our next theorem seems to point in the same direction. Nevertheless, we shall see in §4 that $\mathbf{C}^n$ contains infinite discrete sets that are not tame.

3.7. THEOREM. *If E is an infinite discrete set in $\mathbf{C}^n$, then there is a holomorphic $H: \mathbf{C}^n \to \mathbf{C}^n$ so that H is one-to-one on $\mathbf{C}^n$, $JH \equiv 1$, and $H(E) = N$.*

Note that we do not (in fact, cannot) prove that $H(\mathbf{C}^n) = \mathbf{C}^n$, except when E is tame.

PROOF. Let $E = \{p_1, p_2, p_3, \ldots\}$. In the proof of Theorem 1.1 we constructed a holomorphic $F: \mathbf{C}^n \to \mathbf{C}^n$ which was one-to-one, had $JF \equiv 1$, so that the restriction of π_1 to the set $F(E)$ was one-to-one, and $\pi_1(F(E))$ was discrete in $\mathbf{C}$. The case $k = 1$ of Theorem 3.5 shows therefore that $F(E)$ is very tame in $\mathbf{C}^n$. Put $H = G \circ F$, where $G \in \operatorname{Aut}(\mathbf{C}^n)$, $JG \equiv 1$, and $G(F(E)) = N$.

Here is another application of Theorem 3.5. It gives rise to an interesting open question. (See Question 4.)

3.8. THEOREM. *Suppose that E is an infinite discrete set in $\mathbf{C}^n$ $(n > 1)$ and that all coordinates z_i of every $z = (z_1, \ldots, z_n) \in E$ satisfy $|z_i| \geq 1$. Then E is very tame in $\mathbf{C}^n$.*

PROOF. Put $P(z) = z_1 z_2 \cdots z_n$. Let $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ be an enumeration of the set $P(E) \subset \mathbf{C}$, and put

$$(1) \qquad E_t = \{z \in E \colon P(z) = \lambda_t\} \qquad (t = 1, 2, 3, \ldots).$$

Then $\{\lambda_t\}$ is discrete in $\mathbf{C}$, and each E_t is finite. Choose a holomorphic $f \colon \mathbf{C} \to \mathbf{C}$ so that

$$(2) \qquad f(\lambda_t) = t \qquad (t = 1, 2, 3, \ldots).$$

Define a map $w = \Phi(z)$ by

$$(3) \qquad w_1 = z_1 e^{f(P(z))}, \qquad w_2 = z_2 e^{-f(P(z))}, \qquad w_i = z_i \quad \text{for } 3 \leq i \leq n.$$

Then $\Phi \in \operatorname{Aut}(\mathbf{C}^n)$, $J\Phi \equiv 1$, and if $z \in E_t$ then the first coordinate w_1 of $\Phi(z)$ satisfies $|w_1| \geq e^t$. This shows that $\Phi(E)$ satisfies the hypotheses of Theorem 3.5, with $k = 1$. Hence $\Phi(E)$ is very tame, and the same is then of course true of E.

We conclude this section with an analogue of Theorem 3.5, without any finiteness assumption.

3.9. THEOREM. *With $n = k + m$ as in Theorem 3.5, assume that E is an infinite discrete set in $\mathbf{C}^n$ and that $\pi'(E)$ is discrete in $\mathbf{C}^k$. Then E is tame in $\mathbf{C}^n$.*

PROOF. Assume, without loss of generality, that $z' \neq 0$ and $z'' \neq 0$ for every $z' + z'' \in E$. Let $\{p_j\}$ be an enumeration of $\pi'(E)$ and put

$$(1) \qquad \delta_j = \min\{|z''| \colon p_j + z'' \in E\}.$$

Then $\delta_j > 0$ for $j = 1, 2, 3, \ldots$ and therefore there is a holomorphic $f \colon \mathbf{C}^k \to \mathbf{C}$ so that

$$(2) \qquad \operatorname{Re} f(p_j) > \log(|p_j|/\delta_j)$$

for all j. Put $g = \exp(f)$ and define $\Phi \in \operatorname{Aut}(\mathbf{C}^n)$ by

$$(3) \qquad \Phi(z' + z'') = z' + g(z')z''.$$

If $p_j + z'' \in E$, then

$$(4) \qquad |g(p_j)z''| \geq |g(p_j)|\delta_j > |p_j|.$$

Each point of $\Phi(E)$ thus has the form $p_j + w''$ with $|w''| > |p_j|$. This shows that π'' is finite-to-one on $\Phi(E)$ and that $\pi''(\Phi(E))$ is discrete in $\mathbf{C}^m$. By Theorem 3.5, $\Phi(E)$ is tame in $\mathbf{C}^n$, hence so is E. This completes the proof.

Note that $(J\Phi)(z) = g(z')$, not a constant. We shall see in §6 that the hypotheses of Theorem 3.9 do not force E to be very tame.

4. Unavoidable sets.

4.1. DEFINITION. If Γ is some class of holomorphic maps from $\mathbf{C}^n$ into $\mathbf{C}^n$, we say that a set $E \subset \mathbf{C}^n$ is Γ-*unavoidable* or that *E is unavoidable by members of* Γ, if E intersects $F(\mathbf{C}^n)$ for every $F \in \Gamma$.

(Examples: If $n = 1$ and Γ is the class of all nonconstant entire functions, then every 2-point set in $\mathbf{C}$ is Γ-unavoidable. For nondegenerate holomorphic maps in $\mathbf{C}^n$, Gruman [**22**] has found unavoidable sets of real dimension n.)

In the present section we show, for $n > 1$, that *tame sets are avoidable by biholomorphic maps* (Proposition 4.2) but that there exist discrete sets $E \subset \mathbf{C}^n$ that are unavoidable by nondegenerate holomorphic maps $F\colon \mathbf{C}^n \to \mathbf{C}^n$ (Theorem 4.5).

These discrete sets are therefore not tame.

In §6 we construct sets $E \subset \mathbf{C}^n$, for all $n > 1$, which are tame, hence avoidable by biholomorphic maps, but which are unavoidable by holomorphic maps with constant (nonzero) Jacobian.

4.2. PROPOSITION. *If $n > 1$ and E is tame in $\mathbf{C}^n$ then there is a biholomorphic map $F\colon \mathbf{C}^n \to \mathbf{C}^n$ so that $F(\mathbf{C}^n)$ does not intersect E.*

If E is very tame in $\mathbf{C}^n$, then the above-mentioned F can be chosen so that $JF \equiv 1$.

PROOF. There is a biholomorphic $G\colon \mathbf{C}^n \to \mathbf{C}^n$, with $JG \equiv 1$ and $G(\mathbf{C}^n) \neq \mathbf{C}^n$. (This is the well-known Fatou-Bieberbach phenomenon; see [**2**], or §9 of the present paper.) Put $G(\mathbf{C}^n) = \Omega$. Since Ω is homeomorphic to $\mathbf{C}^n$ but $\Omega \neq \mathbf{C}^n$, we see that $\mathbf{C}^n \backslash \Omega$ is unbounded and therefore contains a subset E_0 to which Theorem 3.5 can be applied. Thus there exists a very tame set $E_0 \subset \mathbf{C}^n \backslash \Omega$.

Since E and E_0 are both tame, there is a $\Phi \in \mathrm{Aut}(\mathbf{C}^n)$ with $\Phi(E_0) = E$. If E is very tame, then Φ can be chosen so as to have $J\Phi \equiv 1$. In either case, $F = \Phi \circ G$ has the desired properties.

The next lemma will involve two spaces, $\mathbf{C}^k$ and $\mathbf{C}^n$, with $1 \leq k \leq n$. The case $k = n$ is all that will be used in the present section, but in §6 we will need $k = n-1$.

For the sake of clarity, we shall write B_k and B_n for the corresponding open unit balls. As is frequently done, we shall use the symbol ∂ to denote boundaries as well as partial derivatives.

4.3. LEMMA. *Given $0 < a_1 < a_2$, $0 < r_1 < r_2$, $c > 0$, let Γ be the class of all holomorphic maps*

$$F = (f_1, \dots, f_k)\colon a_2 B_n \to r_2 B_k \tag{1}$$

such that

$$|F(0)| \leq \tfrac{1}{2} r_1 \tag{2}$$

and

$$\left| \frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_k)} \right| \geq c \text{ at some point of } a_1 \bar{B}_n. \tag{3}$$

Then there is a finite set

$$E = E(a_1, a_2, r_1, r_2, c) \subset \partial(r_1 B_k) \tag{4}$$

with the following property:

If $F \in \Gamma$ and $F(a_1 B_n)$ intersects $\partial(r_1 B_k)$ then $F(a_2 B_n)$ intersects E.

PROOF. Let $E_1 \subset E_2 \subset E_3 \subset \cdots$ be finite subsets of $\partial(r_1 B_k)$ whose union is dense in $\partial(r_1 B_k)$. Assume, to reach a contradiction, that no E_j does what the lemma claims.

This means that there exist $F_j \in \Gamma$ and $z_j \in a_1 B_n$ ($j = 1, 2, 3, \dots$), with $F_j(z_j) \in \partial(r_1 B_k)$, so that

$$F_j(a_2 B_n) \cap E_j = \varnothing. \tag{5}$$

Note that Γ is a normal family in $a_2 B_n$. Hence, passing to a subsequence if necessary, we have $z_j \to w \in a_1 \bar{B}_n$, $F_j \to F \in \Gamma$ as $j \to \infty$, uniformly on compact subsets of $a_2 B_n$, and

$$F(w) = \lim_{j \to \infty} F_j(z_j) \in \partial(r_1 B_k). \tag{6}$$

Since $F \in \Gamma$, (3) shows that the set

$$\Omega = \{z \in a_2 B_n \colon \operatorname{rank} F'(z) = k\} \tag{7}$$

is not empty. Hence Ω is a connected open set that is dense in $a_2 B_n$, so that $F(\Omega)$ is connected, open, and dense in $F(a_2 B_n)$.

Since $|F(w)| = r_1$ and $w \in a_2 B_n$, the maximum principle shows that $F(a_2 B_n)$ contains points outside $r_1 \bar{B}_k$, hence so does $F(\Omega)$. On the other hand, (2) shows (since $F(\Omega)$ is dense in $F(a_2 B_n)$) that $F(\Omega)$ intersects $r_1 B_k$. Being connected, $F(\Omega)$ must therefore intersect $\partial(r_1 B_k)$.

So there is a point $p \in \Omega$ with $F(p) = q \in \partial(r_1 B_k)$. Since $F'(p)$ has rank k, the rank theorem implies that p lies in a compact set $K \subset a_2 B_n$ so that the restriction of F to K is a one-to-one map from K onto a closed ball β with center at q, radius $\delta > 0$. Let F^{-1} denote the inverse of this restriction. Since $F_j \to F$ uniformly on K, we see that $F_j \circ F^{-1}$ moves no point of β by more than $\delta/3$, for all sufficiently large j, and this implies that $F_j(K) \supset \beta'$, the ball with center q, radius $(2/3)\delta$. Thus $\beta' \subset F_j(a_2 B_n)$. But as soon as j is large enough, β' contains points of E_j, and we have a contradiction to (5).

In the rest of this section we will use the preceding lemma with $k = n$, and we will revert to our earlier notation, writing B for B_n and S for ∂B_n.

4.4. LEMMA. *To each positive integer t corresponds a discrete set $E_t \subset \mathbf{C}^n \setminus tB$ so that the assumptions*

(i) $F \colon tB \to \mathbf{C}^n$ *is holomorphic,*
(ii) $|F(0)| \le t/2$,
(iii) $|(JF)(z)| \ge 1/t$ *at some point of* $\frac{1}{2} t \bar{B}$,
(iv) $F(tB) \cap E_t = \varnothing$

imply that $F(\frac{1}{2} tB) \subset tB$.

PROOF. Choose $\{a_j\}$ and $\{r_j\}$ so that

$$\tfrac{1}{2} t = a_1 < a_2 < a_3 < \cdots < \tfrac{3}{4} t, \tag{1}$$

$$t = r_1 < r_2 < r_3 < \cdots \tag{2}$$

and $r_j \to \infty$ as $j \to \infty$. Define

$$E_t = \bigcup_{j=1}^{\infty} E(a_j, a_{j+1}, r_j, r_{j+1}, 1/t), \tag{3}$$

using the notation of Lemma 4.3. Since E_t is the union of finite sets lying on the spheres $r_j S$, and $r_j \to \infty$, we see that E_t is discrete.

Assume now that F satisfies (i)–(iv). Then F is bounded on $(3/4)tB$; hence

$$F(a_{j+1}B) \subset r_{j+1}B \tag{4}$$

for some j. By (iv) and (3), $F(a_{j+1}B)$ does not intersect $E(a_j, a_{j+1}, r_j, r_{j+1}, 1/t)$. Lemma 4.3 shows therefore that $F(a_jB)$ does not intersect r_jS. Thus

$$F(a_jB) \subset r_jB. \tag{5}$$

We can now repeat the argument that led from (4) to (5) until we reach

$$F(a_1B) \subset r_1B \tag{6}$$

which is the desired conclusion.

4.5. THEOREM. *If $n > 1$, then there is a discrete set $D \subset \mathbf{C}^n$ which is unavoidable by nondegenerate holomorphic maps from $\mathbf{C}^n$ into $\mathbf{C}^n$.*

As explained in §4.1, this D is not tame in $\mathbf{C}^n$.

PROOF. We define

$$D = \bigcup_{t=1}^{\infty} E_t, \tag{1}$$

where E_t is as in Lemma 4.4. Then D is discrete, because E_t lies outside the ball tB, so that $D \cap (tB)$ is finite, for each t.

Assume now that $F\colon \mathbf{C}^n \to \mathbf{C}^n$ is holomorphic, $JF \not\equiv 0$, and (to reach a contradiction) that $F(\mathbf{C}^n)$ does not intersect D. For large enough t, F satisfies the hypotheses of Lemma 4.4. Hence $F(\frac{1}{2}tB) \subset tB$ for all large enough t. This growth restriction forces F to be a polynomial map, of degree 1. In other words, F is affine. But affine maps from $\mathbf{C}^n$ into $\mathbf{C}^n$ whose Jacobian is not identically zero are automorphisms of $\mathbf{C}^n$. Thus $F(\mathbf{C}^n) = \mathbf{C}^n \supset D$, and we have our contradiction.

4.6. REMARK. It was of course quite unimportant to have *integers t* in the preceding construction. In fact, given any sequence $\{t_j\}$ of positive numbers that tends to ∞ (and $\{t_j\}$ can be arbitrarily "thin") one can construct D as in Theorem 4.5 so that D lies on the spheres whose radii are in $\{t_j\}$. (Note that the r's that occur in Lemma 4.4 can also be put into $\{t_j\}$.) Thus D can lie outside of large prescribed regions.

Also, we used spheres just for convenience. It is clear that many other configurations can serve equally well.

4.7. Here is another way of seeing that unavoidable sets can be confined to certain relatively small regions in $\mathbf{C}^n$.

Let $g\colon \mathbf{C} \to \mathbf{C}$ be entire and let A be the set of all $z \in \mathbf{C}$ for which $|g(z)| < 1$. *We claim that there is a discrete set*

$$D_0 \subset \mathbf{C}^2 \backslash (A \times A) \tag{1}$$

which is unavoidable by nondegenerate holomorphic maps $F\colon \mathbf{C}^2 \to \mathbf{C}^2$.

To see this, pick D as in Theorem 4.5, so that

$$D \subset \mathbf{C}^2 \backslash U^2, \tag{2}$$

where U^2 is the unit bidisc in $\mathbf{C}^2$, put

$$\Phi(z,w) = (g(z), g(w)) \qquad ((z,w) \in \mathbf{C}^2) \tag{3}$$

and define $D_0 = \Phi^{-1}(D)$.

Then D_0 is discrete, (2) and (3) show that (1) holds, and if F avoided D_0 then $\Phi \circ F$ would avoid D, a contradiction.

The point is that A can be very large:

For example $\mathbf{C}\backslash A$ could lie in the half-strip consisting of all $x + iy$ with $x > 0$ and $|y| < 1$ [**18**, p. 334, Example 11].

Or, using Arakelian's theorem [**3**], one can find g so that $\mathbf{C}\backslash A$ lies in the set of all $x + iy$ with $x > 0$ and $|y| < \varepsilon(x)$, where ε is an arbitrary preassigned positive continuous function on $[0, \infty)$ that has $\lim_{x\to\infty} \varepsilon(x) = 0$.

We conclude this section with a result that should be compared to Theorem 3.7:

4.8. THEOREM. *If $E \subset \mathbf{C}^n$ is unavoidable by one-to-one holomorphic maps from $\mathbf{C}^n$ into $\mathbf{C}^n$, then no holomorphic $F\colon \mathbf{C}^n \to \mathbf{C}^n$ with $F(N) = E$ can be one-to-one on $\mathbf{C}^n$.*

PROOF. By Proposition 4.2, there is a biholomorphic $G\colon \mathbf{C}^n \to \mathbf{C}^n\backslash N$. If F is one-to-one on $\mathbf{C}^n$ and $F(N) = E$, then $F \circ G$ avoids E.

5. Rigid sets. We shall now use a more elaborate version of the construction that yielded Theorem 4.5 to prove the following result.

5.1. THEOREM. *There is a discrete set $D \subset \mathbf{C}^n$ with the following property: If $F\colon \mathbf{C}^n \to \mathbf{C}^n$ is holomorphic, $JF \not\equiv 0$, and*

$$F(\mathbf{C}^n\backslash D) \subset \mathbf{C}^n\backslash D \tag{1}$$

then $F(z) = z$ for every $z \in \mathbf{C}^n$.

COROLLARY. *No automorphism of $\mathbf{C}^n$, other than the identity, can map D onto D.*

This is the reason for calling such sets D *rigid.*

PROOF. The set D will be constructed so that

(a) every nondegenerate holomorphic F that satisfies (1) is affine, and

(b) the identity map is the only affine F (with $JF \neq 0$) that satisfies (1).

Of course, it is easy to achieve (b). However, we do it in detail, by starting with certain finite sets Λ_p in the coordinate axes $L_p = \{\lambda e_p \colon \lambda \in \mathbf{C}\}$, where $\{e_1, \ldots, e_n\}$ is the usual basis of $\mathbf{C}^n$.

Let Λ_p be a set of $p + 3$ points in $B \cap L_p$, so located that no affine map of L_p into L_p (other than the identity) permutes Λ_p, for $p = 1, 2, \ldots, n$, and put

$$D_0 = \Lambda_1 \cup \cdots \cup \Lambda_n. \tag{2}$$

Moreover, D_0 is to be so chosen that

$$\#(D_0 \cap L) \leq 2 \tag{3}$$

for every complex line L in $\mathbf{C}^n$ which is not one of $L_1, \ldots, L_n$.

(The symbol $\#$ indicates the cardinality of the set that follows it.)

Suppose now that t is a positive integer and that discrete sets $D_0, \ldots, D_{t-1}$ have been chosen. Put

$$m_t = \#((D_0 \cup \cdots \cup D_{t-1}) \cap tB). \tag{4}$$

Let E_t be as in Lemma 4.4. Apply $1+m_t$ unitary transformations to E_t so that the resulting sets $E_{t,i}$ $(1 \le i \le 1+m_t)$ are pairwise disjoint. Define

$$D_t = \bigcup_{i=1}^{1+m_t} E_{t,i}, \qquad D = \bigcup_{t=0}^{\infty} D_t. \tag{5}$$

There was a great deal of choice in the proofs of Lemmas 4.3 and 4.4. In particular, the set E in Lemma 4.3 can be so chosen that no complex line contains more than two points of E, and the same can be achieved for E_t in Lemma 4.4 (by suitably rotating each summand in 4.4(3)). The sets $E_{t,i}$ can then be so placed that no D_t with $t \ge 1$ intersects any of the lines $L_1, \dots, L_n$, and so that the final set D has no more than two points on any other complex line in $\mathbf{C}^n$.

Note that D is discrete because each $E_{t,i}$ lies outside tB, so that

$$\#(D \cap tB) = m_t < \infty \qquad (t = 1, 2, 3, \dots). \tag{6}$$

We now turn to our given mapping F. For large t we have $|F(0)| \le t/2$, and $|JF| \ge 1/t$ at some point of $\frac{1}{2}t\bar{B}$. Since (6) holds, and F maps no point of $(tB)\backslash D$ into D, it follows that $F(tB)$ misses at least one of the $1+m_t$ sets $E_{t,i}$. Lemma 4.4 shows therefore that

$$F\left(\tfrac{1}{2}tB\right) \subset tB \tag{7}$$

for all sufficiently large t.

As in the proof of Theorem 4.5, the growth restriction (7), combined with the hypothesis $JF \not\equiv 0$, forces F to be an affine automorphism of $\mathbf{C}^n$.

It remains to be shown that F must be the identity map. Define

$$\mu(L) = \#(D \cap L) \tag{8}$$

for complex lines L in $\mathbf{C}^n$. Since F is one-to-one, it follows from (1) that $F(D) \supset D$. Hence $F(D \cap L) = F(D) \cap F(L) \supset D \cap F(L)$, from which it follows that

$$\mu(L) \ge \mu(F(L)) \tag{9}$$

for every complex line L in $\mathbf{C}^n$.

Since F is an affine automorphism, it permutes the set of complex lines. So there is an L for which $F(L) = L_n$, and (9) gives

$$\mu(L) \ge \mu(L_n) = n+3. \tag{10}$$

But L_n is the only line that has as many as $n+3$ points in common with D. Thus $L = L_n$. Because of the way in which Λ_n was chosen at the start of the proof, we see that F fixes every point of L_n.

The same reasoning can now be applied successively to $L_{n-1}, \dots, L_1$. Since $\mathbf{C}^n$ is spanned by $\{L_1, \dots, L_n\}$ and F is linear, we conclude that F is the identity map on $\mathbf{C}^n$.

5.2. REMARK. Let D be as in the preceding proof, and let D', D'' be modifications of D, obtained by moving just one point of Λ_1 to different spots on $L_1 \cap B$, in such a way that D' and D'' have all the properties of D that were used in that proof.

If we now assume that $F \in \mathrm{Aut}(\mathbf{C}^n)$ and $F(D') = D''$, the preceding argument can be repeated, almost word for word, to give the conclusion that $F(z) = z$ for all $z \in \mathbf{C}^n$. But this is absurd if $D' \ne D''$.

Therefore no $F \in \mathrm{Aut}(\mathbf{C}^n)$ *maps* D' *into* D''.

Since there are continuum many choices for D' (any two differing by only one point) we have proved the following result:

5.3. COROLLARY. *If* $n \geq 1$ *then* $\mathbf{C}^n$ *contains continuum many discrete sets no two of which are* $\mathrm{Aut}(\mathbf{C}^n)$*-equivalent.*

6. Tame sets that are not very tame. Theorem 6.4 will show that such sets exist in $\mathbf{C}^n$ for all $n > 1$. We begin with a lemma in linear algebra.

6.1. LEMMA. *Suppose that*

(a) A *is a linear operator in* $\mathbf{C}^n$, $\det A = 1$,

(b) P *is a linear projection in* $\mathbf{C}^n$, $\operatorname{rank} P = n-1$,

(c) $u \in \mathbf{C}^n$, $|u| = 1$, $Pu = 0$.

Then $|A^{-1}u| \leq ||PA||^{n-1}$.

(The norm is the usual operator norm, relative to the Euclidean metric on $\mathbf{C}^n$.)

PROOF. Choose an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbf{C}^n$ so that $A^{-1}u = \lambda v_1$ for some $\lambda > 0$, and use this basis to identify linear operators on $\mathbf{C}^n$ with matrices. If D is diagonal, with entries $(\lambda, 1, \dots, 1)$ down the main diagonal, then $ADv_1 = u$, so that the columns of AD are

$$u, Av_2, \dots, Av_n. \tag{1}$$

Since $P^2 = P$, each vector $Av_j - PAv_j$ lies in the null-space of P, hence is a multiple of u. The columns $Av_2, \dots, Av_n$ can therefore be replaced by $PAv_2, \dots, PAv_n$, without changing the determinant of AD. It follows now from Hadamard's inequality that

$$\begin{aligned} |A^{-1}u| &= \lambda = \det D = \det(AD) \\ &= \det[u, PAv_2, \dots, PAv_n] \\ &\leq |u| \cdot |PAv_2| \cdots |PAv_n| \leq ||PA||^{n-1} \end{aligned}$$

because $|u| = |v_2| = \cdots = |v_n| = 1$.

As in §4, we will now use the notations B_n, B_{n-1} for the open unit balls in $\mathbf{C}^n$, $\mathbf{C}^{n-1}$, $n > 1$.

6.2. LEMMA. *Given* $0 < a_1 < a_2$, $r > 0$, *there is a* $\delta > 0$, *namely*

$$\delta = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{(a_2 - a_1)^n}{r^{n-1}} \tag{1}$$

with the following property:

If $F\colon a_2 B_n \to (rB_{n-1}) \times \mathbf{C}$ *is holomorphic, with* $JF \equiv 1$, *then* $F(a_2 B_n)$ *contains the disc*

$$\{F(z) + \lambda e_n \colon |\lambda| < \delta\} \tag{2}$$

for every $z \in a_1 B_n$.

PROOF. Choose α so that $a_1 < \alpha < a_2$. (The best choice will be indicated at the end of the proof.) Fix $z \in a_1 B_n$. Put $u = e^{i\theta} e_n$ and let γ be the straight line interval defined by

$$\gamma(t) = F(z) + tu \qquad (0 \leq t \leq T), \tag{3}$$

where T is the smallest number for which $\gamma(T)$ lies outside $F(\alpha B_n)$.

Our objective is to find a good lower bound for T.

There is a path Γ in $\alpha\bar{B}_n$, starting at z and ending at some point of $\partial(\alpha B_n)$, so that

$$F(\Gamma(t)) = \gamma(t) \qquad (0 \leq t \leq T). \tag{4}$$

By the chain rule,

$$F'(\Gamma(t))\Gamma'(t) = \gamma'(t) = u \qquad (0 \leq t \leq T). \tag{5}$$

Now fix some $t \in [0, T]$, put $w = \Gamma(t)$, $A = F'(w)$, and let P be the orthogonal projection in $\mathbf{C}^n$ whose null-space is spanned by e_n. Then PF maps the ball with center w and radius $a_2 - \alpha$ into rB_{n-1} (because $|w| \leq \alpha$). The Schwarz lemma [**19**, p. 161] shows therefore that

$$||PA|| = ||P(F'(w))|| = ||(PF)'(w)|| \leq r/(a_2 - \alpha). \tag{6}$$

Since $Pu = 0$, and (5) shows that $\Gamma'(t) = A^{-1}u$, we conclude from (6) and Lemma 6.1 that

$$|\Gamma'(t)| \leq \{r/(a_2 - \alpha)\}^{n-1} \qquad (0 \leq t \leq T). \tag{7}$$

Since $|\Gamma(0)| \leq a_1$ and $|\Gamma(T)| = \alpha$, it follows that

$$\alpha - a_1 \leq \int_0^T |\Gamma'(t)|\, dt \leq \{r/(a_2 - \alpha)\}^{n-1} T. \tag{8}$$

Hence

$$T \geq r^{1-n}(\alpha - a_1)(a_2 - \alpha)^{n-1}. \tag{9}$$

If we choose α so as to maximize the right side of (9), we obtain $T \geq \delta$, where δ is given by (1).

6.3. LEMMA. *Given $0 < a_1 < a_2$, $0 < r_1 < r_2$, there is a discrete set*

$$E = E(a_1, a_2, r_1, r_2) \subset \partial(r_1 B_{n-1}) \times \mathbf{C} \tag{1}$$

so that the assumptions

(i) $F\colon a_2 B_n \to \mathbf{C}^n$ *is holomorphic,*
(ii) $|F(0)| \leq \frac{1}{2} r_1$, $F'(0) = I$, $JF \equiv 1$,
(iii) $F(a_2 B_n) \cap E = \varnothing$,
(iv) $(PF)(a_1 B_n)$ *intersects* $\partial(r_1 B_{n-1})$

imply that $(PF)(a_2 B_n)$ *intersects* $\partial(r_2 B_{n-1})$.

Here P is the same projection that was used in the proof of Lemma 6.2, and I denotes the identity operator in $\mathbf{C}^n$.

PROOF. Choose t, $a_1 < t < a_2$. Use Lemma 4.3, with $k = n - 1$, $c = 1$ (because $F'(0) = I$), and (a_1, t, r_1, r_2) in place of (a_1, a_2, r_1, r_2), and pick a finite set $E' \subset \partial(r_1 B_{n-1})$ in accordance with Lemma 4.3.

Next, pick $\delta > 0$ as in Lemma 6.2, but with (t, a_2, r_2) in place of (a_1, a_2, r), and let E'' be a discrete set in $\mathbf{C}$ which intersects every open disc of radius δ.

Put $E = E' \times E''$.

Assume now that F satisfies (i)–(iv) but that

$$(PF)(a_2 B_n) \subset r_2 B_{n-1}. \tag{2}$$

Then $(PF)(tB_n) \subset r_2 B_{n-1}$ and $(PF)(a_1 B_n)$ intersects $\partial(r_1 B_{n-1})$, so that Lemma 4.3 shows that

$$(3) \qquad (PF)(tB_n) \text{ intersects } E'.$$

By Lemma 6.2, our choice of δ leads from (3) to

$$(4) \qquad F(tB_n) \text{ intersects } E$$

which contradicts (iii).

6.4. THEOREM. *There is a tame set D in $\mathbf{C}^n$ which is unavoidable by holomorphic maps $F\colon \mathbf{C}^n \to \mathbf{C}^n$ with constant (nonzero) Jacobian.*

Proposition 4.2 shows that this tame set D is not very tame.

PROOF. Put $s_k = k/(k+1)$ and define

$$D = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{k} E(j + s_k, j + s_{k+1}, k, k+1)$$

using the notation of Lemma 6.3.

Suppose $F\colon \mathbf{C}^n \to \mathbf{C}^n$ has $JF \equiv c \neq 0$. Let $A = F'(0)$ and define $\Phi = F \circ A^{-1}$. Then $\Phi'(0) = I$ and $J\Phi \equiv 1$. Since $F(\mathbf{C}^n) = \Phi(\mathbf{C}^n)$ it suffices to prove that $\Phi(\mathbf{C}^n)$ intersects D. Assume, to reach a contradiction, that $\Phi(\mathbf{C}^n) \cap D = \varnothing$. Fix $j > |\Phi(0)|$ and make the induction hypothesis

$$(\mathrm{H}_k) \qquad (P\Phi)((j + s_k)B_n) \text{ intersects } \partial(kB_{n-1}).$$

Since

$$\frac{\partial(\varphi_1, \ldots, \varphi_{n-1})}{\partial(z_1, \ldots, z_{n-1})}(0) = 1,$$

where $\Phi = (\varphi_1, \ldots, \varphi_{n-1}, \varphi_n)$, the Schwarz lemma shows that (H_j) holds. The set $\Phi(\mathbf{C}^n)$ is assumed to miss $E(j + s_k, j + s_{k+1}, k, k+1)$. Lemma 6.3 shows therefore that (H_k) implies (H_{k+1}). Hence (H_k) holds for all $k \geq j$. But this is absurd, since $s_k < 1$ for all k, and $P\Phi$ is bounded on $(j+1)B_n$.

Theorem 3.9 shows that D is tame in $\mathbf{C}^n$.

6.5. REMARK. Let D be a set as in Theorem 6.4. Proposition 4.2 shows

There exist regions $\Omega \subset \mathbf{C}^n$ (in fact, $\Omega \subset \mathbf{C}^n \backslash D$) so that $\Omega = F(\mathbf{C}^n)$ for some biholomorphic F, but so that no such F can have constant Jacobian.

As mentioned in the Introduction, this phenomenon was first discovered by Nishimura [**12**, **13**] in the case $n = 2$.

PART II. HOLOMORPHIC IMAGES OF $\mathbf{C}^n$

In the next three sections we describe various ways in which holomorphic images of $\mathbf{C}^n$ in $\mathbf{C}^n$ can be small when $n > 1$.

7. Entire maps whose ranges have finite volume. We shall use the notation vol(E) for the $(2n)$-dimensional Lebesgue measure of a set $E \subset \mathbf{C}^n$. By a *cube* in $\mathbf{C}^n$ we shall mean the Cartesian product of n equal squares in $\mathbf{C}$ whose sides are parallel to the coordinate axes.

7.1. THEOREM. *Suppose $n > 1$, Q is a cube in $\mathbf{C}^n$, and $\varepsilon > 0$. Then there exists a holomorphic map F from $\mathbf{C}^n$ into $\mathbf{C}^n$, with $JF \equiv 1$, so that*

$$\text{(1)} \qquad \operatorname{vol}(F^{-1}(\mathbf{C}^n \backslash Q)) < \varepsilon.$$

In other words, there is a set $X \subset \mathbf{C}^n$ so that $F(X) \subset Q$ and $\operatorname{vol}(\mathbf{C}^n \backslash X) < \varepsilon$. Since $JF \equiv 1$, it follows that

$$\text{(2)} \qquad \operatorname{vol}(F(\mathbf{C}^n)) < \varepsilon + \operatorname{vol}(Q).$$

Nevertheless (and this seems quite remarkable) the range of every such F must intersect the discrete set D that was constructed in the proof of Theorem 6.4.

We begin with the case $n = 2$. The proof will be completed in §7.3 and §7.4.

7.2. LEMMA. *Suppose that Q_0 and Q_1 are concentric open cubes in $\mathbf{C}^2$, with $\bar{Q}_0 \subset Q_1$, and that P is a cube in $\mathbf{C}^2$.*

To every $\varepsilon > 0$ corresponds then a holomorphic map $\Phi\colon \mathbf{C}^2 \to \mathbf{C}^2$ so that

(i) $J\Phi \equiv 1$,

(ii) $|z - \Phi(z)| < \varepsilon$ *in* Q_0, *and*

(iii) $\operatorname{vol}\{z \in Q_1 \backslash Q_0 \colon \Phi(z) \notin P\} < \varepsilon$.

PROOF. Let $Q_0 = \Delta_0 \times \Delta_0$, $Q_1 = \Delta_1 \times \Delta_1$, so that Δ_0 and Δ_1 are squares in $\mathbf{C}$.

Let P_0 be the cube with the same center as P but with half the diameter.

Choose ε so small that $w \in Q_1$ if $z \in Q_0$ and $|z - w| < \varepsilon$.

For each $\alpha > 0$, the map $T_\alpha\colon \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$\text{(1)} \qquad T_\alpha(z) = ((e^{\alpha z_1} - 1)/\alpha, z_2 e^{-\alpha z_1})$$

has $JT_\alpha \equiv 1$ and has period $2\pi i/\alpha$ in z_1. As $\alpha \to 0$, $T_\alpha(z) \to z$, uniformly on compact sets. Hence there exists $\alpha > 0$, fixed from now on, so small that

$$\text{(2)} \qquad |z - T_\alpha(z)| < \varepsilon/2 \quad \text{on } Q_1$$

and

$$\text{(3)} \qquad T_\alpha(P_0) \subset P.$$

Next we put finitely many disjoint closed squares γ_j into Δ_1, in such a way that

(a) the diameter of each γ_j is $< 1/10$ of the diameter of P_0,

(b) no γ_j intersects the boundary of Δ_0, and

(c) the union of the sets $\Gamma_{jk} = \gamma_j \times \gamma_k$ covers all of Q_1 except for a set of volume $< \varepsilon$.

The desired map Φ will carry each Γ_{jk} in $Q_1 \backslash Q_0$ into P, thus assuring conclusion (iii) of the lemma, and will have the form

$$\text{(4)} \qquad \Phi = T_\alpha \circ \Psi$$

where $\Psi = \sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$ will be the composition of four shears σ_i, chosen so that

$$\text{(5)} \qquad |z - \Psi(z)| < \varepsilon/2 \quad \text{if } z \in Q_0,$$

which implies that $\Psi(Q_0) \subset Q_1$, and so that

$$\text{(6)} \qquad \Psi(\Gamma_{jk}) \subset \left(\frac{2\pi i}{\alpha} m_{jk}\right) e_1 + P_0$$

for each $\Gamma_{jk} \subset Q_1 \backslash Q_0$; here m_{jk} is some integer.

Note that T_α maps the translates of P_0 in (6) into P, by (3) and the periodicity of T_α. Hence Φ given by (4) will satisfy the lemma, because of (2) and (5).

To complete the proof, we have to describe the σ_i. The Figure may make it easier to visualize their action.

Let W be the collection of all $\Gamma_{jk} \subset Q_1 \backslash Q_0$.

Let W_1 be the collection of all $\Gamma_{jk} = \gamma_j \times \gamma_k$ that have $\gamma_k \subset \Delta_1 \backslash \Delta_0$.

Runge's approximation theorem will be tacitly used in the construction of each σ_i to give us certain holomorphic functions $\varphi_i \colon \mathbf{C} \to \mathbf{C}$. Recall the projections π_1 and π_2 defined by $\pi_1(z_1, z_2) = z_1$, $\pi_2(z_1, z_2) = z_2$.

Put $\sigma_1(z_1, z_2) = (z_1 + \varphi_1(z_2), z_2)$, where φ_1 is almost 0 on $\bar{\Delta}_0$, φ_1 is almost constant on each γ_k outside Δ_0, and these constants are so chosen that the projections $\pi_1(\sigma_1(\Gamma_{jk}))$, for $\Gamma_{jk} \in W_1$, are disjoint from each other and are far from $\pi_1(\sigma_1(\bar{\Delta}_1 \times \bar{\Delta}_0))$.

Put $\sigma_2(z_1, z_2) = (z_1, z_2 + \varphi_2(z_1))$. Again, φ_2 is almost 0 on $\pi_1(\sigma_1(\bar{Q}_0))$, φ_2 is almost constant on each projection $\pi_1(\sigma_1(\Gamma_{jk}))$, this time for all $\Gamma_{jk} \in W$, and these constants are so chosen that the projections $(\pi_2 \circ \sigma_2 \circ \sigma_1)(\Gamma_{jk})$ are disjoint from each other and are far from $(\pi_2 \circ \sigma_2 \circ \sigma_1)(\bar{Q}_0)$.

Setting $\Gamma'_{jk} = \sigma_2(\sigma_1(\Gamma_{jk}))$ and $Q'_0 = \sigma_2(\sigma_1(Q_0))$, we have now reached the following position: *Q'_0 is almost the same as Q_0, the sets Q'_0 and Γ'_{jk} (for $\Gamma_{jk} \in W$) have disjoint π_2-images, and each Γ'_{jk} differs from a translate of Γ_{jk} by a very small distortion.*

Now let $c = (c_1, c_2)$ be the common center of P_0 and P.

Put $\sigma_3(z_1, z_2) = (z_1 + \varphi_3(z_2), z_2)$, where φ_3 is almost 0 on $\pi_2(Q'_0)$, φ_3 is almost constant on each $\pi_2(\Gamma'_{jk})$, and these constants are so chosen that $\sigma_3 \circ \sigma_2 \circ \sigma_1$ moves the center of each $\Gamma_{jk} \in W$ to a point

$$(c_1 + (2\pi i/\alpha) m_{jk}, w_{jk}), \tag{7}$$

where the m_{jk} are distinct large positive integers.

Finally,

$$\sigma_4(z_1, z_2) = (z_1, z_2 + \varphi_4(z_1)),$$

where φ_4 is almost 0 on $\pi_1(\sigma_3(Q'_0))$, and φ_4 is almost equal to the constant $c_2 - w_{jk}$ on $\pi_1(\sigma_3(\Gamma'_{jk}))$.

If all approximations implicit in "almost" are sufficiently close, then $\Psi = \sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$ will satisfy (5) and (6).

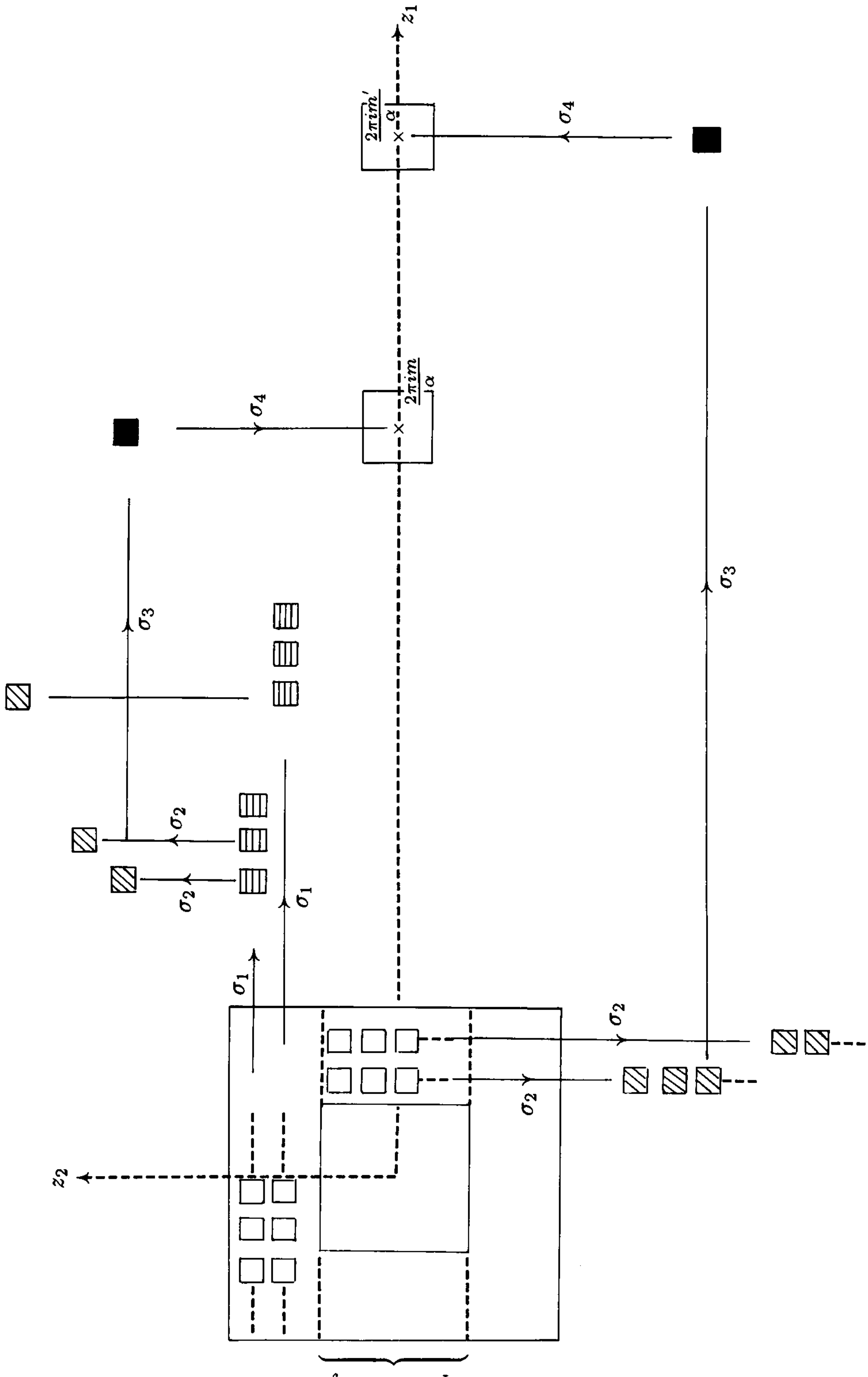
7.3. PROOF OF THEOREM 7.1 WHEN $n = 2$. There are concentric open cubes Q_k in $\mathbf{C}^2$ so that $\bar{Q}_k \subset Q_{k+1}$ for $k = 0, 1, 2, \ldots$, Q_1 is our given cube Q, and $\mathbf{C}^2 = Q_0 \cup Q_1 \cup Q_2 \cup \cdots$.

Assume, without loss of generality, that ε is so small that $w \in Q_1$ if $z \in Q_0$ and $|w - z| < \varepsilon$. Choose $\varepsilon_k > 0$ so that $\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \cdots < \varepsilon$.

Put $F_0(z) = z$. Assume, for some $k \geq 0$, that we have a holomorphic map $F_k \colon \mathbf{C}^2 \to \mathbf{C}^2$, with $JF_k \equiv 1$, and that there is a cube $P_k \subset Q_k$ so that $F_k(P_k) \subset Q_0$. (This induction hypothesis holds when $k = 0$, with $P_0 = Q_0$.) Choose $\delta_k > 0$ so that

$$|F_k(z') - F_k(z'')| < \varepsilon_k \tag{1}$$

whenever $z' \in Q_k$ and $|z'' - z'| < \delta_k$.


z_1
z_2
σ_1
σ_2
σ_3
σ_4
$\frac{2\pi i m}{\alpha}$
$\frac{2\pi i m'}{\alpha}$
$\sigma_1 \simeq$ identity

Lemma 7.2 furnishes a holomorphic map $\Phi_k: \mathbf{C}^2 \to \mathbf{C}^2$ and a closed set $Y_k \subset Q_{k+1} \backslash Q_k$ with $\mathrm{vol}(Y_k) < \varepsilon_k$, so that

(i) $J\Phi_k \equiv 1$,

(ii) $|z - \Phi_k(z)| < \delta_k$ if $z \in Q_k$, and

(iii) $\Phi_k((Q_{k+1}\backslash Q_k)\backslash Y_k) \subset P_k$.

Define $F_{k+1} = F_k \circ \Phi_k$, and let P_{k+1} be some cube in $(Q_{k+1}\backslash Q_k)\backslash Y_k$. Then

$$F_{k+1}(P_{k+1}) \subset F_k(P_k) \subset Q_0. \tag{2}$$

This completes the induction step.

For all $z \in Q_k$ we have

$$|F_{k+1}(z) - F_k(z)| = |F_k(\Phi_k(z)) - F_k(z)| < \varepsilon_k \tag{3}$$

by (ii) and our choice of δ_k. Hence there exists

$$F = \lim_{k \to \infty} F_k \tag{4}$$

uniformly on compact subsets of $\mathbf{C}^2$, $JF \equiv 1$, and

$$|F(z) - F_k(z)| \leq \sum_{j=k}^{\infty} |F_{j+1}(z) - F_j(z)| < \sum_{k}^{\infty} \varepsilon_j < \varepsilon \tag{5}$$

for all $z \in Q_k$, by (3).

If $z \in Q_0$ then (5) implies that $|F(z) - z| < \varepsilon$; hence $F(z) \in Q_1 = Q$.

If $z \in (Q_{k+1}\backslash Q_k)\backslash Y_k$ for some $k \geq 0$, then (iii) gives

$$F_{k+1}(z) = F_k(\Phi_k(z)) \in F_k(P_k) \subset Q_0. \tag{6}$$

Also, $|F(z) - F_{k+1}(z)| < \varepsilon$, by another application of (5). As before, we conclude that $F(z) \in Q$.

Any other $z \in \mathbf{C}^2$ lies in some Y_k. This completes the proof, because $\sum \mathrm{vol}(Y_k) < \sum \varepsilon_k < \varepsilon$.

7.4. PROOF OF THEOREM 7.1. WHEN $n \geq 3$. The lemma is now to be stated for cubes in $\mathbf{C}^n$ rather than in $\mathbf{C}^2$; in its proof we have to take shears in n directions rather than 2; the only other difference is that we define

$$T_\alpha(z) = ((e^{\alpha z_1} - 1)/\alpha, z_2 e^{-\alpha z_1}, z_3, \ldots, z_n).$$

The theorem follows from the lemma precisely as before.

8. Moving compact convex sets. Given m affine transformations $A_1, \ldots, A_m$ of $\mathbf{C}^n$, with Jacobian 1, and m pairwise disjoint compact sets $K_1, \ldots, K_m$ whose images $A_j(K_j)$ are also pairwise disjoint, what additional information will guarantee that for every $\varepsilon > 0$ there is a polynomial automorphism Φ of $\mathbf{C}^n$ which furnishes the simultaneous approximations

$$|\Phi(z) - A_j(z)| < \varepsilon \quad \text{on } K_j$$

for $j = 1, \ldots, m$?

When $m = 2$ it is enough to assume that K_1 and K_2 are convex. But when $m \geq 3$, this is no longer enough: the fact that $\bigcup K_j$ may have a nontrivial polynomial hull [**8**; **17**; **21**, p. 389] while the polynomial hull of $\bigcup A_j(K_j)$ may be trivial gives rise to an obstruction. (We do not know whether there is another one.) We will, however, obtain a positive result when K_1 and K_2 are convex and K_3 is a point (Theorem 8.1). This will suffice to prove Theorem 8.5.

The proof of Theorem 8.1 shows that generalizations to more than two convex sets are possible, provided that sufficiently strong separation properties are assumed. We will not go into the details of this.

Since every shear, and hence every finite composition of shears, is a limit of polynomial automorphisms (uniformly on compact subsets of $\mathbf{C}^n$), whereas it does not seem to be known whether every polynomial automorphism with Jacobian 1 is a composition of shears when $n \geq 3$ [**1**, p. 299], we state the following theorem in terms of shears.

8.1. THEOREM. *Suppose that $A: \mathbf{C}^n \to \mathbf{C}^n$ is affine, with $JA = 1$, that H and K are compact convex sets in $\mathbf{C}^n$, and that H intersects neither K nor $A(K)$.*
Suppose also that $v, w \in \mathbf{C}^n$, $v \notin H \cup K$, $w \notin H \cup A(K)$.
To every $\varepsilon > 0$ corresponds then a composition Φ of finitely many shears so that

$$|\Phi(z) - z| < \varepsilon \quad \text{on } H, \qquad |\Phi(z) - A(z)| < \varepsilon \quad \text{on } K,$$

and $\Phi(v) = w$.

We break the proof into 3 steps. The points v and w are ignored in the first two of these. Step 1 proves the resulting simpler theorem under an additional separation hypothesis, which is then removed in Step 2.

If $u \in \mathbf{C}^n$, $u \neq 0$, and Λ is a linear functional on $\mathbf{C}^n$ so that $\Lambda u = 0$, we will use the phrase "*σ is a (Λ, u)-shear*" or "σ is a shear in the direction of u" to indicate that $\sigma(z) = z + f(\Lambda z)u$ for some holomorphic $f: \mathbf{C}^n \to \mathbf{C}$.

The symbol $\approx$ will indicate uniform approximation, to whatever degree is needed. Thus, for example, $\varphi \approx \text{id.}$ on K means that, given $\eta > 0$, we can find φ so that $|\varphi(z) - z| < \eta$ on K (and so that φ satisfies whatever else is needed in the particular context).

Just as in §7, Runge's theorem will be tacitly used every time we pick a shear. Although we start with convex sets, the shears that are used in the proof may well destroy their convexity. However, the distortions can be controlled so as to be so small that the separation properties needed to apply Runge's theorem will be satisfied. Or one may begin by replacing H and K by larger sets which are *strictly* convex (see Remark 8.4) but still disjoint. Then convexity can be maintained throughout the construction. We will not say anything further about this in the proof that follows.

8.2. PROOF OF THEOREM 8.1.

Step 1. *If A, H, K are as in the theorem, and there is a linear functional L so that the sets $L(H)$, $L(K)$, $L(A(K))$ are pairwise disjoint, then there exists Φ_1, a finite composition of shears, so that*

$$\Phi_1 \approx \begin{cases} \text{id.} & \text{on } H, \\ A & \text{on } K. \end{cases} \tag{1}$$

PROOF. Pick $u \in \mathbf{C}^n$, $u \neq 0$, so that $Lu = 0$. We will use an (L, u)-shear φ to move H out of the way whole keeping K and $A(K)$ where they are, will approximate A on K by a sequence of shears that does not move $\varphi(H)$ much, and will then move $\varphi(H)$ back to H.

Choose coordinates in $\mathbf{C}^n$ so that $0 \in K$ and

$$u = (1, 1, \dots, 1). \tag{2}$$

Then $A(z) = \Lambda z + p$, where $p = A(0)$, $\Lambda\colon \mathbf{C}^n \to \mathbf{C}^n$ is linear, and $\det \Lambda = 1$. This last fact implies, via elementary matrix manipulations, that there is a decomposition

$$\Lambda = \sigma_m \circ \sigma_{m-1} \circ \cdots \circ \sigma_1 \tag{3}$$

in which each σ_i is a linear shear in the direction of one of the basis vectors e_j.

Choose $r > 0$ so large that $K \subset rB$ and

$$(\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1)(K) \subset rB \qquad (1 \le j \le m). \tag{4}$$

One more observation before we start to move sets around: Since $p \in A(K)$ and L separates K from $A(K)$, we have $Lp \neq 0$. Hence there is a linear functional L' with $L'p = 0$, $L'u = 1$.

Our *first* move is an (L, u)-shear φ so that

$$\varphi(z) \approx \begin{cases} z & \text{on } K \cup A(K), \\ z + tu & \text{on } H, \end{cases} \tag{5}$$

where the constant t is so large that each coordinate projection π_i separates $\varphi(H)$ from $2r\bar{B}$ and L' separates $\varphi(H)$ from $A(K)$. This can be done because of (2) and because $L'u = 1$.

In the *second* move we replace each σ_i in (3) by a shear ψ_i (in the same direction as σ_i) so that

$$\psi_i \approx \begin{cases} \text{id.} & \text{on } \varphi(H), \\ \sigma_i & \text{on } 2r\bar{B}. \end{cases} \tag{6}$$

Setting $\Psi = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_1$ we thus obtain

$$\Psi \circ \varphi \approx \begin{cases} \varphi & \text{on } H, \\ \Lambda & \text{on } K. \end{cases} \tag{7}$$

For the *third* move, note that $L'p = 0$ and that L' separates $\varphi(H)$ from $A(K)$, hence also from

$$A(K) - p = \Lambda(K) \approx \Psi(\varphi(K)).$$

Therefore there is an (L', p)-shear τ so that

$$\tau(z) \approx \begin{cases} z & \text{on } \Psi(\varphi(H)), \\ z + p & \text{on } \Psi(\varphi(K)). \end{cases} \tag{8}$$

The *fourth* move is φ^{-1}.

Then $\Phi_1 = \varphi^{-1} \circ \tau \circ \Psi \circ \varphi$ satisfies (1).

Step 2. Assume now that A, H, K are as in the statement of Theorem 8.1.

There are linear functionals L and L_0 so that L_0 separates H from K, L separates H from $A(K)$, and the pair $\{L, L_0\}$ is linearly independent. Hence there exists $x \in \mathbf{C}^n$ so that $L_0 x = 0$, $Lx = 1$, and there is an (L_0, x)-shear φ_0 so that

$$\varphi_0(z) \approx \begin{cases} z & \text{on } H, \\ z + tx & \text{on } K, \end{cases} \tag{9}$$

where the constant t is so large that the set

$$L(K + tx) = L(K) + t \tag{10}$$

is disjoint from $L(H)$ and from $L(A(K))$.

Thus L separates the sets H, $K + tx$, $A(K)$.

The affine map A' that sends z to $A(z - tx)$ sends $K + tx$ to $A(K)$. We can therefore apply Step 1, with A' in place of A, and obtain Φ_1, a finite composition of shears, so that

$$\Phi_1 \approx \begin{cases} \text{id.} & \text{on } H \approx \varphi_0(H), \\ A\varphi_0^{-1} & \text{on } \varphi_0(K). \end{cases} \tag{11}$$

Then

$$\Phi_2 = \Phi_1 \circ \varphi_0 \approx \begin{cases} \text{id.} & \text{on } H, \\ A & \text{on } K, \end{cases} \tag{12}$$

which proves the theorem, except that the points v and w still have to be taken into account.

Step 3. With Φ_2 as in (12), put $H' = \Phi_2(H)$, $K' = \Phi_2(K)$, $v' = \Phi_2(v)$. The construction that led to (12) can be so controlled that the convex hulls $\text{co}(H')$ and $\text{co}(K')$ are disjoint and so that neither v' nor w is in their union.

Then $\Phi = F \circ \Phi_2$ will satisfy the conclusion of Theorem 8.1 if F is as in the following proposition in which, for simplicity, we have replaced $\text{co}(H')$, $\text{co}(K')$, and v' by H, K, and v.

8.3. PROPOSITION. *If H and K are disjoint compact convex sets in $\mathbf{C}^n$, and v, w, are points in $\mathbf{C}^n$, outside $H \cup K$, then to every $\varepsilon > 0$ corresponds a composition F of finitely many shears, so that $|F(z) - z| < \varepsilon$ on $H \cup K$, $F(v) = w$.*

PROOF. Assume that $H \cap \text{co}(K \cup \{v\}) = \varnothing$. (If this is not the case, then the convex hull of $H \cup \{v\}$ does not intersect K, and the same proof works, with H and K interchanged.)

Choose coordinates so that $w = 0$, $\text{Re}\, z_1 > 0$ on H. Since $0 \notin K - v$, there is a linear functional Λ so that $\text{Re}\, \Lambda z < 0$ on $K - v$, and

$$\Lambda z = c_1 z_1 + \cdots + c_n z_n, \qquad |c_1|^2 + \cdots + |c_n|^2 = 1.$$

There is a unitary matrix U with $(c_1, \ldots, c_n)$ in the top row, and with $\det U = 1$. Then $z \to U(z - v)$ is an affine transformation A, with $A(v) = 0$, that maps K into $\{\text{Re}\, z_1 < 0\}$. Thus H is disjoint from $A(\text{co}(K \cup \{v\}))$, and Step 2 of the preceding proof furnishes an F_1 so that

$$F_1 \approx \begin{cases} \text{id.} & \text{on } H, \\ A & \text{on } \text{co}(K \cup \{v\}) \end{cases}$$

and (by a minor adjustment) $F_1(v) = 0$.

But now $\text{co}(H \cup \{0\})$ does not intersect $F_1(K)$, and we can apply Step 2 again to get F_2 so that

$$F_2 \approx \begin{cases} \text{id.} & \text{on } H \cup \{0\}, \\ A^{-1} & \text{on } F_1(K), \end{cases}$$

and $F_2(0) = 0$.

Finally, $F = F_2 \circ F_1$ does what is needed.

8.4. REMARKS. (a) in Theorem 8.1 we could also prescribe some finite set in H and find a map Φ which, in addition to the conclusion of Theorem 8.1, also fixes every point of this finite set.

(b) As already pointed out, if H and K are *strictly convex bodies* (i.e., if they have defining functions whose Hessian is strictly positive on their boundaries) then, by keeping the second derivatives of all shears in the proof very small on the various images of H and K (note that we only approximated *affine* maps locally) we can obtain Φ so that the conclusions of Theorem 8.1 hold *and* so that $\Phi(H)$ and $\Phi(K)$ are strictly convex.

We shall now apply these approximation theorems to construct certain biholomorphic maps from $\mathbf{C}^n$ into $\mathbf{C}^n$.

8.5. THEOREM. *Suppose $n > 1$, and*

(i) *$K \subset \mathbf{C}^n$ is compact and strictly convex, or K is a point,*

(ii) *E is a countable subset of $\mathbf{C}^n \backslash K$.*

Then there is a biholomorphic $F\colon \mathbf{C}^n \to \mathbf{C}^n$ so that

$$E \subset F(\mathbf{C}^n) \subset \mathbf{C}^n \backslash K, \tag{1}$$

and $JF \equiv 1$.

Note that E could be dense in $\mathbf{C}^n \backslash K$. When K is a point, it follows that $\mathbf{C}^n$ has *dense proper subsets* that are biholomorphic images of $\mathbf{C}^n$.

PROOF. Let $\{w_1, w_2, w_3, \dots\}$ be an enumeration of E. Choose coordinates so that K lies outside the closed unit ball $\bar{B}$ of $\mathbf{C}^n$. Put $G_0(z) = z$, $K_0 = K$.

Now assume, as induction hypothesis, that $j \geq 0$, $G_j \in \operatorname{Aut}(\mathbf{C}^n)$, $K_j = G_j(K)$ is strictly convex (or is a point) and lies outside $(j+1)\bar{B}$, and that

$$G_j(w_i) = z_i \in jB \quad \text{for all } i < j. \tag{2}$$

Choose δ_j, $0 < \delta_j < 1$, so that

$$|G_j^{-1}(z') - G_j^{-1}(z'')| < 2^{-j} \tag{3}$$

for all $z', z'' \in (j+1)\bar{B}$ with $|z' - z''| < \delta_j$.

We can now apply Theorem 8.1 to the convex sets $j\bar{B}$, K_j, and the point $G_j(w_j)$ in place of v, to find Φ_j, a composition of finitely many shears, so that (see Remark 8.4)

(a) Φ_j^{-1} moves no point of $j\bar{B}$ by as much as δ_j,

(b) $\Phi_j(K_j)$ lies outside $(j+2)\bar{B}$ and is strictly convex (or is a point),

(c) $\Phi_j(z_i) = z_i$ for all $i < j$,

(d) $\Phi_j(G_j(w_j)) = z_j$ lies in $(j+1)B$.

As regards (d), note that $w_j \notin K$, hence $G_j(w_j) \notin K_j$, so there is no conflict between (b) and (d). Moreover, if $G_j(w_j) \in j\bar{B}$ we satisfy (d) by choosing $z_j = G_j(w_j)$; otherwise, pick z_j anywhere so that $j < |z_j| < j+1$.

Now put $G_{j+1} = \Phi_j \circ G_j$, and continue.

By (a) and our choice of δ_j,

$$|G_{j+1}^{-1}(z) - G_j^{-1}(z)| = |G_j^{-1}(\Phi_j^{-1}(z)) - G_j^{-1}(z)| < 2^{-j}$$

for all $z \in j\bar{B}$. The limit

$$F = \lim_{j\to\infty} G_j^{-1} \tag{4}$$

exists therefore, uniformly on compact subsets of $\mathbf{C}^n$, and defines a biholomorphic $F\colon \mathbf{C}^n \to \mathbf{C}^n$ with $JF \equiv 1$.

Since $G_j^{-1}(z_i) = w_i$ for all $j > i$, we have $F(z_i) = w_i$ for all i. Thus $E \subset F(\mathbf{C}^n)$.

To finish, assume, to reach a contradiction, that $w = F(z)$ for some $w \in K$, $z \in \mathbf{C}^n$. Let β be a ball with center z. For all sufficiently large j it follows from (4) that there are points $p_j \in \beta$ so that $G_j^{-1}(p_j) = w$, i.e., $p_j = G_j(w)$. But our construction shows that $|G_j(w)| \to \infty$ as $j \to \infty$, because $w \in K$, whereas $\{p_j\}$ is bounded. This contradiction shows that $F(\mathbf{C}^n)$ contains no point of K.

8.6. REMARK. In [**9, 10**], J. A. Morrow has classified the nonsingular compact complex manifolds M of complex dimension 2 that contain a nonempty nowhere dense closed *analytic* subset A so that $M \backslash A$ is biholomorphic to $\mathbf{C}^2$.

One may ask whether "analytic" is redundant in this statement. Theorem 8.5 shows that it is not:

Take $n = 2$, K a point (say $K = \{0\}$), E dense in $\mathbf{C}^2$, and construct F as in the proof of Theorem 8.5, as the limit of a sequence of automorphisms of $\mathbf{C}^2$. This implies that $\Omega = F(\mathbf{C}^2)$ is a Runge domain [**2**, p. 141].

Let L be a complex line in $\mathbf{C}^2 \backslash \{0\}$ which intersects Ω, and put $L_w = \{\lambda w : \lambda \in \mathbf{C}\}$ for each $w \in L \cap \Omega$. Since $0 \notin \Omega$, no L_w lies in Ω. Since Ω is a Runge domain, each component of $\Omega \cap L_w$ is simply connected (otherwise polynomial approximation would fail; see, for example, [**20**]) and its boundary relative to L_w must therefore have positive one-dimensional Hausdorff measure. This holds for each $w \in L \cap \Omega$. A Fubini-type argument shows now that *the Hausdorff dimension of* $\mathbf{C}^2 \backslash \Omega$ *is at least* 3.

We may regard F as a biholomorphic map from $\mathbf{C}^2$ into (for example) complex projective space P^2, with Ω dense in P^2. Put $A = P^2 \backslash \Omega$. Then $A \supset \mathbf{C}^2 \backslash \Omega$, so that A has Hausdorff dimension ≥ 3, and this shows that A is not an analytic subset of P^2. (The Hausdorff dimension of analytic subsets of P^2 is at most 2.)

We thank E. L. Stout for drawing our attention to this question.

9. Regions attracted to a fixed point. We begin with a simple case of the basic theorem that was mentioned in the Introduction.

9.1. THEOREM. *Suppose* $F \in \operatorname{Aut}(\mathbf{C}^n)$, $p \in \mathbf{C}^n$, $F(p) = p$, *and the eigenvalues* λ_i *of* $A = F'(p)$ *satisfy* $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ *and*

$$|\lambda_1|^2 < |\lambda_n|. \tag{1}$$

Define

$$\Omega = \left\{ z \in \mathbf{C}^n : \lim_{k \to \infty} F^k(z) = p \right\}. \tag{2}$$

Then Ω *is a region, and there is a biholomorphic map* Ψ *from* Ω *onto* $\mathbf{C}^n$, *given by*

$$\Psi = \lim_{k \to \infty} A^{-k} F^k. \tag{3}$$

The convergence in (3) *is uniform on compact subsets of* Ω.

Recall that $F^k = F \circ F^{k-1}$, $F^1 = F$. Note that (1) implies that $0 < |\lambda_i| < 1$ for all i. We may describe Ω as *the region that is attracted to* p *by* F.

One immediate consequence of (3) is the functional equation

$$\Psi = A^{-1} \Psi F. \tag{4}$$

Another is that $J\Psi \equiv 1$ whenever JF is constant.

PROOF. Take $p = 0$, without loss of generality. Pick constants α, β_1, β_2, β so that $\alpha < |\lambda_n|$, $|\lambda_1| < \beta_1 < \beta_2 < \beta$, and $\beta^2 < \alpha$. The spectral radius formula gives an m so that $||A^{-N}|| < \alpha^{-N}$ and $||A^N|| < \beta_1^N$ for all $N \geq m$. Approximating F^m by A^m shows that there is an $r > 0$ so that (for our fixed m) $z \in rB$ implies

$$|F^m(z)| \leq \beta_2^m |z|. \tag{5}$$

Put $C = \sup\{|F^j(z)|/|z| \colon 0 \leq j < m, 0 < |z| < r\}$.

If $N = km + j$, $k = 1, 2, 3, \ldots$, $0 \leq j < m$, and if $|z| < r$, then iteration of (5) yields

$$|F^N(z)| = |F^j(F^{km}(z))| \leq C|F^{km}(z)| \leq C\beta_2^{km}|z|.$$

Thus, for all sufficiently large $N \geq N_0$ (where $N_0 \geq m$ depends only on m and r) we have

$$|F^N(z)| < \beta^N \quad \text{for all } z \in rB. \tag{6}$$

It follows from (6) that $rB \subset \Omega$ (because $\beta < 1$), hence that

$$\Omega = \bigcup_{-\infty}^{\infty} F^k(rB). \tag{7}$$

This shows that Ω is a region and that $F(\Omega) = \Omega$.

Now pick a compact set $K \subset \Omega$. For some s, $F^s(K) \subset rB$. Hence (6) shows that

$$|F^N(z)| \leq \beta^{N-s} = a\beta^N \qquad (z \in K,\ N \geq s + N_0) \tag{8}$$

where $a = \beta^{-s}$. Since $(A^{-1}F)'(0) = I$, there is a constant b so that

$$|w - A^{-1}F(w)| \leq b|w|^2 \qquad (|w| \leq a). \tag{9}$$

Thus, if $z \in K$ and if we set $w_N = F^N(z)$, we get the estimate

$$\begin{aligned} |A^{-N}F^N(z) - A^{-N-1}F^{N+1}(z)| &\leq ||A^{-N}|| \cdot |w_N - A^{-1}F(w_N)| \\ &\leq \alpha^{-N} b |w_N|^2 \leq a^2 b (\beta^2/\alpha)^N \end{aligned}$$

for all $N \geq s + N_0$.

Since $\beta^2/\alpha < 1$, it follows that (3) holds. It is clear that Ψ (being a limit of a sequence of automorphisms) is holomorphic and one-to-one in Ω. (Note that $\Psi'(0) = I$.) Since $F(\Omega) = \Omega$ and $\Psi = A^{-1}\Psi F$, we see that Ψ and $A^{-1}\Psi$ have the same range. Since the linear operator A^{-1} is an expansion, it follows that $\Psi(\Omega)$ is all of $\mathbf{C}^n$.

9.2. EXAMPLE. Define $F \in \operatorname{Aut}(\mathbf{C}^2)$ by $F(z, w) = (\alpha z, \beta w + z^2)$, where $0 < \beta < \alpha < 1$. This F fixes the origin, and $A = F'(0,0) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. By induction

$$F^k(z, w) = (\alpha^k z, \beta^k w + \beta^{k-1}(1 + c + \cdots + c^{k-1})z^2),$$

where $c = \alpha^2/\beta$. Thus

$$(A^{-k}F^k)(z, w) = (z, w + \beta^{-1}(1 + c + \cdots + c^{k-1})z^2).$$

The coefficient of z^2 in the second component of $A^{-k}F^k$ tends to infinity, except when $c < 1$, i.e., when $\alpha^2 < \beta$.

Conclusion: *The sequence* (3) *may fail to converge* (*even locally, and even on the level of formal power series*) *if assumption* (1) *of Theorem* 9.1 *is violated.*

The region that is attracted to the origin by this F is all of $\mathbf{C}^2$. To get away from this, put

$$G(z,w) = (\alpha z + (\beta w + z^2)^2, \beta w + z^2).$$

Again, $G \in \operatorname{Aut}(\mathbf{C}^2)$; $G'(0,0) = F'(0,0)$; the coefficients in G^k are at least as large as those in F^k. Therefore $A^{-k}G^k$ will still diverge when $\alpha^2 \geq \beta$. But now the region Ω that is attracted to $(0,0)$ by G is not all of $\mathbf{C}^2$, because G has three other fixed points, given by $z^3 = (1-\alpha)(1-\beta)^2$, $w = z^2/(1-\beta)$.

9.3. NOTATION. In the examples that follow, we shall use the abbreviation *F. B. region* (for Fatou-Bieberbach) to denote regions $\Omega \subset \mathbf{C}^n$, $\Omega \neq \mathbf{C}^n$, which are biholomorphically equivalent to $\mathbf{C}^n$.

Actually, the examples will all be in $\mathbf{C}^2$.

9.4. *Example of an F.B. region $\Omega \subset \mathbf{C}^2$ whose intersection with every complex line is bounded.* Define $F(z,w) = (u,v)$ by

$$(1) \qquad u = \alpha w, \qquad v = \alpha z + w^2$$

for some fixed α, $0 < |\alpha| < 1$. Then $F \in \operatorname{Aut}(\mathbf{C}^2)$, F fixes $(0,0)$, the eigenvalues of $F'(0,0)$ are $\pm\alpha$. Let Ω be the region attracted to $(0,0)$ by F, as in Theorem 9.1.

If (z,w) lies in the set E defined by $|w| > 1 + 2|\alpha| + |z|$, then (1) shows that

$$\begin{aligned}|v| &\geq |w|^2 - |\alpha z| > |w|^2 - |\alpha w| = |w|(|w| - |\alpha|)\\ &> |w|(1 + |\alpha|) > 1 + 2|\alpha| + |u|\end{aligned}$$

so that $(u,v) \in E$. Thus $F(E) \subset E$. This shows that no point of E lies in Ω.

Now let L be a complex line in $\mathbf{C}^2$. Parametrize L by $z = a + b\lambda$, $w = c + d\lambda$, where a, b, c, d are constants and λ ranges over $\mathbf{C}$. If we substitute these expressions for z and w into (1) we see that $F(z,w) \in E$ as soon as $|\lambda|$ is large enough. (Note that $d = 0$ implies $b \neq 0$.) For such λ, it follows that (z,w) is not in Ω.

9.5. EXAMPLE. The automorphism $F(z,w) = (u,v)$ given by

$$(1) \qquad u = z + w, \qquad v = \tfrac{1}{2}(1 - w - e^{z+w})$$

leads to several interesting phenomena.

Its fixed points are

$$(2) \qquad p_m = (2m\pi i, 0),$$

one for each integer m. The eigenvalues of $F'(p_m)$ are $\pm 1/\sqrt{2}$. Theorem 9.1 can therefore be applied:

There exist pairwise disjoint F.B. regions $\Omega_m \subset \mathbf{C}^2$ $(m = 0, \pm 1, \pm 2, \dots)$, attracted to p_m by F, which are translates of each other:

$$(3) \qquad \Omega_m = \Omega_0 + p_m.$$

To see (3), note that $F((z,w) + p_m) = F(z,w) + p_m$. Hence

$$\lim_{k\to\infty} F^k((z,w) + p_m) = p_m \quad \text{if} \quad \lim_{k\to\infty} F^k(z,w) = p_0.$$

It follows from (3) and the disjointness of $\{\Omega_m\}$ that the map E given by

$$(4) \qquad E(u,v) = (e^u, ve^{-u})$$

is one-to-one on each Ω_m, and that

$$(5) \qquad \Omega^* = E(\Omega_m)$$

is independent of m.

This gives an F. B. region Ω^* in $\mathbf{C}^2$ which does not intersect the line $\{z = 0\}$.

Moreover, since $JF \equiv -1/2$ (a constant), the Ω_m's as well as Ω^* are biholomorphic images of $\mathbf{C}^2$ via *volume-preserving maps.* (This is why we defined E by (4), rather than by the simpler formula $E(u, v) = (e^u, v)$.)

It is known [**5**] that the range of a nondegenerate holomorphic map from $\mathbf{C}^2$ into $\mathbf{C}^2$ cannot avoid 3 complex lines. We shall now see that this is not so if complex lines are replaced by translates of R^2. Here R^2 denotes the set of points of $\mathbf{C}^2$ both of whose coordinates are real. Define

$$\Pi_k = R^2 + ((2k+1)\pi i, 0) \tag{6}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Then $F(\Pi_k) = \Pi_k$, and no p_m lies in any Π_k. Therefore no point of any Π_k is attracted to any p_m by F.

Conclusion: *No Π_k intersects any Ω_m.*

Finally, we modify the regions Ω_m so as to obtain disjoint F. B. regions $\tilde{\Omega}_m$ with the following property:

For each m, $\tilde{\Omega}_m \cap \{w = 0\}$ has infinitely many components.

Picard's theorem shows that at most one line $u =$ const. misses Ω_0. Therefore Ω_0 contains points (u_s, v_s) with $u_s = s + iy_s$, $2s\pi < y_s < (2s+1)\pi$, for every integer s. Since the numbers $\exp u_s$ are not real, and no two of them are complex conjugates of each other, there is an entire function $h\colon \mathbf{C} \to \mathbf{C}$ so that $h(R) \subset R$ and $h(\exp(u_s)) = v_s$. Define a shear Φ by

$$\Phi(u, v) = (u, v - h(e^u)) \tag{7}$$

and put $\tilde{\Omega}_m = \Phi(\Omega_m)$.

Since $\Phi(\Pi_k) = \Pi_k$, no Π_k intersects any $\tilde{\Omega}_m$. Each $\tilde{\Omega}_m$ contains the points

$$(u_s + 2m\pi i, v_s - h(e^{u_s})) = (s + (y_s + 2m\pi)i, 0), \tag{8}$$

one in each strip bounded by the (real) lines

$$(x + (2k+1)\pi i, 0) \qquad (-\infty < x < \infty) \tag{9}$$

which lie in Π_k. Thus $\tilde{\Omega}_m$ has at least one component in each of these strips.

9.6. EXAMPLE. We just saw that there exist F. B. regions Ω_m in $\mathbf{C}^2$ which miss infinitely many translates of R^2. *The same can be done with finitely many rotated copies of R^2*:

Let N be a positive integer, put $\alpha = \exp(\pi i/2N)$, and put $E_k = \alpha^k R^2$ for $k = 0, 1, \ldots, 2N-1$. Define $F(z, w) = (u, v)$ by

$$u = z + w, \qquad v = \frac{1}{2N+1}[z + (z+w)^{2N+1}]. \tag{1}$$

Then $F \in \operatorname{Aut}(\mathbf{C}^2)$, $F(E_k) = E_k$ for all k, the fixed points of F are $(0, 0)$ and $p_m = (\alpha^m, 0)$ for *odd* m. The eigenvalues of $F'(p_m)$ are $\pm(2N+1)^{-1/2}$. It follows from Theorem 9.1 that there are N pairwise disjoint F. B. regions Ω_m, attracted to p_m by F, and

$$\Omega_m \subset \mathbf{C}^2 \backslash (E_0 \cup E_2 \cup \cdots \cup E_{2N-2}). \tag{2}$$

Note also that $F(\alpha^2 z, \alpha^2 w) = \alpha^2 F(z, w)$, by (1). From this one can deduce that the rotation $(z, w) \to (\alpha^2 z, \alpha^2 w)$ permutes the regions Ω_m.

9.7. *Example of an F.B. region $\Omega_0 \subset \mathbf{C}^2$ whose closure misses a complex line.* (We do not know whether the region Ω^* in Example 9.5 also has this property.)

Pick $\alpha \in \mathbf{C}$, $0 < |\alpha| < 1$, find an entire function $f\colon \mathbf{C} \to \mathbf{C}$ so that

$$(1) \qquad e^{f(0)} = 1/\alpha, \quad f'(0) = 0, \quad f(1) = 0, \quad f'(1) = (1 + \alpha^2/(1-\alpha^2),$$

and define $F(z,w) = (u,v)$ by

$$(2) \qquad u = 1 - \alpha^2 + \alpha^2 z e^{f(zw)}, \qquad v = we^{-f(zw)}.$$

Then $F \in \mathrm{Aut}(\mathbf{C}^2)$, $JF \equiv \alpha^2$, $F(1,1) = (1,1)$, and the eigenvalues of $F'(1,1)$ are $\pm\alpha i$. Let Ω_0 be the region attracted to $(1,1)$ by F.

Let Ω_1 be the region attracted to the fixed point $(1+\alpha, 0)$, where $F' = \alpha I$. Since

$$(3) \qquad F(z,0) = (1 - \alpha^2 + \alpha z, 0)$$

for all $z \in \mathbf{C}$, we see that Ω_1 contains the line $\{w = 0\}$. Therefore $\bar{\Omega}_0$ does not intersect this line.

This example is quite similar to one of Nishimura's [**11**]. He does not, however, derive it from a theorem about *fixed points* of automorphisms, but from a more difficult one that involves *pointwise fixed analytic subvarieties.*

9.8. REMARK. All the F. B. regions Ω obtained in Examples 9.4 to 9.7 were ranges of biholomorphic maps $\Phi\colon \mathbf{C}^2 \to \Omega$ with $J\Phi \equiv 1$, because the automorphisms that were used in the constructions had constant Jacobians. (Here $\Phi = \Psi^{-1}$, where Ψ is given by Theorem 9.1.)

Our next example will use automorphisms of the kind that we mentioned at the end of the Introduction. That the resulting map Φ does not have constant Jacobian follows from Theorem I of [**13**], which states:

If $\Phi\colon \mathbf{C}^2 \to \mathbf{C}^2$ is holomorphic and one-to-one, $J\Phi \equiv c$, and Φ preserves the lines $\{z = 0\}$ and $\{w = 0\}$, then $\Phi \in \mathrm{Aut}(\mathbf{C}^2)$; in fact

$$\Phi(z,w) = (cze^{f(zw)}, we^{-f(zw)})$$

for some entire $f\colon \mathbf{C} \to \mathbf{C}$.

9.9. *Example of an F. B. region Ω in $\mathbf{C}^2$ which contains the set $\{zw = 0\}$ and is not dense in $\mathbf{C}^2$.* Let $g, h\colon \mathbf{C} \to \mathbf{C}$ be entire functions, so that

$$(1) \qquad \exp g(0) = 2, \qquad \exp h(0) = 1/4$$

and

$$(2) \qquad \exp g(2^{4p}) = 1/2, \qquad \exp h(2^{2p+2}) = 4$$

for $p = 0, 1, 2, \ldots$, define

$$(3) \qquad G(z,w) = (z \exp g(z^3 w), w \exp[-3g(z^3 w)]),$$

$$(4) \qquad H(u,v) = (u \exp h(uv), v \exp[-h(uv)])$$

and put $F = H \circ G$. Then $F \in \mathrm{Aut}(\mathbf{C}^2)$,

$$(5) \qquad F(z,0) = \left(\tfrac{1}{2}z, 0\right), \qquad F(0,w) = \left(0, \tfrac{1}{2}w\right)$$

and

$$(6) \qquad F(2^p, 2^p) = (2^{p+1}, 2^{p+1}) \qquad (p = 0, 1, 2, \ldots).$$

Setting $A = F'(0,0)$, we have $A = \frac{1}{2}I$. Hence (5) shows that each $A^{-k}F^k$ fixes every point of $\{zw = 0\}$. If $\Phi = \Psi^{-1}$, where Ψ is as given by Theorem 9.1, we conclude:

Φ is a biholomorphic map from $\mathbf{C}^2$ onto the region Ω that is attracted to the origin by F; every point of $\{zw = 0\}$ lies in Ω because Φ fixes it; by (6), Ω contains none of the points $(2^p, 2^p)$.

In particular, $\Omega \neq \mathbf{C}^2$.

But we claimed more, namely that Ω is not dense in $\mathbf{C}^2$. To achieve this, we have to choose g and h with more care; specifically, we strengthen (2) by requiring that g and h are almost constant on discs centered at 2^{4p} and 2^{2p+2}, respectively. Here are the details:

Choose constants c_p and c so that

$$0 < c_0 < c_1 < \cdots < c, \qquad (1+c)^4 - 1 < 1/4. \tag{7}$$

Writing $D(a, r)$ for the open disc in $\mathbf{C}$ with center at a and radius r, consider the discs

$$D_p = 2^p D(1, c_p), \qquad X_p = 2^{4p} D\left(1, \tfrac{1}{4}\right), \qquad Y_p = 2^{2p+2} D\left(1, \tfrac{1}{4}\right) \tag{8}$$

and the polydiscs

$$\Delta_p = D_p \times D_p \tag{9}$$

for $p = 0, 1, 2, \ldots$.

The X_p's have disjoint closures; the same is true of the Y_p's. Therefore, given $\varepsilon_p > 0$, we can find entire functions g and h so that (1) holds and

$$\left|\tfrac{1}{2} - e^g\right| < \varepsilon_p \quad \text{on } X_p, \qquad |4 - e^h| < \varepsilon_p \quad \text{on } Y_p, \tag{10}$$

for $p = 0, 1, 2, \ldots$. (The existence of g and h can be proved by repeated applications of Runge's theorem, followed by a passage to the limit.)

Our choice of c in (7) guarantees that $z^3 w \in X_p$ and $4zw \in Y_p$ for all $(z, w) \in \bar{\Delta}_p$.

Therefore, if $(z, w) \in \bar{\Delta}_p$ and $(u, v) = G(z, w)$, then $(u, v) \approx (z/2, 8w)$ since $e^g \approx 1/2$ on X_p. So if ε_p is small enough, it follows that $uv \in Y_p$, and therefore

$$F(z, w) = H(u, v) \approx (4u, v/4) \approx (2z, 2w). \tag{11}$$

We conclude: If ε_p is small enough (depending on the choices made in (7)) then (10) will ensure that

$$F(\Delta_p) \subset \Delta_{p+1} \qquad (p = 0, 1, 2, \ldots). \tag{12}$$

Thus $|F^k(z, w)| \to \infty$ as $k \to \infty$, for (z, w) in any Δ_p. This shows that Ω intersects no Δ_p.

Open questions.

1. Consider the following properties which an infinite discrete set $E \subset \mathbf{C}^n$ may or may not have:

(a) E is tame in $\mathbf{C}^n$.
(b) E is avoidable by biholomorphic maps.
(c) E is permutable: every permutation of E extends to an automorphism of $\mathbf{C}^n$.
(d) E is the set of all fixed points of some automorphism of $\mathbf{C}^n$.

We know that (a) implies the other three. (For (a) $\Rightarrow$ (d) see Example 9.5.) What other implications hold among these four properties?

2. Suppose $\{\Omega_j\}$ is an infinite disjoint collection of F. B. regions in $\mathbf{C}^n$, E is discrete in $\mathbf{C}^n$, and E has exactly one point in each Ω_j. (So E is obviously avoidable by biholomorphic maps.) Must E be tame in $\mathbf{C}^n$?

3. Suppose that the distance between any two points of a set $E \subset \mathbf{C}^n$ is at least 1. Must E be tame in $\mathbf{C}^n$?

4. If E is discrete in $\mathbf{C}^2$ and $|z_1| > 1$ for every $(z_1, z_2) \in E$, must E be tame in $\mathbf{C}^2$? (Compare with Theorem 3.8.) The proof of Theorem 6.4 shows that E need not be very tame.

5. If a discrete set $E \subset \mathbf{C}^n$ is unavoidable (by whatever class of maps), must E stay unavoidable after removal of one point?

6. Is there a biholomorphic map from $\mathbf{C}^n$ into $\mathbf{C}^n$ which is not a limit of automorphisms?

Some related questions: If F is biholomorphic, must $F(\mathbf{C}^n)$ be a Runge domain?

Is the region Ω^* in Example 9.5 a Runge domain?

Is the union of every expanding sequence of F. B. regions an F. B. region?

7. Is there a holomorphic $F\colon \mathbf{C}^2 \to \mathbf{C}^2$ with $JF \equiv 1$ (or with $JF \not\equiv 0$) so that the *closure* of $F(\mathbf{C}^2)$ has finite volume?

8. Is every $F \in \operatorname{Aut}(\mathbf{C}^n)$ with $JF \equiv 1$ a limit of a sequence of compositions of shears?

A more specific question: Is the map $(z, w) \to (ze^{zw}, we^{-zw})$ a limit of a sequence of compositions of shears in $\mathbf{C}^2$?

9. Let $n = 2$ for simplicity. Do the transformations described at the end of the Introduction generate the group Γ of all automorphisms of $\mathbf{C}^2$ that fix every point of $\{z_1 z_2 = 0\}$? (One needs to have $f(0) = 0$.)

Does every $F \in \Gamma$ satisfy

$$(JF)(z_1, z_2) = w_1 w_2 / z_1 z_2$$

if $(w_1, w_2) = F(z_1, z_2)$ and $z_1 z_2 \neq 0$?

Peschl [**14**] claims that the answer to the second question is yes. We believe that there may be a gap in his proof. To be specific, we do not see how one can justify the claim (made on line 10 of p. 1838) that $G_n^m \overset{m}{=} G$.

10. Is there a biholomorphic map from $\mathbf{C}^2$ into the set $\{zw \neq 0\}$, i.e., into the complement of the union of two intersecting complex lines?

(Nishimura's papers [**12** and **13**] contain several results about biholomorphic maps from $\mathbf{C}^2$ into the complement of one complex line.)

11. If Ω is an F. B. region and L is a complex line, is it possible that

(a) $L \cap \Omega$ is connected (and not empty)?

(b) $L \cap \Omega$ has finitely many components?

(c) $L \cap \Omega$ is a circular disc?

12. How many complex lines can an F. B. region in $\mathbf{C}^2$ contain? Examples 9.4, 9.7, and 9.9 show that 0, 1, and 2 are possible.

13. Are there two disjoint F. B. regions in $\mathbf{C}^n$ whose union is dense in $\mathbf{C}^n$? What if "two" is replaced by "finitely many" or by "infinitely many"?

Appendix. As mentioned earlier, it is the purpose of this Appendix to give a proof of the theorem concerning attracting fixed points of automorphisms that was stated in the Introduction.

We begin with some facts about holomorphic maps $G = (g_1, \dots, g_m)$ from $\mathbf{C}^n$ into $\mathbf{C}^n$ of the form

$$\begin{aligned} g_1(z) &= c_1 z_1, \\ g_2(z) &= c_2 z_2 + h_2(z_1), \\ &\vdots \\ g_n(z) &= c_n z_n + h_n(z_1, \dots, z_{n-1}) \end{aligned}$$

where $c_1, \dots, c_n$ are scalars and each h_i is a holomorphic function of $(z_1, \dots, z_{i-1})$ which vanishes at the origin. We call such maps *lower triangular*.

The matrix that represents the linear operator $G'(0)$ is then lower triangular. Thus $G'(0)$ is invertible if and only if no c_i is 0. It follows that G is an automorphism of $\mathbf{C}^n$ (a composition of an invertible linear map and $n-1$ shears) if and only if no c_i is 0.

If $g_1, \dots, g_n$ are polynomials, the degree of $G = (g_1, \dots, g_n)$ is defined to be $\deg G = \max_i \deg g_i$.

LEMMA 1. *Let G be a lower triangular polynomial automorphism of $\mathbf{C}^n$.*

(a) *The degrees of the iterates G^k of G are then bounded, and there is a constant $\beta < \infty$ so that*

$$G^k(U^n) \subset \beta^k U^n \qquad (k = 1, 2, 3, \dots). \tag{1}$$

Here U^n is the unit polydisc in $\mathbf{C}^n$.

(b) *If also $|c_i| < 1$ for $1 \le i \le n$, then $G^k(z) \to 0$, uniformly on compact subsets of $\mathbf{C}^n$, and*

$$\bigcup_{k=1}^{\infty} G^{-k}(V) = \mathbf{C}^n \tag{2}$$

for every neighborhood V of 0.

PROOF. Let $G = (g_1, \dots, g_n)$, $G^k = (g_1^{(k)}, \dots, g_n^{(k)})$, put $\mu_i = \deg g_i$, and let $S(m,k)$ be the statement

$$\deg g_i^{(k)} \le \mu_1 \cdots \mu_i \quad \text{for } 1 \le i \le m. \tag{3}$$

We want to prove $S(n,k)$ for $k = 1, 2, 3, \dots$.

Since $G^{k+1} = G \circ G^k$, we have

$$g_i^{(k+1)} = c_i g_i^{(k)} + h_i(g_1^{(k)}, \dots, g_{i-1}^{(k)}) \qquad (2 \le i \le n). \tag{4}$$

This shows that $S(m, k+1)$ follows from $S(m,k)$ and $S(m-1,k)$. Since $S(1,k)$ and $S(m,1)$ are obviously true for all k and m (note that $\mu_1 = 1$, and $\mu_i \ge 1$ for all i), $S(n,k)$ follows by induction.

Putting $d = \mu_1 \cdots \mu_n$ we have thus proved that $\deg G^k \le d$ for $k = 1, 2, 3, \dots$.

Next, let M be the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ that have $|\alpha| \le d$. (As usual, a multi-index α is an ordered n-tuple of nonnegative integers $\alpha_1, \dots, \alpha_n$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.) Choose $C \geq 1$ so that $|g_i| \leq C$ on U^n for $1 \leq i \leq n$, and put $\beta = M \cdot C^d$. We claim that then

$$(5) \qquad |g_i^{(k)}(z)| \leq \beta^k \qquad (z \in U^n,\ 1 \leq i \leq n,\ k = 1, 2, 3, \ldots).$$

Since $C \leq \beta$, (5) holds when $k = 1$. Assume (5) for some $k \geq 1$. The coefficients a_α in

$$(6) \qquad g_i^{(k)}(z) = \sum_{|\alpha| \leq d} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \sum_{|\alpha| \leq d} a_\alpha z^\alpha$$

are equal to the integrals of $g_i^{(k)}(z)\bar{z}^\alpha$ over the unit torus T^n. Thus (5) implies $|a_\alpha| \leq \beta^k$.

Since $G^{k+1} = G^k \circ G$, (6) shows that

$$(7) \qquad g_i^{(k+1)} = g_i^{(k)}(g_1, \ldots, g_n) = \sum_{|\alpha| \leq d} a_\alpha g_1^{\alpha_1} \cdots g_n^{\alpha_n}.$$

Our choice of M and C implies now that

$$(8) \qquad |g_i^{(k+1)}| \leq M\beta^k C^{|\alpha|} \leq M\beta^k C^d = \beta^{k+1}$$

which is (5) with $k+1$ in place of k.

Thus (1) holds, and part (a) of the lemma is proved.

We turn to (b). Let $E \subset \mathbf{C}^n$ be compact. Note that $g_1^{(k)}(z) = c_1^k z_1$. Thus $\|g_1^{(k)}\|_E \to 0$ as $k \to \infty$. (We use $\|\cdot\|_E$ to denote the sup-norm over E.) Assume now that $1 < i \leq n$ and that

$$(9) \qquad \lim_{k\to\infty} \|g_j^{(k)}\|_E = 0 \quad \text{for } 1 \leq j < i.$$

Since $h_i(0) = 0$, it follows that

$$(10) \qquad \lim_{k\to\infty} \|h_i(g_1^{(k)}, \ldots, g_{i-1}^{(k)}\|_E = 0.$$

Therefore, given $\varepsilon > 0$, (4) shows that

$$(11) \qquad |g_i^{(k+1)}| \leq |c_i|\,|g_i^{(k)}| + \varepsilon$$

on E, for all sufficiently large k. This implies

$$(12) \qquad \limsup_{k\to\infty} \|g_i^{(k)}\|_E \leq \frac{\varepsilon}{1 - |c_i|},$$

for all $\varepsilon > 0$. Hence (9) holds with $i+1$ in place of i.

The first assertion in part (b) follows now by induction on i. The second assertion is an immediate consequence of the first.

This completes the proof of Lemma 1.

From now on we shall deal with a fixed invertible linear transformation $A\colon \mathbf{C}^n \to \mathbf{C}^n$, all of whose eigenvalues λ_i are less than 1 in absolute value. We order them so that

$$0 < |\lambda_n| \leq \cdots \leq |\lambda_1| < 1$$

and then choose coordinates in $\mathbf{C}^n$ in such a way that the matrix representation of A is *lower triangular*: If $A = (a_{ij})$ then $a_{ii} = \lambda_i$ and $a_{ij} = 0$ when $i < j$.

In preparation for our next lemma, we let $\mathscr{H}_m$ denote the vector space of all holomorphic maps $H\colon \mathbf{C}^n \to \mathbf{C}^n$, $H = (h_1, \dots, h_n)$, whose components h_i are homogeneous polynomials of degree m.

A convenient basis $\mathscr{B}$ for $\mathscr{H}_m$ consists of those maps H that have only one component different from 0, and that one, say h_j, is a monomial z^α (with $|\alpha| = m$, of course). Among the members of $\mathscr{B}$ we call those *special* in which this h_j has the form

$$h_j(z) = z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}}$$

and the relation

$$\lambda_j = \lambda_1^{\alpha_1} \cdots \lambda_{j-1}^{\alpha_{j-1}}$$

holds.

This notion of "special" depends of course on our operator A; more precisely, it depends on the *spectrum* of A. Note that no such relation can exist when m is so large that $|\lambda_1|^m < |\lambda_n|$; in that case, no member of $\mathscr{B}$ is special. Note also that the special members of $\mathscr{B}$ are lower triangular.

We let X_m be the subspace of $\mathscr{H}_m$ that is spanned by these special basis elements. ($X_m = \{0\}$ when there are none.)

We let Γ_A be the "commutator map" defined by $\Gamma_A(H) = A \circ H - H \circ A$. For each m, Γ_A is thus a linear operator on $\mathscr{H}_m$.

LEMMA 2. *For $m \geq 2$, $\mathscr{H}_m = X_m + \Gamma_A(\mathscr{H}_m)$.*

PROOF. In place of A, we begin with the diagonal matrix D which has $\lambda_1, \lambda_2, \dots, \lambda_n$ down its main diagonal.

If $H = (0, \dots, 0, z^\alpha, 0, \dots, 0)$ is in $\mathscr{B}$, with z^α in the jth spot, then

$$\Gamma_D(H) = DH - HD = (\lambda_j - \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n})H.$$

This shows that Γ_D annihilates precisely those members of $\mathscr{B}$ that are special, and that Γ_D acts as an invertible linear operator on the space Y_m that is spanned by the other members of $\mathscr{B}$.

(Note that $\lambda_j - \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$ *cannot* be 0 if $\alpha_k > 0$ for some $k \geq j$, because $|\alpha| = m \geq 2$, so that $|\lambda_j| > |\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}|$.)

Let π be the projection in $\mathscr{H}_m$ whose range is X_m and whose nullspace is Y_m. The preceding observations can then be summarized by saying *that $\pi + \Gamma_D$ is an invertible linear operator on $\mathscr{H}_m$.*

We now return to our given A. For any $\varepsilon > 0$, let $S = S_\varepsilon$ be the diagonal matrix that has $\varepsilon^n, \varepsilon^{n-1}, \dots, \varepsilon$ down its main diagonal. Since A is lower triangular, so is $S^{-1}AS$; if $i \geq j$ then $\varepsilon^{i-j}a_{ij}$ stands in the ith row and jth column of $S^{-1}AS$. Thus $S^{-1}AS$ converges to D as $\varepsilon \to 0$. The invertible operators form an open set in the algebra of all linear operators on $\mathscr{H}_m$. We conclude from this that *there is an $\varepsilon > 0$, so small that $\pi + \Gamma_{S^{-1}AS}$ is invertible on $\mathscr{H}_m$.*

In other words, to each $G \in \mathscr{H}_m$ corresponds some $H_0 \in X_m$ and some $H \in \mathscr{H}_m$ so that

$$S^{-1}GS = H_0 + (S^{-1}AS)H - H(S^{-1}AS)$$

or

$$G = SH_0S^{-1} + A(SHS^{-1}) - (SHS^{-1})A.$$

The fact that S is diagonal shows that SHS^{-1} is a scalar multiple of H, for every $H \in \mathscr{B}$. Since $H_0 \in X_m$, it follows that $SH_0S^{-1} \in X_m$. Thus $G \in X_m + \Gamma_A(\mathscr{H}_m)$. This completes the proof of Lemma 2.

LEMMA 3. *Suppose that V is a neighborhood of* 0 *in* $\mathbf{C}^n$, *that* $F\colon V \to \mathbf{C}^n$ *is holomorphic,* $F(0) = 0$, *and that all eigenvalues* λ_i *of* $A = F'(0)$ *satisfy* $0 < |\lambda_i| < 1$.

Then there exist

(i) *a lower triangular polynomial automorphism G of* $\mathbf{C}^n$, *with* $G(0) = 0$, $G'(0) = A$, *and*

(ii) *polynomial maps* $T_m\colon \mathbf{C}^n \to \mathbf{C}^n$, *with* $T_m(0) = 0$, $T'_m(0) = I$, *so that*

$$G^{-1} \circ T_m \circ F - T_m = O(|z|^m) \qquad (m = 2, 3, 4, \dots). \tag{1}$$

In other words, the conclusion is that the power series expansion of the left side of (1), about the origin of $\mathbf{C}^n$, contains no terms of degree less than m.

PROOF. We choose coordinates, as before, so that A is lower triangular and $|\lambda_1| \ge \cdots \ge |\lambda_n|$.

Suppose that the following induction hypothesis holds for some $m \ge 2$: T_m is as in (ii), G_m is a lower triangular polynomial automorphism of $\mathbf{C}^n$ with $G'_m(0) = A$, and

$$T_m \circ F - G_m \circ T_m = O(|z|^m). \tag{2_m}$$

Note that this is true when $m = 2$, with $G_2 = A$, $T_2 = I$.

Now (2_m) can be rewritten in the form

$$T_m \circ F - G_m \circ T_m - P_m = O(|z|^{m+1}) \tag{3_m}$$

for some $P_m \in \mathscr{H}_m$. Lemma 2 allows us to decompose P_m:

$$P_m = Q + A \circ H - H \circ A \tag{4}$$

for some $Q \in X_m$, $H \in \mathscr{H}_m$. Define

$$G_{m+1} = G_m + Q, \qquad T_{m+1} = T_m + H \circ T_m. \tag{5}$$

We have to prove that (2_{m+1}) holds.

Let the symbol $\sim$ indicate that the difference between the two terms on either side of it is $O(|z|^{m+1})$.

Then $Q \circ T_{m+1} \sim Q$, $T_{m+1} - T_m \sim H$, and the difference Δ between the left sides of (2_{m+1}) and (3_m) satisfies therefore

$$\begin{aligned}\Delta &= (H \circ T_m \circ F) + (G_m \circ T_m) - (G_m \circ T_{m+1}) - (Q \circ T_{m+1}) + P_m \\ &\sim (H \circ A) + (G_m \circ T_m) - (G_m \circ (T_m + H)) + (A \circ H) - (H \circ A)\end{aligned}$$

so that

$$-\Delta \sim G_m \circ (T_m + H) - G_m \circ T_m - G'_m(0)H,$$

or, equivalently

$$-\Delta(z) \sim \int_0^1 \{G'_m[T_m(z) + tH(z)] - G'_m(0)\}H(z)\,dt.$$

Observe now that $H(z) = O(|z|^m)$, $T_m(z) = O(|z|)$, and that the norm of the linear operator in $\{\cdots\}$ is therefore $O(|z|)$. This shows that $\Delta(z) = O(|z|^{m+1})$ and proves (2_{m+1}).

As soon as m is large enough, $X_m = \{0\}$, hence $G_{m+1} = G_m$. This gives G, as in (i) satisfying

$$T_m \circ F - G \circ T_m = O(|z|^m) \tag{6}$$

for all $m \geq 2$. (Note that anything that is $O(|z|^m)$ is also $O(|z|^{m-1})$, etc.) Finally we apply G^{-1} to (6) to obtain (1).

We are now ready for the main result:

THEOREM. *Suppose that $F \in \mathrm{Aut}(\mathbf{C}^n)$, $F(0) = 0$, and all eigenvalues λ_i of $F'(0)$ satisfy $|\lambda_i| < 1$.*

Then there exists a biholomorphic map Φ from $\mathbf{C}^n$ onto the region

$$\Omega = \left\{ z \in \mathbf{C}^n \colon \lim_{k \to \infty} F^k(z) = 0 \right\}.$$

Moreover, Φ can be chosen so that $J\Phi \equiv 1$ if JF is constant.

PROOF. As before, we choose coordinates so that $A = F'(0)$ is lower diagonal, and $|\lambda_1| \geq \cdots \geq |\lambda_n|$. We can then find a diagonal operator S, as in the proof of Lemma 2, which makes $A_0 = S^{-1}AS$ so close to being diagonal that $|A_0 z| \leq c|z|$ holds for some $c < 1$ and all $z \in \mathbf{C}^n$. (This uses the assumption $|\lambda_1| < 1$.) If we put $F_0 = S^{-1}FS$ and prove the theorem for F_0, obtaining Φ_0 and Ω_0, then it holds also for F, with $\Phi = S\Phi_0 S^{-1}$ and $\Omega = S(\Omega_0)$.

So we may assume, in addition to the stated hypotheses, that $||A|| < 1$.

Fix α, $||A|| < \alpha < 1$. Then there exists $r > 0$ so that

$$|F(z)| \leq \alpha|z| \quad \text{if } |z| \leq r. \tag{1}$$

It follows, as in the proof of Theorem 9.1, that $rB \subset \Omega$, that Ω is a region, and that $F(\Omega) = \Omega$.

Next, we associate G to F as in Lemma 3, and apply Lemma 1(a) to G^{-1} in place of G to conclude (with the aid of the Schwarz lemma) that there is a constant $\gamma < \infty$ so that

$$|G^{-k}(w) - G^{-k}(w')| \leq \gamma^k |w - w'| \qquad (k = 1, 2, 3, \ldots) \tag{2}$$

for all $w, w' \in \mathbf{C}^n$ with $|w| \leq 1/2$, $|w'| \leq 1/2$.

Fix a positive integer m so that $\alpha^m < 1/\gamma$.

Lemma 3 gives us a polynomial map $T = T_m$, with $T(0) = 0$, $T'(0) = I$, and it gives us constants $\delta > 0$, $C_1 < \infty$, so that $|w| \leq \delta$ implies

$$|G^{-1}TF(w) - T(w)| \leq C_1 |w|^m. \tag{3}$$

Now let $E \subset \Omega$ be compact. Then $F^s(E) \subset rB$ for some integer s. Hence $F^{s+k}(E) \subset F^k(rB) \subset \alpha^k rB$, for all $k \geq 0$, by (1). It follows that there is a constant $C_2 < \infty$ so that

$$|F^k(z)| \leq C_2 \alpha^k < \delta \tag{4}$$

for all $z \in E$ and all $k \geq k_0$. For such z and k, (3) and (4) show that

$$|G^{-1}TF^{k+1}(z) - TF^k(z)| \leq C_1 |F^k(z)|^m \leq C_1 C_2^m \alpha^{mk}. \tag{5}$$

For large k, $|G^{-1}TF^{k+1}(z)|$ and $|TF^k(z)|$ are $< 1/2$, for all $z \in E$. Hence (2) can be applied to (5), and we conclude that for $k \geq k_1$ and for all $z \in E$,

$$|G^{-k-1}TF^{k+1}(z) - G^{-k}TF^k(z)| \leq C_1 C_2^m (\gamma\alpha^m)^k. \tag{6}$$

Since $\gamma\alpha^m < 1$, we have proved:

The limit

$$\Psi(z) = \lim_{k\to\infty} (G^{-k} \circ T \circ F^k)(z) \tag{7}$$

exists, uniformly on every compact subset of Ω, and defines a holomorphic map $\Psi\colon \Omega \to \mathbf{C}^n$ which satisfies $\Psi(0) = 0$, $\Psi'(0) = I$, as well as the functional equation

$$G^{-1} \circ \Psi \circ F = \Psi. \tag{8}$$

Since $F(\Omega) = \Omega$, (8) shows that Ψ has the same range as $G^{-1} \circ \Psi$. Thus

$$\Psi(\Omega) = G^{-1}(\Psi(\Omega)) = \cdots = G^{-k}(\Psi(\Omega)) = \cdots \tag{9}$$

and since $\Psi(\Omega)$ contains a neighborhood of 0, Lemma 1(b) shows that $\Psi(\Omega) = \mathbf{C}^n$.

Assume next that $x, y \in \Omega$ and $\Psi(x) = \Psi(y)$. By (8), $\Psi \circ F = G \circ \Psi$. Hence $\Psi(F(x)) = \Psi(F(y))$. Continuing, we see that $\Psi(F^k(x)) = \Psi(F^k(y))$ for all positive k. But when k is sufficiently large, both $F^k(x)$ and $F^k(y)$ are in a neighborhood of 0 in which Ψ is one-to-one. Thus $F^k(x) = F^k(y)$, and this implies $x = y$. So Ψ is one-to-one in Ω.

We have now proved that Ψ is a biholomorphic map from Ω onto $\mathbf{C}^n$.

The first conclusion of the theorem is therefore satisfied by $\Phi = \Psi^{-1}$.

Finally, assume that JF is constant. Since G is a *polynomial automorphism* of $\mathbf{C}^n$, the polynomial JG has no zero in $\mathbf{C}^n$, hence is also constant. In fact, $JG = JF$ because $G'(0) = F'(0)$. If we apply the chain rule to $\Psi \circ F = G \circ \Psi$, we obtain, for $z \in \Omega$,

$$(J\Psi)(F(z))(JF)(z) = (JG)(\Psi(z))(J\Psi)(z). \tag{10}$$

Hence

$$(J\Psi)(z) = (J\Psi)(F(z)) = \cdots = (J\Psi)(F^k(z)) = \cdots. \tag{11}$$

Since $F^k(z) \to 0$ as $k \to \infty$ we conclude that

$$(J\Psi)(z) = (J\Psi)(0) = 1 \tag{12}$$

for all $z \in \Omega$. Hence $J\Phi \equiv 1$ on $\mathbf{C}^n$.

REMARK. In the generic case, the eigenvalues of $F'(0)$ satisfy none of the relations that give rise to the "special" basis elements of $\mathscr{H}_m$. In that case, $X_m = \{0\}$ for all m, the proof of Lemma 3 gives $G = A$, and the functional equation (8) can be written in the form

$$\Psi \circ F \circ \Psi^{-1} = A. \tag{13}$$

One refers to this as "linearizing" the map F, by a biholomorphic change of variables.

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